Section 2.7 - Projective Morphisms

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In this section we gather together several topics concerned with morphisms of a given scheme to projective space. We will show how a morphism of a scheme X to a projective space is determined by giving an invertible sheaf $\mathscr{L}$ on X and a set of its global sections. We will give some criteria for this morphism to be an immersion. Then we study the closely connected topic of ample invertible sheaves.

Contents

1 Morphisms to $\mathbb{P}^n$

Let A be a fixed ring, and consider the projective space $\mathbb{P}^n_A = Proj A[x_0, \ldots, x_n]$ over A. On $\mathbb{P}^n_A$ we have the invertible sheaf $\mathcal{O}(1)$, and the homogenous coordinates $x_0, \ldots, x_n$ give rise to global sections $\dot{x_0}, \ldots, \dot{x_n} \in \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$. Throughout this section we drop the dot in this notation and just write x_i for the global section of $\mathcal{O}(1)$. One sees easily that the sheaf $\mathcal{O}(1)$ is generated by the global sections $x_0, \ldots, x_n$, i.e., the images of these sections generate the stalk $\mathcal{O}(1)_P$ of the sheaf $\mathcal{O}(1)$ as a module over the local ring $\mathcal{O}_{X,P}$ for each point $P \in \mathbb{P}^n_A$.

Definition 1. Let $f : X \longrightarrow Y$ be a morphism of schemes and $\mathscr{F}$ be a $\mathcal{O}_Y$-module. Let $\eta : \mathscr{F} \longrightarrow f_* f^* \mathscr{F}$ be canonical. Given $s \in \mathscr{F}(V)$ we denote by $f^*(s)$ the section $\eta_V(s) \in f^*\mathscr{F}(f^{-1}V)$. So $f^*(s) = [V, s] \dot{\otimes} 1$ and for $x \in f^{-1}V$ we have $f^*(s)(x) = (f^{-1}V, (\dot{V, s}) \otimes 1)$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of $\mathcal{O}_Y$-modules then $(f^*\phi)_{f^{-1}V}(f^*(s)) = f^*(\phi_V(s))$.

Lemma 1. *Let $f : X \longrightarrow Y$ be a morphism of schemes and let $\mathscr{F}$ be a $\mathcal{O}_Y$-module generated by global sections $x_1, \ldots, x_n \in \mathscr{F}(Y)$. Then the global sections $s_i = f^*(x_i)$ generate $f^*\mathscr{F}$.*

Proof. Our notes on the isomorphism $(f^*\mathscr{F})_x \cong \mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ show that there is a commutative diagram of abelian groups for $x \in X$:

$$\begin{array}{ccc} \mathscr{F}_{f(x)} & \longrightarrow & (f^*\mathscr{F})_x \\ & \searrow & \Downarrow \\ & & \mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \end{array}$$

where the top morphism is compatible with the ring morphism $f_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ and maps $germ_{f(x)} x_i$ to $germ_x s_i$, the right side is an isomorphism of $\mathcal{O}_{X,x}$-modules and the diagonal map is $m \mapsto m \otimes 1$. Since $\mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ is clearly generated by the $germ_{f(x)} x_i \otimes 1$ as a $\mathcal{O}_{X,x}$-module it follows that $f^*\mathscr{F}$ is generated by the global sections s_i, as required. $\square$

Now let X be any scheme over A, and let $\varphi : X \longrightarrow \mathbb{P}^n_A$ be an A-morphism. Then $\mathscr{L} = \varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X and the global sections $s_0, \ldots, s_n$ where $s_i = \varphi^*(x_i)$, $s_i \in \Gamma(X, \mathscr{L})$ generate the sheaf $\mathscr{L}$. Conversely, we will see that $\mathscr{L}$ and the sections s_i determine φ.

Lemma 2. *Let $(X, \mathcal{O}_X)$ be a ringed space and $U \subseteq X$ open. If $s \in \mathcal{O}_X(U)$ is such that $germ_x s \in \mathcal{O}_{X,x}$ is a unit for all $x \in U$, then s is a unit in $\mathcal{O}_X(U)$.*

Proof. For each $x \in U$ we can find an open neighborhood $x \in V_x \subseteq U$ and $t_x \in \mathcal{O}_X(V_x)$ with $s|_{V_x} t_x = 1$. It is clear that the t_x paste together to give an inverse for s. □

Theorem 3. *Let A be a ring and let X be a scheme over A.*

(a) If $\varphi : X \longrightarrow \mathbb{P}^n_A$ is an A-morphism $(n \geq 1)$, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by the global sections $s_i = \varphi^*(x_i)$, $i = 0, 1, \ldots, n$.

(b) Conversely, if $\mathscr{L}$ is an invertible sheaf on X, and if $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ are global sections which generate $\mathscr{L}$ $(n \geq 1)$, then there exists a unique A-morphism $\varphi : X \longrightarrow \mathbb{P}^n_A$ such that $\mathscr{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism.

Proof. Part (a) is clear from the discussion above. To prove (b), suppose we are given $\mathscr{L}$ and the global sections $s_0, \ldots, s_n$ which generate it. The result is trivial if $X = \emptyset$, so assume X is nonempty. For each i let $X_i = \{x \in X \,|\, germ_x s_i \notin \mathfrak{m}_x \mathscr{L}_x\}$. This is an open subset of X (see our Locally Free Sheaves notes) and since the s_i generate $\mathscr{L}$ and $\mathscr{L}_x \cong \mathcal{O}_{X,x} \neq 0$ for all $x \in X$ the open sets X_i must cover X.

For each i and $x \in X_i$ we can choose a basis $\theta \in \mathscr{L}_x$ and write $germ_x s_i = \lambda^x_i \theta$. By definition of X_i, λ^x_i is a unit in the local ring $\mathcal{O}_{X,x}$. For $j \neq i$ if we write $germ_x s_j = \lambda^x_j \theta$ then the quotient $\lambda^x_j / \lambda^x_i \in \mathcal{O}_{X,x}$ is independent of the basis θ chosen. In this way we associate an element λ^x_j/λ^x_i with each point of X_i. We denote by s_j/s_i the unique element of $\Gamma(X_i, \mathcal{O}_X)$ with $germ_x s_j/s_i = \lambda^x_j/\lambda^x_i$ for all $x \in X_i$ (equivalently $germ_x s_j/s_i \cdot germ_x s_i = germ_x s_j$ for all $x \in X_i$).

We define a morphism from X_i to the standard open set $U_i = D_+(x_i)$ of $\mathbb{P}^n_A$ as follows. Recall that $U_i \cong SpecA[x_0/x_i, \ldots, x_n/x_i]$ with x_i/x_i omitted. We define a morphism of A-algebras $A[x_0/x_i, \ldots, x_n/x_i] \longrightarrow \Gamma(X_i, \mathcal{O}_X|_{X_i})$ by sending $x_j/x_i \mapsto s_j/s_i$. This induces morphisms of schemes $\varphi_i : X_i \longrightarrow SpecA[x_0/x_i, \ldots, x_n/x_i] \cong U_i \longrightarrow \mathbb{P}^n_A$ over A. To show that we can glue the φ_i, we need to show that for any $j \neq i$ the following diagram commutes:

$$\begin{array}{ccc} X_i \cap X_j & \longrightarrow & X_i \\ \downarrow & & \downarrow{\scriptstyle \varphi_i} \\ X_j & \xrightarrow[\varphi_j]{} & \mathbb{P}^n_A \end{array} \qquad (1)$$

The morphism $A[x_0/x_i, \ldots, x_n/x_i] \longrightarrow \Gamma(X, \mathcal{O}_X|_{X_i})$ defined above composes with restriction to give $A[x_0/x_i, \ldots, x_n/x_i] \longrightarrow \Gamma(X_i \cap X_j, \mathcal{O}_X|_{X_i \cap X_j})$, which clearly sends x_j/x_i to a unit. So we get a morphism of schemes $X_j \cap X_i \longrightarrow SpecA[x_0/x_i, \ldots, x_n/x_i]_{x_j/x_i}$ over A, and similarly $X_j \cap X_i \longrightarrow SpecA[x_0/x_j, \ldots, x_n/x_j]_{x_i/x_j}$. Our Section 2.2 notes show that there are isomorphisms of A-algebras

$$A[x_0/x_j, \ldots, x_n/x_j]_{x_i/x_j} \cong A[x_0, \ldots, x_n]_{(x_i x_j)} \cong A[x_0/x_i, \ldots, x_n/x_i]_{x_j/x_i}$$

which make the bottom right square in the following diagram of schemes over A commute

$$\begin{array}{ccccc} & & X_i & \longrightarrow & SpecA[x_0/x_i, \ldots, x_n/x_i] \\ & & \uparrow & & \uparrow \\ X_j & \longleftarrow & X_i \cap X_j & \longrightarrow & SpecA[x_0/x_i, \ldots, x_n/x_i]_{x_j/x_i} \\ \downarrow & & \downarrow & & \Downarrow \\ SpecA[x_0/x_j, \ldots, x_n/x_j] & \longleftarrow & SpecA[x_0/x_j, \ldots, x_n/x_j]_{x_i/x_j} & \Longrightarrow & SpecA[x_0, \ldots, x_n]_{(x_i x_j)} \end{array}$$

Commutativity of the other two squares follows easily from the definition of φ_i, φ_j. It also follows from our Section 2.2 notes that the following diagram (and its partner with i, j interchanged) commutes

$$\begin{array}{ccc}
SpecA[x_0/x_i, \ldots, x_n/x_i] & \Longrightarrow & D_+(x_i) \\
\uparrow & & \\
SpecA[x_0/x_i, \ldots, x_n/x_i]_{x_j/x_i} & & \uparrow \\
\Downarrow & & \\
SpecA[x_0, \ldots, x_n]_{(x_i x_j)} & \Longrightarrow & D_+(x_i x_j)
\end{array}$$

Combining these three diagrams shows that (1) commutes, as required. So there exists a unique morphism $\varphi : X \longrightarrow \mathbb{P}^n_A$ of schemes over A restricting to give φ_i on X_i.

