Section 2.7.1 - Blowing Up

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Now we come to the generalised notion of blowing up. In (I, §4) we defined the blowing up of a variety with respect to a point. Now we will define the blowing up of a noetherian scheme with respect to any closed subscheme. Since a closed subscheme corresponds to a coherent sheaf of ideals, we may as well speak of blowing up a coherent sheaf of ideals. The necessary background for this section includes (MRS,Section 1.9), (MOS,Section 2) and (SSA,Section 6).

Lemma 1. *Let X be a noetherian scheme and $\mathscr{J}$ a coherent sheaf of ideals on X. Then $\mathbb{B}(\mathscr{J}) = \bigoplus_{n \geq 0} \mathscr{J}^n$ is a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally finitely generated by $\mathbb{B}^1(\mathscr{J})$ as a $\mathbb{B}^0(\mathscr{J})$-algebra with $\mathbb{B}^0(\mathscr{J}) = \mathcal{O}_X$.*

Proof. The sheaf of graded $\mathcal{O}_X$-algebras $\mathbb{B}(\mathscr{J})$ is clearly quasi-coherent, and has $\mathbb{B}^0(\mathscr{J}) = \mathcal{O}_X$ by construction. The other properties follow from (SSA,Corollary 55). □

Definition 1. Let X be a noetherian scheme and $\mathscr{J}$ a coherent sheaf of ideals on X. We define $\widetilde{X}$ to be the scheme **Proj**$\mathbb{B}(\mathscr{J})$. This is a noetherian scheme with proper structural morphism $\pi : \widetilde{X} \longrightarrow X$ and twisting sheaf $\mathcal{O}(1)$. We call $\widetilde{X}$ the *blowing-up of X with respect to the coherent sheaf of ideals $\mathscr{J}$*. If $Y \longrightarrow X$ is a closed immersion with sheaf of ideals $\mathscr{J}$, then we also call $\widetilde{X}$ the *blowing-up of X along Y* or *with center Y*. If $\mathscr{J} = 0$ then $\widetilde{X}$ is the empty scheme.

Proposition 2. *If X is an integral noetherian scheme and $\mathscr{J}$ a nonzero coherent sheaf of ideals then the blowing-up of $\mathscr{J}$ is an integral scheme.*

Proof. By (TRPC,Proposition 5) it suffices to show that $\mathbb{B}(\mathscr{J})$ is *relevant*, in the sense of (TRPC,Definition 2). It follows from (SSA,Proposition 54) that $\mathbb{B}(\mathscr{J})$ is locally an integral domain. Let Λ be the set of all nonempty affine open subsets $U \subseteq X$ with $\mathscr{J}(U) \neq 0$. Since $\mathscr{J} \neq 0$ the set Λ is clearly nonempty. Given $U, V \in \Lambda$ let $f : U \longrightarrow Spec\mathcal{O}_X(U)$ be the canonical isomorphism. Since X is integral, $U \cap V$ is nonempty and we can find $f \in \mathcal{O}_X(U)$ with $W = D(f)$ nonempty and $W \subseteq U \cap V$. Then $\Gamma(D(f), \mathscr{J}) \cong \Gamma(U, \mathscr{J})_f$. Since $\mathcal{O}_X(U)$ is an integral domain and $f \neq 0$ this latter ideal cannot be zero. Therefore $W \in \Lambda$ and the proof is complete. □

Proposition 3. *Let X be a noetherian scheme, $\mathscr{J}$ a coherent sheaf of ideals, and let $\pi : \widetilde{X} \longrightarrow X$ be the blowing-up of $\mathscr{J}$. Then*

(a) The inverse image ideal sheaf $\widetilde{\mathscr{J}} = \mathscr{J} \cdot \mathcal{O}_{\widetilde{X}}$ is an invertible sheaf on $\widetilde{X}$.

(b) If Y is the closed subset corresponding to $\mathscr{J}$ then $\pi^{-1}U \longrightarrow U$ is an isomorphism, where $U = X \setminus Y$.

Proof. (a) It suffices to show that $\widetilde{\mathscr{J}}$ is invertible on a neighborhood of every point of $\widetilde{X}$. Given $x \in \widetilde{X}$ find an affine open subset $U \subseteq X$ with $\pi(x) \in U$. Using (SSA,Proposition 54) there is an isomorphism of schemes over U

$$\pi^{-1}U \cong Proj\mathbb{B}(\mathscr{J})(U) \cong ProjB(\mathscr{J}(U))$$

By (MRS,Lemma 48) we have $\widetilde{\mathscr{J}}|_{\pi^{-1}U} = \mathscr{J}|_U \cdot \mathcal{O}_{\widetilde{X}}|_{\pi^{-1}U}$ and therefore by (MRS,Lemma 51) it suffices to show that $\mathscr{J}|_U \cdot \mathcal{O}_{ProjB(\mathscr{J}(U))}$ is an invertible sheaf on $ProjB(\mathscr{J}(U))$. In fact we will show that there is a canonical isomorphism of sheaves of modules on $ProjB(\mathscr{J}(U))$

$$\mathscr{J}|_U \cdot \mathcal{O}_{ProjB(\mathscr{J}(U))} \cong \mathcal{O}(1) \tag{1}$$

By (MRS,Lemma 51), (MOS,Lemma 9)(b) and (MOS,Lemma 10), $\mathscr{J}|_U \cdot \mathcal{O}_{ProjB(\mathscr{J}(U))}$ is the sheaf of ideals corresponding to the homogenous ideal $\mathscr{J}(U)B(\mathscr{J}(U))$. The graded modules $\mathscr{J}(U)B(\mathscr{J}(U))$ and $B(\mathscr{J})(U)(1)$ are quasi-isomorphic, so by (MPS,Proposition 18) we obtain the desired isomorphism (1). Note that for $a \in \mathscr{J}(U)$, (1) maps the global section $a_i/1$ of $\mathcal{O}_{ProjB(\mathscr{J}(U))}$ (here a_i has grade zero) to the global section $a_i/1$ of $\mathcal{O}(1)$ (here a_i has grade one).

