Section 2.8 - Differentials

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In this section we will define the sheaf of relative differential forms of one scheme over another. In the case of a nonsingular variety over $\mathbb{C}$, which is like a complex manifold, the sheaf of differential forms is essentially the same as the dual of the tangent bundle in differential geometry. However, in abstract algebraic geometry, we will define the sheaf of differentials first, by a purely algebraic method, and then define the tangent bundle as its dual. Hence we will begin this section with a review of the module of differentials of one ring over another. As applications of the sheaf of differentials, we will give a characterisation of nonsingular varieties among schemes of finite type over a field. We will also use the sheaf of differentials on a nonsingular variety to define its tangent sheaf, its canonical sheaf, and its geometric genus. This latter is an important numerical invariant of a variety.

Contents

1 Kähler Differentials

See (MAT2,Section 2) for the definition and basic properties of Kähler differentials.

Definition 1. Let B be a local ring with maximal ideal $\mathfrak{m}$. A *field of representatives* for B is a subfield L of A which is mapped onto $A/\mathfrak{m}$ by the canonical mapping $A \longrightarrow A/\mathfrak{m}$. Since L is a field, the restriction gives an isomorphism of fields $L \cong A/\mathfrak{m}$.

Proposition 1. *Let B be a local ring with field of representatives k. Then the canonical morphism of k-modules $\delta : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \otimes_B k$ is an isomorphism.*

Proof. The map δ is defined by $x + \mathfrak{m}^2 \longrightarrow d_{B/k}(a) \otimes 1$ (see (MAT2,Theorem 17)). Since $\Omega_{k/k} = 0$ it follows from (MAT2,Theorem 17) that δ is surjective. To show that δ is an isomorphism it suffices by (MAT2,Theorem 17)(ii) to show that the morphism of k-algebras $B/\mathfrak{m}^2 \longrightarrow B/\mathfrak{m}$ has a right inverse. Since every coset of $B/\mathfrak{m}$ contains a unique element of k, this is easily checked. $\square$

Lemma 2. *Let A be a noetherian local domain with residue field k and quotient field K. If M is a finitely generated A-module with $rank_k(M \otimes_A k) = rank_K(M \otimes_A K) = r$ then M is a free A-module of rank r.*

Proof. Since $rank_k(M \otimes_A k) = r$, Nakayama's Lemma tells us that M can be generated by r elements. So there is a surjective map $\varphi : A^r \longrightarrow M$. Let R be its kernel. Then we obtain an exact sequence

$$0 \longrightarrow R \otimes_A K \longrightarrow K^r \longrightarrow M \otimes_A K \longrightarrow 0$$

and since $rank_k(M \otimes_A K) = r$, we have $R \otimes_A K = 0$. But R is torsion-free, so $R = 0$ and therefore M is isomorphic to A^r. □

Lemma 3. *Let k be a field, A a finitely generated k-algebra and suppose that $B = A_{\mathfrak{p}}$ is a domain for some prime ideal $\mathfrak{p}$ of A. Then B is isomorphic as a k-algebra to the localisation of an affine k-algebra at a prime ideal.*

Proof. Recall that an *affine* k-algebra is a finitely generated k-algebra which is a domain. Let A be a finitely generated k-algebra, $\mathfrak{p}$ a prime ideal of A and suppose that $B = A_{\mathfrak{p}}$ is a domain. Let $\mathfrak{q}$ be the kernel of the ring morphism $A \longrightarrow A_{\mathfrak{p}}$. Then $\mathfrak{q}$ is a prime ideal contained in $\mathfrak{p}$, and it is not hard to check that $B \cong (A/\mathfrak{q})_{\mathfrak{p}}$ as k-algebras. □

Theorem 4. *Let B be a local ring with field of representatives k. Assume that k is perfect and that B is isomorphic as a k-algebra to the localisation of a finitely generated k-algebra at a prime ideal. Then B is a regular local ring if and only if $\Omega_{B/k}$ is a free B-module of rank $dimB$.*

Proof. By hypothesis B is noetherian. Suppose that $\Omega_{B/k}$ is a free B-module of rank $dimB$. Then by Proposition 1, $rank_k \mathfrak{m}/\mathfrak{m}^2 = dimB$, so B is a regular local ring. In particular this implies that B is a normal domain (MAT,Theorem 108).

For the converse, suppose that B is a regular local ring of dimension r. Then $rank_k \mathfrak{m}/\mathfrak{m}^2 = r$ so by Proposition 1 we have $rank_k(\Omega_{B/k} \otimes_B k) = r$. On the other hand, let K be the quotient field of B. Then there is an isomorphism of K-modules $\Omega_{B/k} \otimes_B K \cong \Omega_{K/k}$ (MAT2,Corollary 15). Since k is perfect, K is a separably generated extension field of k (H,I.4.8A), so $rank_K \Omega_{K/k} = tr.deg.K/k$ (MAT2,Corollary 20). We claim that $dimB = tr.deg.K/k$. By Lemma 3 we can find an affine k-algebra A and a prime ideal $\mathfrak{p}$ such that $B \cong A_{\mathfrak{p}}$ as k-algebras. Since k is a field of representatives of B we have $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong k$ as k-algebras. Using (H, I.1.8A) and the fact that $A/\mathfrak{p}$ is an affine k-algebra with quotient field k-isomorphic to $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, we have

$$coht.\mathfrak{p} = dim(A/\mathfrak{p}) = tr.deg.(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})/k = tr.deg.k/k = 0$$

Applying (H, I.1.8A) to A and using the fact that the quotient field of A is k-isomorphic to K, we have $dimB = dimA = tr.deg.K/k$ as claimed. It now follows from (MAT2,Corollary 16) and Lemma 2 that $\Omega_{B/k}$ is a free B-module of rank $dimB$. □

The next result says intuitively that in a regular local ring (thought of as the local ring of a nonsingular variety) regular systems of parameters define bases for the module of differentials.

Corollary 5. *Let B be a regular local ring satisfying the hypothesis of Theorem 4 with $n = dimB \geq 1$. If $x_1, \ldots, x_n$ is a regular system of parameters then $\Omega_{B/k}$ is a free B-module on the basis $\{dx_1, \ldots, dx_n\}$.*

Proof. Let $x_1, \ldots, x_n$ be a regular system of parameters. That is, the elements $x_i + \mathfrak{m}^2$ are a k-basis for $\mathfrak{m}/\mathfrak{m}^2$ (notation of the proof of Theorem 4). It follows from Proposition 1 that the elements $d_{B/k}(x_i) \otimes 1$ are a k-basis for $\Omega_{B/k} \otimes_B k$ and therefore by Nakayama the elements $d_{B/k}(x_i)$ generate $\Omega_{B/k}$ as a B-module. The proof of Lemma 2 shows that this is a basis. □

2 Sheaves of Differentials

Lemma 6. *If $f : X \longrightarrow Y$ is a morphism of schemes then the diagonal $\Delta : X \longrightarrow X \times_Y X$ is an emmersion. In particular Δ gives a homeomorphism of X with the locally closed subset $\Delta(X)$ of $X \times_Y X$.*

Proof. See (SI,Definition 6) for the definition of an *emmersion*. Let $p_1, p_2 : X \times_Y X \longrightarrow X$ be the canonical projections. Since $p_1 \circ \Delta = 1$ it is clear that Δ gives a continuous bijection between X and $\Delta(X)$. If $U \subseteq X$ is open then $p_1^{-1}U \cap p_2^{-1}U$ is a pullback $U \times_Y U$ and since $\Delta^{-1}(p_1^{-1}U \cap p_2^{-1}U) = U$ we have a pullback diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & U \times_Y U \arrow[d] \\
X \arrow[r, "\Delta"'] & X \times_Y X
\end{tikzcd}$$

Using (SPM,Proposition 11) we see that $\Delta(U) = \Delta(X) \cap (U \times_Y U)$ is an open subset of $\Delta(X)$, which shows that Δ induces a homeomorphism $X \longrightarrow \Delta(X)$.

To see that $\Delta(X)$ is locally closed, let $x \in X$ be given and find an affine open neighborhood S of $f(x)$ in Y. Let U be an affine open neighborhood of x contained in $f^{-1}S$, and observe that $p_1^{-1}U \cap p_2^{-1}U$ is a pullback $U \times_S U$ and the induced morphism $U \longrightarrow U \times_S U$ is the diagonal. Since U, S are affine this is a closed immersion, so $\Delta(U)$ is closed in the open susbet $U \times_S U$ of $X \times_Y X$. This shows that $\Delta(X)$ is a locally closed subset of $X \times_Y X$.

Since the composite $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_Y X, \Delta(x)} \longrightarrow \mathcal{O}_{X,x}$ is the identity it is clear that the local maps $\mathcal{O}_{X \times_Y X, \Delta(x)} \longrightarrow \mathcal{O}_{X,x}$ are all surjective, so by (SI,Proposition 12) the morphism Δ is an emmersion. $\square$

Definition 2. Let $f : X \longrightarrow Y$ be a morphism of schemes, $\Delta : X \longrightarrow X \times_Y X$ the diagonal and W_f the largest open subset of $X \times_Y X$ containing $\Delta(X)$ as a closed subset (SI,Definition 4). Then by (SI,Remark 2) we have a canonical factorisation of Δ as a closed immersion followed by an open immersion

$$\begin{tikzcd}
& W_f \arrow[dr] & \\
X \arrow[ur, "g"] \arrow[rr, "\Delta"'] & & X \times_Y X
\end{tikzcd}$$

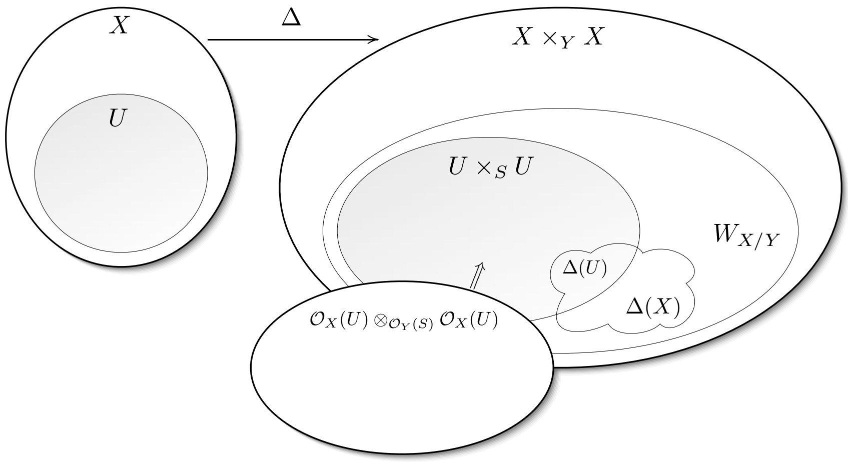
Let $\mathscr{I}_f$ be the quasi-coherent sheaf of ideals on W_f corresponding to the closed immersion g. Then we define the *sheaf of relative differentials* of f to be $\Omega_f = g^*(\mathscr{I}_f/\mathscr{I}_f^2)$. This is a quasi-coherent sheaf of $\mathcal{O}_X$-modules (H,5.8), (MOS,Corollary 12), (MOS,Definition 1). We write $\Omega_{X/Y}$ for Ω_f, $\mathscr{I}_{X/Y}$ for $\mathscr{I}_f$ and $W_{X/Y}$ for W_f if no confusion seems likely. If Y is noetherian and f is a morphism of finite type then $X \times_Y X$ is also noetherian, so $\Omega_{X/Y}$ is coherent.

Proposition 7. *Let $f : X \longrightarrow Y$ be a morphism of schemes. For affine open $S \subseteq Y$ and $U \subseteq f^{-1}S$ we have $U \times_S U \subseteq W_{X/Y}$ and the canonical ring isomorphism*

$$\mathcal{O}_{W_{X/Y}}(U \times_S U) \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(S)} \mathcal{O}_X(U) \tag{1}$$

identifies $\mathscr{I}_{X/Y}(U \times_S U)$ with the kernel of the product morphism. In fact if $\psi : U \times_S U \longrightarrow Spec(\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(S)} \mathcal{O}_X(U))$ is the canonical isomorphism we have a canonical isomorphism of sheaves of modules

$$\theta_{X/Y,U/S} : (\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(S)})^{\sim} \longrightarrow \psi_*(\mathscr{I}_{X/Y}/\mathscr{I}_{X/Y}^2)|_{U \times_S U}$$



Proof. By $U \times_S U$ we mean the open subset $p_1^{-1}U \cap p_2^{-1}U$ of $X \times_Y X$, which together with the canonical morphisms is a pullback. To show that $U \times_S U \subseteq W_{X/Y}$ it is enough by (SI,Lemma 10) to show that $\Lambda(U) = \Lambda(X) \cap (U \times_S U)$ is closed in $U \times_S U$. But this follows from the fact that morphisms of affine schemes are separated.