Next we have to show that $\varphi^*\mathcal{O}(1) \cong \mathscr{L}$. Set $Y = \mathbb{P}^n_A$ and suppose we are given $x \in X$. Set $\mathfrak{p} = \varphi(x)$ and identify $\mathcal{O}(1)_{\varphi(x)}$ with $S(1)_{(\mathfrak{p})}$ and $\mathcal{O}_{Y,\varphi(x)}$ with $S_{(\mathfrak{p})}$. If $x \in X_i$ then $\varphi(x) \in U_i$ so $x_i/1$ is a $\mathcal{O}_{Y,\varphi(x)}$-basis of $\mathcal{O}(1)_{\varphi(x)}$, which gives an isomorphism $\alpha_i : \mathcal{O}(1)_{\varphi(x)} \cong \mathcal{O}_{Y,\varphi(x)}$. By definition of X_i, $germ_x s_i$ is a $\mathcal{O}_{X,x}$-basis of $\mathscr{L}_x$, which gives an isomorphism $\beta_i : \mathcal{O}_{X,x} \cong \mathscr{L}_X$. Together these give an isomorphism of $\mathcal{O}_{X,x}$-modules

$$\begin{aligned}
\kappa_x : (\varphi^*\mathcal{O}(1))_x &\cong \mathcal{O}(1)_{\varphi(x)} \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{O}_{X,x} \\
&\cong \mathcal{O}_{Y,\varphi(x)} \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{O}_{X,x} \\
&\cong \mathcal{O}_{X,x} \cong \mathscr{L}_x
\end{aligned}$$

If also $x \in X_j$ then we get another isomorphism, and we claim it is the same as the one obtained using i. We have to show that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{O}(1)_{\varphi(x)} \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{O}_{X,x} & \xRightarrow{\alpha_i \otimes 1} & \mathcal{O}_{Y,\varphi(x)} \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{O}_{X,x} \\
{\scriptstyle \alpha_j \otimes 1}\Downarrow & & \Downarrow \\
\mathcal{O}_{Y,\varphi(x)} \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{O}_{X,x} & & \mathcal{O}_{X,x} \\
\Downarrow & & \Downarrow{\scriptstyle \beta_i} \\
\mathcal{O}_{X,x} & \xRightarrow[\beta_j]{} & \mathscr{L}_x
\end{array}$$

We invert $\alpha_j \otimes 1$ and track $a \otimes b \in \mathcal{O}_{Y,\varphi(x)} \otimes \mathcal{O}_{X,x}$ both ways around the diagram. We end up having to check that $\varphi_x(ax_j/x_i)b \cdot germ_x s_i$ and $\varphi_x(a)b \cdot germ_x s_j$ are the same element of $\mathscr{L}_x$, which follows immediately from the fact that $\varphi_x(x_j/x_i) = germ_x s_j/s_i$.

Next we define

$$\kappa : \varphi^*\mathcal{O}(1) \longrightarrow \mathscr{L}$$
$$germ_x \kappa_U(s) = \kappa_x(germ_x s)$$

Given $s \in \Gamma(U, \varphi^*\mathcal{O}(1))$ we have to check that the germs $\kappa_x(germ_x s) \in \mathscr{L}_x$ actually belong to a section of $\mathscr{L}$. We reduce easily to the case $s = (\dot{T}, a) \dot{\otimes} b$ where $U \subseteq X_i$ for some i and $(T, a) \in \varinjlim_{T \supseteq \varphi(U)} \Gamma(T, \mathcal{O}(1))$, $b \in \mathcal{O}_X(U)$. We can assume that $T \subseteq U_i$, so that $a = \mu \cdot x_i/1$ for some $\mu \in \mathcal{O}_X(T)$. Let $Q = \varphi^{-1}T \cap U$. Then for $y \in Q$ we have $germ_{\varphi(y)} a = germ_{\varphi(y)} \mu \cdot x_i/1$ in $\mathcal{O}(1)_{\varphi(y)}$ and therefore

$$\kappa_y(germ_y s) = \varphi_y(germ_{\varphi(y)}\mu) germ_y b \cdot germ_y s_i = germ_y t$$

where $t = \varphi^\#_T(\mu)|_Q b|_Q \cdot s_i|_Q$. This shows that κ is well-defined, and it is not hard to check it is an isomorphism of $\mathcal{O}_X$-modules. It is clear from the definition that the isomorphism κ maps the global section $\varphi^*(x_i)$ to s_i for all $0 \leq i \leq n$.

To prove uniqueness, let $\psi : X \longrightarrow \mathbb{P}^n_A$ be another A-morphism for which there exists an isomorphism $\kappa' : \psi^*\mathcal{O}(1) \longrightarrow \mathscr{L}$ with $\psi^*(x_i) = s_i$. Notice that for $x \in X$ the isomorphism

$$\mathcal{O}(1)_{\psi(x)} \otimes_{\mathcal{O}_{Y,\psi(x)}} \mathcal{O}_{X,x} \cong (\psi^*\mathcal{O}(1))_x \cong \mathscr{L}_x \tag{2}$$

sends the element $x_i/1 \otimes 1$ is mapped to $germ_x s_i$. Write $\mathfrak{p} = \psi(x)$ and suppose that $\mathfrak{p} \notin U_i$. Let j be such that $\mathfrak{p} \in U_j$, and note that in $\mathcal{O}_{Y,\mathfrak{p}}$ we have $x_i/1 = x_i/x_j \cdot x_j/1$ (identifying $\mathcal{O}_{Y,\mathfrak{p}}$ with $S_{(\mathfrak{p})}$ and $\mathcal{O}(1)_\mathfrak{p} \cong S(1)_{(\mathfrak{p})}$). Since $x_i \in \mathfrak{p}$ it follows that $germ_x s_i \in \mathfrak{m}_x \mathscr{L}_x$. This shows that $\psi(X_i) \subseteq U_i$. Similarly one shows that if $x \in X_j$ and $\psi(x) \in U_i$ then $x \in X_i$, so we conclude that $X_i = \psi^{-1}U_i$. Let ψ_i be the restriction of ψ to X_i, and let $X_i \longrightarrow U_i$ be the unique morphism fitting into the following pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \mathbb{P}^n_A \\ \uparrow & & \uparrow \\ X_i & \longrightarrow & U_i \end{array} \tag{3}$$

This induces a morphism of A-algebras $A[x_0/x_i, \dots, x_n/x_i] \longrightarrow \Gamma(X_i, \mathcal{O}_X|_{X_i})$. To show that this morphism maps x_j/x_i to s_j/s_i it suffices to show that $\psi_x(x_j/x_i) = \lambda^x_j/\lambda^x_i$ for $x \in X_i$. Since $germ_x s_i$ is a basis for $\mathscr{L}_x$ it would be enough to show that $\psi_x(x_j/x_i) \cdot germ_x s_i = germ_x s_j$. But

$$\begin{aligned} \psi_x(x_j/x_i) \cdot germ_x s_i &= \psi_x(x_j/x_i) \cdot \kappa'_x(germ_x \psi^*(x_i)) \\ &= \kappa'_x(\psi_x(x_j/x_i) \cdot germ_x \psi^*(x_i)) \\ &= \kappa'_x(germ_x \psi^*(x_j)) = germ_x s_j \end{aligned}$$

as required. It follows that $\psi_i = \varphi_i$ for all i, and consequently $\varphi = \psi$, which completes the proof. It is not difficult to see that not only do we have $\varphi = \psi$, but also $\kappa' = \kappa$. So the isomorphism $\mathscr{L} \cong \varphi^*\mathcal{O}(1)$ is uniquely determined by the data $\mathscr{L}, s_0, \dots, s_n$. □

Definition 2. Let X be a scheme and fix $n \geq 1$. We say two tuples $(\mathscr{L}, s_0, \dots, s_n)$ and $(\mathscr{E}, t_0, \dots, t_n)$ of invertible sheaves and generating global sections are *equivalent* if there is an isomorphism of sheaves of modules $\mathscr{L} \cong \mathscr{E}$ under which $s_i = t_i$ for all i. The class of the equivalence classes under this relation is denoted $IG_n(X)$. We will see in a moment that this is actually small. If $f : X \longrightarrow Y$ is a morphism of schemes then there is a well-defined map

$$\begin{gathered} IG_n(f) : IG_n(Y) \longrightarrow IG_n(X) \\ (\mathscr{L}, s_0, \dots, s_n) \longrightarrow (f^*\mathscr{L}, f^*(s_0), \dots, f^*(s_n)) \end{gathered}$$

With this definition $IG_n(-) : \mathbf{Sch} \longrightarrow \mathbf{Sets}$ is a contravariant functor by (MRS,Remark 8) and (MRS,Proposition 110).

