Section 2.6 - Divisors

Daniel Murfet

October 5, 2006

Contents

1 Weil Divisors

Definition 1. We say a scheme X is *regular in codimension one* (or sometimes *nonsingular in codimension one*) if every local ring $\mathcal{O}_{X,x}$ of X of dimension one is regular.

Recall that a regular local ring is a noetherian local ring with dimension equal to $dim_k \mathfrak{m}/\mathfrak{m}^2$. A regular local ring of dimension one is precisely a discrete valuation ring. If V is a nosingular variety then $t(V)$ is a scheme with all local rings regular, so $t(V)$ is clearly regular in codimension one. In this section we will consider schemes satisfying the following condition: $(*)$ X is a noetherian integral separated scheme which is regular in codimension one.

Before defining a divisor we recall some results proved earlier in notes:

- Let X be a scheme. Two closed immersions $Z \longrightarrow X, Z' \longrightarrow X$ of reduced schemes determine the same closed subscheme iff. they have the same image in X (FCI,Corollary 9). In particular there is a bijection between reduced closed subschemes of X and closed subsets of X given by associating a closed subset with the corresponding induced reduced scheme structure.

- Extending the previous result, for any scheme X there is a bijection between *integral* closed subschemes of X and *irreducible* closed subsets of X, where we associate the induced reduced scheme structure with an irreducible closed subset.

- We proved in Ex 3.6 that if X is a scheme and $Y \subseteq X$ a closed irreducible subset with generic point η, then $dim \mathcal{O}_{X,\eta} = codim(Y, X)$.

- If X is an integral scheme with generic point ξ then $K = \mathcal{O}_{X,\xi}$ is a field and for every $x \in X$ there is an injection of rings $\mathcal{O}_{X,x} \longrightarrow K$ such that K is the quotient field of all these local rings. Consequently if two sections $s \in \mathcal{O}_X(U), t \in \mathcal{O}_X(V)$ (with U, V nonempty) have the same germ at ξ then $s|_{U\cap V} = t|_{U\cap V}$.

  If $f \in K$ we call f a *rational function* on X. The *domain of definition* of f is the union of all open sets U occurring in the first position of tuples in the equivalence class f. This is a nonempty open set. If the domain of definition of f is V, there is a unique section of $\mathcal{O}_X(V)$ whose germ at ξ belongs to f. We also denote this section by f.

Definition 2. Let X satisfy $(*)$. A *prime divisor* on X is a closed irreducible subset $Y \subseteq X$ of codimension one. Equivalently, a prime divisor is an integral closed subscheme of codimension one, and given a prime divisor $Y \subseteq X$ we associate it with the corresponding induced reduced closed subscheme.

A *Weil divisor* is an element of the free abelian group $DivX$ generated by the prime divisors ($\text{Div}X = 0$ if no prime divisors exist). We write a divisor as $D = \sum n_i Y_i$ where the Y_i are prime divisors, and the n_i are integers, and only finitely many n_i are different from zero. If the n_i are all ≥ 0 we say that D is *effective*.

If Y is a prime divisor on X, let $\eta \in Y$ be its generic point. Then $dim\mathcal{O}_{X,\eta} = codim(Y, X) = 1$ so $\mathcal{O}_{X,\eta}$ is a discrete valuation ring. We call the corresponding discrete valuation v_Y on K the *valuation of* Y. If $f \in K^*$ is a nonzero rational function on X then $v_Y(f)$ is an integer. If it is positive, we say that f has a *zero* along Y of that order; if it is negative, we say f has a *pole* along Y, of order $-v_Y(f)$.

We make the following remarks:

- If Y is a prime divisor on X with generic point η and $f \in K^*$ then $v_Y(f) \geq 0$ if and only if η belongs to the domain of definition of f, and $v_Y(f) > 0$ if and only if $f(\eta) = 0$ (that is, $germ_\eta f \in \mathfrak{m}_\eta$).

- If X satisfies $(*)$ then so does any open subset $U \subseteq X$. If Y is a prime divisor of X then $Y \cap U$ is a prime divisor of U, provided it is nonempty. Moreover the assignment $Y \mapsto Y \cap U$ is *injective* since $Y = \overline{Y \cap U}$ (provided of course $Y \cap U \neq \emptyset$). If Y is a prime divisor of U then $\overline{Y}$ is a prime divisor of X and $\overline{Y} \cap U = Y$, so in fact there is a *bijection* between prime divisors of U and prime divisors of X meeting U.

- If A is a normal noetherian domain then $X = SpecA$ has the property $(*)$, since any affine scheme is separated (see (MAT,Definition 18) for the definition of a normal domain). A prime divisor $Y \subseteq X$ is $V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of height 1. If Q is the quotient field of A there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X,\mathfrak{p}} \arrow[r] \arrow[d, Rightarrow] & \mathcal{O}_{X,0} \arrow[d, Rightarrow] \\ A_{\mathfrak{p}} \arrow[r] & Q \end{tikzcd}$$

  The discrete valuation v_Y on Q with valuation ring $A_{\mathfrak{p}}$ is defined on elements of $A_{\mathfrak{p}}$ by $v_Y(a/s) =$ largest $k \geq 0$ such that $a/s \in \mathfrak{p}^k A_{\mathfrak{p}}$.

- If Q is a field then the scheme $X = SpecQ$ has the property $(*)$ but there are no prime divisors on X, so the case $DivX = 0$ can occur. But this is the *only* way it can occur. If $X = SpecA$ is an affine scheme satisfying $(*)$ with no prime divisors, then it follows from $(I, 1.11A)$ that A must be a field.