Set $\mathscr{I} = \mathscr{I}_{X/Y}$. Since $\mathscr{I}|_{U \times_S U}$ is the ideal sheaf of the closed immersion $U \longrightarrow U \times_S U$ the scheme isomorphism $U \times_S U \cong Spec(\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(S)} \mathcal{O}_X(U))$ identifies $\mathscr{I}|_{U \times_S U}$ with the ideal sheaf $I^{\sim}$ where I is the kernel of the product morphism $\varepsilon : \mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(S)} \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)$. It is now clear that the ring isomorphism (1) identifies $\mathscr{I}(U \times_S U)$ with I and that we have an isomorphism of sheaves of modules

$$\begin{aligned}\theta_{X/Y,U/S} : (I/I^2)^{\sim} &\cong I^{\sim}/(I^2)^{\sim} \cong \psi_*(\mathscr{I}|_{U \times_S U})/\psi_*(\mathscr{I}|^2_{U \times_S U})\\ &\cong \psi_*(\mathscr{I}|_{U \times_S U}/\mathscr{I}|^2_{U \times_S U}) \cong \psi_*(\mathscr{I}/\mathscr{I}^2)|_{U \times_S U}\end{aligned}$$

which completes the proof. □

Remark 1. With the notation of Proposition 7 if $T \subseteq S$ and $Q \subseteq U \cap f^{-1}T$ are affine open sets then $Q \times_T Q \subseteq U \times_S U$. Put $k = \mathcal{O}_Y(S), k' = \mathcal{O}_Y(T), A = \mathcal{O}_X(U), A' = \mathcal{O}_X(Q)$ and $B = A \otimes_k A, B' = A' \otimes_{k'} A'$. It is not hard to check that the following diagram commutes, where the bottom row is induced by the ring morphism $a \otimes b \mapsto a|_Q \otimes b|_Q$

$$\begin{tikzcd} Q \times_T Q \arrow[r] \arrow[d, Rightarrow, "\psi_Q"'] & U \times_S U \arrow[d, Rightarrow, "\psi_U"] \\ Spec(B') \arrow[r, "\rho_{Q,U}"'] & Spec(B) \end{tikzcd} \tag{2}$$

The restriction map $A \longrightarrow A'$ induces a canonical morphism of A-modules

$$\begin{gathered}(-)|_Q : \Omega_{A/k} \longrightarrow \Omega_{A'/k'}\\ (a \otimes b + I^2) \mapsto a|_Q \otimes b|_Q + I'^2\end{gathered} \tag{3}$$

We claim that the following canonical morphism of B'-modules is an isomorphism

$$\begin{gathered}\mu_{X/Y,U/S,Q/T} : \Omega_{A/k} \otimes_B B' \longrightarrow \Omega_{A'/k'}\\ (a \otimes b + I^2) \otimes (c \otimes e) \mapsto ca|_Q \otimes eb|_Q + I'^2\end{gathered} \tag{4}$$

To show this, we first observe that by Proposition 7 we have canonical isomorphisms of sheaves of modules

$$\begin{aligned}\theta_{X/Y,U/S} &: (\Omega_{A/k})^{\sim} \longrightarrow (\psi_U)_*(\mathscr{I}/\mathscr{I}^2)|_{U \times_S U}\\ \theta_{X/Y,Q/T} &: (\Omega_{A'/k'})^{\sim} \longrightarrow (\psi_Q)_*(\mathscr{I}/\mathscr{I}^2)|_{Q \times_T Q}\end{aligned}$$

Using (MRS,Proposition 108),(MRS,Proposition 110) we have an isomorphism of sheaves of modules

$$\begin{aligned}(\Omega_{A/k} \otimes_B B')^\sim &\cong (\rho_{Q,U})^*(\Omega_{A/k})^\sim \\ &\cong (\rho_{Q,U})^*(\psi_U)_*(\mathscr{I}/\mathscr{I}^2)|_{U\times_S U} \\ &\cong (\psi_Q)_*(\mathscr{I}/\mathscr{I}^2)|_{Q\times_T Q} \\ &\cong (\Omega_{A'/k'})^\sim\end{aligned}$$

one checks that the corresponding morphism of modules is (4), which is therefore an isomorphism. By construction we have a commutative diagram of sheaves of modules on $Spec(B')$

$$\begin{tikzcd}
(\rho_{Q,U})^*(\Omega_{A/k})^\sim \arrow[r, Rightarrow, "\rho^*_{Q,U}(\theta_{X/Y,U/S})"] \arrow[d, Rightarrow] & (\rho_{Q,U})^*(\psi_U)_*(\mathscr{I}/\mathscr{I}^2)|_{U\times_S U} \arrow[dd, Rightarrow] \\
(\Omega_{A/k} \otimes_B B')^\sim \arrow[d, Rightarrow, "(\mu_{X/Y,U/S,Q/T})^\sim"'] & \\
(\Omega_{A'/k'})^\sim \arrow[r, Rightarrow, "\theta_{X/Y,Q/T}"'] & (\psi_Q)_*(\mathscr{I}/\mathscr{I}^2)|_{Q\times_T Q}
\end{tikzcd} \tag{5}$$

which expresses the naturality of θ in the affine open set U.

Let $L \subseteq U\times_S U$ be open and let $L' = \psi_U(L)$ be the corresponding open subset of $Spec(B)$. Suppose we have $a \in \Omega_{A/k}$ and $s \in B$ with $L' \subseteq D(s)$. Then $a\dot{/}s$ defines an element of $\Gamma(L', (\Omega_{A/k})^\sim)$ which we can identify with a section of $\Gamma(L, \mathscr{I}/\mathscr{I}^2)$ via the isomorphism $\theta_{X/Y,U/S}$. Commutativity of (5) says that this identification behaves under restriction. That is, if $L \subseteq Q \times_T Q$ then $a\dot{/}s$ and $a|_Q\dot{/}s|_Q$ denote the same section of $\mathscr{I}/\mathscr{I}^2$ (using ψ_Q to identify the section $a|_Q\dot{/}s|_Q$ of $\Omega_{A'/k'}$ with a section of $\mathscr{I}/\mathscr{I}^2$).

Lemma 8. *Let k be a ring, A a k-algebra and I the kernel of the product morphism $A\otimes_k A \longrightarrow A$. Then there is a canonical isomorphism of A-modules $I/I^2 \otimes_{A\otimes_k A} A \cong \Omega_{A/k}$.*

Proof. By definition $\Omega_{A/k}$ is I/I^2 with the A-module structure induced by the ring morphism $A \longrightarrow A \otimes_k A$ with $a \mapsto a \otimes 1$. It is not hard to see that the map $\Omega_{A/k} \longrightarrow I/I^2 \otimes A$ defined by $t \mapsto t \otimes 1$ is an isomorphism of A-modules. □

Proposition 9. *Let $f : X \longrightarrow Y$ be a morphism of schemes, $S \subseteq Y$ and $U \subseteq f^{-1}S$ affine open subsets. If $\varphi : U \longrightarrow Spec\mathcal{O}_X(U)$ is the canonical isomorphism then there is a canonical isomorphism of sheaves of modules*

$$\vartheta_{X/Y,U/S} : (\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(S)})^\sim \longrightarrow \varphi_*(\Omega_{X/Y}|_U)$$

In particular there is an isomorphism $\Omega_{X/Y}(U) \cong \Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(S)}$ of $\mathcal{O}_X(U)$-modules.

Proof. Set $k = \mathcal{O}_Y(S), A = \mathcal{O}_X(U), B = A\otimes_k A$ and observe that we have a commutative diagram with the right hand square a pullback

$$\begin{tikzcd}
SpecA \arrow[d, "\Delta"'] & U \arrow[l, Rightarrow, "\varphi"'] \arrow[r] \arrow[d, "\Delta"] & X \arrow[d, "g"] \\
Spec(A \otimes_k A) & U \times_S U \arrow[l, Rightarrow, "\psi_U"] \arrow[r] & W_{X/Y}
\end{tikzcd}$$

Using (MRS,Proposition 111), (MRS,Proposition 108), the isomorphism $\theta_{X/Y,U/S}$ of Proposition

7 and Lemma 8 we have an isomorphism of sheaves of modules

$$\begin{aligned}\varphi_*(\Omega_{X/Y}|_U) &= \varphi_* g^*(\mathscr{I}_{X/Y}/\mathscr{I}^2_{X/Y})|_U \\ &\cong \varphi_*(\Delta^*(\mathscr{J}_{X/Y}/\mathscr{J}^2_{X/Y})|_{U\times_S U}) \\ &\cong \Delta^*(\psi_U)_*(\mathscr{J}_{X/Y}/\mathscr{J}^2_{X/Y})|_{U\times_S U} \\ &\cong \Delta^*(\Omega_{A/k})^\sim \\ &\cong (\Omega_{A/k}\otimes_B A)^\sim \\ &\cong (\Omega_{A/k})^\sim\end{aligned}$$

as required. □

Remark 2. With the notation of Proposition 9 let $T \subseteq S$ and $Q \subseteq U \cap f^{-1}T$ be affine open subsets and put $k' = \mathcal{O}_Y(T), A' = \mathcal{O}_X(Q)$. There is a canonical morphism of A'-modules $\Omega_{A/k}\otimes_A A' \longrightarrow \Omega_{A'/k'}$ defined by $d(a)\otimes c \mapsto c\cdot d(a|_Q)$. Let $\tau_{Q,U} : SpecA \longrightarrow SpecA'$ be the canonical morphism and note that we have a canonical isomorphism $(\tau_{Q,U})^*(\varphi_U)_*\Omega_{X/Y}|_U \cong (\varphi_Q)_*\Omega_{X/Y}|_Q$ (MRS,Proposition 108), (MRS,Proposition 110) which we claim makes the following diagram commute

$$\begin{tikzcd} (\tau_{Q,U})^*(\Omega_{A/k})^\sim \arrow[r, Rightarrow, "\tau^*_{Q,U}(\vartheta_{X/Y,U/S})"] \arrow[d, Rightarrow] & (\tau_{Q,U})^*(\varphi_U)_*\Omega_{X/Y}|_U \arrow[dd, Rightarrow] \\ (\Omega_{A/k}\otimes_A A')^\sim \arrow[d] & \\ (\Omega_{A'/k'})^\sim \arrow[r, Rightarrow, "\vartheta_{X/Y,Q/T}"'] & (\varphi_Q)_*\Omega_{X/Y}|_Q \end{tikzcd}$$

one checks this by reducing to special sections and using Remark 1.

Suppose we are given an open subset $L \subseteq U$ and let $L' = \varphi_U(L)$ be the corresponding open subset of $SpecA$. Suppose we have $a \in \Omega_{A/k}$ and $s \in \mathcal{O}_X(U)$ with $L' \subseteq D(s)$. Then $a\dot{/}s$ defines an element of $\Gamma(L', (\Omega_{A/k})^\sim)$ which we can identify with a section of $\Gamma(L, \Omega_{X/Y})$ via the isomorphism $\vartheta_{X/Y,U/S}$. We have $a\dot{/}s = [\psi^{-1}D(s\otimes 1), a\dot{/}(s\otimes 1)]\dot{\otimes}1$ where we identify $U\times_S U$ with $Spec(A\otimes_k A)$ as in Remark 1.

The identification of the previous paragraph commutes with restriction in the following sense: if $L \subseteq Q$ then $a|_Q \in \Omega_{A'/k'}$ together with $s|_Q \in \mathcal{O}_X(Q)$ determine a section of $(\Omega_{A'/k'})^\sim$ over $\varphi_Q(L) \subseteq SpecA'$. The corresponding section $a|_Q\dot{/}s|_Q \in \Gamma(L, \Omega_{X/Y})$ (using $\vartheta_{X/Y,Q/T}$) is equal to the section denoted $a\dot{/}s$ in the previous paragraph.