Corollary 4. *For any scheme X over a ring A and $n \geq 1$ there is a bijection*

$$\begin{gathered} Hom_A(X, \mathbb{P}^n_A) \longrightarrow IG_n(X) \\ \varphi \mapsto (\varphi^*\mathcal{O}(1), \varphi^*(x_0), \dots, \varphi^*(x_n)) \end{gathered}$$

If $f : X \longrightarrow Y$ a morphism of schemes over A then the following diagram commutes

$$\begin{array}{ccc} Hom_A(Y, \mathbb{P}^n_A) & \Longrightarrow & IG_n(Y) \\ \downarrow & & \downarrow{\scriptstyle IG_n(f)} \\ Hom_A(X, \mathbb{P}^n_A) & \Longrightarrow & IG_n(X) \end{array}$$

Finally if φ corresponds to $(\mathscr{L}, s_0, \dots, s_n)$ then $\varphi^{-1}U_i = X_i$ and (3) is a pullback where $U_i = D_+(x_i)$ and $X_i = X_{s_i} = \{x \in X \mid germ_x s_i \notin \mathfrak{m}_x \mathscr{L}_x\}$.

Proof. Note that if $(\mathscr{L}, s_0, \dots, s_n) \sim (\mathscr{E}, t_0, \dots, t_n)$ the isomorphism $\mathscr{L} \cong \mathscr{E}$ with the desired property is unique, since the s_i and t_i generate the respective invertible sheaves. Theorem 3 shows that the map is well-defined and surjective. The uniqueness part of (b) shows that the map is also injective, so we have the desired bijection. To prove naturality, let $\varphi : Y \longrightarrow \mathbb{P}^n_A$ be a morphism of schemes over A. Then we need to show that the following tuples determine the same equivalence class of $IG_n(X)$

$$(f^*g^*\mathcal{O}(1), f^*g^*(x_0), \dots, f^*g^*(x_n)), \quad ((gf)^*\mathcal{O}(1), (gf)^*(x_0), \dots, (gf)^*(x_n))$$

But this follows immediately from the fact that the canonical isomorphism of sheaves of modules $(gf)^*\mathcal{O}(1) \cong f^*g^*\mathcal{O}(1)$ (MRS,Remark 8) identifies $f^*g^*(x_i)$ with $(gf)^*(x_i)$. □

Corollary 5. *For any scheme X and $n \geq 1$ the conglomerate $IG_n(X)$ is small.*

Proof. Any scheme is a scheme over $\mathbb{Z}$, so in particular $IG_n(X) \cong Hom(X, \mathbb{P}^n_{\mathbb{Z}})$ is small. □

Definition 3. Fix $n \geq 1$ and let $A = (a_{ij})$ be an invertible $(n+1) \times (n+1)$ matrix over a field k. For convenience we use the indices $0 \leq i, j \leq n$. Let X be a scheme over k and $\mathscr{L}$ an invertible sheaf on X together with generating global sections $s_0, \dots, s_n$. It is not hard to see that the global sections $s_i' = \sum_{j=0}^n a_{ij} s_j$ also generate $\mathscr{L}$ and that this gives a well-defined function

$$IG_n^A : IG_n(X) \longrightarrow IG_n(X) \tag{4}$$

$$(\mathscr{L}, s_0, \dots, s_n) \mapsto (\mathscr{L}, \sum_j a_{0j} s_j, \dots, \sum_j a_{nj} s_j) \tag{5}$$

If $B \in GL_{n+1}(k)$ is another invertible matrix then $IG_n^B \circ IG_n^A = IG_n^{BA}$ and $IG_n^{I_{n+1}} = 1$. This shows that (4) is a bijection of sets. The matrix A also determines an isomorphism of k-algebras

$$\varphi_A : k[x_0, \dots, x_n] \longrightarrow k[x_0, \dots, x_n]$$
$$x_i \mapsto \sum_{j=0}^n a_{ij} x_j$$

which gives rise to an isomorphism of k-schemes $\Phi_A = Proj \varphi_A : \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$. If $A, B \in GL_{n+1}(k)$ then $\varphi_B \varphi_A = \varphi_{AB}$ so it is clear that $\Phi_A \Phi_B = \Phi_{AB}$. Moreover If $A = \lambda I_{n+1}$ for some nonzero $\lambda \in k$ then $\Phi_A = 1$. In particular $\Phi_{\lambda A} = \Phi_A$ for any $A \in GL_{n+1}(k)$.

Proposition 6. *Let X be a scheme over a field k and $A \in GL_{n+1}(k)$ for some $n \geq 1$. Then the following diagram commutes*

$$\begin{array}{ccc} Hom_k(X, \mathbb{P}^n_k) & \Longrightarrow & IG_n(X) \\ \Phi_A \circ - \Big\Downarrow & & \Big\Downarrow IG_n^A \\ Hom_k(X, \mathbb{P}^n_k) & \Longrightarrow & IG_n(X) \end{array}$$

Proof. Given a morphism of k-schemes $\varphi : X \longrightarrow \mathbb{P}^n_k$ it suffices to observe that the canonical isomorphism $(\Phi_A \varphi)^* \mathcal{O}(1) \cong \varphi^* \Phi_A^* \mathcal{O}(1) \cong \varphi^* \mathcal{O}(1)$ maps $(\Phi_A \varphi)^* x_i$ to $\sum_{j=0}^n a_{ij} \varphi^* x_i$. This follows from (MRS,Remark 8) and (MPS,Proposition 13). □

Proposition 7. *Let $\varphi : X \longrightarrow \mathbb{P}^n_A$ be a morphism of schemes over A, corresponding to an invertible sheaf $\mathscr{L}$ on X and sections $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$ as above. Then φ is a closed immersion if and only if*

(1) Each open set $X_i = X_{s_i}$ is affine;

(2) For each i, the A-algebra morphism $A[y_0, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_X)$ defined by $y_j \mapsto s_j/s_i$ is surjective.

Proof. First we note that if $(\mathscr{L}, s_0, \dots, s_n), (\mathscr{E}, t_0, \dots, t_n)$ are equivalent (in the sense of Corollary 4) then $(1), (2)$ are satisfied for one iff. they are satisfied for the other. This is clear in the case of (1) since $X_i = X_{s_i} = X_{t_i}$, and it is not hard to see that s_j/s_i determines the same section of $\Gamma(X_i, \mathcal{O}_X)$ as t_j/t_i, so it is clear in the case of (2) as well.

Now suppose that φ is a closed immersion. For each $0 \le i \le n$ we have a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{P}^n_A \\ \uparrow & & \uparrow \\ X_i & \longrightarrow & U_i \end{array}$$

Since $X_i = \varphi^{-1}U_i$ it follows from our proof of Ex 4.3 that X_i is an affine open subset. Moreover by Ex 3.11(a) the morphism $X_i \longrightarrow U_i$ is a closed immersion, and therefore the induced morphism $A[y_0, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_X)$ is surjective by (5.10). Conversely, suppose that (1) and (2) are satisfied. Then for each i the morphism $X_i \longrightarrow U_i$ is a closed immersion, and since the property of being a closed immersion is local on the base (see p.14 of our Section 2.3 notes) it follows that φ is a closed immersion. ☐

Remark 1. Let k be an algebraically closed field, set $S = k[x_0, \dots, x_n]$ for $n \ge 1$ and let $Y = \mathbb{P}^n_k = Proj S$. Then there is a canonical isomorphism of schemes $Y \cong t(\mathbb{P}^n)$ (VS,Corollary 8). Under this isomorphism a point $P \in \mathbb{P}^n$ (which corresponds to a closed point of $t(\mathbb{P}^n)$) corresponds to the homogenous prime ideal $I(P)$ of S. If $f \in S$ is homogenous of degree $d > 0$ then $f \in I(P)$ if and only if $f(P) = 0$. Consider the corresponding global section $\dot{f} \in \Gamma(Y, \mathcal{O}_Y(d))$ and the associated open subset of Y

$$X_{\dot{f}} = \{x \in Y \,|\, germ_x f \notin \mathfrak{m}_x \mathcal{O}_Y(d)_x\} = \{\mathfrak{p} \in Proj S \,|\, f \notin \mathfrak{p}\} = D_+(f)$$

Therefore the closed points of the closed set $Y \setminus X_{\dot{f}}$ correspond under the isomorphism $Y \cong t(\mathbb{P}^n)$ to the points of the linear variety $Z(f)$ of $\mathbb{P}^n$. In other words, the closed points of $V(f)$ are homeomorphic to the variety $Z(f)$. By (AAMPS,Proposition 15) the map $f \mapsto \dot{f}$ defines a bijection between S_d and $\Gamma(Y, \mathcal{O}_Y(d))$. In particular we can identify nonzero elements of $\Gamma(Y, \mathcal{O}_Y(1))$ with hyperplanes in $\mathbb{P}^n$ (LV,Definition 2).