  If X is a scheme which satisfies $(*)$ and has no prime divisors, then any point $x \in X$ must have an open neighborhood $V \cong SpecA$ where A is a field. Since every open subset of X has to contain the generic point, this is only possible if X is a closed point (therefore isomorphic to the spectrum of its function field).

Lemma 1. *If X satisfies $(*)$ then so does the scheme $Spec(\mathcal{O}_{X,x})$ for any $x \in X$. If $f : Spec(\mathcal{O}_{X,x}) \longrightarrow X$ is canonical then $Y \mapsto f^{-1}Y$ gives an injective map from the set of prime divisors of X containing x to the set of prime divisors of $Spec(\mathcal{O}_{X,x})$.*

Proof. Suppose X is a scheme satisfying $(*)$. Then $\mathcal{O}_{X,x}$ is a noetherian integral domain, so $Spec(\mathcal{O}_{X,x})$ is clearly noetherian, integral and separated. Let $V \cong SpecB$ be an open affine neighborhood of x and suppose $\mathfrak{p}$ is the prime of B corresponding to x. Then $\mathcal{O}_{X,x} \cong B_{\mathfrak{p}}$ and any local ring of the scheme $Spec(\mathcal{O}_{X,x})$ is isomorphic to $(B_{\mathfrak{p}})_{\mathfrak{q}B_{\mathfrak{p}}} \cong B_{\mathfrak{q}}$ for some prime $\mathfrak{q} \subseteq \mathfrak{p}$. It follows that $Spec(\mathcal{O}_{X,x})$ is regular in codimension one.

Let Z be the set of all generisations of x and let Y be a prime divisor of X with $x \in Y$. If η, ξ are the generic points of Y, X respectively then $\xi \in Z$ but $\xi \notin Y$ while $\eta \in Z$ so $Y \cap Z$ is a nonempty proper closed subset of Z. In fact $Y \cap Z$ is the closure in Z of the point η, so Z is a closed irreducible subset of Z. The usual argument shows that Z has codimension one. The map f gives a homeomorphism of $Spec(\mathcal{O}_{X,x})$ with Z. So $f^{-1}Z$ is a prime divisor on $Spec(\mathcal{O}_{X,x})$, and it is clear that this assignment is injective. $\square$

Lemma 2. *Let X satisfy $(*)$ and let $x \in X$. Then x is contained in no prime divisor of X if and only if x is the unique generic point of X.*

Proof. It is clear that the generic point is contained in no prime divisor of X. For the converse, let $x \in X$ be a point not contained in any prime divisor of X, and let $U \cong SpecA$ be an affine open neighborhood of x, with prime $\mathfrak{p}$ corresponding to x. By assumption $\mathfrak{p}$ does not contain any prime ideal of height 1. Suppose $\mathfrak{p} \neq 0$, and let $0 \neq a \in \mathfrak{p}$. Then $\mathfrak{p}$ can be shrunk to a prime ideal minimal containing (a), which must have height 1 by $(I, 1.11A)$, which is a contradiction. Hence $\mathfrak{p} = 0$ and x is the generic point of X. $\square$

Lemma 3. *Let X satisfy $(*)$, and let $f \in K^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

Proof. Let U be the domain of definition of f. Then $Z = X \setminus U$ is a proper closed subset of X. Since X is noetherian we can write $Z = Z_1 \cup \cdots \cup Z_n$ for closed irreducible Z_i. Thus Z can contain at most a finite number of prime divisors of X, since any closed irreducible subset of Z of codimension one in X must be one of the Z_i. So it suffices to show that there are only finitely many prime divisors $Y \subseteq X$ meeting U with $v_Y(f) \neq 0$. If $Y \cap U \neq \emptyset$ then U must contain the generic point η of Y. Hence $v_Y(f) \geq 0$. Now $v_Y(f) > 0$ iff. $\eta \in U - X_f$, which is a proper closed subset of U ($f \neq 0$ and therefore cannot be nilpotent by Ex 2.16 since X noetherian and hence U is quasi-compact). Hence $v_Y(f) > 0$ iff. $Y \cap U \subseteq U - X_f$. But $U - X_f$ is a proper closed subset of the noetherian space U, hence contains only finitely many closed irreducible subsets of codimension one of U. $\square$

Definition 3. Let X satisfy $(*)$ and let $f \in K^*$. We define the *divisor* of f, denoted (f), by

$$(f) = \sum v_Y(f) \cdot Y$$

where the sum is taken over all prime divisors of X. This is well-defined by the lemma, and we use the convention that if no divisors exist the sum is zero. Any divisor which is equal to the divisor of a function is called a *principal divisor*.

Note that if $f, g \in K^*$ then $(f/g) = (f) - (g)$ because of the properties of valuations. Therefore sending a function f to its divisor (f) gives a homomorphism of multiplicative group K^* to the additive group $DivX$, and the image, which consists of the principal divisors, is a subgroup of $DivX$.

Definition 4. Let X satisfy $(*)$. Two divisors D and D' are said to be *linearly equivalent*, written $D \sim D'$, if $D - D'$ is a principal divisor. The group $DivX$ of all divisors divided by the subgroup of principal divisors is called the *divisor class group* of X and is denoted ClX.

It is not hard to check that if two schemes X, Y satisfy $(*)$ and $X \cong Y$ then $DivX \cong DivY$ and $ClX \cong ClY$ as abelian groups. The divisor class group of a scheme is a very interesting invariant. In general it is not easy to calculate. However, in the following propositions and examples we will calculate a number of special cases to give some idea of what it is like. First let us study divisors on affine schemes.