(b) By the closed subset corresponding to $\mathscr{J}$ we mean $Y = Supp(\mathcal{O}_X/\mathscr{J})$, which is the closed image of the closed subscheme of X with ideal sheaf $\mathscr{J}$. To show $\pi^{-1}U \longrightarrow U$ is an isomorphism it suffices to show that for every open affine $V \subseteq U$, $\pi^{-1}V \longrightarrow V$ is an isomorphism. Using the argument of part (a) we see that $\pi^{-1}V$ is isomorphic as a scheme over V to $ProjB(\mathscr{J}(V))$. So it suffices to show that the structural morphism $ProjB(\mathscr{J}(V)) \longrightarrow Spec\mathcal{O}_X(V)$ is an isomorphism. But since $Y = Supp(\mathcal{O}_X/\mathscr{J})$ we have $\mathscr{J}|_U = \mathcal{O}_X|_U$ (MRS,Lemma 8) and therefore $\mathscr{J}(V) = \mathcal{O}_X(V)$. But $B(\mathcal{O}_X(V)) \cong \mathcal{O}_X(V)[x]$ as graded $\mathcal{O}_X(V)$-algebras (SSA,Proposition 56) so $ProjB(\mathscr{J}(V)) \longrightarrow Spec\mathcal{O}_X(V)$ is an isomorphism, as required. ☐

Remark 1. If X is a noetherian scheme and $\mathscr{J}$ a coherent sheaf of ideals, then we associate with an affine open subset $U \subseteq X$ the open immersion $ProjB(\mathscr{J}(U)) \longrightarrow \widetilde{X}$ given in the previous result. This is the composite $ProjB(\mathscr{J}(U)) \cong Proj\mathbb{B}(\mathscr{J})(U) \longrightarrow \widetilde{X}$, so there is a pullback diagram

$$\begin{array}{ccc} ProjB(\mathscr{J}(U)) & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

Lemma 4. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{J}$ a sheaf of ideals on Y generated by global sections $a_1, \ldots, a_n$. Then the global sections $f_Y^\#(a_1), \ldots, f_Y^\#(a_n)$ generate the ideal sheaf $\mathscr{J} \cdot \mathcal{O}_X$.*

Proof. We know from (PM,Lemma 1) that the global sections $[Y, a_i] \dot{\otimes} 1$ generate the sheaf of modules $f^*\mathscr{J}$ on X. By construction there is an epimorphism of sheaves of modules $f^*\mathscr{J} \longrightarrow \mathscr{J} \cdot \mathcal{O}_X$ mappng $[Y, a_i] \dot{\otimes} 1$ to $f_Y^\#(a_i)$, so the latter sections clearly generate $\mathscr{J} \cdot \mathcal{O}_X$. ☐

Lemma 5. *Let X be scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. Let $U \subseteq X$ be an open affine subset and $\{a_i\}_{i \in I}$ a nonempty subset of $\mathscr{F}(U)$. Then the elements a_i generate $\mathscr{F}(U)$ as an $\mathcal{O}_X(U)$-module if and only if the global sections a_i generate $\mathscr{F}|_U$.*

Proof. We may assume $U = X$. Let $f : X \longrightarrow Spec\mathcal{O}_X(X)$ be the canonical isomorphism. Then since $f_*\mathscr{F} \cong \mathscr{F}(X)^\sim$ it suffices to show the global sections $a_i/1$ generate $\mathscr{F}(X)^\sim$ if and only if the a_i generate $\mathscr{F}(X)$ as a $\mathcal{O}_X(X)$-module. The reverse implication is clear, so suppose that the $a_i/1$ generate $\mathscr{F}(X)^\sim$ and let G be the $\mathcal{O}_X(X)$-submodule of $\mathscr{F}(X)$ generated by the a_i. By hypothesis for every prime ideal $\mathfrak{p}$ we have $G_\mathfrak{p} = \mathscr{F}(X)_\mathfrak{p}$, and therefore $G = \mathscr{F}(X)$ as required. ☐

Lemma 6. *Let X be a ringed space and $\mathscr{J}$ an invertible sheaf of ideals on X. Then for any $d > 1$ there is a canonical isomorphism of sheaves of modules*

$$\mathscr{J}^{\otimes d} \longrightarrow \mathscr{J}^d$$
$$a_1 \dot{\otimes} \cdots \dot{\otimes} a_d \mapsto a_1 \cdots a_d$$

In particular the ideal sheaf $\mathscr{J}^d$ is invertible.

Proof. We prove the case $d = 2$, with other cases following by a simple induction. Since $\mathscr{J}$ is invertible it is flat, so $\mathscr{J}^2$ is the image of the monomorphism $\mathscr{J} \otimes \mathscr{J} \longrightarrow \mathcal{O}_X \otimes \mathscr{J} \cong \mathscr{J}$. Therefore $\mathscr{J} \otimes \mathscr{J} \cong \mathscr{J}^2$, and so $\mathscr{J}^2$ is invertible as required. ☐

We need two more small results before embarking on the proof of the next Theorem. Intuitively the Lemmas prove various naturality properties of the morphisms produced by (PM,Corollary 4). The first proves naturality with respect to adding superfluous generators while the second proves naturality with respect to restriction.

Lemma 7. *Let A be a ring, set $X = SpecA$ and let $f : Z \longrightarrow X$ a scheme over A. Let $\mathscr{L}$ be an invertible sheaf of ideals on Z generated by global sections $a_0, \dots, a_n \in A$. Suppose $b_1, \dots, b_m \in A$ are linear combinations of the a_i*

$$b_j = \lambda_{j0}a_0 + \cdots + \lambda_{jn}a_n \qquad \lambda_{jk} \in A$$

and let ϕ, ϕ', ϕ'' be the morphisms of A-schemes determined by the following tuples

$$\begin{aligned} \phi &: (\mathscr{L}, a_0, \dots, a_n) \\ \phi' &: (\mathscr{L}, a_0, \dots, a_n, b_1, \dots, b_m) \\ \phi'' &: (\mathscr{L}, b_1, \dots, b_m, a_0, \dots, a_n) \end{aligned}$$

Then the following diagrams commute

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & \mathbb{P}^n_A \\ & \searrow^{\phi'} & \downarrow \Psi \\ & & \mathbb{P}^{n+m}_A \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{\phi} & \mathbb{P}^n_A \\ & \searrow^{\phi''} & \downarrow \Psi' \\ & & \mathbb{P}^{n+m}_A \end{array}$$

where the vertical morphisms are induced by the following morphisms of graded A-algebras

$$\psi : A[x_0, \dots, x_n, y_1, \dots, y_m] \longrightarrow A[x_0, \dots, x_n] \qquad x_i \mapsto x_i, y_j \mapsto \sum_{i=0}^{n} \lambda_{ji}x_i$$

$$\psi' : A[y_1, \dots, y_m, x_0, \dots, x_n] \longrightarrow A[x_0, \dots, x_n] \qquad y_j \mapsto \sum_{i=0}^{n} \lambda_{ji}x_i, x_i \mapsto x_i$$

Proof. The correspondence between tuples and morphisms we refer to is the one defined in (PM,Corollary 4). We show that the first diagram commutes, since the proof for the second diagram is the same. It suffices to show that there is an isomorphism of sheaves $(\Psi\phi)^*\mathcal{O}(1) \cong \mathscr{L}$ that identifies $(\Psi\phi)^*(x_i)$ with a_i and $(\Psi\phi)^*(y_j)$ with b_j. The isomorphism is given by

$$(\Psi\phi)^*\mathcal{O}(1) \cong \phi^*\Psi^*\mathcal{O}(1) \cong \phi^*\mathcal{O}(1) \cong \mathscr{L}$$

Using (MRS,Remark 8), (MPS,Proposition 13) and the unique isomorphism $\alpha : \phi^*\mathcal{O}(1) \cong \mathscr{L}$ identifying $\phi^*(x_i)$ with a_i, it is not hard to see this isomorphism has the desired property. $\square$