With more hypotheses we can give a more local criterion.

Lemma 8. *Let $f : X \longrightarrow Y$ be a closed morphism of schemes of finite type over a field k. Then f is injective if and only if it is injective on closed points.*

Proof. We know from (VS,Proposition 19) that f is a morphism of finite type which maps closed points to closed points. One implication is clear, so suppose that f is injective on closed points and that $f(x) = f(y)$ for some $x, y \in X$. To show that $x = y$ it suffices by symmetry and (VS,Proposition 14) to show that every closed point z of $\overline{\{x\}}$ belongs to $\overline{\{y\}}$. Since f is a closed mapping

$$f(\overline{\{y\}}) = \overline{\{f(y)\}} = \overline{\{f(x)\}} = f(\overline{\{x\}})$$

This shows that $f(z) \in f(\overline{\{y\}})$. Therefore $f^{-1}f(z) \cap \overline{\{y\}}$ is a nonempty closed set, which must by (VS,Proposition 14) contain a closed point q. Since f is injective on closed points, it follows that $z = q$ and so $z \in \overline{\{y\}}$ as required. ☐

Proposition 9. *Let k be an algebraically closed field, let X be a projective scheme over k, and let $\varphi : X \longrightarrow \mathbb{P}^n_k$ be a k-morphism corresponding to $\mathscr{L}$ and $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$ as above. Let $V \subseteq \Gamma(X, \mathscr{L})$ be the k-subspace generated by the s_i. Then φ is a closed immersion if and only if*

(1) Elements of V separate points. *That is, for any two distinct closed points $P, Q \in X$ there is $s \in V$ such that $germ_P s \in \mathfrak{m}_P \mathscr{L}_P$ but $germ_P s \notin \mathfrak{m}_Q \mathscr{L}_Q$ or vice versa, and*

(2) Elements of V separate tangent vectors. *That is, for each closed point $P \in X$, the set $\{germ_P s + \mathfrak{m}_P^2 \mathscr{L}_P \,|\, s \in V \text{ and } germ_P s \in \mathfrak{m}_P \mathscr{L}_P\}$ spans the k-vector space $\mathfrak{m}_P \mathscr{L}_P / \mathfrak{m}_P^2 \mathscr{L}_P$.*

Proof. For the moment just let X be a scheme over a field k. Conditions (1), (2) are properties of a tuple $(\mathscr{L}, s_0, \ldots, s_n)$ consisting of an invertible sheaf $\mathscr{L}$ and global sections s_i which generate $\mathscr{L}$. This property is invariant under the equivalence relation on tuples defined in Definition 2. It is also invariant under isomorphisms of schemes, in the sense that if $f : X' \longrightarrow X$ is an isomorphism of schemes over k then $(\mathscr{L}, s_0, \ldots, s_n)$ satisfies (1), (2) if and only if $(f^*\mathscr{L}, f^*(s_0), \ldots, f^*(s_n))$ does.

Suppose that φ is a closed immersion. Then using (H,5.16) we can reduce to the case where $X = Proj(S/I)$ for a homogenous ideal I of $S = k[x_0, \ldots, x_n]$, $\varphi : X \longrightarrow \mathbb{P}^n_k$ is the canonical morphism of k-schemes induced by the surjection $S \longrightarrow S/I$, and $\mathscr{L} = \mathcal{O}_X(1)$ with s_i equal to the global section corresponding to $x_i + I \in (S/I)_1$ (MPS,Proposition 13). The closed points of X are then homeomorphic to the closed set $Z(I)$ of the variety $\mathbb{P}^n$, and we freely identify the two. (1) Let two distinct closed points $P, Q \in X$ be given and let $f \in S_1$ be a nonzero polynomial with $P \in Z(f)$ and $Q \notin Z(f)$ (LV,Lemma 13) (we need the hypothesis k algebraically closed for this step). If we take $s \in V$ to be the global section of $\mathcal{O}_X(1)$ corresponding to $f + I$ then it is clear s has the right property. (2) Let $P = (a_0, \ldots, a_n)$ be a closed point of X, with say $a_i \neq 0$ and therefore $I(P) + I \in D_+(x_i + I)$. Using (TPC,Lemma 21) we have an isomorphism of schemes

$$g : D_+(x_i + I) \cong Spec(S/I)_{(x_i+I)} \cong Spec(S_{(x_i)}/(IS_{x_i} \cap S_{(x_i)})) \cong Spec(T/I')$$

where $T = k[x_0/x_i, \ldots, x_n/x_i]$ and I' is an ideal. There is an isomorphism $g_*\mathcal{O}_X(1)|_{D_+(x_i+I)} \cong \mathcal{O}_{Spec(T/I')}$ of sheaves of modules which identifies the global section corresponding to $x_j + I$ with the global section corresponding to $x_j/x_i + I'$. The homogenous prime $I(P)$ corresponds to $\mathfrak{m} + I'$ where $\mathfrak{m}$ is the maximal ideal $(x_0/x_i - a_0/a_i, \ldots, x_n/x_i - a_n/a_i)$ of T, so there is an isomorphism of k-vector spaces

$$\mathfrak{m}_P\mathscr{L}_P/\mathfrak{m}_P^2\mathscr{L}_P \cong (\mathfrak{m} + I')/(\mathfrak{m} + I')^2 \cong \mathfrak{m}/(\mathfrak{m}^2 + I') \tag{6}$$

Where we use the fact that for a ring A with maximal ideal $\mathfrak{n}$ there is a canonical isomorphism $\mathfrak{n}/\mathfrak{n}^2 \cong \mathfrak{n}A_\mathfrak{n}/\mathfrak{n}^2A_\mathfrak{n}$ (see our Hartshorne Ch. 1 Section 5 notes). For $j \neq i$ the global section $a_ix_j - a_jx_i + I$ of $\mathcal{O}_X(1)$ is identified with the global section $a_ix_j/x_i - a_j + I'$ of $Spec(T/I')$, so using (6) to see that $\mathfrak{m}_P\mathscr{L}_P/\mathfrak{m}_P^2\mathscr{L}_P$ is generated as a k-module by the images of elements of V, it suffices to show that the images of $x_j/x_i - a_j/a_i$ generate $\mathfrak{m}/(\mathfrak{m}^2 + I')$ as a k-vector space. After applying an automorphism of $k[x_0/x_i, \ldots, x_n/x_i]$ we are essentially trying to show that in a polynomial ring $k[y_1, \ldots, y_n]$ the k-vector space $(y_1, \ldots, y_n)/(y_1, \ldots, y_n)^2$ is generated by the y_i, which is trivial.

For the converse, let $\varphi : X \longrightarrow \mathbb{P}^n_k$ be a k-morphism such that $(\mathscr{L}, \varphi^*x_0, \ldots, \varphi^*x_n)$ satisfies (1), (2) where $\mathscr{L} = \varphi^*\mathcal{O}(1)$. Observe that the subspace V is precisely the image of $\Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$ under the map $\varphi^*(-) : \Gamma(\mathbb{P}^n_k, \mathcal{O}(1)) \longrightarrow \Gamma(X, \varphi^*\mathcal{O}(1))$ (AAMPS,Proposition 15). Since X is projective over k, it is proper over k (H,4.9). It follows from (SPM,Proposition 13) that φ is proper and therefore a closed morphism. By (VS,Proposition 20) if $P \in X$ is a closed point we have an isomorphism of k-vector spaces

$$\mathcal{O}(1)_{\varphi(P)}/\mathfrak{m}_{\varphi(P)}\mathcal{O}(1)_{\varphi(P)} \longrightarrow \mathscr{L}_P/\mathfrak{m}_P\mathscr{L}_P \tag{7}$$

$$germ_{\varphi(P)}x_i + \mathfrak{m}_{\varphi(P)}\mathcal{O}(1)_{\varphi(P)} \mapsto germ_P\varphi^*x_i + \mathfrak{m}_P\mathscr{L}_P \tag{8}$$