A ring A is *normal* if $A_{\mathfrak{p}}$ is a normal domain for every prime ideal $\mathfrak{p}$. We say that a scheme X is *normal* if all its local rings are normal domains, so the scheme $SpecA$ is normal iff. A is a normal ring.

Proposition 4. *Let A be a noetherian domain. Then A is a unique factorisation domain if and only if $X = SpecA$ is normal and $ClX = 0$.*

Proof. It is well-known that a UFD is normal, so X will be normal. On the other hand, A is a UFD if and only if every prime ideal of height 1 is principal (I, 1.12A). So what we must show is that if A is a normal noetherian domain, then every prime ideal of height 1 is principal if and only if $Cl(SpecA) = 0$.

One way is easy: if every prime ideal of height 1 is principal, consider a prime divisor $Y \subseteq X = SpecA$. Then Y corresponds to a prime ideal $\mathfrak{p}$ of height 1. If $\mathfrak{p}$ is generated by an element $f \in A$, then considering f as a global section of X we claim that $(f) = Y$. By the preceding notes for any divisor $Z = V(\mathfrak{q})$, $v_Z(f) =$ largest $k \geq 0$ such that $f/1 \in \mathfrak{q}^k A_{\mathfrak{q}}$. It is easy to see that for $Z \neq Y$ we must have $v_Z(f) = 0$. Since $f/1 \in \mathfrak{p}A_{\mathfrak{p}}$ we have $v_Y(f) \geq 1$. Suppose $f/1 \in \mathfrak{p}^2 A_{\mathfrak{p}}$. Then $tf \in \mathfrak{p}^2$ for some $t \notin \mathfrak{p}$. But then $tf = af^2$ for some $a \in A$, implying that $t = af \in \mathfrak{p}$, a contradiction. Hence $v_Y(f) = 1$ and consequently $(f) = 1 \cdot Y$, as desired. Thus every prime divisor is principal, so $ClX = 0$.

For the converse, suppose $ClX = 0$. Let $\mathfrak{p}$ be a prime ideal of height 1, and let Y be the corresponding prime divisor. If Q is the quotient field of A then there is $f \in Q^*$ with $(f) = Y$. We will show that in fact $f \in A$ and f generates $\mathfrak{p}$. Since for any prime divisor Z we have $v_Z(f) \geq 0$ it follows that $f \in A_{\mathfrak{q}}$ for all primes $\mathfrak{q}$ of height 1. It follows from the following algebraic result that $f \in A$. Since $v_Y(f) \geq 1$ we have $f \in \mathfrak{p}A_{\mathfrak{p}}$ and in fact f generates the ideal $\mathfrak{p}A_{\mathfrak{p}}$. To show that f generates $\mathfrak{p}$, let g be any other nonzero element of $\mathfrak{p}$. Then $v_Y(g) \geq 1$ and $v_Z(g) \geq 0$ for all $Z \neq Y$. Hence $v_Z(g/f) \geq 0$ for *all* prime divisors Z (including Y). Thus $g/f \in A_{\mathfrak{q}}$ for *all* $\mathfrak{q}$ of height 1, so again $g/f \in A$. In other words $g \in (f)$ and thus $\mathfrak{p}$ is principal. □

Proposition 5. *Let A be a normal noetherian domain. Then*

$$A = \bigcap_{ht\mathfrak{p}=1} A_{\mathfrak{p}}$$

where the intersection is taken over all prime ideals of height 1.

Proof. See (MAT,Theorem 112). □

Corollary 6. *Let A be a normal noetherian domain and set $X = SpecA$. If Q is the quotient field of A and $f \in Q^*$ then (f) is an effective divisor if and only if $f \in A$.*

Example 1. Let $A = k[x_1, \ldots, x_n]$ for a field k and $n \geq 1$, so $X = SpecA = \mathbb{A}^n_k$. Then A is a normal noetherian domain since it is a UFD. A prime divisor $Y \subseteq X$ is $V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of height 1, which by $(I, 1.12A)$ is principal. So any prime divisor is $V(f)$ for an irreducible polynomial f, which is unique up to multiplication by a unit. Conversely any irreducible polynomial f determines a prime ideal (f) which has height 1 by $(I, 1.11A)$, so $V(f)$ is a prime divisor. So there is a bijection between prime divisors of $\mathbb{A}^n_k$ and associate classes of irreducible polynomials $f \in k[x_1, \ldots, x_n]$ (i.e. $f \sim g$ iff. exists $u \in k$ with $f = gu$). By Proposition 4 we have $ClX = 0$. Given an irreducible polynomial f and $\mathfrak{p} = (f)$, what is the valuation on K corresponding to the discrete valuation ring $A_{\mathfrak{p}}$?

Definition 5. Let $A = k[x_1, \ldots, x_n]$ for a field k and $n \geq 1$. Given a nonzero polynomial g and an irreducible polynomial f let $v_f(g)$ be the largest integer $i \geq 0$ such that f^i divides g (of course $1 = f^0$ always divides g). For any $a \in k$ it is clear that $v_f(a) = 0$ for all irreducible f. It is also clear that if f, f' are irreducible polynomials with $f \sim f'$ then $v_f(g) = v_{f'}(g)$.

Given a nonzero nonconstant polynomial $g \in k[x_1, \ldots, x_n]$, there is an essentially unique factorisation of g of the following form:

$$g = uf_1 \cdots f_n$$

where u is a unit and the f_i are irreducible polynomials. The uniqueness of this factorisation means that the number of times a particular associate class of irreducible polynomials is represented among the f_i is well-defined, and in fact for an irreducible polynomial f the integer $v_f(g)$ is the number of elements of $\{f_1, \ldots, f_n\}$ associated to f.