Lemma 8. *Let $\alpha : B \longrightarrow A$ be a morphism of rings, X a scheme over B and $U \subseteq X$ an open subset which is a scheme over A in such a way that the following diagram commutes*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ SpecA & \longrightarrow & SpecB \end{array}$$

Let $\mathscr{L}$ be an invertible sheaf of ideals on X generated by global sections $b_0, \dots, b_n \in B$ and let $\phi : X \longrightarrow \mathbb{P}^n_B$ and $\phi' : U \longrightarrow \mathbb{P}^n_A$ be the morphisms determined by the following tuples

$$\begin{aligned} \phi &: (\mathscr{L}, b_0, \dots, b_n) \\ \phi' &: (\mathscr{L}|_U, \alpha(b_0), \dots, \alpha(b_n)) \end{aligned}$$

Then the following diagram commutes

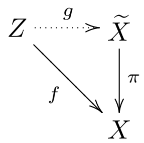
$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ \phi' \downarrow & & \downarrow \phi \\ \mathbb{P}^n_A & \xrightarrow[\gamma]{} & \mathbb{P}^n_B \end{array}$$

Proof. Both ϕi and $\gamma\phi'$ are morphisms of schemes over B and we show they both correspond to the tuple $(\mathscr{L}|_U, \alpha(b_0), \dots, \alpha(b_n))$, which is enough by (PM,Corollary 4) to show they are equal. We have isomorphisms of sheaves of modules

$$\begin{aligned}(\phi i)^*\mathcal{O}(1) &\cong i^*\phi^*\mathcal{O}(1) \\ &\cong (\phi^*\mathcal{O}(1))|_U \\ &\cong \mathscr{L}|_U\end{aligned} \qquad \begin{aligned}(\gamma\phi')^*\mathcal{O}(1) &\cong (\phi')^*\gamma^*\mathcal{O}(1) \\ &\cong (\phi')^*\mathcal{O}(1) \\ &\cong \mathscr{L}|_U\end{aligned}$$

In both cases we use (MRS,Remark 8) in the first step. On the left we then apply (MRS,Proposition 110), while on the right we use (MPS,Proposition 13). Then we use the isomorphisms $\phi^*\mathcal{O}(1) \cong \mathscr{L}$ and $(\phi')^*\mathcal{O}(1) \cong \mathscr{L}|_U$ expressing the fact that ϕ, ϕ' are determined by the tuples given in the statement of the Lemma. Using the aforementioned results, it is not hard to check that these isomorphisms have the necessary properties. □

Theorem 9. *Let X be a noetherian scheme, $\mathscr{J}$ a coherent sheaf of ideals, and $\pi : \widetilde{X} \longrightarrow X$ the blowing-up with respect to $\mathscr{J}$. If $f : Z \longrightarrow X$ is any morphism such that $\mathscr{J} \cdot \mathcal{O}_Z$ is an invertible sheaf of ideals on Z, then there exists a unique morphism $g : Z \longrightarrow \widetilde{X}$ making the following diagram commute*



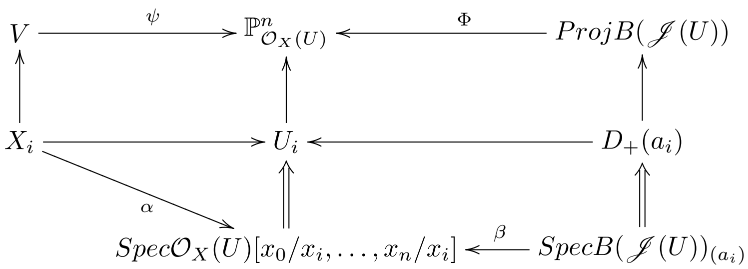
Proof. We construct the morphism $g : Z \longrightarrow \widetilde{X}$ locally and then glue. Let $U \subseteq X$ be an affine open subset, $a_0, \dots, a_n \in \mathscr{J}(U)$ generators for the ideal $\mathscr{J}(U)$, and V the scheme $f^{-1}U$. Using (MRS,Lemma 48), Lemma 4 and Lemma 5 we see that $\mathscr{J}|_U \cdot \mathcal{O}_Z|_V$ is an invertible sheaf generated by the global sections $f_U^{\#}(a_i)$. There is a surjective morphism of graded $\mathcal{O}_X(U)$-algebras $\varphi : \mathcal{O}_X(U)[x_0, \dots, x_n] \longrightarrow B(\mathscr{J}(U))$ given by sending x_i to a_i, considered as an element of degree one in $B(\mathscr{J}(U))$. This induces a closed immersion $\Phi : ProjB(\mathscr{J}(U)) \longrightarrow \mathbb{P}^n_{\mathcal{O}_X(U)}$ of $\mathcal{O}_X(U)$-schemes with ideal sheaf $\mathscr{J}_{Ker\varphi}$ (SIPS,Lemma 4). It is easy to check that

$$Ker\varphi = \{F(x_0, \dots, x_n) \,|\, F_i(a_0, \dots, a_n) = 0 \text{ for all } i \geq 0\}$$

The invertible sheaf $\mathscr{L} = \mathscr{J}|_U \cdot \mathcal{O}_Z|_V$ together with the generating global sections $f_U^{\#}(a_i)$ induces a morphism of schemes $\psi : V \longrightarrow \mathbb{P}^n_{\mathcal{O}_X(U)}$ over $\mathcal{O}_X(U)$ with

$$\psi^{-1}U_i = X_i = \{x \in V \,|\, germ_x f_U^{\#}(a_i) \notin \mathfrak{m}_x \mathscr{L}_x\}$$

Let $\mathscr{K}$ be the ideal sheaf of ψ. Then to show that ψ factors through Φ, it suffices to show that $\mathscr{J}_{Ker\varphi} \subseteq \mathscr{K}$ (FCI,Corollary 3). Since the open sets $U_i = D_+(x_i)$ cover $\mathbb{P}^n_{\mathcal{O}_X(U)}$ is suffices to show that $\mathscr{J}_{Ker\varphi}|_{U_i} \subseteq \mathscr{K}|_{U_i}$. Consider the following commutative diagram



Where α corresponds to the morphism of $\mathcal{O}_X(U)$-algebras

$$\begin{aligned} a : \mathcal{O}_X(U)[x_0/x_i, \dots, x_n/x_i] &\longrightarrow \mathcal{O}_Z(X_i) \\ x_j/x_i &\mapsto f_U^{\#}(a_j)/f_U^{\#}(a_i)\end{aligned}$$

and β corresponds to the morphism of $\mathcal{O}_X(U)$-algebras

$$b : \mathcal{O}_X(U)[x_0/x_i, \dots, x_n/x_i] \longrightarrow B(\mathscr{J}(U))_{(a_i)}$$
$$x_j/x_i \mapsto a_j/a_i$$