Therefore if $f \in S_1$ we have $germ_P\varphi^*(f) \in \mathfrak{m}_P\mathscr{L}_P$ if and only if $f(\varphi(P)) = 0$. Using (1) it is now immediate that φ is injective on closed points, and therefore injective by Lemma 8. This shows that φ gives a homeomorphism with the closed subset $\varphi(X)$ of $\mathbb{P}^n_k$. To show that φ is a closed immersion, it suffices by (VS,Proposition 17) to show that for each closed point $P \in X$ the ring morphism $\mathcal{O}_{Y,\varphi(P)} \longrightarrow \mathcal{O}_{X,P}$ is surjective (we write $Y = \mathbb{P}^n_k$). Both local rings have the same residue field (VS,Corollary 11) and the isomorphism (7) together with condition (2) implies that the image of the maximal ideal $\mathfrak{m}_{Y,\varphi(P)}$ generates $\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$ as a k-vector space, so the canonical morphism $\mathfrak{m}_{Y,\varphi(P)} \longrightarrow \mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$ is surjective. It follows from (SEM,Corollary 12) that φ is projective, and then from (H,5.20) that $\varphi_*\mathcal{O}_X$ is a coherent sheaf of modules. Therefore $\mathcal{O}_{X,P}$ is a finitely generated $\mathcal{O}_{Y,\varphi(P)}$-module, so our result is a consequence of the following Lemma. □

Lemma 10. *Let $f : A \longrightarrow B$ be a local morphism of local noetherian rings, such that*

(1) $A/\mathfrak{m}_A \longrightarrow B/\mathfrak{m}_B$ is an isomorphism of rings,

(2) $\mathfrak{m}_A \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, and

(3) B is a finitely generated A-module.

Then we claim f is surjective.

Proof. Consider the ideal $\mathfrak{a} = \mathfrak{m}_A B$ of B. We have $\mathfrak{a} \subseteq \mathfrak{m}_B$ and by (2), $\mathfrak{m}_B = \mathfrak{a} + \mathfrak{m}_B^2$. It follows from Nakayama's Lemma that $\mathfrak{a} = \mathfrak{m}_B$. Now apply Nakayama's Lemma to the A-module B. By (3), B is a finitely generated A-module. The element $1 \in B$ gives a generator for $B/\mathfrak{m}_A B = B/\mathfrak{m}_B \cong A/\mathfrak{m}_A$ by (1), so we conclude that 1 also generates B as an A-module. That is, f is surjective. □

2 The Duple Embedding

Let A be a ring and S a graded A-algebra. Then for $n \geq 1$ morphisms of graded A-algebras $A[x_0, \dots, x_n] \longrightarrow S$ are in bijection with elements of S_1^{n+1}. Let $\varphi : A[x_0, \dots, x_n] \longrightarrow S$ be the morphism determined by elements $s_0, \dots, s_n \in S_1$ and let $\Phi : U \longrightarrow \mathbb{P}_A^n$ be the corresponding morphism of schemes over A, where U is a certain open subset of $X = ProjS$. Then it follows from (5.12c) that this is the morphism corresponding to the invertible sheaf $\mathcal{O}_X(1)|_U$ with global sections $\widetilde{s_0}, \dots, \widetilde{s_n} \in \Gamma(U, \mathcal{O}_X(1)|_U)$. This raises the question of what morphisms into $\mathbb{P}_A^n$ correspond to the invertible sheaves $\mathcal{O}_X(d)$ and their global sections? We will answer this question, but first we need various results to prepare the way.

Lemma 11. *Let S be a graded ring generated by elements $s_0, \dots, s_n \in S_1$ as an S_0-algebra and set $X = ProjS$. For $d \geq 1$ let $M_0, \dots, M_N$ be the elements of S given by all monomials of degree d in the s_i. Then the sheaf of modules $\mathcal{O}_X(d)$ is generated by the sections $\widetilde{M_i} \in \Gamma(X, \mathcal{O}_X(d))$.*

Proof. By the same argument as Example 5.16.3. □

Proposition 12. *Let S be a graded ring and let $e > 0$. The morphism of graded rings $\varphi : S^{[e]} \longrightarrow S$ induces an isomorphism of schemes $\Phi : ProjS \longrightarrow ProjS^{[e]}$ natural in S. If S is generated by S_1 as an S_0-algebra then for $n \geq 1$ there is a canonical isomorphism of sheaves of modules on $ProjS$*

$$\zeta : \Phi^* \mathcal{O}(ne) \longrightarrow \mathcal{O}(ne)$$
$$\zeta_Q([W, \dot{a/s}] \dot{\otimes} \dot{b/t}) = \dot{ab/st}$$

Where $Q \subseteq ProjS, W \supseteq \Phi(Q)$ are open, $a \in S_{(n+m)e}, s \in S_{me}$ for some $m \geq 0$ and $b, t \in S_r$ for some $r \geq 0$ such that $Q \subseteq D_+(t)$ and $W \subseteq D_+(s)$.

Proof. We have already shown in (TPC,Proposition 9) that Φ is an isomorphism of schemes. To prove the second statement, suppose that S is generated by S_1 as an S_0-algebra and for $n \geq 1$ and a homogenous prime $\mathfrak{p} \in ProjS$ define a ring morphism $S^{[e]}(ne)_{(\mathfrak{p} \cap S^{[e]})} \longrightarrow S(ne)_{(\mathfrak{p})}$ by $a/s \mapsto a/s$. If $a/s = 0$ in $S(ne)_{(\mathfrak{p})}$ then $qa = 0$ for some homogenous $q \notin \mathfrak{p}$, so $q^e a = 0$ which shows that $a/s = 0$ in $S^{[e]}(ne)_{(\mathfrak{p} \cap S^{[e]})}$. To see this map is surjective, find $f \in S_1$ with $f \notin \mathfrak{p}$ (which is possible since S is generated by S_1) and given $a/s \in S(ne)_{(\mathfrak{p})}$ we can pad the numerator and denominator until both are homogenous of a degree divisible by e. Therefore this is an isomorphism of rings. Set $X = ProjS$, $Y = ProjS^{[e]}$, $\mathfrak{q} = \mathfrak{p} \cap S^{[e]}$ and let $\zeta_\mathfrak{p}$ be the following isomorphism of abelian groups, which is compatible with the ring isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong S_{(\mathfrak{p})}$

$$\begin{aligned} \zeta_\mathfrak{p} : \Phi^* \mathcal{O}(ne)_\mathfrak{p} &\cong \mathcal{O}(ne)_\mathfrak{q} \otimes_{\mathcal{O}_{Y,\mathfrak{q}}} \mathcal{O}_{X,\mathfrak{p}} \\ &\cong \mathcal{O}(ne)_\mathfrak{q} \\ &\cong S^{[e]}(ne)_{(\mathfrak{q})} \\ &\cong S(ne)_{(\mathfrak{p})} \end{aligned}$$

It is not difficult to check $\zeta : \Phi^*\mathcal{O}(ne) \longrightarrow \mathcal{O}(ne)$ defined by $\zeta_U(s)(\mathfrak{p}) = \zeta_{\mathfrak{p}}(germ_{\mathfrak{p}} s)$ is an isomorphism of sheaves of modules with the required property. □

Definition 4. Let S be a graded ring and $e > 0$. Let $S|e$ denote the graded ring with the same ring structure as S, but with the *inflated* grading of (TPC,Definition 2). We have an equality of schemes $ProjS = Proj(S|e)$ and it is not hard to see that $S(n)^\sim$ and $(S|e)(ne)^\sim$ are the same sheaf of modules on this scheme for $n \geq 1$. In particular we have $ProjS^{(e)} = ProjS^{[e]}$ and under this equality the sheaf of modules $(S^{(e)}(n))^\sim$ is equal to $(S^{[e]}(ne))^\sim$.

Corollary 13. *Let S be a graded ring generated by S_1 as an S_0-algebra, and let $e > 0$. There is a canonical isomorphism of schemes $\Psi : ProjS \longrightarrow ProjS^{(e)}$ natural in S and a canonical isomorphism of sheaves of modules on $ProjS$ for $n \geq 1$*

$$\zeta : \Psi^*\mathcal{O}(n) \longrightarrow \mathcal{O}(ne)$$
$$\zeta_Q([W, \dot{a/s}] \dot{\otimes} \dot{b/t}) = \dot{ab/st}$$

Where $Q \subseteq ProjS, W \supseteq \Phi(Q)$ are open, $a \in S_{me+ne}, s \in S_{me}$ for some $m \geq 0$ and $b, t \in S_r$ for some $r \geq 0$ such that $Q \subseteq D_+(t)$ and $W \subseteq D_+(s)$.