Proposition 7. *Let $A = k[x_1, \ldots, x_n]$ for a field k and $n \geq 1$. Let f be an irreducible polynomial and $\mathfrak{p} = (f)$. The discrete valuation on K with valuation ring $A_\mathfrak{p}$ is defined for nonzero $g, h \in A$ by $v_\mathfrak{p}(g/h) = v_f(g) - v_f(h)$.*

Proof. Since $v_\mathfrak{p}(g/h) = v_\mathfrak{p}(g) - v_\mathfrak{p}(h)$ it suffices to show that $v_\mathfrak{p}(g) = v_f(g)$ for nonzero $g \in A$. But we know that $v_\mathfrak{p}(g)$ is the largest $k \geq 0$ with $g/1 \in \mathfrak{p}^k A_\mathfrak{p}$. So it suffices to show that for $k \geq 1$ $g/1 \in \mathfrak{p}^k A_\mathfrak{p}$ iff. f^k divides g. Since $\mathfrak{p}^k = (f^k)$ one implication is obvious. The other implication follows from the following fact, which is a simple induction on k: for $k \geq 1, s \notin \mathfrak{p}, 0 \neq g$ and any irreducible f, if $sg = af^k$ then f^k divides g. □

Let $X = \mathbb{A}^n_k$ be affine n-space over a field k and let K be the quotient field of $A = k[x_1, \ldots, x_n]$. Given nonzero $g, h \in A$ the corresponding principal divisor is

$$(g/h) = \sum_{ht.\mathfrak{p}=1} v_\mathfrak{p}(g/h) \cdot V(\mathfrak{p}) = \sum_f (v_f(g) - v_f(h)) \cdot V(f)$$

where the second sum is over the set of equivalence classes of irreducible polynomials under the associate relation, and we pick a single f from each class. For example if g is a nonconstant polynomial with factorisation $g = up_1^{n_1} \cdots p_r^{n_r}$ with the p_i non-associate irreducible polynomials and $n_i \geq 1$, then

$$(g) = n_1 \cdot V(p_1) + \cdots + n_r \cdot V(p_r)$$

This makes it obvious why $Cl(\mathbb{A}^n_k) = 0$. If $D = \sum_{i=1}^r n_i \cdot Y_i$ is any nonzero effective divisor ($n_i > 0$), write $Y_i = V(p_i)$ for irreducible p_i. Then $D = (p_1^{n_1} \cdots p_r^{n_r})$. So any effective divisor is principal. But any divisor D can be written as a difference $D_1 - D_2$ of two effective divisors, so *every* divisor is principal and thus $Cl(\mathbb{A}^n_k) = 0$.

Generalising from the case of the polynomial ring (which is a UFD) there is a similar formula for any Dedekind domain, which we develop over the next couple of results:

Proposition 8. *Let A be a Dedekind domain. For a nonzero element $a \in A$ the unique factorisation of (a) as a product of prime ideals is given by*

$$(a) = \prod_{0 \neq \mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(a)}$$

Proof. The scheme $X = SpecA$ satisfies $(*)$ since A is a normal noetherian domain. It follows from Lemma 3 that $v_\mathfrak{p}(a) > 0$ for only finitely many nonzero primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, so the product is at least defined. If we write v_i for the integer $v_{\mathfrak{p}_i}(a)$ then we need to show that

$$(a) = \mathfrak{p}_1^{v_1} \cdots \mathfrak{p}_n^{v_n}$$

Since the $\mathfrak{p}_i$ are distinct the powers are all coprime, so we need to show that (a) is the intersection ${\mathfrak{p}_1}^{v_1} \cap \cdots \cap {\mathfrak{p}_n}^{v_n}$. By the previous result we know that $a \in \mathfrak{p}_i^{v_i}$ for all i, so we have $\subseteq$. Now suppose that $b \in A$ also belongs to this intersection. Then for all nonzero primes $\mathfrak{p}$ we have $v_\mathfrak{p}(b) \geq v_\mathfrak{p}(a)$ and hence $v_\mathfrak{p}(b/a) \geq 0$. By Proposition 5 it follows that $b/a \in A$ and hence $b \in (a)$, as required. □

Corollary 9. *Let A be a Dedekind domain with quotient field K. For a nonzero element $x \in K$ the unique factorisation of the fractional ideal (x) as a product of prime powers is given by*

$$(x) = \prod_{0 \neq \mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(x)}$$

Proof. As before the nonzero primes with $v_\mathfrak{p}(x) \neq 0$ form a finite set $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, the difference being that now the integers $v_i = v_{\mathfrak{p}_i}(x)$ may be negative. Suppose $x = a/b$ with a, b nonzero elements of A. Then $v_\mathfrak{p}(x) = v_\mathfrak{p}(a) - v_\mathfrak{p}(b)$ for any nonzero prime $\mathfrak{p}$. Then

$$(x) = (a)(b)^{-1} = \prod_{0 \neq \mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(a)} \prod_{0 \neq \mathfrak{p}} \mathfrak{p}^{-v_\mathfrak{p}(b)} = \prod_{0 \neq \mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(a) - v_\mathfrak{p}(b)} = \prod_{0 \neq \mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(x)}$$