Using this diagram we reduce to showing that the ideal sheaf of β is contained in the ideal sheaf of α. Therefore by (SIAS,Lemma 5) and (SIAS,Corollary 6) it is enough to show $Ker(b) \subseteq Ker(a)$. Suppose $F(x_0/x_i, \dots, x_n/x_i)$ is a polynomial in $\mathcal{O}_X(U)[x_0/x_i, \dots, x_n/x_i]$ with

$$F(a_0/a_i, \dots, a_n/a_i) = 0 \in B(\mathscr{J}(U))_{(a_i)}$$

Then one sees fairly easily that there is an integer $M > 1$ with the property that in $\mathcal{O}_X(U)$ we have

$$\sum_\alpha F(\alpha) a_0^{\alpha_0} \cdots a_n^{\alpha_n} a_i^{M-\alpha_0-\cdots-\alpha_n} = 0 \tag{2}$$

We now have to show that $a(F) = 0$. For $0 \le j \le n$ with $j \neq i$ the quotient $z_{ji} = f_U^\#(a_j)/f_U^\#(a_i)$ is by definition the unique section of $\mathcal{O}_Z(X_i)$ with $germ_x z_{ji} \cdot germ_x f_U^\#(a_i) = germ_x f_U^\#(a_j)$ in $\mathscr{L}_x$ for all $x \in X_i$. We have to show that in $\mathcal{O}_Z(X_i)$

$$\sum_\alpha F(\alpha) z_{0i}^{\alpha_0} \cdots z_{ni}^{\alpha_n} = 0 \tag{3}$$

By Lemma 6 the sheaf of ideals $\mathscr{L}^M$ on V is invertible, and by (MOS,Lemma 38) we have $X_i = \{x \in V \,|\, germ_x f_U^\#(a_i^M) \notin \mathfrak{m}_x \mathscr{L}_x^M\}$. Therefore to establish (3) it suffices to show that for $x \in X_i$ the following equation holds in the ring $\mathcal{O}_{Z,x}$

$$\sum_\alpha F(\alpha)(germ_x z_{0i})^{\alpha_0} \cdots (germ_x z_{ni})^{\alpha_n} (germ_x f_U^\#(a_i))^M = 0$$

This is straightforward, using (2) and the defining property of the z_{ji}. Therefore $Ker(b) \subseteq Ker(a)$ and hence $\mathscr{J}_{Ker\varphi} \subseteq \mathscr{K}$. It now follows that there is a unique morphism $g_U : f^{-1}U \longrightarrow ProjB(\mathscr{J}(U))$ making the following diagram commute

$$\begin{array}{ccc} f^{-1}U & \xrightarrow{\psi} & \mathbb{P}^n_{\mathcal{O}_X(U)} \\ & \searrow^{g_U} & \uparrow \Phi \\ & & ProjB(\mathscr{J}(U)) \end{array}$$

The construction of g_U involved choosing a sequence of generators $a_0, \dots, a_n$. We show that in fact g_U does not depend on this choice. Let $b_0, \dots, b_m$ be another set of generators for $\mathscr{J}(U)$ as an $\mathcal{O}_X(U)$-module and let

$$\psi' : f^{-1}U \longrightarrow \mathbb{P}^m_{\mathcal{O}_X(U)}$$
$$\Phi' : ProjB(\mathscr{J}(U)) \longrightarrow \mathbb{P}^m_{\mathcal{O}_X(U)}$$
$$g'_U : f^{-1}U \longrightarrow ProjB(\mathscr{J}(U))$$

be the associated morphisms. For each j we can write $b_j = \sum_i \lambda_{ji} a_i$ for coefficients $\lambda_{ji} \in A$. Similarly we can write $a_i = \sum_j \mu_{ij} b_j$. Define morphisms of graded $\mathcal{O}_X(U)$-algebras

$$\omega : \mathcal{O}_X(U)[x_0, \dots, x_n, y_0, \dots, y_m] \longrightarrow \mathcal{O}_X(U)[x_0, \dots, x_n] \qquad x_i \mapsto x_i, y_j \mapsto \sum_i \lambda_{ji} x_i$$

$$\omega' : \mathcal{O}_X(U)[x_0, \dots, x_n, y_0, \dots, y_m] \longrightarrow \mathcal{O}_X(U)[y_0, \dots, y_m] \qquad x_i \mapsto \sum_j \mu_{ij} y_j, y_j \mapsto y_j$$

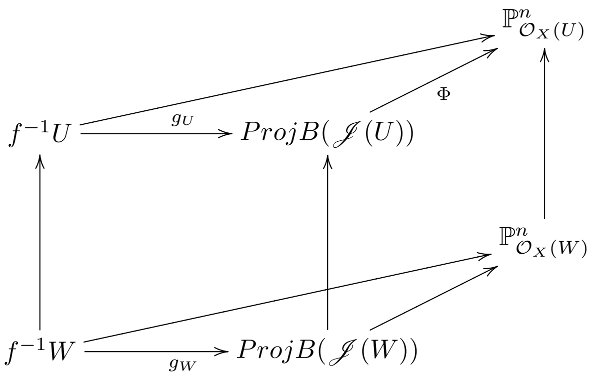
$$\omega'' : \mathcal{O}_X(U)[x_0, \dots, x_n, y_0, \dots, y_m] \longrightarrow B(\mathscr{J}(U)) \qquad x_i \mapsto a_i, y_j \mapsto b_j$$

These morphisms are all surjective, so the induced morphisms of schemes $\Omega, \Omega', \Omega''$ are closed immersions. By construction we have $\Omega\Phi = \Omega'' = \Omega'\Phi'$. To show that $g_U = g'_U$ it suffices to show that $\Omega'' g_U = \Omega'' g'_U$. Since $\Phi g_U = \psi$ and $\Phi' g'_U = \psi'$ by construction, we reduce to showing that $\Omega\psi = \Omega'\psi'$. But by Lemma 7 these morphisms both determine the tuple $(\mathscr{L}, f_U^\#(a_0), \ldots, f_U^\#(a_n), f_U^\#(b_0), \ldots, f_U^\#(b_m))$ and are therefore equal. This shows that the morphism $g_U : f^{-1}U \longrightarrow ProjB(\mathscr{J}(U))$ of U-schemes does not depend on the choice of generators. Our next task is prove two properties of the morphisms g_U:

(i) The morphisms g_U for affine open $U \subseteq X$ are natural in U. That is, we claim that for an affine open subset $W \subseteq U$ the following diagram commutes

$$\begin{array}{ccc} f^{-1}U & \xrightarrow{g_U} & ProjB(\mathscr{J}(U)) \\ \uparrow & & \uparrow \\ f^{-1}W & \xrightarrow[g_W]{} & ProjB(\mathscr{J}(W)) \end{array} \tag{4}$$

If we choose generators $a_0, \ldots, a_n$ for $\mathscr{J}(U)$ then we can choose the elements $a_0|_W, \ldots, a_n|_W$ as our generating set for $\mathscr{J}(W)$ by Lemma 5. Let the additional morphisms in the following diagram be defined with respect to these choices



(5)

To show that (4) commutes it is enough to show that both legs agree when composed with the closed immersion Φ. This amounts to checking that the other two square faces in (5) commute, which is straightforward in light of our special choice of generators for $\mathscr{J}(W)$ and Lemma 8.