Corollary 10. *Let X be a scheme over a ring k. Then for $x \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules $(\Omega_{X/k})_x \cong \Omega_{\mathcal{O}_{X,x}/k}$.*

Proof. Set $Y = Speck$, let U be an affine open neighborhood of x, set $A = \mathcal{O}_X(U)$ and let $\varphi : U \longrightarrow Spec\mathcal{O}_X(U)$ be the canonical isomorphism. Let $\mathfrak{p} = \varphi(x)$ and use $\vartheta_{X/Y,U/Y}$ together with (MAT2,Corollary 15) to obtain an isomorphism of $\mathcal{O}_{X,x}$-modules

$$\begin{aligned}(\Omega_{X/Y})_x &\cong (\Omega_{X/Y}|_U)_x \\ &\cong \varphi_*(\Omega_{X/Y}|_U)_\mathfrak{p} \\ &\cong (\Omega_{A/k})^\sim{}_\mathfrak{p} \\ &\cong (\Omega_{A/k})_\mathfrak{p} \\ &\cong \Omega_{A_\mathfrak{p}/k} \\ &\cong \Omega_{\mathcal{O}_{X,x}/k}\end{aligned}$$

If $a, s \in \mathcal{O}_X(U)$ determines a section $a\dot{/}s \in \Gamma(Q', (\Omega_{A/k})^\sim)$ for some open neighborhood $Q' = \varphi(Q)$ of $\mathfrak{p}$ then this map sends $d_{A/k}(a)\dot{/}s$ to $(Q, s^{-1})\cdot d_{\mathcal{O}_{X,x}/k}(U, a)$. It is straightforward to check that this isomorphism is independent of the chosen affine open neighborhood U. □

Lemma 11. *Let $\varphi : A \longrightarrow B$ be a morphism of rings and $\Phi : SpecB \longrightarrow SpecA$ the induced morphism of schemes. Then there is a canonical isomorphism $(\Omega_{B/A})^{\sim} \cong \Omega_{SpecB/SpecA}$ of sheaves of modules on $SpecB$.*

Proof. We can assume that the diagonal is the morphism of schemes $\Delta : SpecB \longrightarrow Spec(B \otimes_A B)$ corresponding to the product morphism $\varepsilon : B \otimes_A B \longrightarrow B$. This is a closed immersion with ideal sheaf $I^{\sim}$ where I is the kernel of ε. Therefore there is a canonical isomorphism of Ω_Φ with the associated sheaf of the B-module $I/I^2 \otimes_{B \otimes_A B} B$. So the result is a consequence of Lemma 8. $\square$

Proposition 12. *Suppose we have a commutative diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \tag{6}$$

Then there are canonical morphisms of sheaves of modules

$$\nu : p^*(\Omega_{X/Y}) \longrightarrow \Omega_{X'/Y'}$$
$$[M, a\dot{/}s] \dot{\otimes} b\dot{/}t \mapsto b \cdot u_{U/S,P/V}(a)\dot{/}\phi(s)t$$

$$\upsilon : \Omega_{X/Y} \longrightarrow p_*(\Omega_{X'/Y'})$$
$$a\dot{/}s \mapsto u_{U/S,P/V}(a)\dot{/}\phi(s)$$

Moreover if (6) is a pullback then ν is an isomorphism.

Proof. Let $S \subseteq Y, U \subseteq f^{-1}S, V \subseteq g^{-1}S$ and $P \subseteq q^{-1}V \cap p^{-1}U$ be affine open subsets. Then we have a commutative diagram

$$\begin{tikzcd}
Spec\mathcal{O}_{X'}(P) \arrow[rrrrr, "z"] \arrow[ddddd] & & & & & Spec\mathcal{O}_X(U) \arrow[ddddd] \\
& P \arrow[ul, Rightarrow, "\varphi"'] \arrow[rrr, "r"] \arrow[dr] \arrow[ddd] & & & U \arrow[ur, Rightarrow, "\psi"] \arrow[dl] \arrow[ddd] & \\
& & X' \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] & & \\
& & Y' \arrow[r, "g"'] & Y & & \\
& V \arrow[ur] \arrow[rrr, "h"'] \arrow[dl, Rightarrow] & & & S \arrow[ul] \arrow[dr, Rightarrow] & \\
Spec\mathcal{O}_{Y'}(V) \arrow[rrrrr, "i"'] & & & & & Spec\mathcal{O}_Y(S)
\end{tikzcd}$$

The outside diagram of commutative rings induces the following morphisms (MAT2,Definition 7)

$$u_{U/S,P/V} : \Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(S)} \longrightarrow \Omega_{\mathcal{O}_{X'}(P)/\mathcal{O}_{Y'}(V)}$$
$$v_{U/S,P/V} : \Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(S)} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(P) \longrightarrow \Omega_{\mathcal{O}_{X'}(P)/\mathcal{O}_{Y'}(V)}$$

We have a canonical morphism of sheaves of modules

$$\begin{aligned}
\varphi_* p^* \Omega_{X/Y}|_P &\cong \varphi_* r^* \Omega_{X/Y}|_U \\
&\cong z^* \psi_* \Omega_{X/Y}|_U \\
&\cong z^* (\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(S)})^\sim \\
&\cong (\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(S)} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(P))^\sim \\
&\longrightarrow (\Omega_{\mathcal{O}_{X'}(P)/\mathcal{O}_{Y'}(V)})^\sim \\
&\cong \varphi_* \Omega_{X'/Y'}|_P
\end{aligned}$$

Using (MRS,Proposition 111), (MRS,Proposition 108), Proposition 9, the morphism $(v_{U/S,P/V})^\sim$ and then Proposition 9 once more. So finally we have a morphism of sheaves of modules

$$\begin{gathered}
\nu^{U,V,S,P} : p^*(\Omega_{X/Y})|_P \longrightarrow \Omega_{X'/Y'}|_P \\
[M, a\dot{/}s] \dot{\otimes} b\dot{/}t \mapsto b \cdot u_{U/S,P/V}(a)\dot{/}\phi(s)t
\end{gathered} \tag{7}$$

Where $\phi : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X'}(P)$ is the ring morphism induced by z. We claim that the morphisms $\nu^{U,V,S,P}$ glue together, as S ranges over all affine open subsets of Y and U, V, P over all affine open subsets of $f^{-1}S, g^{-1}S$ and $p^{-1}U \cap q^{-1}V$ respectively. In the usual way (see for example the proof of (TRPC,Proposition 10)) this follows from Remark 2 and the explicit form of (7). Since the affine sets P form an open cover of X' there is a unique morphism of sheaves of modules $\nu : p^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}$ with $\nu|_P = \nu^{U,V,S,P}$ for all U, V, S, P. It follows from (MAT2,Proposition 11) that if (6) is a pullback then ν is an isomorphism. By adjointness there is a morphism of sheaves of modules $v : \Omega_{X/Y} \longrightarrow p_*\Omega_{X'/Y'}$ corresponding to ν. □

Remark 3. Let $m : X \longrightarrow Y$ and $n : Y \longrightarrow Z$ be morphisms of schemes. Using the special case of Proposition 12 where p is the identity, we have a morphism of sheaves of modules on X

$$\begin{gathered}
v : \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \\
a\dot{/}s \mapsto u_{U/S,U/V}(a)\dot{/}s
\end{gathered}$$

Using the special case where g is the identity, we have a morphism of sheaves of modules on X

$$\begin{gathered}
\nu : m^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \\
[M, a\dot{/}s] \dot{\otimes} b\dot{/}t \mapsto b \cdot u_{V/S,U/S}(a)\dot{/}\phi(s)t
\end{gathered}$$

where $\phi : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(U)$ is the canonical ring morphism.

Lemma 13. *Let X be a scheme over a ring k. If $U \subseteq X$ is an open subset then there is a canonical isomorphism of sheaves of modules on U*

$$\begin{gathered}
\lambda : \Omega_{U/k} \longrightarrow (\Omega_{X/k})|_U \\
a\dot{/}s \mapsto a\dot{/}s
\end{gathered}$$

Proof. Let $j : U \longrightarrow X$ be the inclusion. The desired morphism is the composite of $\nu : j^*\Omega_{X/k} \longrightarrow \Omega_{U/k}$ with the canonical isomorphism $j^*\Omega_{X/k} \cong (\Omega_{X/k})|_U$. It follows from Corollary 10 that this is an isomorphism on stalks, and therefore an isomorphism. □

Proposition 14. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of schemes. Then there is an exact sequence of sheaves of modules on X*

$$f^*\Omega_{Y/Z} \xrightarrow{\nu} \Omega_{X/Z} \xrightarrow{v} \Omega_{X/Y} \longrightarrow 0 \tag{8}$$

Proof. To show that this sequence is exact, it suffices to show that for affine open $S \subseteq Z, V \subseteq g^{-1}S$ and $U \subseteq f^{-1}V$ the following sequence is exact (MRS,Lemma 38)

$$f^*\Omega_{Y/Z}|_U \longrightarrow \Omega_{X/Z}|_U \longrightarrow \Omega_{X/Y}|_U \longrightarrow 0 \tag{9}$$

Let $\varphi : U \longrightarrow Spec\mathcal{O}_X(U)$ be the canonical isomorphism. Then exactness of (9) follows from (MAT2,Theorem 13) and commutativity of the following diagram

$$\begin{tikzcd}
\varphi_*(f^*\Omega_{Y/Z}|_U) \arrow[r] \arrow[d, Rightarrow] & \varphi_*(\Omega_{X/Z}|_U) \arrow[r] \arrow[d, Rightarrow] & \varphi_*(\Omega_{X/Y}|_U) \arrow[r] \arrow[d, Rightarrow] & 0 \\
(\Omega_{\mathcal{O}_Y(V)/\mathcal{O}_Z(S)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U))^{\sim} \arrow[r] & (\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Z(S)})^{\sim} \arrow[r] & (\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)})^{\sim} \arrow[r] & 0
\end{tikzcd} \quad (10)$$

This shows that (8) is exact, as required. □

Remark 4. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of schemes. Commutativity of (10) implies that if f is an isomorphism then $\nu : f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z}$ is an isomorphism. Similarly if g is an isomorphism then $\upsilon : \Omega_{X/Z} \longrightarrow \Omega_{X/Y}$ is an isomorphism.

Definition 3. Let $\alpha : \mathscr{S} \longrightarrow \mathscr{S}'$ be a morphism of sheaves of rings on a topological space X, which is an epimorphism of $\mathscr{S}$-modules. Let $\mathscr{J}$ denote the kernel of α and let $\mathscr{F}$ be a $\mathcal{O}_S$-module. Then for $x \in X$ we have an epimorphism of rings $\alpha_x : \mathscr{S}_x \longrightarrow \mathscr{S}'_x$ with kernel $\mathscr{J}_x$ and an isomorphism $\mathscr{S}_x$-modules $(\mathscr{F}/\mathscr{J}\mathscr{F})_x \cong \mathscr{F}_x/\mathscr{J}_x\mathscr{F}_x$ (MRS,Lemma 45). Therefore $(\mathscr{F}/\mathscr{J}\mathscr{F})_x$ becomes a $\mathscr{S}'_x$-module in a canonical way, and therefore $\mathscr{F}/\mathscr{J}\mathscr{F}$ is canonically a $\mathscr{S}'$-module. In particular for $n \geq 1$ the $\mathscr{S}$-module $\mathscr{J}^n/\mathscr{J}^{n+1}$ is canonically a $\mathscr{S}'$-module.

Definition 4. Let $f : Y \longrightarrow X$ be a morphism of ringed spaces with the property that the corresponding morphism of sheaves of rings $\theta : f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ is an epimorphism of $f^{-1}\mathcal{O}_X$-modules. Let $\mathscr{I}_f$ denote the kernel of θ, so that we have an exact sequences of $f^{-1}\mathcal{O}_X$-modules

$$0 \longrightarrow \mathscr{I}_f \longrightarrow f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and therefore a canonical isomorphism $f^{-1}\mathcal{O}_X/\mathscr{I}_f \cong \mathcal{O}_Y$ of sheaves of rings. The $f^{-1}\mathcal{O}_X$-module $\mathscr{I}_f/\mathscr{I}_f^2$ is then canonically a $\mathcal{O}_Y$-module. The ring isomorphism $(f^{-1}\mathcal{O}_X)_y \cong \mathcal{O}_{X,f(y)}$ identifies $\mathscr{I}_{f,y}$ with the kernel of the ring morphism $\mathcal{O}_{X,f(y)} \longrightarrow \mathcal{O}_{Y,y}$ for $y \in Y$.