as required. □

Example 2. Let A be a Dedekind domain with quotient field K. The nonzero fractional ideals of A form an abelian group I under multiplication, isomorphic to the free abelian group on the set of nonzero prime ideals. These are precisely the prime ideals of height 1, which are in bijection with the prime divisors of $X = SpecA$. So there is an isomorphism of abelian groups $DivX \cong I$ defined by

$$\sum_{0 \neq \mathfrak{p}} n_{\mathfrak{p}} Y_{\mathfrak{p}} \mapsto \prod_{0 \neq \mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$$

where $Y_{\mathfrak{p}} = V(\mathfrak{p})$ is the prime divisor corresponding to $\mathfrak{p}$. A fractional ideal is *principal* if it is of the form (u) for some nonzero $u \in K$. Corresponding to this $u \in K^*$ is a global section of X which determines the principal divisor $\sum v_{\mathfrak{p}}(u) Y_{\mathfrak{p}}$, which by the preceding Corollary corresponds to (u) under the isomorphism $DivX \cong I$. So the principal divisors and principal fractional ideals are identified by this isomorphism.

The *ideal class group* of A is the quotient of I by the principal fractional ideals, so it follows that the ideal class group of A is isomorphic as an abelian group to the divisor class group $Cl(X)$ of $SpecA$.

Now we turn to the scheme $X = ProjS$ for a graded ring S. We assume that the ideal S_+ is maximal (i.e. S_0 is a field). This has the consequence that for any proper homogenous ideal $\mathfrak{b}$ with $V(\mathfrak{b}) \neq \emptyset$, the radical $\sqrt{\mathfrak{b}}$ is the intersection of all the homogenous primes $\mathfrak{p} \in V(\mathfrak{b})$ (since every proper homogenous ideal is contained in S_+). Associated to any homogenous ideal $\mathfrak{a}$ is the closed set $V(\mathfrak{a})$, and associated to a subset $Y \subseteq X$ (not necessarily closed) is the radical homogenous ideal $I(Y) = \cap_{\mathfrak{p} \in Y} \mathfrak{p}$ (so $I(\emptyset) = S$). Clearly $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$ for homogenous $\mathfrak{a}$. We make the following claims:

(i) For subsets $Y \subseteq W$ of X we have $I(W) \subseteq I(Y)$.

(ii) For homogenous ideals $\mathfrak{a} \subseteq \mathfrak{b}$ we have $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.

(iii) For a homogenous ideal $\mathfrak{a}$, $V(\mathfrak{a}) = \emptyset$ iff. either $\sqrt{a} = S$ or $\sqrt{a} = S_+$.

(iv) For a homogenous ideal $\mathfrak{a}$ with $V(\mathfrak{a}) \neq \emptyset$, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. This follows from the fact that S_+ is maximal by our earlier comment.

(v) For any subset $Y \subseteq X$, $V(I(Y)) = \overline{Y}$. Since the case $Y = \emptyset$ is trivial we assume $Y \neq \emptyset$. Clearly $Y \subseteq V(I(Y))$ so $\overline{Y} \subseteq V(I(Y))$. Let $W = V(\mathfrak{b})$ be closed and suppose $Y \subseteq W$ for a homogenous radical ideal $\mathfrak{b}$. Then $\mathfrak{b} = IV(\mathfrak{b}) = I(W)$ since we can assume $W \neq \emptyset$. Consequently $\mathfrak{b} \subseteq I(Y)$ and so $V(I(Y)) \subseteq W$. Hence $V(I(Y)) = \overline{Y}$.

(vi) The operations $V(-)$ and $I(-)$ set up an inclusion reversing bijection between homogenous radical ideals of S other than S_+ and closed subsets of $ProjS$:

$$\begin{tikzcd}
S \arrow[d, leftrightarrow] & S_+ \arrow[dl] & \mathfrak{a} \arrow[dl, leftrightarrow] & \mathfrak{b} \arrow[dl, leftrightarrow] & \cdots \\
\emptyset & V(\mathfrak{a}) & V(\mathfrak{b}) & \cdots &
\end{tikzcd}$$

(vii) This bijection identifies homogenous prime ideals other than S_+ with the irreducible closed subsets of $ProjS$. If $\mathfrak{a}$ is a radical homogenous ideal and $V(\mathfrak{a})$ is irreducible then $\mathfrak{a}$ is proper and if a, b are homogenous with $ab \in \mathfrak{a}$ then $(a)(b) \subseteq \mathfrak{a}$ so $V(\mathfrak{a}) \subseteq V(a) \cup V(b)$. Since $V(\mathfrak{a})$ is irreducible either $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$. Conversely if $\mathfrak{p}$ is a homogenous prime then $V(\mathfrak{p})$ is nonempty and if $V(\mathfrak{p}) = V(\mathfrak{b}) \cup V(\mathfrak{c}) = V(\mathfrak{bc})$ for homogenous radical ideals $\mathfrak{b}, \mathfrak{c}$ then $\mathfrak{p} = \sqrt{\mathfrak{p}} = \sqrt{\mathfrak{bc}} = \sqrt{\mathfrak{b}} \cap \sqrt{\mathfrak{c}} = \mathfrak{b} \cap \mathfrak{c}$ (we can assume $V(\mathfrak{b}), V(\mathfrak{c})$ are nonempty). Hence $\mathfrak{p} = \mathfrak{a}$ or $\mathfrak{p} = \mathfrak{b}$, as required.