(ii) We claim that $g_U : f^{-1}U \longrightarrow ProjB(\mathscr{J}(U))$ is the *unique* morphism of U-schemes $f^{-1}U \longrightarrow ProjB(\mathscr{J}(U))$. Pick generators $a_0, \ldots, a_n$ for $\mathscr{J}(U)$ and define morphisms Φ, ψ as above. If $h : f^{-1}U \longrightarrow ProjB(\mathscr{J}(U))$ is another morphism of schemes over U, then to show $h = g_U$ it suffices to show that the morphism of $\mathcal{O}_X(U)$-schemes $\Phi h : f^{-1}U \longrightarrow \mathbb{P}^n_{\mathcal{O}_X(U)}$ corresponds to the tuple $(\mathscr{L}, f_U^\#(a_0), \ldots, f_U^\#(a_n))$ under (PM,Corollary 4). Here $\mathscr{L}$ is the invertible sheaf $\mathscr{J}|_U \cdot \mathcal{O}_Z|_{f^{-1}U}$, which by (MRS,Lemma 50) can also be written as

$$\mathscr{L} = (\mathscr{J}|_U \cdot \mathcal{O}_{ProjB(\mathscr{J}(U))}) \cdot \mathcal{O}_Z|_{f^{-1}U}$$

By definition there is an epimorphism of sheaves of modules

$$\tau : h^*(\mathscr{J}|_U \cdot \mathcal{O}_{ProjB(\mathscr{J}(U))}) \longrightarrow \mathscr{L}$$

In the proof of Proposition 3 we showed that $\mathscr{J}|_U \cdot \mathcal{O}_{ProjB(\mathscr{J}(U))}$ is actually invertible, so τ is an isomorphism (MRS,Lemma 57). Therefore we have an isomorphism of sheaves of modules

$$\begin{aligned}(\Phi h)^*\mathcal{O}(1) &\cong h^*\Phi^*\mathcal{O}(1) \cong h^*\mathcal{O}(1) \\ &\cong h^*(\mathscr{J}|_U \cdot \mathcal{O}_{ProjB(\mathscr{J}(U))}) \cong \mathscr{L}\end{aligned}$$

Using (MRS,Remark 8), (MPS,Proposition 13), (1) and τ it is readily checked that this isomorphism identifies $(\Phi h)^*(x_i)$ with $f_U^{\#}(a_i)$. Therefore $h = g_U$ and we have the desired uniqueness of g_U.

Now we are ready to define the factorisation $g : Z \longrightarrow \widetilde{X}$. Let g'_U be the following morphism of X-schemes

$$f^{-1}U \longrightarrow ProjB(\mathscr{J}(U)) \cong Proj\mathbb{B}(\mathscr{J})(U) \cong \pi^{-1}U \longrightarrow \widetilde{X}$$

It follows from commutativity of (4) and naturality of (SSA,Proposition 54) in U that the morphisms g'_U for affine open $U \subseteq X$ can be glued to give a morphism of X-schemes $g : Z \longrightarrow \widetilde{X}$ unique with the property that $g|_{f^{-1}U} = g'_U$ for every affine open $U \subseteq X$. The fact that g is unique satisfying $\pi g = f$ follows from the uniqueness property of the morphisms g_U proved in (ii) above. $\square$

Remark 2. In the situation of Theorem 9 the factorisation $g : Z \longrightarrow \widetilde{X}$ can be described explicitly as follows. For every open affine subset $U \subseteq X$ the induced morphism $g_U : f^{-1}U \longrightarrow \pi^{-1}U \cong ProjB(\mathscr{J}(U))$ is the *unique* morphism of schemes over U. It is also unique with the property that for any choice of generators $a_0, \dots, a_n$ for $\mathscr{J}(U)$ as a $\mathcal{O}_X(U)$-module, the composition of g_U with the morphism $ProjB(\mathscr{J}(U)) \longrightarrow \mathbb{P}^n_{\mathcal{O}_X(U)}$ determined by the a_i is the morphism $f^{-1}U \longrightarrow \mathbb{P}^n_{\mathcal{O}_X(U)}$ determined by the tuple

$$(\mathscr{J}|_U \cdot \mathcal{O}_Z|_{f^{-1}U}, f_U^{\#}(a_0), \dots, f_U^{\#}(a_n))$$

Proposition 10. *A morphism of schemes $f : Y \longrightarrow X$ is a closed immersion if and only if for every affine open subset $U \subseteq X$, $f^{-1}U$ is an affine open subset of Y and $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(f^{-1}U)$ is surjective.*

Proof. Suppose that f is a closed immersion. Then our solution to (H, Ex.4.3) shows that the inverse image of affine open sets are affine. Since the closed immersion property is local, if $U \subseteq X$ is affine then the induced morphism $f^{-1}U \longrightarrow U$ is a closed immersion, and therefore $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(f^{-1}U)$ is surjective. Conversely we can cover X with open affines U with the property that $f^{-1}U \longrightarrow U$ is a closed immersion, and therefore f is a closed immersion. $\square$

Lemma 11. *Let $f : Y \longrightarrow X$ be a morphism of schemes, $\mathscr{J}$ a quasi-coherent sheaf of ideals on X and $U \subseteq X$ an affine open subset such that $f^{-1}U$ is also affine. If $s \in (\mathscr{J} \cdot \mathcal{O}_Y)(f^{-1}U)$ then $s = s_1 + \cdots + s_n$ where each s_i is of the form $r f_U^{\#}(a)$ for $r \in \mathcal{O}_Y(f^{-1}U)$ and $a \in \mathscr{J}(U)$.*

Proof. We can reduce to the case where $X = U$ and $Y = f^{-1}U$, so $X \cong Spec\mathcal{O}_X(X)$ and $Y \cong Spec\mathcal{O}_Y(Y)$ and the result follows (MOS,Lemma 9)(a) and the fact that $\mathscr{J}$ is quasi-coherent. $\square$

Corollary 12. *Let $f : Y \longrightarrow X$ be a closed immersion and $\mathscr{J}$ a quasi-coherent sheaf of ideals on X. If $U \subseteq X$ is an affine open subset then $\mathscr{J}(U) \longrightarrow (\mathscr{J} \cdot \mathcal{O}_Y)(f^{-1}U)$ is surjective.*

Proof. By (H,Ex.4.3), $f^{-1}U$ is an affine open subset of Y, and by Proposition 10 the map $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(f^{-1}U)$ is surjective, so the result follows from Lemma 11. $\square$

Corollary 13. *Let $f : Y \longrightarrow X$ be a morphism of noetherian schemes, and let $\mathscr{J}$ be a coherent sheaf of ideals on X. Let $\widetilde{X}$ be the blowing-up of $\mathscr{J}$ and let $\widetilde{Y}$ be the blowing-up of the inverse image ideal sheaf $\mathscr{K} = \mathscr{J} \cdot \mathcal{O}_Y$. Then there is a unique morphism $\widetilde{f} : \widetilde{Y} \longrightarrow \widetilde{X}$ making the following diagram commute*

$$\begin{array}{ccc} \widetilde{Y} & \overset{\widetilde{f}}{\dashrightarrow} & \widetilde{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \underset{f}{\longrightarrow} & X \end{array}$$

Moreover, if f is a closed immersion, so is $\widetilde{f}$.