Proposition 15. *Let $f : Y \longrightarrow X$ be an emmersion of schemes with ideal sheaf $\mathscr{K}$. Then there is a canonical isomorphism of sheaves of modules on Y*

$$\begin{aligned} \eta : \mathscr{I}_f/\mathscr{I}_f^2 &\longrightarrow f^*\mathscr{K} \\ [V,s] \dotplus \mathscr{I}_f^2(U) &\mapsto [V,s] \dot{\otimes} 1 \end{aligned}$$

Proof. By assumption f gives a homeomorphism of Y with a locally closed subset of X so for $y \in Y$ the ideal $\mathscr{K}_{f(y)}$ is the kernel of the surjective ring morphism $\mathcal{O}_{X,f(y)} \longrightarrow \mathcal{O}_{Y,y}$. The induced morphism $f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ is an epimorphism, so we have the sheaf of modules $\mathscr{I}_f/\mathscr{I}_f^2$ of Definition 4. There is an isomorphism of $\mathcal{O}_{Y,y}$-modules (MRS,Proposition 20)

$$\begin{aligned} \eta_y : (\mathscr{I}_f/\mathscr{I}_f^2)_y &\cong \mathscr{I}_{f,f(y)}/\mathscr{I}_{f,f(y)}^2 \\ &\cong \mathscr{K}_{f(y)}/\mathscr{K}_{f(y)}^2 \\ &\cong \mathscr{K}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{X,f(y)}/\mathscr{K}_{f(y)} \\ &\cong \mathscr{K}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y} \\ &\cong (f^*\mathscr{K})_y \end{aligned}$$

Let U be an open neighborhood of y, V an open subset containing $f(U)$ and $s \in \mathscr{K}(V)$. Then $[V,s] \in \mathscr{I}_f(U) \subseteq (f^{-1}\mathcal{O}_X)(U)$ and η_y is defined by

$$germ_y([V,s] \dotplus \mathscr{I}_f^2(U)) \mapsto germ_y([V,s] \dot{\otimes} 1)$$

It is not hard to check that $germ_y \eta_U(s) = \eta_y(germ_y s)$ gives a well-defined isomorphism of sheaves of modules $\mathscr{I}_f/\mathscr{I}_f^2 \longrightarrow f^*\mathscr{K}$. □

Corollary 16. *Let $f : X \longrightarrow S$ be a morphism of schemes with diagonal $\Delta : X \longrightarrow X \times_S X$. Then there is a canonical isomorphism of sheaves of modules on X*

$$\xi : \mathscr{J}_\Delta / \mathscr{J}_\Delta^2 \longrightarrow \Omega_{X/S}$$

Proof. The diagonal Δ is an emmersion so by Proposition 15 there is a canonical isomorphism $\mathscr{J}_\Delta / \mathscr{J}_\Delta^2 \longrightarrow \Delta^*(\mathscr{I})$ where $\mathscr{I}$ is the ideal sheaf of Δ. It is not hard to see that if $g : X \longrightarrow W_{X/S}$ is the canonical closed immersion then the ideal sheaf of g is $\mathscr{I}_{X/S} = \mathscr{I}|_{W_{X/S}}$. Let ξ be the composite of $\Delta^*(\mathscr{I}) \cong g^*(\mathscr{I}_{X/S})$ with the epimorphism $g^*(\mathscr{I}_{X/S}) \longrightarrow g^*(\mathscr{I}_{X/S}/\mathscr{I}_{X/S}^2) = \Omega_{X/S}$. To show that ξ is an isomorphism, it suffices to show that this latter morphism is an isomorphism. By passing to stalks, this follows from from (MRS,Proposition 20) and the fact that if A is a ring, M an A-module, $\mathfrak{a}$ an ideal and $N = M/\mathfrak{a}M$, then $N/\mathfrak{a}N = N$. □

Remark 5. This shows that the definition of $\Omega_{X/S}$ adopted in Definition 2 agrees with the one given in EGA IV$_4$(16.3.1). In other words, the three sheaves of modules $\mathscr{J}_\Delta / \mathscr{J}_\Delta^2, \Omega_{X/S}, \Delta^*(\mathscr{I})$ are all isomorphic, where $\mathscr{I}$ is the ideal sheaf of the diagonal Δ.

Definition 5. Let X be a ringed space and $\mathscr{F}$ an $\mathcal{O}_X$-module. The presheaf coproduct $\mathcal{O}_X \oplus \mathscr{F}$ becomes a presheaf of $\mathcal{O}_X$-algebras with the product on $\mathcal{O}_X(U) \oplus \mathscr{F}(U)$ defined by $(a, x)(b, y) = (ab, ay + bx)$. Sheafifying gives the sheaf coproduct $\mathcal{O}_X \oplus \mathscr{F}$ a canonical $\mathcal{O}_X$-algebra structure, which we denote by $\mathscr{D}_{\mathcal{O}_X}(\dot{\mathscr{F}})$. There is a canonical morphism of sheaves of $\mathcal{O}_X$-algebras $\varepsilon : \mathscr{D}_{\mathcal{O}_X}(\dot{\mathscr{F}}) \longrightarrow \mathcal{O}_X$ with $(a, x) \mapsto a$, and a canonical morphism of $\mathcal{O}_X$-modules $\rho : \mathscr{D}_{\mathcal{O}_X}(\dot{\mathscr{F}}) \longrightarrow \mathscr{F}$ with $(a, x) \mapsto x$. Note that the presheaf coproduct $\mathcal{O}_X \oplus \mathscr{F}$ is a sheaf (since finite coproducts are finite products, which are computed pointwise) so every element of $\Gamma(U, \mathscr{D}_{\mathcal{O}_X}(\dot{\mathscr{F}}))$ is of the form (a, x).

Definition 6. Let $f : X \longrightarrow S$ be a morphism of ringed spaces. For an $\mathcal{O}_X$-module $\mathscr{F}$ a *S-derivation of $\mathcal{O}_X$ to $\mathscr{F}$* is a morphism of sheaves of abelian groups $D : \mathcal{O}_X \longrightarrow \mathscr{F}$ which satisfies the following conditions

(a) For open $V \subseteq X$ and $a, b \in \mathcal{O}_X(V)$ we have $D(ab) = aD(b) + bD(a)$.

(b) For open $U \subseteq S, V \subseteq f^{-1}U$ and $t \in \mathcal{O}_X(V), s \in \mathcal{O}_S(U)$ we have $D((s|_V)t) = (s|_V)D(t)$.

It is equivalent to say that for $x \in X$ the morphism of abelian groups $D_x : \mathcal{O}_{X,x} \longrightarrow \mathscr{F}_x$ is an $\mathcal{O}_{S,f(x)}$-derivation. The set of all S-derivations of $\mathcal{O}_X$ to $\mathscr{F}$ form a $\mathcal{O}_X(X)$-module $Der_S(\mathcal{O}_X, \mathscr{F})$.

Proposition 17. *Let $f : X \longrightarrow S$ be a morphism of ringed spaces and $\mathscr{F}$ a $\mathcal{O}_X$-module. Then there is a bijection between $Der_S(\mathcal{O}_X, \mathscr{F})$ and morphisms of $\mathcal{O}_S$-algebras $\varphi : \mathcal{O}_X \longrightarrow \mathscr{D}_{\mathcal{O}_X}(\dot{\mathscr{F}})$ with $\varepsilon\varphi = 1$.*

Proof. If $\mathscr{S}, \mathscr{T}$ are $\mathcal{O}_X$-algebras and $\varphi : \mathscr{S} \longrightarrow \mathscr{T}$ is a morphism of sheaves of rings on X, then we say φ is a morphism of $\mathcal{O}_S$-algebras if $f_*\varphi$ is a morphism of $\mathcal{O}_S$-algebras. Given $D \in Der_S(\mathcal{O}_X, \mathscr{F})$ we define

$$1 + D : \mathcal{O}_X \longrightarrow \mathscr{D}_{\mathcal{O}_X}(\dot{\mathscr{F}})$$
$$(1 + D)_U(a) = (a, D_U(a))$$

It is straightforward to check that this is a morphism of $\mathcal{O}_S$-algebras with $\varepsilon\varphi = 1$.

Now suppose we are given a morphism $\varphi : \mathcal{O}_X \longrightarrow \mathscr{D}_{\mathcal{O}_X}(\dot{\mathscr{F}})$ of $\mathcal{O}_S$-algebras with $\varepsilon\varphi = 1$. The morphism of sheaves of abelian groups $D = \rho\varphi : \mathcal{O}_X \longrightarrow \mathscr{F}$ is easily checked to be a S-derivation of $\mathcal{O}_X$ to $\mathscr{F}$. This defines the required bijection. □

Definition 7. Let $f : X \longrightarrow S$ be a morphism of schemes and $\Delta : X \longrightarrow X \times_S X$ the diagonal with projections $p_1, p_2 : X \times_S X \longrightarrow X$. Then we define the following S-derivation of $\mathcal{O}_X$ to $\Omega_{X/S}$

$$d : \mathcal{O}_X \longrightarrow \Omega_{X/S}$$
$$d_U(s) = [U \times_S U, (p_2)_U^{\#}(s)|_{U \times_S U} - (p_1)_U^{\#}(s)|_{U \times_S U} \dotplus \mathscr{I}_{X/S}^2(U \times_S U)] \dot{\otimes} 1$$

We write $d_{X/S}$ for d and call it the *canonical derivation*. Suppose that $V \subseteq S$ and $U \subseteq f^{-1}S$ are affine open subsets, and set $A = \mathcal{O}_X(U), k = \mathcal{O}_S(V)$. It is straightforward to check that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, Rightarrow] \arrow[d, "d_{A/k}"'] & \Gamma(U, \mathcal{O}_X) \arrow[d, "d_U"] \\ \Omega_{A/k} \arrow[r, Rightarrow] & \Gamma(U, \Omega_{X/S}) \end{tikzcd}$$

where the bottom isomorphism is defined in Proposition 7.

Remark 6. In the special case where $S = Speck$ is affine for a ring k, it is not hard to check that the following diagram commutes for $x \in X$

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "d_{\mathcal{O}_{X,x}/k}"] \arrow[dr, "(d_{X/k})_x"'] & \Omega_{\mathcal{O}_{X,x}/k} \arrow[d, Rightarrow] \\ & (\Omega_{X/k})_x \end{tikzcd}$$

where the vertical isomorphism is the one defined in Corollary 10.

Remark 7. Let $f : X \longrightarrow Y$ be a morphism of schemes and $j : Z \longrightarrow X$ a closed immersion with ideal sheaf $\mathscr{K}$. It is not hard to check that the following defines a morphism of sheaves of modules on X

$$\widehat{\delta} : \mathscr{K} \longrightarrow j_* j^* \Omega_{X/Y}$$
$$\widehat{\delta}_U(s) \mapsto [U, d_U(s)] \dot{\otimes} 1$$

where $d = d_{X/Y} : \mathcal{O}_X \longrightarrow \Omega_{X/Y}$ is the canonical derivation. By adjointness this corresponds to a morphism of sheaves of modules $\delta : j^*\mathscr{K} \longrightarrow j^*\Omega_{X/Y}$ defined by $[U, s] \dot{\otimes} b \mapsto [U, d_U(s)] \dot{\otimes} b$.

Proposition 18. *Let $f : X \longrightarrow Y$ be a morphism of schemes and $j : Z \longrightarrow X$ a closed immersion with ideal sheaf $\mathscr{K}$. Then there is an exact sequence of sheaves of modules on Z*

$$j^*\mathscr{K} \xrightarrow{\delta} j^*\Omega_{X/Y} \xrightarrow{\nu} \Omega_{Z/Y} \longrightarrow 0 \tag{11}$$

Proof. By using the canonical isomorphism $j^*\mathscr{K} \cong \mathscr{J}_j/\mathscr{J}_j^2$ of Proposition 15 we can put the statement of the result into the form that it takes in EGA IV_4(16.4.21). Here ν is as defined in Remark 3. To show that the sequence is exact, it suffices to show that for affine open $S \subseteq Z, V \subseteq g^{-1}S$ and $U = j^{-1}V$ the following sequence is exact (since j is a closed immersion, U is affine)

$$(j^*\mathscr{K})|_U \longrightarrow (j^*\Omega_{X/Y})|_U \longrightarrow \Omega_{Z/Y}|_U \longrightarrow 0 \tag{12}$$

Set $A = \mathcal{O}_X(V), B = \mathcal{O}_Z(U), k = \mathcal{O}_Y(S)$ and observe that $\mathfrak{a} = \mathscr{K}(U)$ is the kernel of the surjective ring morphism $A \longrightarrow B$. Let $\varphi : U \longrightarrow SpecB$ be the canonical isomorphism. Then exactness of (12) follows from (MAT2,Theorem 17) and commutativity of the following diagram

$$\begin{tikzcd} \varphi_*(j^*\mathscr{K}|_U) \arrow[r] \arrow[d, Rightarrow] & \varphi_*(j^*\Omega_{X/Y}|_U) \arrow[r] \arrow[d, Rightarrow] & \varphi_*(\Omega_{Z/Y}|_U) \arrow[r] \arrow[d, Rightarrow] & 0 \\ (\mathfrak{a}/\mathfrak{a}^2)^\sim \arrow[r] & (\Omega_{A/k} \otimes_A B)^\sim \arrow[r] & (\Omega_{B/k})^\sim \arrow[r] & 0 \end{tikzcd}$$

This shows that (11) is exact, as required. ☐

Definition 8. If Y is a scheme then $\mathbb{A}^n_Y$ denotes a pullback $\mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$ where $\mathbb{A}^n_{\mathbb{Z}} = Spec(\mathbb{Z}[x_1, \ldots, x_n])$. There is a canonical ring morphism $\mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \Gamma(\mathbb{A}^n_Y)$ and we denote the image of x_i once again by x_i. In particular if A is a ring then $Spec(A[x_1, \ldots, x_n])$ together with the canonical morphisms to $\mathbb{A}^n_{\mathbb{Z}}$ and $Spec(A)$ is such a pullback, and in this case the global section x_i is just $\dot{x_i/1}$, so $\mathbb{A}^n_{SpecA} = \mathbb{A}^n_A$.