Lemma 10. *If A is a nonzero ring, then $dim(SpecA) = dimA$. If A is a nonzero noetherian ring, then $dim(\mathbb{A}^n_A) = dim(\mathbb{P}^n_A) = dimA + n$ for any $n \geq 1$.*

Proof. We proved the first equality in our Section 3 notes. The second equality follow from Corollary 5.6.5 of our A & M dimension theory notes. In particular $dimA = \infty$ iff. $dim(\mathbb{A}^n_A) = \infty$. Now $\mathbb{P}^n_A = ProjA[x_0, \ldots, x_n]$ and this scheme is covered by the affine opens $D_+(x_i) \cong SpecA[x_0, \ldots, x_n]_{(x_i)} \cong SpecA[x_1, \ldots, x_n]$. The dimension of each of these affine opens is $dimA + n$, so by $(I, Ex.1.10)$ we have $dim(\mathbb{P}^n_A) = dimA + n$ also. So $dimA = \infty$ iff. $dim(\mathbb{P}^n_A) = \infty$. $\square$

Let k be a field and $S = k[x_0, \ldots, x_n]$ $(n \geq 1)$. Then $\mathbb{P}^n_k$ is an integral scheme of finite type over k, so the conclusions of Ex 3.20 apply. So for any irreducible closed subset $Y \subseteq \mathbb{P}^n_k$ we have

$$dimY + codim(Y, X) = dim(\mathbb{P}^n_k) = n$$

From our notes on projective space (TPC,Proposition 22) we know that $dimY = dim(S/I(Y)) - 1$. It follows that $ht.I(Y) = codim(Y, X)$ and $coht.I(Y) = dimY + 1$. We know that $\mathbb{P}^n_k$ is a noetherian integral separated scheme (TPC,Corollary 5). Regularity in codimension one follows from the fact that $\mathbb{P}^n_k$ is covered by affine opens $Speck[y_1, \ldots, y_n]$. Hence $\mathbb{P}^n_k$ satisfies the condition $(*)$ and we can talk about Weil divisors on $\mathbb{P}^n_k$.

The bijection between closed irreducible subsets of $\mathbb{P}^n_k$ and homogenous primes of S other than S_+ identifies the prime divisors of $\mathbb{P}^n_k$ with the homogenous primes of height 1 (since $ht.S_+ = ht.(x_0, \ldots, x_n) = n + 1 > 1$). By $(I, 1.12A)$ every prime ideal of height 1 in S is principal, and if $\mathfrak{p}$ is generated by an element f, then f is necessarily an irreducible homogenous polynomial (see our Chapter 1, Section 2 notes). Conversely every irreducible homogenous polynomial generates a homogenous prime of height 1 by $(I, 1.11A)$. So the map $f \mapsto V(f)$ sets up a bijection between associate classes of irreducible homogenous polynomials $f \in k[x_0, \ldots, x_n]$ with the prime divisors of $\mathbb{P}^n_k$. This means that the following definition makes sense:

Definition 6. For a field k and $n \geq 1$ let $Y \subseteq \mathbb{P}^n_k$ be a prime divisor. The *degree* of Y, denoted $degY$, is the degree of the associated irreducible homogenous polynomial (so $degY \geq 1$). For any divisor $D = \sum_i n_i Y_i$ the *degree* of D is $degD = \sum_i n_i \cdot degY_i$. Similarly we define the degree of a prime divisor $Y \subseteq \mathbb{A}^n_k$ to be the degree of the associated irreducible polynomial, and the degree of a divisor on $\mathbb{A}^n_k$ as above. It is clear that for divisors D, D' on affine or projective space $deg(D + D') = degD + degD'$.

Let $X = \mathbb{P}^n_k$ for a field k and $n \geq 1$, and let $Y \subseteq X$ be a prime divisor. So $Y = V(\mathfrak{p})$ for a homogenous prime of height 1. There is a commutative diagram of rings:

$$\begin{tikzcd}
\mathcal{O}_{X,\mathfrak{p}} \arrow[r] \arrow[d, Rightarrow] & \mathcal{O}_{X,0} \arrow[d, Rightarrow] \\
S_{(\mathfrak{p})} \arrow[r] & S_{((0))}
\end{tikzcd}$$

The vertical isomorphisms are defined in Proposition 2.5, and $S = k[x_0, \ldots, x_n]$. Thus $S_{((0))}$ is the quotient field of $S_{(\mathfrak{p})}$ and the question is: what is the valuation v_Y on $S_{((0))}$ corresponding to the prime divisor Y? With the notation of Definition 5 the result is the one we expect:

Proposition 11. *Let $S = k[x_0, \ldots, x_n]$ for a field k and $n \geq 1$. Let f be an irreducible homogenous polynomial and $\mathfrak{p} = (f)$. The discrete valuation on the field $S_{((0))}$ with valuation ring $S_{(\mathfrak{p})}$ is defined for nonzero homogenous $g, h \in S$ of the same degree by $v_{(\mathfrak{p})}(g/h) = v_f(g) - v_f(h)$.*

Proof. Let T be the multiplicatively closed subset of S consisting of all homogenous elements not in $\mathfrak{p}$. Then the maximal ideal of $S_{(\mathfrak{p})}$ is $\mathfrak{m} = \mathfrak{p}T^{-1}S \cap S_{(\mathfrak{p})}$. Of course in a discrete valuation ring the maximal ideal is principal, and $\mathfrak{p} = (f)$, so we can find homogenous $b, s \in S$ with $s \notin \mathfrak{p}$ and $\mathfrak{m} = (bf/s)$ (we can assume $b \notin \mathfrak{p}$ also). Given nonzero homogenous $c, t \in S$ of the same degree with $t \notin \mathfrak{p}$, $v_{(\mathfrak{p})}(c/t)$ is the largest $k \geq 0$ with $c/t \in \mathfrak{m}^k = (b^k f^k / s^k)$. By assumption $t \notin (f)$, so to show $v_{(\mathfrak{p})}(c/t) = v_f(c) - v_f(t)$ it suffices to show that for $k \geq 1$ $c/t \in \mathfrak{m}^k$ iff. f^k divides c. It is not hard to see that if $c/t \in \mathfrak{m}^k$ then f^k divides c. For the converse, suppose $c = hf^k$. Then h is homogenous and $hs^k/b^kt \in S_{(\mathfrak{p})}$. Clearly $(b^k f^k/s^k)(hs^k/b^kt) = c/t$, so $c/t \in \mathfrak{m}^k$ as required.