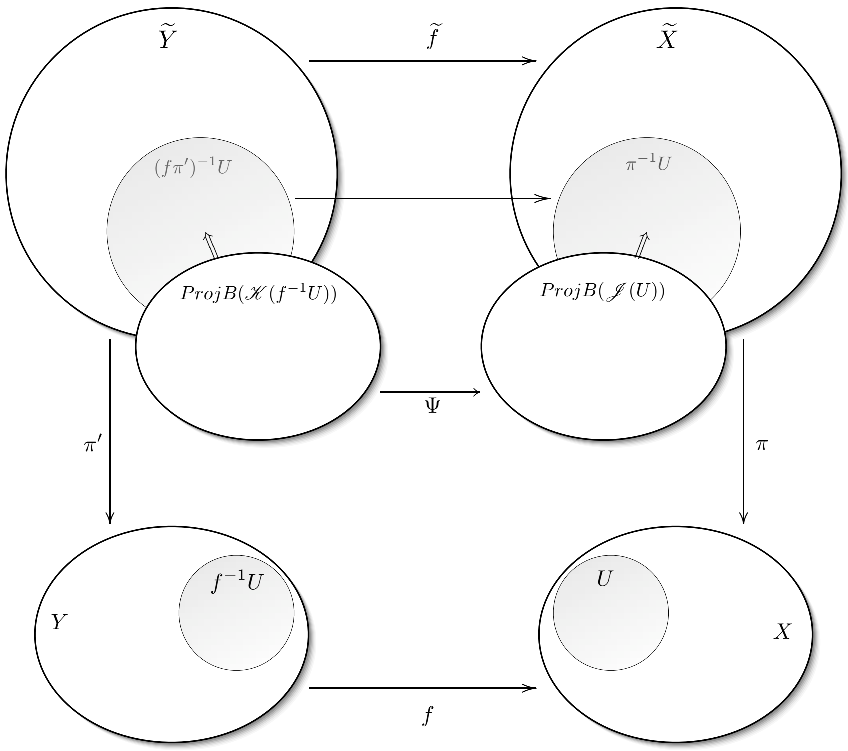
Proof. It follows from (H,5.8) and (H,5.7) that $\mathscr{K}$ is a coherent sheaf of ideals on Y. The existence and uniqueness of $\widetilde{f}$ follows immediately from Theorem 9. Now assume that f is a closed immersion. Then by our solution to (H, Ex. 4.3), $f^{-1}U$ is an affine open set for any affine open $U \subseteq X$. Our first claim is that for affine open $U \subseteq X$ the following diagram is a pullback

$$\begin{array}{ccc} ProjB(\mathscr{K}(f^{-1}U)) & \xrightarrow{\Psi} & ProjB(\mathscr{J}(U)) \\ \downarrow & & \downarrow \\ \widetilde{Y} & \xrightarrow[\widetilde{f}]{} & \widetilde{X} \end{array} \tag{6}$$

where Ψ is induced by the following morphism of graded rings

$$\psi : B(\mathscr{J}(U)) \longrightarrow B(\mathscr{K}(f^{-1}U))$$
$$(r, a_1, a_2, \ldots) \mapsto (f_U^{\#}(r), f_U^{\#}(a_1), f_U^{\#}(a_2), \ldots)$$

By Corollary 12 this is surjective, so in fact Ψ is a closed immersion. To show that (6) commutes it suffices to show that the inside square involving Ψ in the following diagram commutes



By construction $(f\pi')^{-1}U \longrightarrow \pi^{-1}U \cong ProjB(\mathscr{J}(U))$ is the unique morphism of U-schemes between its domain and codomain, so commutativity of (6) follows from the fact that Ψ is a morphism of schemes over U. Since $\widetilde{f}^{-1}\pi^{-1}U = (f\pi')^{-1}U$ it is clear that (6) is a pullback, and therefore $\widetilde{f}$ is a closed immersion since the $\pi^{-1}U$ for open affine $U \subseteq X$ cover $\widetilde{X}$. $\square$

Definition 2. If $f : Y \longrightarrow X$ is a closed immersion of noetherian schemes and $\mathscr{J}$ a coherent sheaf of ideals on X, then we call the closed immersion $\widetilde{f} : \widetilde{Y} \longrightarrow \widetilde{X}$ the *strict transform* of f (or less precisely Y) under the blowing-up $\pi : \widetilde{X} \longrightarrow X$.

1 Blowing up Varieties

Now we will study blowing up in the special case that X is a variety. Recall from our notes on Varieties as Schemes that a variety over a field k is an integral separated scheme of finite type over k.

Definition 3. We say a morphism of schemes $f : X \longrightarrow Y$ is *birational* if there is a nonempty open subset $V \subseteq Y$ such that the induced morphism $f^{-1}V \longrightarrow V$ is an isomorphism. This property is stable under composition with isomorphisms.

Proposition 14. *Let X be a variety over a field k, $\mathscr{J}$ a nonzero coherent sheaf of ideals on X and $\pi : \widetilde{X} \longrightarrow X$ be the blowing-up with respect to $\mathscr{J}$. Then:*

(a) $\widetilde{X}$ is also a variety;

(b) π is a birational, proper, surjective morphism;

(c) If X is quasi-projective or projective over k so is $\widetilde{X}$, and in either case π is a projective morphism.

Proof. (a) We know from the definition and Proposition 2 that $\widetilde{X}$ is an integral noetherian scheme and that π is proper, and therefore separated and of finite type. Using $(4.6b)$ and $(Ex3.13c)$ we see that $\widetilde{X}$ is separated of finite type over k, and is therefore a variety.

(b) Since $\mathscr{J} \neq 0$ the corresponding closed subset Y is proper (use (FCI,Lemma 4) and the fact that X is integral), and if $U = X \setminus Y$ then $\pi^{-1}U \longrightarrow U$ is an isomorphism by Proposition 3, so π is birational. Since π is proper it is closed, so $\pi(\widetilde{X})$ is a closed set containing U, which must be all of X since X is irreducible. Thus π is surjective. If X is quasi-projective then it trivially admits a very ample invertible sheaf, and therefore also an ample invertible sheaf (PM,Theorem 19), and therefore π is a projective morphism (TRPC,Proposition 64). Therefore $\widetilde{X}$ is quasi-projective and if X is projective then so is $\widetilde{X}$ (SEM,Proposition 8). □

Lemma 15. *Let $f : X \longrightarrow Y$ be a birational morphism of integral schemes. If $\mathscr{K}_X, \mathscr{K}_Y$ are the respective sheaves of total quotient rings on X, Y then there is a canonical isomorphism of sheaves of $\mathcal{O}_Y$-algebras $f_*\mathscr{K}_X \longrightarrow \mathscr{K}_Y$.*