Lemma 19. *If Y is any scheme and $X = \mathbb{A}^n_Y$ then $\Omega_{X/Y}$ is free $\mathcal{O}_X$-module of rank n, with basis $\{d_{X/Y}(x_1), \dots, d_{X/Y}(x_n)\}$.*

Proof. Let $f : X \longrightarrow Y$ be the structural morphism, and $u_i : \mathcal{O}_X \longrightarrow \Omega_{X/Y}$ the morphism of $\mathcal{O}_X$-modules corresponding to the global section $d_{X/Y}(x_i)$. We are claiming that the u_i are a coproduct. By (MRS,Proposition 40) it suffices to show that the morphisms $u_i|_U : \mathcal{O}_X|_U \longrightarrow \Omega_{X/Y}|_U$ are a coproduct for every affine open $V \subseteq Y$, where $U = f^{-1}V$. Since $U = V \times_{\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$ it is affine, so we can reduce to the case where $Y = SpecA$ and $X = Spec(A[x_1, \dots, x_n])$. This follows from (MAT2,Lemma 9). □

Next we will give an exact sequence relating the sheaf of differentials on a projective space to sheaves we already know. This is a fundamental result, upon which we will base all future calculations involving differentials on projective varieties.

Theorem 20. *Let A be a ring, $Y = SpecA$ and let $X = \mathbb{P}^n_A$. Then there is an exact sequence of sheaves of modules on X*

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Proof. We write $(-)^{n+1}$ to indicate a coproduct of $n+1$ copies. Let $S = A[x_0, \dots, x_n]$ and let E be the graded S-module $S(-1)^{n+1}$ with basis $e_0, \dots, e_n$ in degree 1. By (GRM,Lemma 19) there is a morphism of graded S-modules $\alpha : E \longrightarrow S$ sending e_i to x_i, which is clearly a quasi-epimorphism. Let M be the kernel of α. Using (MPS,Corollary 19) we have an exact sequence of sheaves of modules on X

$$0 \longrightarrow \widetilde{M} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

We will now show that $\widetilde{M} \cong \Omega_{X/Y}$. Let $U_i = D_+(x_i)$ be the canonical affine open subsets and let $\varphi : Spec(S_{(x_i)}) \longrightarrow U_i$ be the canonical isomorphism. It is straightforward to check that $\alpha_{(x_i)} : E_{(x_i)} \longrightarrow S_{(x_i)}$ is surjective and that $E_{(x_i)}$ is a free $S_{(x_i)}$-module on the basis $\{e_0/x_i, \dots, e_i/x_i, \dots, e_n/x_i\}$. We claim that $M_{(x_i)}$ is free on the basis $\{(x_ie_j - x_je_i)/x_i^2 \,|\, j \neq i\}$. In other words, $K = Ker\alpha_{(x_i)}$ is free on the basis $\{e_j/x_i - (x_j/x_i)e_i/x_i \,|\, j \neq i\}$. Let $t \in K$ be given and write

$$t = \frac{s_0}{x_i^m}\frac{e_0}{x_i} + \dots + \frac{s_n}{x_i^m}\frac{e_n}{x_i}$$

where each $s_k \in S_m$ and $m \geq 1$. Since $\alpha_{(x_i)}(t) = 0$ it follows that $s_0x_0 + \dots + s_nx_n = 0$ in S. Therefore

$$\frac{s_0}{x_i^m}\left(\frac{e_0}{x_i} - \frac{x_0}{x_i}\frac{e_i}{x_i}\right) + \dots + \frac{s_n}{x_i^m}\left(\frac{e_n}{x_i} - \frac{x_n}{x_i}\frac{e_i}{x_i}\right) = t - \frac{(s_0x_0 + \dots + s_nx_n)e_i}{x_i^{m+2}} = t$$

So the elements $e_j/x_i - (x_j/x_i)e_i/x_i$ for $j \neq i$ at least generate K as an $S_{(x_i)}$-module. It is easy to check that they are also linearly independent, and therefore a basis.

There is a canonical isomorphism of A-algebras $S_{(x_i)} \cong A[x_0/x_i, \dots, x_n/x_i]$, so $\Omega_{S_{(x_i)}/A}$ is free on the basis $\{d(x_j/x_i) \,|\, j \neq i\}$ (MAT2,Lemma 9). Therefore we have an isomorphism of $S_{(x_i)}$-modules

$$q_i : \Omega_{S_{(x_i)}/A} \longrightarrow M_{(x_i)}$$
$$d(x_j/x_i) \mapsto (x_ie_j - x_je_i)/x_i^2$$

Composing $\varphi_*(\widetilde{q_i})$ with the isomorphism $\varphi_*((\Omega_{S_{(x_i)}/A})\tilde{}) \cong \Omega_{X/Y}|_{U_i}$ of (DIFF,Proposition 9) and the canonical isomorphism $\varphi_*(M_{(x_i)}\tilde{}) \cong (M\tilde{})|_{U_i}$ we have an isomorphism of sheaves of modules on U_i

$$Q_i : (\Omega_{X/Y})|_{U_i} \longrightarrow \widetilde{M}|_{U_i}$$
$$d(x_j/x_i)\dot{/}(a/x_i^n) \mapsto x_i^n(x_ie_j - x_je_i)\dot{/}x_i^2a$$

To check compatibility of these isomorphisms, it suffices to check they agree on sections $z \in \Gamma(W, \Omega_{X/Y})$ of the following form: let $W \subseteq U_i \cap U_j$ be an affine open subset and $z = d_{\mathcal{O}_X(U_i)}(\dot{x_k/x_i})$ where $\dot{x_k/x_i} \in \mathcal{O}_X(U_i)$ (using $\vartheta_{X/Y,U_i/Y}$). By Remark 2 this is equal to $d_{\mathcal{O}_X(W)}(\dot{x_k/x_i})$ (using $\vartheta_{X/Y,W/Y}$). But in $\mathcal{O}_X(W)$ we have $\dot{x_k/x_i} = \dot{x_k/x_j} \cdot \dot{x_j/x_i}$ and therefore

$$\begin{aligned} d_{\mathcal{O}_X(W)}(\dot{x_k/x_i}) &= \dot{x_k/x_j} \cdot d_{\mathcal{O}_X(W)}(\dot{x_j/x_i}) + \dot{x_j/x_i} \cdot d_{\mathcal{O}_X(W)}(\dot{x_k/x_j}) \\ &= -\dot{x_j x_k/x_i^2} \cdot d_{\mathcal{O}_X(W)}(\dot{x_i/x_j}) + \dot{x_j/x_i} \cdot d_{\mathcal{O}_X(W)}(\dot{x_k/x_j}) \end{aligned}$$

It follows that $z = -\dot{x_j x_k/x_i^2} \cdot d_{\mathcal{O}_X(U_j)}(\dot{x_i/x_j}) + \dot{x_j/x_i} \cdot d_{\mathcal{O}_X(U_j)}(\dot{x_k/x_j})$ (using $\vartheta_{X/Y,U_j/Y}$). Using these representations it is easy to check that $(Q_i)_W$ and $(Q_j)_W$ both give $\dot{(x_i e_k - x_k e_i)/x_i^2}$ on z. This shows that the Q_i glue, so there is a unique isomorphism $\mu : \Omega_{X/Y} \longrightarrow M^\sim$ of sheaves of modules with $\mu|_{U_i} = Q_i$. Composing μ with $M^\sim \longrightarrow \mathcal{O}_X(-1)^{n+1}$ gives the desired exact sequence. $\square$

Corollary 21. *Let A be a ring, $Y = SpecA$ and $X = \mathbb{P}^n_A$. If X is nonempty, then $\Omega_{X/Y}$ is a locally free sheaf of rank n.*

Proof. The open sets $U_i = D_+(x_i)$ cover X, and by the proof of Theorem 20 we have for each i a canonical isomorphism of sheaves of modules $\Omega_{X/Y}|_{U_i} \cong \varphi_*(M_{(x_i)}{}^\sim)$ where $M_{(x_i)}$ is a free $S_{(x_i)}$-module of rank n. Therefore $\Omega_{X/Y}$ is a locally free sheaf of rank n. $\square$

Definition 9. Let A be a ring, $Y = SpecA$ and let $X = \mathbb{P}^n_A$ for some $n \geq 1$. Then the sheaf modules $\omega_{X/Y} = \bigwedge^n \Omega_{X/Y}$ is invertible (SSA,Corollary 28). We call $\omega_{X/Y}$ the *canonical sheaf* and just write ω_X if there is no chance of confusion.

Corollary 22. *Let A be a ring, $Y = SpecA$ and let $X = \mathbb{P}^n_A$. There is a canonical isomorphism of sheaves of modules $\omega_X \cong \mathcal{O}_X(-n-1)$.*

Proof. If X is empty this is trivial, so assume otherwise. By Theorem 20 we have a canonical exact sequence of sheaves of modules

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

By Corollary 21 the sheaf $\Omega_{X/Y}$ is locally free of rank n, so this is a short exact sequence of locally free sheaves of finite rank. Taking highest exterior powers (SSA,Proposition 31) and using (SSA,Corollary 35),(AAMPS,Lemma 12) we have a canonical isomorphism

$$\omega_{X/Y} = \bigwedge^n \Omega_{X/Y} \cong \bigwedge^n \Omega_{X/Y} \otimes \mathcal{O}_X \cong \bigwedge^{n+1} \mathcal{O}_X(-1)^{n+1} \cong \mathcal{O}_X(-1)^{\otimes(n+1)} \cong \mathcal{O}_X(-n-1)$$

as required. $\square$

3 Nonsingular Varieties

Our principal application of the sheaf of differentials is to nonsingular varieties. Recall that a *variety* X over a field k is an integral separated scheme of finite type over k. We say that X is a *nonsingular variety* if all the local rings of X are regular. Throughout this section k denotes an algebraically closed field. The connection between nonsingularity and differentials is given by the following result.

Theorem 23. *Let X be a variety of dimension n over k. Then $\Omega_{X/k}$ is a locally free sheaf of rank n if and only if X is nonsingular.*

Proof. We know from (H,Ex3.20) that $n = dimX$ is an integer $0 \leq n < \infty$ and that for any closed point $P \in X$ the local ring $B = \mathcal{O}_{X,x}$ has dimension n, field of representatives k, and is a localisation of a finitely generated k-domain. So it follows from Theorem 4 that B is a regular local ring if and only if $\Omega_{B/k}$ is a free B-module of rank n. By Corollary 10 we have an isomorphism of

B-modules $(\Omega_{X/k})_x \cong \Omega_{B/k}$. So it follows from (VS,Lemma 29) that X is nonsingular if and only if $(\Omega_{X/k})_x$ is free of rank n for every closed point $x \in X$. But (MRS,Corollary 90), (MOS,Lemma 34) and (VS,Proposition 14) imply that $\Omega_{X/k}$ is locally free of rank n if and only if $(\Omega_{X/k})_x$ is a free $\mathcal{O}_{X,x}$-module of rank n for every closed point x, which completes the proof (recall that since X is of finite type over k, $\Omega_{X/k}$ is coherent). $\square$

The next result gives a new proof of (H,I.5.3).

Corollary 24. *If X is a variety over k, then there is an open dense subset U of X which is nonsingular.*

Proof. If $n = dimX$ then we know from (H,Ex3.20) that the function field K of X has transcendence degree n over k, and it is a finitely generated extension field, which is separably generated by (H,I.4.8A). Therefore by (MAT2,Corollary 20), $\Omega_{K/k}$ is free of rank n. But if ξ is the generic point of X then by Corollary 10 there is an isomorphism of K-modules $\Omega_{K/k} \cong (\Omega_{X/k})_\xi$. Therefore combining (MRS,Corollary 90), (MOS,Lemma 34) we see that $\Omega_{X/k}$ is free of rank n on some nonempty open set $U \subseteq X$. Theorem 23 and Lemma 13 now imply that U is a nonsingular variety over k. $\square$

Remark 8. Let X be an integral scheme and $\mathscr{L}$ a locally free sheaf of modules. Since every nonempty open set contains the generic point, the free $\mathcal{O}_{X,x}$-modules $\mathscr{L}_x$ have the same rank for every $x \in X$, so $\mathscr{L}$ is locally free of some rank $n \in \{0, 1, 2, \ldots, \infty\}$.