Proof. The isomorphism is the isomorphism $\Phi : ProjS \longrightarrow ProjS^{[e]}$ of Proposition 12 followed by the equality $ProjS^{(e)} = ProjS^{[e]}$. The sheaf of modules $(S^{(e)}(1))^\sim$ on $ProjS^{(e)}$ is equal to the the sheaf of modules $(S^{[e]}(e))^\sim$ on $ProjS^{[e]}$, so the claim follows directly from Proposition 12. □

Proposition 14. *Let S be a graded A-algebra generated by elements $s_0, \ldots, s_n \in S_1$ as an A-algebra and set $X = ProjS$. Let $M_0, \ldots, M_N$ be the monomials of degree d in the s_i for some $d \geq 1$. Then the invertible sheaf $\mathcal{O}_X(d)$ with generating global sections $\widetilde{M_0}, \ldots, \widetilde{M_N}$ corresponds to a closed immersion of A-schemes $\nu : X \longrightarrow \mathbb{P}^N_A$ called the d-*uple embedding*, which is given by the following composite*

$$X \overset{\Psi}{\Longrightarrow} ProjS^{(e)} \overset{\Lambda}{\longrightarrow} \mathbb{P}^N_A$$

where Ψ is the isomorphism defined in Corollary 13 and Λ corresponds to the following morphism of graded A-algebras

$$\lambda : A[y_0, \ldots, y_N] \longrightarrow S^{(e)}$$
$$y_i \mapsto M_i$$

Proof. It is easy to check that λ is a surjective morphism of graded A-algebras, so Λ is a closed immersion of A-schemes by (Ex.3.12), and therefore so is ν. We have to show there is an isomorphism $\nu^*\mathcal{O}(1) \cong \mathcal{O}_X(d)$ which identifies $v^*(y_i)$ with $\widetilde{M_i}$. But using (5.12c), Corollary 13 and our notes on composing inverse image functors, we have

$$\begin{aligned} \nu^*\mathcal{O}(1) &\cong \Psi^*\Lambda^*\mathcal{O}(1) \\ &\cong \Psi^*\mathcal{O}(1) \\ &\cong \mathcal{O}(d) \end{aligned}$$

So it is a matter of checking that this isomorphism has the right properties. Using the explicit maps given in our notes this is tedious but straightforward (see Section 3 of our notes on Modules over Projective Space and our note on Inverse and Direct Images). □

3 Ample Invertible Sheaves

Now that we have seen that a morphism of a scheme X to a projective space can be characterised by giving an invertible sheaf on X and a suitable set of its global sections, we can reduce the study of varieties in projective space to the study of schemes with certain invertible sheaves and given global sections. Recall that in Section 5 we made the following definition:

Definition 5. If X is a scheme over Y then an invertible sheaf $\mathscr{L}$ on X is *very ample relative to* Y if there is an immersion $i : X \longrightarrow \mathbb{P}^n_Y$ of schemes over Y for some $n \geq 1$ such that $\mathscr{L} \cong i^*\mathcal{O}(1)$. This property is stable under isomorphism of $\mathcal{O}_X$-modules and also under isomorphisms of schemes, in the sense that if $f : X \longrightarrow X'$ is an isomorphism of schemes over Y then $f_*\mathscr{L}$ is very ample relative to Y. If $Y \longrightarrow Y'$ is an isomorphism of schemes then $\mathscr{L}$ is very ample relative Y'.

The most obvious example of very ample sheaves are the usual twisting sheaves.

Lemma 15. *Let A be a ring and S a graded A-algebra finitely generated as an A-algebra by S_1, and set $X = ProjS$. Then the invertible sheaf $\mathcal{O}_X(1)$ is very ample relative to $SpecA$.*

Proof. We know from (TPC,Lemma 19) that the structural morphism $X \longrightarrow SpecA$ is projective, and it follows from (MPS,Proposition 13) that $\mathcal{O}_X(1)$ is very ample relative to $SpecA$. □

Lemma 16. *If X is a scheme over A then an invertible sheaf $\mathscr{L}$ is very ample relative to A if and only if there exists $n \geq 1$ and a set of global sections $s_0, \ldots, s_n$ which generate $\mathscr{L}$ such that the corresponding morphism $X \longrightarrow \mathbb{P}^n_A$ is an immersion.*

We have also seen (5.17) that if $\mathscr{L}$ is a very ample invertible sheaf on a projective scheme X over a noetherian ring A, then for any coherent sheaf $\mathscr{F}$ on X, there is an integer $n_0 > 0$ such that for all $n \geq n_0$, $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. We will use this last property of being generated by global sections to define the notion of an ample invertible sheaf, which is more general, and in many ways is more convenient to work with than the notion of very ample sheaf.

Definition 6. An invertible sheaf $\mathscr{L}$ on a noetherian scheme X is said to be *ample* if for every coherent sheaf $\mathscr{F}$ on X, there is an integer $n_0 > 0$ (depending on $\mathscr{F}$) such that for every $n \geq n_0$, the sheaf $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. This property is stable under isomorphism of $\mathcal{O}_X$-modules and also under isomorphisms of schemes, in the sense that if $f : X \longrightarrow X'$ is an isomorphism of schemes then $f_*\mathscr{L}$ is ample.

Remark 2. Note that "ample" is an absolute notion, i.e., it depends only on the scheme X, whereas "very ample" is a relative notion, depending on a morphism $X \longrightarrow Y$. Notice that since $\mathcal{O}_X$ is coherent, if $\mathscr{L}$ is ample then there is an integer $n_0 > 0$ such that for all $n \geq n_0$, $\mathscr{L}^{\otimes n}$ is generated by global sections. The sheaf $\mathcal{O}_X$ is ample iff. every coherent sheaf is generated by global sections.

Example 1. If $X = SpecA$ for a noetherian ring A then any invertible sheaf is ample. This follows from the fact that any invertible sheaf is coherent (see our Locally Free Sheaves notes), and by (5.16.2) any coherent sheaf on X is generated by global sections. Since X is noetherian the tensor product of coherent sheaves is coherent (p.47 of our Section 2.5 notes), so if $\mathscr{F}$ is a coherent sheaf then $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is coherent and therefore generated by global sections for all $n \geq 1$.

Remark 3. Serre's theorem (5.17) asserts that a very ample sheaf $\mathscr{L}$ on a projective scheme X over a noetherian ring A is ample. The converse is false, but we will see below (7.6) that if $\mathscr{L}$ is ample, then some tensor power $\mathscr{L}^{\otimes m}$ of $\mathscr{L}$ is very ample. Thus "ample" can be viewed as a stable version of "very ample".

Proposition 17. *Let $\mathscr{L}$ be an invertible sheaf on a noetherian scheme X. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is ample;*

(ii) *$\mathscr{L}^{\otimes m}$ is ample for all $m > 0$;*

(iii) *$\mathscr{L}^{\otimes m}$ is ample for some $m > 0$.*

Proof. $(i) \Rightarrow (ii)$ is immediate from the definition of ample. $(ii) \Rightarrow (iii)$ is trivial. To prove $(iii) \Rightarrow (i)$ we note that if $\mathscr{L}^{\otimes m}$ is ample and $\mathscr{F}$ coherent, there is n_0 such that for all $n \geq n_0$ the module $\mathscr{F} \otimes (\mathscr{L}^{\otimes m})^{\otimes n} \cong \mathscr{F} \otimes \mathscr{L}^{\otimes (mn)}$ is generated by global sections. So $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections for $n = mn_0, mn_0 + m, mn_0 + 2m, \ldots$ and so on. We simply need to fill in these gaps. Consider the coherent sheaves $\mathscr{F} \otimes \mathscr{L}^{\otimes k}$ for $k = 1, 2, \ldots, m-1$. There must exist integers $n_k > 0$ such that the following module is generated by global sections for $n \geq n_k$

$$(\mathscr{F} \otimes \mathscr{L}^{\otimes k}) \otimes (\mathscr{L}^{\otimes m})^{\otimes n} \cong \mathscr{F} \otimes \mathscr{L}^{\otimes (k+mn)}$$

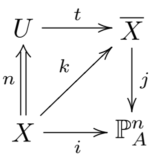
If we take $N = m \cdot max\{n_0, n_1, \ldots, n_{m-1}\}$ then it is clear that $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections for $n \geq N$, as required. □

Lemma 18. *Let $X \longrightarrow Y$ be an immersion with X noetherian. Then there is a closed subscheme structure on the closure $\overline{X}$ such that $X \longrightarrow Y$ factors uniquely through $\overline{X} \longrightarrow Y$ via an open immersion $X \longrightarrow \overline{X}$.*

Proof. In the case where $X \longrightarrow Y$ is an open immersion we have already shown this in our Open Subscheme notes. For a general immersion, write $X \longrightarrow Y$ as an open immersion $X \longrightarrow Z$ followed by a closed immersion $Z \longrightarrow Y$ and use the fact that taking the closure of X in Z is the same as taking the closure in Y. □

Theorem 19. *Let X be a scheme of finite type over a noetherian ring A, and let $\mathscr{L}$ be an invertible sheaf on X. Then $\mathscr{L}$ is ample if and only if $\mathscr{L}^{\otimes m}$ is very ample over $SpecA$ for some $m > 0$.*