Proof. Let ξ be the generic point of X and η the generic point of Y. Since f is birational it is not hard to see that $f(\xi) = \eta$ and that the morphism of rings $f_\eta : \mathcal{O}_{Y,\eta} \longrightarrow \mathcal{O}_{X,\xi}$ is an isomorphism of rings. Using this isomorphism and (DIV,Lemma 30) we obtain the desired isomorphism of sheaves of algebras. □

Remark 3. Let us take a moment to avoid some potential confusion. Let X be a scheme, $\mathscr{K}$ the sheaf of total quotient rings and $\mathscr{F}$ a submodule of $\mathscr{K}$. We consider $\mathcal{O}_X$ as a submodule of $\mathscr{K}$ via the canonical monomorphism $\mathcal{O}_X \longrightarrow \mathscr{K}$ (DIV,Lemma 24). In this way, every sheaf of ideals on X can be considered as a submodule of $\mathscr{K}$. In particular the sheaf of ideals $(\mathcal{O}_X : \mathscr{F})$ defined in (MRS,Definition 13) can be considered as a submodule of $\mathscr{K}$, which we must take care to distinguish from the submodule $(\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})$ defined in (SOA,Definition 8). The distinction is intuitively the distinction between the following A-submodules of the quotient field K of an integral domain A for an A-submodule M of K

$$(A : M) = \{r \in A \,|\, rM \subseteq A\}$$
$$(A :_K M) = \{k \in K \,|\, kM \subseteq A\}$$

If $\mathscr{J}$ is a sheaf of ideals on X then $\mathscr{J}\mathscr{F}$ (see (MRS,Definition 11)) agrees with the product $\mathscr{J}\mathscr{F}$ of (SOA,Definition 7).

Lemma 16. *Let X be an integral noetherian scheme and $\mathscr{F}$ a coherent $\mathcal{O}_X$-submodule of $\mathscr{K}$. Then $(\mathcal{O}_X : \mathscr{F})$ is a nonzero coherent sheaf of ideals on X.*

Proof. Since $\mathscr{K}$ is a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras (DIV,Proposition 29) this follows immediately from (MOS,Corollary 16). To see that $(\mathcal{O}_X : \mathscr{F})$ is nonzero we use (DIV,Lemma 30) and the proof of (MOS,Corollary 16) to reduce to the following algebra problem: A is an integral domain with quotient field Q, M is a finitely generated A-submodule of Q and we need to show that $(A : M)$ is nonzero. But if M is generated by $a_1/s_1, \dots, a_n/s_n$ then $s_1 \cdots s_n$ is a nonzero element of $(A : M)$, so the proof is complete. □

Lemma 17. *If X is a quasi-projective scheme over a noetherian ring A, then there is an ample invertible sheaf $\mathscr{M}$ on X.*

Proof. Since $X \longrightarrow Spec(A)$ is quasi-projective there is a very ample invertible sheaf $\mathscr{M}$ on X, and it follows from (PM,Theorem 19) that this sheaf is ample. □

Lemma 18. *Let X be a scheme and $\mathscr{L}$ an invertible submodule of $\mathscr{K}$. Then $\mathscr{L}\mathscr{K} = \mathscr{K}$ and the canonical morphism $\mathscr{L} \otimes \mathscr{K} \longrightarrow \mathscr{K}$ is an isomorphism of sheaves of modules on X.*

Proof. It suffices to show that $(\mathscr{L}\mathscr{K})_x = \mathscr{K}_x$ for all $x \in X$ (MRS,Lemma 8). But for any $x \in X$ we have $(\mathscr{L}\mathscr{K})_x = \mathscr{L}_x\mathscr{K}_x$ (SOA,Lemma 24) and the $\mathcal{O}_{X,x}$-submodule $\mathscr{L}_x$ of $\mathscr{K}_x$ is generated by a unit of the ring $\mathscr{K}_x$ (DIV,Corollary 38), so this is not difficult to check. We already know that there is an isomorphism of sheaves of modules $\mathscr{L} \otimes \mathscr{K} \longrightarrow \mathscr{L}\mathscr{K}$ (DIV,Proposition 40), so the proof is complete. □

We can now generalise Lemma 6 to submodules of the sheaf of total quotients.

Lemma 19. *Let X be a scheme and $\mathscr{L}$ an invertible submodule of $\mathscr{K}$. Then for any $d > 1$ there is a canonical isomorphism of sheaves of modules*

$$\mathscr{L}^{\otimes d} \longrightarrow \mathscr{L}^d$$
$$a_1 \dot{\otimes} \cdots \dot{\otimes} a_d \mapsto a_1 \cdots a_d$$

In particular the submodule $\mathscr{L}^d$ is invertible.

Proof. We prove the case $d = 2$, with other cases following by a simple induction. We can factor the canonical morphism $\mathscr{L} \otimes \mathscr{L} \longrightarrow \mathscr{K}$ as $\mathscr{L} \otimes \mathscr{L} \longrightarrow \mathscr{L} \otimes \mathscr{K}$ followed by $\mathscr{L} \otimes \mathscr{K} \longrightarrow \mathscr{K}$. The former is a monomorphism since $\mathscr{L}$ is invertible and therefore flat, and the latter is an isomorphism by Lemma 18. Therefore $\mathscr{L} \otimes \mathscr{L} \longrightarrow \mathscr{K}$ is a monomorphism, which implies that $\mathscr{L} \otimes \mathscr{L} \longrightarrow \mathscr{L}^2$ is an isomorphism of sheaves of modules, as required. □

Theorem 20. *Let X be a quasi-projective variety over a field k. If Z is another variety over k and $f : Z \longrightarrow X$ is any birational projective morphism, then there exists a coherent sheaf of ideals $\mathscr{I}$ on X such that Z is X-isomorphic to the blowing-up $\widetilde{X}$ of X with respect to $\mathscr{I}$.*

Proof. The first part of the proof consists of setting up some notation and making some important reductions.

First reduction By (TRPC,Corollary 57) we can assume that Z is $\mathbf{Proj}\mathscr{T}$ for some commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{T}$ locally finitely generated by $\mathscr{T}_1$ as an $\mathcal{O}_X$-algebra with $\mathscr{T}_0 = \mathcal{O}_X$, and that f is the structural morphism $\mathbf{Proj}\mathscr{T} \longrightarrow X$. In fact, we can assume $\mathscr{T}$ is $\mathcal{O}_X[x_0, \dots, x_n]/\mathscr{Z}$ for some $n \geq 1$ and $\mathscr{Z}$ is a quasi-coherent sheaf of homogenous $\mathcal{O}_X[x_0, \dots, x_n]$-ideals with $\mathscr{Z}_0 = 0$.