Proposition 25. *Let X be a noetherian scheme and $\mathscr{F}$ a coherent sheaf of modules on X. If $\mathscr{F}_x$ can be generated by r elements as an $\mathcal{O}_{X,x}$-module then there is an open neighborhood U of x such that $\mathscr{F}|_U$ is generated by r elements of $\mathscr{F}(U)$.*

Proof. In particular the statement for $r = 0$ reads that if $\mathscr{F}_x = 0$ then $\mathscr{F}|_U = 0$ on some open neighborhood U of x. The case $r = 0$ follows directly from (MRS,Corollary 90), (MOS,Lemma 34) so assume $r \geq 1$. Suppose that $\mathscr{F}_x$ is generated by germs $(V, x_1), \ldots, (V, x_r)$. We can reduce easily to the case where $V = X$, so the x_i are global sections of $\mathscr{F}$. Let $u_i : \mathcal{O}_X \longrightarrow \mathscr{F}$ be the morphism of sheaves of modules corresponding to x_i and let $f : \mathcal{O}_X^r \longrightarrow \mathscr{F}$ be the induced morphism out of the coproduct. Set $\mathscr{G} = Imf$ and let $i : \mathscr{G} \longrightarrow \mathscr{F}$ be the inclusion. To complete the proof, it suffices to show that $\mathscr{G}|_U = \mathscr{F}|_U$ for some open neighborhood U of x. But $\mathscr{G}$ is coherent and by assumption $i_x : \mathscr{G}_x \longrightarrow \mathscr{F}_x$ is an isomorphism. It follows from (MRS,Corollary 91) that $i|_U$ is an isomorphism for some open neighborhood U of x, which is exactly what we wanted to show. $\square$

Corollary 26 (Nakayama). *Let X be a noetherian scheme, $j : Z \longrightarrow X$ a closed immersion and $\mathscr{K}$ a coherent sheaf of modules on X. If $j^*\mathscr{K}$ is a locally free sheaf of finite rank $r \geq 0$ then for every $z \in Z$ there is an open neighborhood $j(z) \in U \subseteq X$ such that $\mathscr{K}|_U$ is generated by r global sections.*

Proof. By Proposition 25 it is enough to show that $\mathscr{K}_{j(z)}$ can be generated by r elements as an $\mathcal{O}_{X,j(z)}$-module for every $z \in Z$. Let $\mathscr{I}$ be the ideal sheaf of j, and observe that by definition for $z \in Z$, $\mathscr{I}_{j(z)}$ is a proper ideal of $\mathcal{O}_{X,j(z)}$. By assumption $(j^*\mathscr{K})_z \cong \mathscr{K}_{j(z)} \otimes_{\mathcal{O}_{X,j(z)}} \mathcal{O}_{Z,z} \cong \mathscr{K}_{j(z)}/\mathscr{I}_{j(z)}\mathscr{K}_{j(z)}$ is a free $\mathcal{O}_{Z,z}$-module of rank $r \geq 0$. It follows from Nakayama's Lemma that $\mathscr{K}_{j(z)}$ can be generated by r elements, as required. $\square$

It is worth writing down the case $r = 0$ of the previous result separately.

Lemma 27. *Let X be a noetherian scheme, $j : Z \longrightarrow X$ a closed immersion and $\mathscr{K}$ a coherent sheaf of modules on X. If $j^*\mathscr{K} = 0$, then there is an open subset U containing Z with $\mathscr{K}|_U = 0$.*

In this section only, we say that a k-algebra B is *wholesome* if it is a regular local ring isomorphic as a k-algebra to the localisation of a finitely generated k-algebra at a maximal ideal. Observe that if B is wholesome then the residue field of B is a finite extension field of k, and is therefore equal to k. In other words, k is a field of representatives for B.

Proposition 28. *Let A be a wholesome k-algebra, $\mathfrak{a} \subseteq A$ an ideal such that $B = A/\mathfrak{a}$ is also wholesome. Then the sequence of (MAT2,Theorem 17) is exact on the left also*

$$0 \longrightarrow \mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{\nu} \Omega_{B/k} \longrightarrow 0$$

Proof. Since a regular local ring is a domain (MAT,Theorem 108), $\mathfrak{a}$ must be prime. It suffices to show that δ is injective (MAT2,Theorem 17). Set $n = dimA, q = dimB, r = ht.\mathfrak{a}$ and observe that since A is Cohen-Macaulay (MAT,Theorem 108) we have $n = r + q$ (MAT,Theorem 90). If $r = 0$ then $\mathfrak{a} = 0$ and the result is trivial, so assume $r \geq 1$. If $q = 0$ then $\mathfrak{a}$ is maximal and δ is an isomorphism by Proposition 1, so we can assume $q \geq 1$. Then by (MAT,Theorem 108) (5) there is a regular system of parameters $x_1, \ldots, x_n$ for A such that $\mathfrak{a} = (x_1, \ldots, x_r)$. In that case $x_{r+1} + \mathfrak{a}, \ldots, x_n + \mathfrak{a}$ is a regular system of parameters for B.

By Corollary 5 the set $\{d_{A/k}(x_i) \otimes 1 \,|\, 1 \leq i \leq n\}$ is a B-basis for $\Omega_{A/k} \otimes_A B$ and similarly the set $\{d_{B/k}(x_j + \mathfrak{a}) \,|\, r+1 \leq j \leq n\}$ is a B-basis for $\Omega_{B/k}$. So it is clear that $Ker\nu$ is the free submodule generated by the set $\{d_{A/k}(x_i) \otimes 1 \,|\, 1 \leq i \leq r\}$. By construction $\mathfrak{a}/\mathfrak{a}^2$ is generated as a B-module by the residues of $x_1, \ldots, x_r$, and since $\delta(x_i + \mathfrak{a}^2) = d_{A/k}(x_i) \otimes 1$ is not hard to see that δ must be injective. $\square$

Theorem 29. *Let X be a nonsingular variety over k and let $j : Y \longrightarrow X$ be an integral closed subscheme with ideal sheaf $\mathscr{K}$. Then Y is nonsingular if and only if*

(1) $\Omega_{Y/k}$ is locally free, and

(2) The sequence (11) is exact on the left also

$$0 \longrightarrow j^*\mathscr{K} \longrightarrow j^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

Furthermore, in this case, $\mathscr{K}$ is locally generated by $r = codim(Y, X)$ elements, and $j^*\mathscr{K}$ is a locally free sheaf of rank r on Y.

Proof. By an integral closed subscheme we mean a closed immersion $j : Y \longrightarrow X$ with Y an integral scheme. There is a bijection between integral closed subschemes of X and irreducible closed subsets of X. First suppose that (1) and (2) hold. Then for every $y \in Y$ we have an exact sequence of modules over the local noetherian domain $\mathcal{O}_{Y,y}$

$$0 \longrightarrow (j^*\mathscr{K})_y \longrightarrow \Omega_{\mathcal{O}_{X,j(y)}/k} \otimes_{\mathcal{O}_{X,j(y)}} \mathcal{O}_{Y,y} \longrightarrow \Omega_{\mathcal{O}_{Y,y}/k} \longrightarrow 0 \tag{13}$$

By hypothesis X is nonsingular, so Theorem 23 implies that $\Omega_{\mathcal{O}_{X,x}/k}$ is a free module of rank $n = dimX$ for every $x \in X$. In particular $\Omega_{\mathcal{O}_{Y,y}/k}$ in the above sequence must be finitely generated, and therefore by (1) a free module of finite rank $q \leq n$. Therefore by Remark 8, $\Omega_{Y/k}$ is locally free of finite rank q. It follows from (MAT,Proposition 24) that $(j^*\mathscr{K})_y$ must be free of rank $n - q$, and so $j^*\mathscr{K}$ is locally free of rank $n - q$ (MOS,Proposition 35).

The argument of Corollary 8 applies here to show that $q = dimY$, which shows that Y is nonsingular. We showed in (H,Ex.3.20) that $r = codim(Y, X) = dimX - dimY = n - q$ so $j^*\mathscr{K}$ is indeed a locally free sheaf of rank r. Combining Corollary 26 with the fact that $\mathscr{K}|_{X\setminus Y} = \mathcal{O}_X|_{X\setminus Y}$ we see that every point $x \in X$ has an open neighborhood U with $\mathscr{K}|_U$ generated by r global sections (observe that $r = 0$ iff. $Y = X$, which is iff. $\mathscr{K} = 0$).

Conversely, assume that Y is nonsingular of dimension $q = dimY = n - r$. Then by Theorem 23, $\Omega_{Y/k}$ is a locally free sheaf of rank q, so (1) is immediate. By Proposition 18 we have an exact sequence

$$j^*\mathscr{K} \xrightarrow{\delta} j^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

To show that δ is a monomorphism it suffices by (VS,Proposition 17) to show that δ_z is injective for every closed point $z \in Y$. Since X, Y are both nonsingular varieties over k the local rings $\mathcal{O}_{X,j(z)}, \mathcal{O}_{Z,z}$ are wholesome, so injectivity of δ_z follows from Lemma 10 and Proposition 28. This establishes (2) and completes the proof. $\square$

Definition 10. Let X be a variety over k. A *closed subvariety* of X is a closed immersion $Y \longrightarrow X$ of k-schemes where Y is also a variety over k. If $Z \longrightarrow X$ is a closed immersion with Z any integral scheme then Z is closed subvariety of X.

Definition 11. Let $f : Y \longrightarrow X$ be a morphism of schemes with ideal sheaf $\mathscr{K}$. The sheaf of modules $f^*\mathscr{K}$ is called the *conormal sheaf* of Y in X. Its dual $\mathscr{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}(f^*\mathscr{K}, \mathcal{O}_Y)$ is called the *normal sheaf* of Y in X. If X is a nonsingular variety over k and $f : Y \longrightarrow X$ a nonsingular closed subvariety then by Theorem 29 both $f^*\mathscr{K}$ and $\mathscr{N}_{Y/X}$ are locally free sheaves of rank $r = codim(Y, X)$ (MRS,Lemma 82).

4 Rational Maps

Throughout this section k denotes an arbitrary field by default. We make some preliminary comments

- Let X be a projective variety over k. Then X is proper over k, so by (H,Ex4.5) every valuation ring R of $K(X)/k$ dominates the local ring $\mathcal{O}_{X,x}$ for a unique point $x \in X$, called the *center* of R. It may be possible for two distinct valuation rings R, S to have the same center.
- Let X be a nonsingular variety over k. Given a point $x \in X$ we say that x *has codimension* 1 if $dim\mathcal{O}_{X,x} = 1$ (equivalently, the irreducible closed subset $Y = \overline{\{x\}}$ has codimension 1 in X). If x is such a point then $\mathcal{O}_{X,x}$ is a discrete valuation ring with quotient field $K(X)$, so $\mathcal{O}_{X,x}$ is a discrete valuation ring of $K(X)/k$.
- If X is a nonsingular projective variety over k then $x \mapsto \mathcal{O}_{X,x}$ defines an injective map from the set of points of codimension 1 to $C_{K(X)/k}$, the set of all discrete valuations of $K(X)/k$ (this is surjective if X is a curve (DIV,Proposition 16)).

Proposition 30. *Let X, Y be varieties over k with generic points ξ, η and assume X nonsingular and Y projective. If $\rho : K(Y) \longrightarrow K(X)$ is a morphism of k-algebras and $x \in X$ a point of codimension 1 then there is an open neighborhood U of x and a morphism of k-schemes $f : U \longrightarrow Y$ with $f(\xi) = \eta$ which induces ρ on the function fields.*

Proof. Identify $K(Y)$ with a subring of $K(X)$. Let $x \in X$ be a point of codimension 1 with corresponding discrete valuation ring $\mathcal{O}_{X,x} \subseteq K(X)$. Then $\mathcal{O}_{X,x} \cap K(Y)$ is a valuation ring of $K(Y)/k$ which dominates $\mathcal{O}_{Y,y}$ for a unique $y \in Y$ (SPM,Remark 3),(SPM,Proposition 16). So associated to every point $x \in X$ of codimension 1 in X is a canonical point $y \in Y$.

Now assume that such a point $x \in X$ and its associated point $y \in Y$ are fixed. Let $V \cong SpecA$ ($A = \mathcal{O}_Y(V)$) be an affine open neighborhood of y, so that A is a finitely generated k-domain. Choose generators $g_1, \dots, g_n$ for A over k. Write $\rho(V, g_i) = (U_i, f_i)$ where U_i is the domain of definition of $\rho(V, g_i)$ (see comments preceding (DIV,Definition 2)). By construction $\mathcal{O}_Y(V) \subseteq \mathcal{O}_{Y,y}$ as subrings of $K(Y)$, so $\rho(V, g_i) \in \mathcal{O}_{X,x}$ and therefore $x \in U = U_1 \cap \dots \cap U_n$. Clearly $\rho(\mathcal{O}_Y(V)) \subseteq \mathcal{O}_X(U)$ so there is a well-defined morphism of k-algebras $A \longrightarrow \mathcal{O}_X(U)$ sending g_i to $f_i|_U$. This induces a morphism of k-schemes $f : U \longrightarrow SpecA \cong V \longrightarrow Y$ with $f(x) = y, f(\xi) = \eta$ which induces ρ on the function fields. □

Lemma 31. *Let X, Y be varieties over k and let $f, g : X \longrightarrow Y$ be morphisms of k-schemes. If f, g agree on a nonempty open subset of X then they are equal.*

Proof. This is a special case of (SPM,Proposition 10). □

Definition 12. Let X, Y be varieties over k. A *rational morphism over* k $\varphi : X \longrightarrow Y$ is an equivalence class of pairs (U, φ_U) where U is a nonempty open subset of X and $\varphi_U : U \longrightarrow Y$ is a morphism of k-schemes. The equivalence relation says that $(U, \varphi_U) \sim (V, \varphi_V)$ if φ_U and φ_V restrict to give the same morphism $U \cap V \longrightarrow Y$ (equivalently by Lemma 31 they agree on some nonempty open subset of $U \cap V$). If there is no chance of confusion we will say that φ is a *rational morphism* from X to Y.