Proof. First suppose that $\mathscr{L}^{\otimes m}$ is very ample for some $m > 0$. Then there is an immersion $i : X \longrightarrow \mathbb{P}^n_A$ for some $n \geq 1$ such that $\mathscr{L}^{\otimes m} \cong i^*\mathcal{O}(1)$. Let $\overline{X}$ be the closure of $i(X)$ in $\mathbb{P}^n_A$ and let $j : \overline{X} \longrightarrow \mathbb{P}^n_A$ be a closed immersion with the property that i factors through j via an open immersion $k : X \longrightarrow \overline{X}$. Let U be the image of this open immersion, and let $n : X \longrightarrow U$ be the isomorphism in the following commutative diagram



Then $\overline{X}$ is a projective scheme over A, so $\overline{X}$ is noetherian and by (5.17) the invertible sheaf $\mathcal{O}_{\overline{X}}(1) = j^*\mathcal{O}(1)$ is ample on $\overline{X}$. For any integer $\ell \in \mathbb{Z}$ we write $\mathcal{O}_{\overline{X}}(\ell)$ for $\mathcal{O}_{\overline{X}}(1)^{\otimes \ell}$. Now given any coherent sheaf $\mathscr{F}$ the sheaf $n_*\mathscr{F}$ extends by (Ex 5.15) to a coherent sheaf $\overline{\mathscr{F}}$ on $\overline{X}$. If for some integer $\ell > 0$ the sheaf $\overline{\mathscr{F}} \otimes \mathcal{O}_{\overline{X}}(\ell)$ is generated by global sections, then so is the sheaf (writing m for n^{-1})

$$\begin{aligned} m_*(\overline{\mathscr{F}} \otimes \mathcal{O}_{\overline{X}}(\ell))|_U &\cong m_*(n_*\mathscr{F} \otimes (\mathcal{O}_{\overline{X}}(1)|_U)^{\otimes \ell}) \\ &\cong \mathscr{F} \otimes m_*(\mathcal{O}_{\overline{X}}(1)|_U)^{\otimes \ell} \\ &\cong \mathscr{F} \otimes (m_*j^*\mathcal{O}(1)|_U)^{\otimes \ell} \end{aligned}$$

But $m_* \cong n^*$ and $|_U \cong t^*$, so $m_*j^*\mathcal{O}(1)|_U \cong n^*j^*t^*\mathcal{O}(1) \cong i^*\mathcal{O}(1)$. Therefore $\mathscr{F} \otimes (\mathscr{L}^{\otimes m})^{\otimes \ell}$ is generated by global sections. Since $\mathcal{O}_{\overline{X}}(1)$ is ample it follows that $\mathscr{L}^{\otimes m}$ is ample on X, and therefore by (7.5) so is $\mathscr{L}$.

For the converse, suppose that $\mathscr{L}$ is ample on X. Given any $P \in X$, let U be an affine open neighborhood of P such that $\mathscr{L}|_U$ is free on U. Let Y be the closed set $X \setminus U$ and let $\mathscr{J}_Y$ the corresponding sheaf of ideals (i.e. the sheaf of ideals of the reduced induced scheme structure). Then $\mathscr{J}_Y$ is a coherent sheaf on X, so for some $n > 0$ the module $\mathscr{J}_Y \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. Since

$$Y = Supp(\mathcal{O}_X / \mathscr{J}_Y) = \{x \in X \mid \mathscr{J}_{Y,x} \neq \mathcal{O}_{X,x}\}$$

and $(\mathscr{J}_Y \otimes \mathscr{L}^{\otimes n})_P \cong \mathscr{J}_{Y,P} \cong \mathcal{O}_{X,P}$ it follows from Nakayama's Lemma that there is a section $s \in \Gamma(X, \mathscr{J}_Y \otimes \mathscr{L}^{\otimes n})$ such that $germ_P s \notin \mathfrak{m}_P(\mathscr{J}_Y \otimes \mathscr{L}^{\otimes n})_P$. In particular $germ_P s$ is a basis for $(\mathscr{J}_Y \otimes \mathscr{L}^{\otimes n})_P$. Since $\mathscr{L}^{\otimes n}$ is invertible it is flat, so there is a monomorphism $\phi : \mathscr{J}_Y \otimes \mathscr{L}^{\otimes n} \longrightarrow$

$\mathcal{O}_X \otimes \mathscr{L}^{\otimes n} \cong \mathscr{L}^{\otimes n}$ and we consider s as a global section of $\mathscr{L}^{\otimes n}$. The following commutative diagram shows that ϕ_P is an isomorphism

$$\begin{array}{ccccc} (\mathscr{J}_Y \otimes \mathscr{L}^{\otimes n})_P & \longrightarrow & (\mathcal{O}_X \otimes \mathscr{L}^{\otimes n})_P & \Longrightarrow & \mathscr{L}_P^{\otimes n} \\ \Downarrow & & \Downarrow & \nearrow\!\!\!\nearrow & \\ \mathscr{J}_{Y,P} & \Longrightarrow & \mathcal{O}_{X,P} \otimes \mathscr{L}_P^{\otimes n} & & \end{array}$$

Therefore $germ_P s$ is a basis for $\mathscr{L}_P^{\otimes n}$ and so if X_s denotes the open set $\{Q \in X \,|\, germ_Q s \notin \mathfrak{m}_Q \mathscr{L}_Q^{\otimes n}\}$ it is clear that $P \in X_s$. If $Q \notin U$ then since $\mathscr{J}_{Y,Q}$ is a proper ideal it must be contained in $\mathfrak{m}_Q$, and therefore $germ_Q s \in \mathfrak{m}_Q \mathscr{L}_Q^{\otimes n}$. Hence $P \in X_s \subseteq U$. Now U is affine, and $\mathscr{L}|_U$ is trivial, so s induces an element $f \in \Gamma(U, \mathcal{O}_U)$ and then $X_s = U_f$ is also affine.

Thus we have shown that for any point $P \in X$, there is an $n > 0$ and a section $s \in \Gamma(X, \mathscr{L}^{\otimes n})$ such that $P \in X_s$ and X_s is affine. Since X is quasi-compact, we can cover X by a finite number of such open affines, corresponding to sections $s_i \in \Gamma(X, \mathscr{L}^{\otimes n_i})$. Replacing each s_i by a suitable power $s_i^k \in \Gamma(X, \mathscr{L}^{\otimes k n_i})$, which doesn't change X_{s_i}, we may assume that all n_i are equal to one n. Finally, since $\mathscr{L}^{\otimes n}$ is also ample, and since we are only trying to show that some power of $\mathscr{L}$ is very ample, we may replace $\mathscr{L}$ by $\mathscr{L}^{\otimes n}$. Thus we may assume now that we have global sections $s_1, \ldots, s_k \in \Gamma(X, \mathscr{L})$ such that each $X_i = X_{s_i}$ is affine, and the X_i cover X.

Now for each i, let $B_i = \Gamma(X_i, \mathcal{O}_{X_i})$. Since X is a scheme of finite type over A, each B_i is a finitely generated A-algebra (Ex 3.3). So let $\{b_{ij} \,|\, j = 1, \ldots, k\}$ be a set of generators for B_i as an A-algebra. By (5.14) for each i, j there is an integer n such that $b_{ij} s_i^n$ extends to a global section $c_{ij} \in \Gamma(X, \mathscr{L}^{\otimes n})$. We can take one n large enough to work for all i, j. Now we take the invertible sheaf $\mathscr{L}^{\otimes n}$ on X, and the sections $\{s_i^n \,|\, 1 \le i \le k\}$ and $\{c_{ij}, \,|\, 1 \le i, j \le k\}$ and use all these sections to define a morphism (over A) $\varphi : X \longrightarrow \mathbb{P}_A^N$ as in (7.1) above. This makes sense since X is covered by the X_i, so the sections s_i^n already generate $\mathscr{L}^{\otimes n}$.

Let $\{x_i \,|\, 1 \le i \le k\}$ and $\{x_{ij} \,|\, 1 \le i, j \le k\}$ be the homogenous coordinates of $\mathbb{P}_A^N$ corresponding to the above sections of $\mathscr{L}^{\otimes n}$. For each $i = 1, \ldots, k$ let $U_i \subseteq \mathbb{P}_A^n$ be the open subset $D_+(x_i)$. Then $\varphi^{-1} U_i = X_i$ and the corresponding morphism of A-algebras

$$A[\{y_i\}; \{y_{ij}\}] \longrightarrow B_i$$

is surjective, because $y_{ij} \mapsto c_{ij}/s_i^n = b_{ij}$, and we chose the b_{ij} so as to generate B_i as an A-algebra. Thus $X_i \longrightarrow U_i$ is a closed immersion, and so the factorisation of φ through $\bigcup_i U_i \subseteq \mathbb{P}_A^n$ is a closed immersion. It follows from our Subschemes and Immersions notes that $X \longrightarrow \mathbb{P}_A^n$ is an immersion, as required. $\square$

Example 2. Let $X = \mathbb{P}_k^n$ where k is a field. Then $\mathcal{O}(1)$ is very ample over k by definition. For any $d > 0$, $\mathcal{O}(d)$ corresponds to the d-uple embedding of Proposition 14, so $\mathcal{O}(d)$ is very ample over k. Hence $\mathcal{O}(d)$ is ample for all $d > 0$. If $\ell < 0$ then the module $\mathcal{O}(\ell)$ has no global sections (see our $\Gamma_* \mathcal{O}_X$ notes) and therefore can not be generated by global sections. But then $\mathcal{O}(\ell)$ can not be ample, since if it were there would be some $n_0 > 0$ with $\mathcal{O}(\ell)^{\otimes n} \cong \mathcal{O}(n\ell)$ generated by global sections for all $n \ge n_0$. Since there exist coherent modules which are not ample, $\mathcal{O}_X$ can not be ample. So on $\mathbb{P}_k^n$, we have $\mathcal{O}(\ell)$ is ample $\Leftrightarrow$ very ample $\Leftrightarrow$ $\ell > 0$.