By (TRPC,Corollary 36) the canonical morphism of sheaves of graded $\mathcal{O}_X$-algebras $\eta : \mathscr{T} \longrightarrow \Gamma_*(\mathcal{O}_Z)'$ is a quasi-isomorphism. Let $\mathscr{B}$ the sheaf of graded $\mathcal{O}_X$-algebras with $\mathscr{B}_0 = \mathcal{O}_X$ and $\mathscr{B}_d = \Gamma_*(\mathcal{O}_Z)'_d$ for $d \geq 1$. This is a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras (see the proof of (TRPC,Proposition 53)) and η induces a quasi-isomorphism of sheaves of graded $\mathcal{O}_X$-algebras $\lambda : \mathscr{T} \longrightarrow \mathscr{B}$. By (TRPC,Proposition 55) there is an integer $E > 0$ such that for all $e \geq E$, $\lambda^{(e)} : \mathscr{T}^{(e)} \longrightarrow \mathscr{B}^{(e)}$ is an isomorphism of sheaves of graded $\mathcal{O}_X$-algebras. So to complete the proof it suffices to find a coherent sheaf of ideals $\mathscr{I}$ on X, an invertible sheaf $\mathscr{J}$ and an isomorphism of sheaves of graded $\mathcal{O}_X$-algebras $\mathscr{S}_{(\mathscr{J})} \cong \mathbb{B}(\mathscr{I})$ where $\mathscr{S} = \mathscr{B}^{(E)}$ (TRPC,Corollary 52),(TRPC,Proposition 59).

Let $\mathscr{K}_Z$ and $\mathscr{K}_X$ be the sheaves of total quotient rings on Z, X respectively (DIV,Definition 10). Since Z is integral there is a monomorphism $\psi : \mathcal{O}(E) \longrightarrow \mathscr{K}_Z$ (DIV,Proposition 44) and we let $\mathscr{L}$ denote the image of this morphism. Therefore by (H,5.20) $f_*\mathscr{L}$ is a coherent submodule of $f_*\mathscr{K}_Z$ and using the isomorphism $f_*\mathscr{K}_Z \cong \mathscr{K}_X$ of Lemma 15 we identify $f_*\mathscr{L}$ with a coherent submodule of $\mathscr{K}_X$.

Let $\mathscr{J}'$ be the nonzero coherent sheaf of ideals $(\mathcal{O}_X : f_*\mathscr{L})$ on X of Lemma 16. Using the composite $\mathscr{J}' \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}_X$ we identify $\mathscr{J}'$ with a coherent submodule of $\mathscr{K}_X$. It is clear that $\mathscr{J}' \cdot f_*\mathscr{L} \subseteq \mathcal{O}_X$ using the product of (SOA,Definition 7). Our next task is to replace $\mathscr{J}'$ by an invertible submodule. Since X is quasi-projective it admits an ample invertible sheaf $\mathscr{M}$ by Lemma 17, which we can assume is a submodule of $\mathscr{K}_X$. Since $\mathscr{J}'$ is nonzero and coherent the sheaf $\mathscr{J}' \otimes \mathscr{M}^{\otimes n}$ admits a nonzero global section x' for some $n > 1$. Using Lemma 19 and (DIV,Proposition 40) there is a canonical isomorphism of sheaves of modules

$$\mathscr{J}' \otimes \mathscr{M}^{\otimes n} \cong \mathscr{J}' \otimes \mathscr{M}^n \cong \mathscr{J}'\mathscr{M}^n$$

Let x be the nonzero global section of $\mathscr{J}'\mathscr{M}^n$ corresponding to x'. Then $(x) \subseteq \mathscr{J}'\mathscr{M}^n$ is an invertible submodule of $\mathscr{K}$ (DIV,Proposition 41)(e). Multiplying both sides of this inclusion by $\mathscr{M}^{-n}$ (see (DIV,Proposition 41)) we have $(x)\mathscr{M}^{-n} \subseteq \mathscr{J}'$. The product of invertible submodules of $\mathscr{K}$ is invertible (DIV,Proposition 40), (MRS,Lemma 56), so $\mathscr{J} = (x)\mathscr{M}^{-n}$ is an invertible submodule of $\mathscr{J}'$, as required. The submodule $\mathscr{J} \cdot f_*\mathscr{L} \subseteq \mathcal{O}_X$ corresponds to a coherent sheaf of ideals $\mathscr{I}$ on X (SOA,Proposition 28).

This is the required ideal sheaf, as we will now show that Z is X-isomorphic to the blowing up of X with respect to $\mathscr{I}$. For $d \geq 1$ we have a monomorphism of sheaves of modules

$$\begin{aligned} \kappa^d : \mathscr{I}^d &\cong (f_*\mathscr{L})^d \cdot \mathscr{J}^d \cong (f_*\mathscr{L})^d \otimes \mathscr{J}^{\otimes d} \longrightarrow f_*(\mathscr{L}^d) \otimes \mathscr{J}^{\otimes d} \\ &\cong f_*(\mathscr{L}^{\otimes d}) \otimes \mathscr{J}^{\otimes d} \cong f_*(\mathcal{O}(E)^{\otimes d}) \otimes \mathscr{J}^{\otimes d} \cong f_*(\mathcal{O}(dE)) \otimes \mathscr{J}^{\otimes d} \\ &\cong f_*(\mathcal{O}_Z \otimes \mathcal{O}(dE)) \otimes \mathscr{J}^{\otimes d} \cong (\mathscr{S}_{(\mathscr{J})})_d \end{aligned}$$

Using (DIV,Proposition 40), (SOA,Lemma 26), Lemma 19, (TRPC,Lemma 17) and the fact that invertible modules are flat. Clearly κ^1 is an isomorphism. Taking the coproduct we have a monomorphism of sheaves of modules on X

$$\kappa = 1 \oplus \bigoplus_{d \geq 1} \kappa^d : \mathbb{B}(\mathscr{I}) \longrightarrow \mathscr{S}_{(\mathscr{J})}$$

To check that this is a morphism of sheaves of graded $\mathcal{O}_X$-algebras it suffices to show that for $d, e > 0$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{I}^d \otimes \mathscr{I}^e & \longrightarrow & \mathscr{I}^{d+e} \\ {\scriptstyle \kappa^d \otimes \kappa^e} \downarrow & & \downarrow {\scriptstyle \kappa^{d+e}} \\ (\mathscr{S}_{(\mathscr{J})})_d \otimes (\mathscr{S}_{(\mathscr{J})})_e & \longrightarrow & (\mathscr{S}_{(\mathscr{J})})_{d+e} \end{array}$$

This is tedious but straightforward. To show that κ is an isomorphism of sheaves of graded $\mathcal{O}_X$-algebras it suffices to show that $\kappa_U : \mathbb{B}(\mathscr{I})(U) \longrightarrow \mathscr{S}_{(\mathscr{J})}(U)$ is a surjective morphism of graded $\mathcal{O}_X(U)$-algebras for nonempty affine open $U \subseteq X$ (MOS,Lemma 2). This follows from the fact that κ^1_U is an isomorphism and $\mathscr{S}_{(\mathscr{J})}(U)$ is generated by $\mathscr{S}_{(\mathscr{J})}(U)_1$ as an $\mathcal{O}_X(U)$-algebra. The isomorphism κ induces the desired isomorphism of X-schemes $\mathbf{Proj}\mathscr{T} \cong \widetilde{X}$, where $\widetilde{X}$ is the blowing-up of X with respect to $\mathscr{I}$. $\square$