A morphism of k-schemes $X \longrightarrow Y$ is *dominant* if it sends the generic point of X to the generic point of Y (equivalently, the image is dense in Y). A rational morphism φ is *dominant* if for some (hence every) representative (U, φ_U) the morphism φ_U is dominant.

Remark 9. Let $\varphi : X \longrightarrow Y$ be a rational morphism of varieties over k and let $\{(U_i, \varphi_i)\}_{i \in I}$ be the set of elements of the equivalence class φ. Set $U = \bigcup_{i \in I} U_i$. By gluing the φ_i it is easy to see that there is a unique morphism of k-schemes $\varphi : U \longrightarrow Y$ with $\varphi|_{U_i} = \varphi_i$ for every $i \in I$. Therefore (U, φ) is a representative for the rational morphism, and we call U the *domain of definition* of the rational morphism. In other words, every rational morphism can be represented uniquely by a morphism of k-schemes on its domain of definition.

Lemma 32. *Let $f, g : X \longrightarrow Y$ be dominant morphisms of varieties over k. If f, g induce the same morphism of k-algebras $K(Y) \longrightarrow K(X)$ then they are equal.*

Proof. Suppose that f, g induce the same morphism of k-algebras $K(Y) \longrightarrow K(X)$ and identify $K(Y)$ with a subfield of $K(X)$. Given $x \in X$, let R be a valuation ring of $K(X)$ dominating $\mathcal{O}_{X,x}$ (H,6.1A). Since the morphisms $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}, \mathcal{O}_{Y,g(x)} \longrightarrow \mathcal{O}_{X,x}$ are local, $\mathcal{O}_{X,x}$ dominates the subrings $\mathcal{O}_{Y,f(x)}, \mathcal{O}_{Y,g(x)}$ of $K(X)$. By transitivity R also dominates these subrings, so by (SPM,Proposition 16) we have $f(x) = g(x)$ for every $x \in X$. Given $x \in X$ set $y = f(x) = g(x)$ and observe that the morphisms $\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$ induced by f, g both fit into a commutative diagram

$$\begin{tikzcd} K(Y) \arrow[r] & K(X) \\ \mathcal{O}_{Y,y} \arrow[u] \arrow[r] & \mathcal{O}_{X,x} \arrow[u] \end{tikzcd}$$

This shows that f, g agree on points and have the same local morphisms at every point, so it is clear that $f = g$ as required. $\square$

Theorem 33. *Let X, Y be varieties over an algebraically closed field k with Y quasi-projective. Then there is a canonical bijection between dominant rational morphisms $X \longrightarrow Y$ and morphisms of k-algebras $K(Y) \longrightarrow K(X)$.*

Proof. If $\varphi : X \longrightarrow Y$ is a dominant rational morphism, then it induces a well-defined morphism of k-algebras $K(Y) \longrightarrow K(X)$. We claim that this correspondence is a bijection between dominant rational morphisms $X \longrightarrow Y$ and morphisms of k-algebras $K(Y) \longrightarrow K(X)$. It is injective by Lemma 32, so it only remains to show that every morphism of k-algebras is induced by some rational morphism.

By Corollary 24 we can find an open dense subset $U \subseteq X$ which is nonsingular and an open immersion of k-schemes $i : Y \longrightarrow Z$ where Z is a projective variety over k. Given a morphism of k-algebras $\rho : K(Y) \longrightarrow K(X)$ there is an induced morphism of k-algebras $\rho' : K(Z) \cong K(Y) \longrightarrow K(X) \cong K(U)$. Set $n = dimX, m = dimY$ and let ξ, η, η' be the generic points of X, Y, Z respectively (clearly $i(\eta) = \eta'$). We divide into cases

Case $m = 0, n = 0$**.** By (VS,Corollary 12), X, Y are closed points and the structural morphisms $X \longrightarrow Speck, Y \longrightarrow Speck$ are isomorphisms. So there is precisely one dominant rational morphism $X \longrightarrow Y$ (the unique isomorphism of k-schemes) and precisely one morphism of k-algebras $K(Y) \longrightarrow K(X)$ (which is an isomorphism), so the result is true in this case.

Case $m = 0, n > 0$**.** It follows from (VS,Corollary 12) that ξ is a closed point of X but η is not a closed point of Y, so there can exist no dominant rational morphisms $X \longrightarrow Y$ (VS,Corollary 23). Since $n = tr.deg.K(Y)/k$ we have $K(Y) \neq k$ and so there can exist no k-algebra morphism $K(Y) \longrightarrow k$, which proves the result in this case.

With these cases out of the way, we can assume $m > 0$. As we observed in comments following (DIV,Definition 2), the nonsingular variety U must admit a prime divisor (since it can't be a closed point) and therefore a point $x \in U$ of codimension 1. By Proposition 30 there is an open neighborhood $x \in W \subseteq U$ and a morphism of k-schemes $f : W \longrightarrow Z$ with $f(\xi) = \eta'$ which induces ρ' on the function fields. If we set $W' = f^{-1}Y$ then $f|_{W'}$ factors uniquely through i, with factorisation $f' : W' \longrightarrow Y$. Then f' is a dominant morphism of k-schemes which induces ρ on the function fields, which shows that every morphism of k-algebras is induced by some dominant rational morphism and completes the proof. $\square$

Lemma 34. *Let X, Y be varieties over a field k with X nonsingular and Y projective. If $\varphi : X \longrightarrow Y$ is a dominant rational morphism with domain of definition $V \subset X$ and $Z = X \setminus V$, then $codim(Z, X) \geq 2$.*

Proof. Let $\varphi : X \longrightarrow Y$ be a dominant rational morphism with proper domain of definition $V \subset X$. Since V is nonempty we can exclude the case $codim(Z, X) = 0$ (FPOS,Proposition 2)(v), (H,Ex1.10d). It is now enough to show that $codim(Z, X) \neq 1$, and so by (FPOS,Proposition 2)(iv) it suffices to show that if $z \in Z$ then $dim\mathcal{O}_{X,z} \neq 1$. That is, we have to show that V contains every point of X of codimension one. But this follows immediately from Proposition 30 and Lemma 32. $\square$

Definition 13. Let $\varphi : X \longrightarrow Y, \psi : Y \longrightarrow Z$ be dominant rational morphisms of varieties over k, represented by pairs (U, φ_U) and (V, ψ_V) respectively. Let φ'_U be the induced morphism $\varphi_U^{-1}V \longrightarrow V$. Then $(\varphi_U^{-1}V, \psi_V\kappa)$ is a dominant rational morphism $X \longrightarrow Z$. This defines the category of varieties over k and dominant rational morphisms. If a dominant rational morphism $\varphi : X \longrightarrow Y$ is an isomorphism in this category, we call it a *birational equivalence* and say that X, Y are *birationally equivalent* (or just *birational*).

Definition 14. A variety X over k is *rational* if it is birationally equivalent to $\mathbb{P}^n_k$ for some $n \geq 1$. This property is stable under isomorphism of varieties over k.

Proposition 35. *Let $\varphi : X \longrightarrow Y$ be a dominant rational morphism of varieties over k represented by a morphism of k-schemes $\varphi_U : U \longrightarrow Y$. Then φ is a birational equivalence in the sense of Definition 13 if and only φ_U is a birational morphism in the sense of (BU,Definition 3).*

Proof. Suppose that there is a dominant rational morphism $\psi : Y \longrightarrow X$ represented by (V, ψ_V) with $\varphi\psi = 1$ and $\psi\varphi = 1$ as rational morphisms. In particular for $x \in \varphi_U^{-1}V$ we have $\psi_V\varphi_U(x) = x$ and for $y \in \psi_V^{-1}U$ we have $\varphi_U\psi_V(x) = x$. Let W be the nonempty open subset $\psi_V^{-1}(\varphi_U^{-1}V)$ of $\psi_V^{-1}U \subseteq V$ and W' the nonempty open subset $\varphi_U^{-1}(\psi_V^{-1}U)$ of $\varphi_U^{-1}V \subseteq U$. It is not difficult to check that $\varphi_U^{-1}W = W'$ and $\psi_V^{-1}W' = W$ and therefore the morphism of k-schemes $\varphi_U^{-1}W \longrightarrow W$ induced by φ_U is an isomorphism, which shows that φ_U is birational in the sense of (BU,Definition 3).

Conversely, suppose that there exists a nonempty open subset $W \subseteq Y$ with the induced morphism $\sigma : \varphi_U^{-1}W \longrightarrow W$ an isomorphism of schemes. Set $W' = \varphi_U^{-1}W$ and let $\tau : W \longrightarrow W'$ be the inverse, τ' the composite of τ with the inclusion $W' \longrightarrow X$. Then it is not hard to see that $(W, \tau'), (W', \varphi_U|_{W'})$ are mutually inverse dominant rational morphisms. Since $(W', \varphi_U|_{W'}) = (U, \varphi_U)$, this shows that φ is a birational equivalence in the sense of Definition 13. $\square$

Corollary 36. *Let X, Y be quasi-projective varieties over an algebraically closed field k. Then the following conditions are equivalent*

(i) X, Y are birationally equivalent.

(ii) There are nonempty open subsets $U \subseteq X, V \subseteq Y$ with $U \cong V$ as varieties over k.

(iii) There is an isomorphism of k-algebras $K(Y) \cong K(X)$.

Proof. $(i) \Leftrightarrow (ii)$ follows from Proposition 35 (in fact we only need X, Y to be varieties over an arbitrary field k) while $(i) \Leftrightarrow (iii)$ follows from Theorem 33. $\square$

Corollary 37. *Let X, Y be varieties over an algebraically closed field k (in the classical sense). Then X, Y are birational if and only if $t(X), t(Y)$ are birational.*

Remark 10. Let X be a quasi-projective variety of dimension $r \geq 1$ over an algebraically closed field k. Then X is rational if and only if X is birationally equivalent to $\mathbb{P}^r$.

5 Applications

Throughout this section k denotes an algebraically closed field.

Definition 15. Let X be a nonsingular variety over k. We call $\mathscr{T}_{X/k} = \mathscr{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ the *tangent sheaf of X over k* and just write $\mathscr{T}_X$ if there is no chance of confusion. If $n = dimX$ then $\mathscr{T}_{X/k}$ is a locally free sheaf of rank n by Theorem 23 and (MRS,Lemma 82). We call $\omega_{X/k} = \bigwedge^n \Omega_{X/k}$ the *canonical sheaf* and just write ω_X if there is no chance of confusion. This is an invertible sheaf (SSA,Corollary 28). If X is projective then we define the *geometric genus* of X to be $p_g = rank_k \Gamma(X, \omega_{X/k})$, which is a finite nonnegative integer (H,5.19).

Lemma 38. *Let X be a nonsingular variety over k and $U \subseteq X$ a nonempty open subset. Then there is a canonical isomorphism of sheaves of modules on U*

$$\zeta : \omega_{U/k} \longrightarrow \omega_{X/k}|_U$$
$$a_1 \dot{/} s_1 \dot{\wedge} \cdots \dot{\wedge} a_1 \dot{/} s_1 \mapsto a_1 \dot{/} s_1 \dot{\wedge} \cdots \dot{\wedge} a_1 \dot{/} s_1$$

Proof. If X has dimension n then so does U, by (H,Ex.3.20). Using (SSA,Proposition 22) and Lemma 13 we have an isomorphism of sheaves of modules

$$\omega_{X/k}|_U = (\bigwedge^n \Omega_{X/k})|_U \cong \bigwedge^n (\Omega_{X/k}|_U) \cong \bigwedge^n \Omega_{U/k} = \omega_{U/k}$$

as required. $\square$

Remark 11. Let $f : X \longrightarrow Y$ be a morphism of nonsingular varieties over k. Suppose that X, Y are both of the same dimension n. The composite $f^*(\bigwedge^n \Omega_{Y/k}) \cong \bigwedge^n (f^*\Omega_{Y/k}) \longrightarrow \bigwedge^n \Omega_{X/k}$ defines a canonical morphism of sheaves of modules on X (SSA,Proposition 29)

$$\omega_{f/k} : f^*\omega_{Y/k} \longrightarrow \omega_{X/k}$$
$$[Q, a_1 \dot{/} s_1 \dot{\wedge} \cdots \dot{\wedge} a_n \dot{/} s_n] \dot{\otimes} 1 \mapsto u(a_1) \dot{/} \phi(s_1) \dot{\wedge} \cdots \dot{\wedge} u(a_n) \dot{/} \phi(s_n)$$

where $V \subseteq Y, U \subseteq f^{-1}V$ are affine, $a_i \in \Omega_{\mathcal{O}_Y(V)/k}, s_i \in \mathcal{O}_Y(V)$, $\phi : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(U)$ the canonical morphism of k-algebras, $u : \Omega_{\mathcal{O}_Y(V)/k} \longrightarrow \Omega_{\mathcal{O}_X(U)/k}$ the induced morphism of k-modules. If f is an isomorphism then by Remark 4 both $\omega_{f/k}$ and the adjoint partner $\widehat{\omega}_{f/k} : \omega_{Y/k} \longrightarrow f_*\omega_{X/k}$ are isomorphisms of sheaves of modules. Therefore isomorphic nonsingular projective varieties over k have the same geometric genus.