This takes care of elements of $S_{(\mathfrak{p})}$. For general nonzero homogenous $g, h \in S$ of the same degree we write $g = g' f^{v_f(g)}$ and $h = h' f^{v_f(h)}$. Since $v_{(\mathfrak{p})}(h/g) = -v_{(\mathfrak{p})}(g/h)$ we can assume that $v_f(g) \geq v_f(h)$ (of course, one or both may be zero). Then $g/h = (g' f^{v_f(g) - v_f(h)})/h' \in S_{(\mathfrak{p})}$ so $v_{(\mathfrak{p})}(g/h) = v_f(g) - v_f(h)$ as required. □

Let $X = \mathbb{P}^n_k$ be projective n-space over a field k for $n \geq 1$. Given nonzero homogenous $g, h \in S$ of the same degree the corresponding principal divisor is

$$(g/h) = \sum_{\substack{\mathfrak{p} \text{ homogenous} \\ ht.\mathfrak{p}=1}} v_{(\mathfrak{p})}(g/h) \cdot V(\mathfrak{p}) = \sum_f (v_f(g) - v_f(h)) \cdot V(f)$$

where the second sum is over the set of equivalence classes of irreducible homogenous polynomials under the associate relation, and we pick a single f from each class.

Proposition 12. *Let X be the projective space $\mathbb{P}^n_k$ over a field k ($n \geq 1$). Let H be the hyperplane $x_0 = 0$. Then:*

(a) If D is any divisor of degree d, then $D \sim dH$;

(b) For any $f \in K^*$, $deg(f) = 0$;

(c) The degree function gives an isomorphism of abelian groups $deg : ClX \longrightarrow \mathbb{Z}$.

Proof. Let $S = k[x_0, \ldots, x_n]$ and K the function field $\mathcal{O}_{X,0}$ of X. If $f \in K^*$ then f corresponds to a quotient g/h of two nonzero homogenous polynomials $g, h \in S$ of the same degree. If we factor g, h as $g = u p_1^{n_1} \cdots p_r^{n_r}$ and $h = v p_1^{m_1} \cdots p_r^{m_r}$ for $u, v \in k$ and irreducible polynomials p_i, then the p_i must be homogenous (we allow some zero indices to get the occurring p_i the same in both cases) and by Proposition 11 the principal divisor (f) is defined by

$$(f) = \sum_{i=1}^r (n_i - m_i) \cdot Y_i, \qquad Y_i = V(p_i)$$

Hence

$$deg(f) = \sum_{i=1}^r (n_i - m_i) deg(p_i) = \sum_{i=1}^r n_i deg(p_i) - \sum_{i=1}^r m_i deg(p_i) = deg(g) - deg(h) = 0$$

Which proves (b). To prove (a), let $D = \sum_{i=1}^r n_i \cdot D(p_i)$ be any nonzero effective divisor of degree d with $n_i > 0$ and the p_i homogenous irreducibles. Then $p_1^{n_1} \cdots p_r^{n_r}/x_0^d \in S_{((0))}$ and the corresponding principal divisor is $D - dH$, where $H = V(x_0)$, which shows that $D \sim dH$ (of course there is nothing special about x_0, we could have used any x_i). It follows immediately that any divisor of degree zero is principal.

Taking degrees defines a morphism of abelian groups $DivX \longrightarrow \mathbb{Z}$, and we have just shown the kernel of this map consists of the principal divisors. Since $deg(dH) = d$ for any $d \in \mathbb{Z}$ we obtain the required isomorphism $ClX \longrightarrow \mathbb{Z}$. □

Proposition 13. *Let X satisfy $(*)$ and let U be a nonempty open subset of X, and let $Z = X \setminus U$. Then:*

(a) There is a surjective morphism of groups $ClX \longrightarrow ClU$ defined by $\sum_i n_i \cdot Y_i \mapsto \sum_i n_i \cdot (Y_i \cap U)$ where we ignore those $Y_i \cap U$ which are empty;

(b) If $codim(Z, X) \geq 2$ then $ClX \longrightarrow ClU$ is an isomorphism;

(c) If Z is an irreducible subset of codimension 1, then there is an exact sequence of abelian groups

$$\mathbb{Z} \longrightarrow ClX \longrightarrow ClU \longrightarrow 0$$

where the first map is defined by $1 \mapsto 1 \cdot Z$.

Proof. (a) We have already noted that if X satisfies $(*)$ so does U, so the group ClU is defined. We have also noted that $Y \mapsto Y \cap U$ gives a bijection between the prime divisors of X meeting U and the prime divisors of U. Since Z is a proper closed subset of the noetherian space X, we can write it as a union $Z_1 \cup \cdots \cup Z_n$ of irreducible components, so there is only a finite number of prime divisors of X *not* meeting U.