4 Linear Systems

We will see in a minute how global sections of an invertible sheaf correspond to effective divisors on a variety. Thus giving an invertible sheaf and a set of its global sections is the same as giving a certain set of effective divisors, all linearly equivalent to each other. This leads to the notion of linear system, which is the historically older notion. For simplicity, we will employ this terminology only when dealing with nonsingular projective varieties over an algebraically closed field. Over more general schemes the geometric intuition associated with the concept of linear system may lead one astray, so it is safer to deal with invertible sheaves and their global sections in that case.

So let X be a nonsingular projective variety over an algebraically closed field k. In this case the notions of Weil divisor and Cartier divisor are equivalent (DIV,Proposition 33) in such a way that the effective divisors also agree (DIV,Proposition 51). Furthermore, we have a bijection between linear equivalence classes of divisors and isomorphism classes of invertible sheaves (DIV,Proposition 44). Another useful fact in this situation is that for any invertible sheaf $\mathscr{L}$ on X, the global sections $\Gamma(X, \mathscr{L})$ form a finite-dimensional k-vector space (H,5.19).

Lemma 20. *Let X be an integral scheme with generic point ξ and $\mathscr{L}$ an invertible sheaf on X. Then*

(i) For $x \in X$ the canonical morphism of abelian groups $\mathscr{L}_x \longrightarrow \mathscr{L}_\xi$ is injective.

(ii) If U, V are nonempty open subsets of X and $s \in \mathscr{L}(U), t \in \mathscr{L}(V)$ then $s|_{U\cap V} = t|_{U\cap V}$ if and only if $germ_\xi s = germ_\xi t$.

(iii) If $V \subseteq X$ is a nonempty open subset then $\mathscr{L}(V) \longrightarrow \mathscr{L}_\xi$ is injective.

(iv) For nonempty open sets $W \subseteq V$ the restriction map $\mathscr{L}(V) \longrightarrow \mathscr{L}(W)$ is injective.

Proof. We use invertibility of $\mathscr{L}$ and (POIS,Lemma 1) to prove (i), from which all the other claims follow. □

Let $\mathscr{L}$ be an invertible sheaf on X and let $s \in \Gamma(X, \mathscr{L})$ be a nonzero global section of $\mathscr{L}$. We define an effective divisor $D = (s)_0$, the *divisor of zeros* of s, as follows. Choose an open cover $\{U_i\}_{i\in I}$ of X by nonempty affine open subsets U_i on which $\mathscr{L}$ is trivial. That is, for each $i \in I$ there is an isomorphism of sheaves of modules $\varphi_i : \mathscr{L}|_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}$. Then $\varphi_i(s)$ is a nonzero element of $\Gamma(U_i, \mathcal{O}_X)$ by Lemma 20(iv), which is therefore invertible as an element of $\mathscr{K}(U_i)$ by (POIS,Lemma 1)(iv) and (DIV,Lemma 30). It is straightforward to check that $\{(U_i, \varphi_i(s))\}_{i\in I}$ determines an effective Cartier divisor D on X, which depends only on $\mathscr{L}$ and s (not on the choice of cover or local isomorphisms).

Proposition 21. *Let X be a nonsingular projective variety over an algebraically closed field k. Let D_0 be a divisor on X and let $\mathscr{L} = \mathscr{L}(D_0)$ be the corresponding invertible sheaf. Then*

(a) For each nonzero $s \in \Gamma(X, \mathscr{L})$ the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D_0.

(b) Every effective divisor linearly equivalent to D_0 is $(s)_0$ for some nonzero $s \in \Gamma(X, \mathscr{L})$.

(c) Two nonzero sections $s, s' \in \Gamma(X, \mathscr{L})$ have the same divisor of zeros if and only if there is $\lambda \in k^*$ such that $s' = \lambda \cdot s$.

Proof. (a) Since $\mathscr{L}$ is a submodule of $\mathscr{K}$, s is a nonzero global section of $\mathscr{K}$. Suppose that D_0 is defined as a Cartier divisor by the family $\{(U_i, f_i)\}_{i\in I}$. We can assume that each U_i is a nonempty affine open set. Then for each $i \in I$ there is an isomorphism $\mathscr{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ corresponding to the basis f_i^{-1} of $\mathscr{L}(U_i)$. The element $s|_{U_i}$ of $\mathscr{L}(U_i)$ is mapped to the product $f_i s|_{U_i}$ in $\mathscr{K}(U_i)$, which belongs to $\mathcal{O}_X(U_i)$. Therefore $(s)_0$ is the Cartier divisor determined by the family $\{(U_i, f_i s|_{U_i})\}_{i\in I}$ which is the sum $D_0 + (s)$. Therefore $(s)_0 - D_0 = (s)$ and $(s)_0$ is linearly equivalent to D_0.

(b) Let D be an effective divisor and $s \in \mathscr{K}^*(X)$ such that $D - D_0 = (s)$. Then $(s) + D_0 \geq 0$ and therefore $(s) \geq -D_0$. Therefore $(s^{-1}) = \mathscr{L}((s)) \supseteq \mathscr{L}(-D_0)$ or equivalently $(s) \subseteq \mathscr{L}(D_0)$ (DIV,Proposition 50). Therefore s is a nonzero element of $\Gamma(X, \mathscr{L}(D_0))$ and we showed in (a) that $(s)_0 = D_0 + (s)$. It follows that $D = D_0 + (s) = (s)_0 - (s) + (s) = (s)_0$, as required.

(c) Given nonzero sections $s, s' \in \Gamma(X, \mathscr{L})$ we have $(s)_0 = (s')_0$ if and only if $(s) = (s')$, which by (DIV,Proposition 41)(e) is if and only if $s/s' \in \mathcal{O}^*(X)$. But since X is a projective variety over an algebraically closed field k, we have $\Gamma(X, \mathcal{O}_X) = k$ (H,I.3.4),(VS,Theorem 27) and so $s/s' \in k^*$. □

Definition 7. Let D_0 be a divisor on X and let $|D_0|$ denote the set of all effective divisors linearly equivalent to D_0 (this may be empty). We call any set of divisors of X arising in this way a *complete linear system* on X. We see from Proposition 21 that there is a bijection of $|D_0|$ with the set $(\Gamma(X, \mathscr{L}(D_0)) - \{0\})/k^*$. Provided $|D_0|$ is nonempty it acquires a canonical topology making it homeomorphic to $\mathbb{P}^{n-1}$ where $n \geq 1$ is the dimension of $\Gamma(X, \mathscr{L}(D_0))$ as a k-vector space (TVS,Lemma 1) (by convention $\mathbb{P}^0$ is just a point).

Example 3. Set $X = \mathbb{P}^n_k$ for an algebraically closed field k, and fix an integer $d > 0$. Let H be the hyperplane $x_\ell = 0$ for some $0 \leq \ell \leq n$. Then a divisor D of X is linearly equivalent to dH if and only if $deg(D) = d$ (DIV,Proposition 12). In this case $\mathscr{L}(H)$ is canonically isomorphic to $\mathcal{O}(1)$ (DIV,Lemma 34) (DIV,Lemma 46), so the complete linear system $|H|$, which is the set of all prime divisors of degree 1, is in bijection with

$$\left(\Gamma(X, \mathscr{L}(H)) - \{0\}\right)/k^* \cong \left(\Gamma(X, \mathcal{O}(1)) - \{0\}\right)/k^*$$

which is in bijection with the set of all associate classes of nonzero homogenous polynomials of degree 1 in $k[x_0, \ldots, x_n]$.

With the notation of the example

Lemma 22. *The bijection $(S_1 - \{0\})/k^* \longrightarrow |H|$ sends a polynomial f to the prime divisor $V(f)$.*

Proof. From the proof of Proposition 21(b) we know that $V(f) \in |H|$ corresponds to the global section f/x_ℓ of $\Gamma(X, \mathscr{L}(H))$. Going in the other direction, it is easy to check that $f \in S_1$ goes to $f/x_\ell \in \Gamma(X, \mathscr{L}(H))$, so the proof is complete. □