Remark 12. Let $f : X \longrightarrow Y$ be a morphism of nonsingular varieties over k. Suppose that X, Y have the same dimension n. If $W \subseteq Y$ is a nonempty open subset with induced morphism $g : f^{-1}W \longrightarrow W$ then we claim that the following diagram commutes

$$\begin{tikzcd}
(f^*\omega_{Y/k})|_{f^{-1}W} \arrow[r, "\omega_{f/k}|_{f^{-1}W}"] \arrow[d, Rightarrow] & \omega_{X/k}|_{f^{-1}W} \arrow[d, Rightarrow] \\
g^*(\omega_{W/k}) \arrow[r, "\omega_{g/k}"'] & \omega_{f^{-1}W/k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\omega_{Y/k}|_W \arrow[r, "\widehat{\omega}_{f/k}|_W"] \arrow[d, Rightarrow] & (f_*\omega_{X/k})|_W \arrow[d, Rightarrow] \\
\omega_{W/k} \arrow[r, "\widehat{\omega}_{g/k}"'] & g_*(\omega_{f^{-1}W/k})
\end{tikzcd} \tag{14}$$

one checks this by reducing to special sections.

Theorem 39. *Let X, Y be two birationally equivalent nonsingular projective varieties over k. Then $p_g(X) = p_g(Y)$.*

Proof. Let $\varphi : X \longrightarrow Y$ be a birational equivalence represented by a dominant morphism of k-schemes $f : V \longrightarrow Y$ on its domain of definition V. By Proposition 35 there is a nonempty open subset $W \subseteq Y$ such that the induced morphism $f^{-1}W \longrightarrow W$ is an isomorphism of k-schemes. It is therefore clear that X, Y have the same dimension n so there is a canonical morphism of sheaves of modules $\omega_Y \longrightarrow f_*\omega_V$. In fact commutativity of (14) implies that the following diagram commutes

$$\begin{tikzcd} \Gamma(W, \omega_Y) \arrow[r] \arrow[d, Rightarrow] & \Gamma(f^{-1}W, \omega_V) \arrow[d, Rightarrow] \\ \Gamma(W, \omega_W) \arrow[r, Rightarrow] & \Gamma(f^{-1}W, \omega_{f^{-1}W}) \end{tikzcd}$$

Therefore $\Gamma(W, \omega_Y) \longrightarrow \Gamma(f^{-1}W, \omega_V)$ is an isomorphism. From commutativity of the following diagram and (PM,Lemma 20)(iv) we conclude that there is an injective morphism of k-modules $\Gamma(Y, \omega_Y) \longrightarrow \Gamma(V, \omega_V)$ and therefore $rank_k\Gamma(Y, \omega_Y) \le rank_k\Gamma(V, \omega_V)$

$$\begin{tikzcd} \Gamma(Y, \omega_Y) \arrow[r] \arrow[d] & \Gamma(V, \omega_V) \arrow[d] \\ \Gamma(W, \omega_Y) \arrow[r, Rightarrow] & \Gamma(f^{-1}W, \omega_V) \end{tikzcd}$$

We claim that the canonical morphism of k-modules $\Gamma(X, \omega_X) \longrightarrow \Gamma(V, \omega_X) \cong \Gamma(V, \omega_V)$ is a bijection. It is injective by (PM,Lemma 20)(iv), so it suffices to show that it is surjective. If $V = X$ this is trivial, so assume V is proper. Then by Lemma 34 we have $codim(Z, X) \ge 2$ where $Z = X \setminus V$. To show that $\Gamma(X, \omega_X) \longrightarrow \Gamma(V, \omega_X)$ is surjective it suffices to show that every point $x \in X$ has an affine open neighborhood U for which the restriction map $\Gamma(U, \omega_X) \longrightarrow \Gamma(U \cap V, \omega_X)$ is surjective.

Given a point $x \in X$ let U be an affine open neighborhood of x small enough that $\omega_X|_U \cong \mathcal{O}_X|_U$. We have to show that the restriction map $\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U \cap V, \mathcal{O}_X)$ is surjective. So we have reduced to the following algebra problem: A is a regular noetherian domain, V a proper nonempty open subset of $X = SpecA$ whose complement Z has codimension ≥ 2 in X and we want to show that $\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(V, \mathcal{O}_X)$ is a bijection. Since $codim(Z, X) \ge 2$, every prime $\mathfrak{p} \subseteq A$ of height 1 must belong V, and therefore $\Gamma(V, \mathcal{O}_X) \subseteq \bigcap_{ht.\mathfrak{p}=1} \mathcal{O}_{X,\mathfrak{p}}$ as subrings of $K(X)$. By (MAT,Theorem 108), A is normal so we have $A = \bigcap_{ht.\mathfrak{p}=1} A_\mathfrak{p}$ in the quotient field K of A (MAT,Theorem 112). Identifying K with $K(X)$ and A with $\Gamma(X, \mathcal{O}_X)$ we see that $\Gamma(X, \mathcal{O}_X) = \Gamma(V, \mathcal{O}_X)$ as subrings of $K(X)$, which implies that the restriction map $\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(V, \mathcal{O}_X)$ is bijective, as required.

Returning to the original problem, we have established that there is a canonical isomorphism of k-modules $\Gamma(X, \omega_X) \longrightarrow \Gamma(V, \omega_V)$, and therefore

$$p_g(X) = rank_k\Gamma(X, \omega_X) = rank_k\Gamma(V, \omega_V) \ge rank_k\Gamma(Y, \omega_Y) = p_g(Y)$$

We obtain the reverse inclusion by symmetry, and thus conclude that $p_g(X) = p_g(Y)$. $\square$

Example 1. Let $X = \mathbb{P}^n_k$, which is a nonsingular variety of dimension n over k. By Corollary 22 we have $\omega_{X/k} \cong \mathcal{O}_X(-n-1)$. Since $\mathcal{O}_X(\ell)$ has no global sections for $\ell < 0$ (AAMPS,Proposition 15), we find that $p_g(\mathbb{P}^n_k) = 0$ for any $n \ge 1$. We conclude from Theorem 39 that if X is a nonsingular projective *rational* variety over k, then $p_g(X) = 0$. This fact will allow us to demonstrate the existence of nonrational varieties in all dimensions.

Proposition 40. *Let $f : Y \longrightarrow X$ be a nonsingular closed subvariety of codimension $r \ge 1$ in a nonsingular variety X over k. Then there is a canonical isomorphism of sheaves of modules*

$$\omega_Y \cong f^*\omega_X \otimes \bigwedge^r \mathscr{N}_{Y/X}$$

Proof. Let $\mathscr{K}$ be the ideal sheaf of f. By Theorem 29 we have an exact sequence

$$0 \longrightarrow f^*\mathscr{K} \longrightarrow f^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

Set $n = dimX, q = dimY$ so that $r = n - q$. Taking highest exterior powers (SSA,Proposition 31) and using (SSA,Proposition 29) we have a canonical isomorphism of sheaves of modules

$$f^*\omega_X \cong \bigwedge^n (f^*\Omega_{X/k}) \cong \bigwedge^r (f^*\mathscr{K}) \otimes \bigwedge^q \Omega_{Y/k} = \bigwedge^r (f^*\mathscr{K}) \otimes \omega_Y$$

Tensoring both sides with the dual of $\bigwedge^r(f^*\mathscr{K})$ and using the canonical isomorphism

$$\bigwedge^r (f^*\mathscr{K})^\vee \cong \big(\bigwedge^r (f^*\mathscr{K})\big)^\vee$$

of (SSA,Proposition 32) we have a canonical isomorphism

$$\begin{aligned} f^*\omega_X \otimes \bigwedge^r \mathscr{N}_{Y/X} = f^*\omega_X \otimes \bigwedge^r (f^*\mathscr{K})^\vee &\cong f^*\omega_X \otimes \big(\bigwedge^r (f^*\mathscr{K})\big)^\vee \\ &\cong \big(\bigwedge^r (f^*\mathscr{K})\big)^\vee \otimes \bigwedge^r (f^*\mathscr{K}) \otimes \omega_Y \cong \omega_Y \end{aligned}$$

as required. $\square$

6 Some Local Algebra

See (MAT,Definition 15) and (MAT,Definition 16) for the definition of a Cohen-Macaulay ring.

Theorem 41. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Then*

(a) If A is regular, then it is Cohen-Macaulay.

(b) If A is Cohen-Macaulay, then any localisation of A at a prime ideal is Cohen-Macaulay.

(c) If A is Cohen-Macaulay, then a set of elements $x_1, \ldots, x_r \in \mathfrak{m}$ is an A-regular sequence if and only if $dim(A/(x_1, \ldots, x_r)) = dimA - r$.

(d) If A is Cohen-Macaulay, and $x_1, \ldots, x_r \in \mathfrak{m}$ is an A-regular sequence, then $A/(x_1, \ldots, x_r)$ is Cohen-Macaulay.

Proof. (a) (MAT,Theorem 108). (b) (MAT,Corollary 87). (c) follows by combining (MAT,Theorem 90) $(i), (iii)$. (d) (MAT,Corollary 86). □

Theorem 42 (Serre). *A nonzero noetherian ring A is normal if and only if it satisfies the following two conditions for every prime ideal $\mathfrak{p} \subseteq A$*

(1) If $ht.\mathfrak{p} \leq 1$ then $A_\mathfrak{p}$ is regular (hence a field or a discrete valuation ring).

(2) If $ht.\mathfrak{p} \geq 2$ then $depth(A_\mathfrak{p}) \geq 2$.

Proof. This is the content of (MAT,Theorem 116). □

Definition 16. A scheme X is *Cohen-Macaulay* if all of its local rings are Cohen-Macaulay. This property is stable under isomorphism. A nonsingular scheme is Cohen-Macaulay.

Lemma 43. *Let X be a scheme locally of finite type over a field. Then X is Cohen-Macaulay if and only if $\mathcal{O}_{X,z}$ is Cohen-Macaulay for every* closed *point $z \in X$.*

Proof. This follows from (VS,Corollary 16) and (MAT,Corollary 87). □

Definition 17. Let X be a nonsingular variety over a field k. If Y is a closed subvariety of X then we say that Y is a *local complete intersection* in X if the ideal sheaf $\mathscr{J}$ of $Y \longrightarrow X$ can be locally generated by $r = codim(Y, X)$ elements.

Example 2. If k is algebraically closed and Y nonsingular, then it is a local complete intersection in X by Theorem 29.

Proposition 44. *Let Y be a local complete intersection subscheme of a nonsingular variety X over a field k. Then*

(a) Y is Cohen-Macaulay.

(b) Y is normal if and only if it is regular in codimension 1.

Proof. (a) Set $n = dimX, m = dimY$ and $r = codim(Y, X) = n - m$. If $r = 0$ this is trivial, so assume $r > 0$. Let $\mathscr{J}$ be the ideal sheaf of the closed immersion $j : Y \longrightarrow X$. For each closed point $y \in Y$ there is a ring isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,j(y)}/\mathscr{J}_{j(y)}$. Since y is a closed point, we have (MAT,Theorem 90)

$$ht.\mathscr{J}_{j(y)} = dim\mathcal{O}_{X,j(y)} - dim(\mathcal{O}_{X,j(y)}/\mathscr{J}_{j(y)}) = n - m = r$$

By assumption $\mathscr{J}_{j(y)}$ can be generated by elements $x_1, \ldots, x_r$, and by (MAT,Theorem 90) this must be a regular sequence. It follows from Theorem 41(d) that $\mathcal{O}_{Y,y}$ is Cohen-Macaulay. Therefore by Lemma 43 the scheme Y is Cohen-Macaulay

(b) We already know that normal implies regular in codimension 1. Suppose that Y is regular in codimension 1 and let $y \in Y$ be given. Find an affine open neighborhood $U \cong SpecA$ of y, so A is a Cohen-Macaulay domain with $A_\mathfrak{p}$ regular for every prime ideal $\mathfrak{p}$ of height 1. We have to show that A is normal, which follows from Theorem 42. □