Let $\varphi : DivX \longrightarrow DivU$ be the morphism of abelian groups induced by sending those Y with $Y \cap U \neq \emptyset$ to $Y \cap U$ and all other Y to zero (if $DivX = 0$ we just take the zero map). We claim that φ sends principal divisors to principal divisors. It clearly sends *any* divisor whose support does not include prime divisors meeting U to zero, so suppose ξ is the generic point of X, $K = \mathcal{O}_{X,\xi}$ and $f \in K^*$ with $v_Y(f) \neq 0$ for some prime divisor Y with $Y \cap U \neq \emptyset$. For such Y with generic point η there is a commutative diagram

$$\begin{tikzcd} Q \arrow[r, equal] & K \\ \mathcal{O}_{U,\eta} \arrow[u] \arrow[r, equal] & \mathcal{O}_{X,\eta} \arrow[u] \end{tikzcd}$$

If g is the image of f in Q^* then $v_{Y \cap U}(g) = v_Y(f)$ so it is clear that the image under φ of the principal divisor (f) is the principal divisor (g), as required (in fact φ maps the principal divisors of X *onto* the principal divisors of U). So we induce the desired surjective morphism of abelian groups $ClX \longrightarrow ClU$.

(b) Let $Z = Z_1 \cup \ldots \cup Z_n$ be the irreducible components of Z. If a prime divisor of X does not meet U then it must be one of the Z_i. Since $codim(Z, X) = min_i\{codim(Z_i, X)\}$ by assuming $codim(Z, X) \geq 2$ we are excluding this possibility. Thus *every* prime divisor of X must meet U, so there is a bijection between the prime divisors of U and X and so it is easy to see that $ClX \longrightarrow ClU$ is an isomorphism.

(c) If there are no prime divisors in U then Z is the only prime divisor of X, so $Im(\mathbb{Z} \longrightarrow ClX) = ClX = Ker(ClX \longrightarrow ClU)$ so the sequence is exact. Otherwise it is easy to see that the kernel of $ClX \longrightarrow ClU$ consists of divisors whose support is contained in $\{Z\}$, so the sequence is exact. ☐

Example 3. For a field k the prime divisors of $\mathbb{P}^2_k$ are the irreducible curves. Given an irreducible curve Y of degree d, $Cl(\mathbb{P}^2_k \setminus Y) \cong \mathbb{Z}/d\mathbb{Z}$. This follows immediately from (12), (13). If $d = 1$ then Y is a line and $Cl(\mathbb{P}^2_k \setminus Y) = 0$. Otherwise $Y \neq V(x_0)$ and we can describe the surjective morphism $\mathbb{Z} \longrightarrow ClU$ with kernel (d) by $n \mapsto n \cdot (V(x_0) \cap U)$ where $U = \mathbb{P}^2_k \setminus Y$.

2 Divisors on Curves

We will illustrate the notion of the divisor class group further by paying special attention to the case of divisors on curves. We will define the degree of a divisor on a curve, and we will show that on a complete nonsingular curve, the degree is stable under linear equivalence. Further study of divisors on curves will be found in Chapter IV. In this section k denotes an algebraically closed field.

To begin with, we need some preliminary information about curves and morphisms of curves. Recall our conventions about terminology from the end of Section 4:

Definition 7. A scheme X is *nonsingular* if all the local rings of X are regular local rings. A *variety* over k is an integral separated scheme X of finite type over k. A *curve* is a variety of dimension one. If X is proper over k, we say that X is *complete*. If Y is a nonsingular curve in the sense of Chapter 1, then $t(Y)$ is a nonsingular curve in the present sense.

Lemma 14. *Let X be a variety over k. Then a point $x \in X$ is closed if and only if its residue field is k, and any morphism $f : X \longrightarrow Y$ of varieties over k maps closed points to closed points.*

Proof. The first claim is (VS,Corollary 11), while the second claim follows from (VS,Proposition 19). ☐

Lemma 15. *Let X be a curve over k. Then*

(i) $x \in X$ is closed if and only if $dim\mathcal{O}_{X,x} = 1$ and the only nonclosed point is the generic point.

(ii) X has the finite complement topology.

(iii) $K(X)$ is a finitely generated field extension of k of transcendence degree 1.

Proof. (i) If x is closed then by Ex 3.20, $dim\mathcal{O}_{X,x} = dimX = 1$. Conversely if $dim\mathcal{O}_{X,x} = 1$, let $Y = \overline{\{x\}}$. This is an irreducible closed subset of X, so again by Ex 3.20 we have $dimY + codim(Y,X) = 1$. But $codim(Y,X) = dim\mathcal{O}_{X,x}$ by Ex 3.6 so $dimY = 0$. Since an irreducible closed subset of a scheme has a unique generic point, it follows that $Y = \{x\}$, as required. If $y \in X$ is any point, it follows from Ex 3.20 that $dim\mathcal{O}_{X,y} = dimX - dim\overline{\{y\}} \leq dimX$ so if y is not closed, $dim\mathcal{O}_{X,y} = 0$. But then $dim\overline{\{y\}} = dimX$ and by Ex $I, 1.10$ this is only possible if $\overline{\{y\}} = X$. Hence $y = \xi$ is the only nonclosed point. (ii) X is an irreducible space of dimension 1, so if Y is a proper closed irreducible subset then $dimY = 0$. Since $\xi \notin Y$ it follows from (i) that Y must be a point. Since X is noetherian any closed subset is either X or a finite union of points. (iii) Follows immediately from Ex 3.20. ☐

In particular, if Y is a curve in the sense of Chapter 1, then when we form the scheme $t(Y)$ we only add one point: the generic point, which corresponds to the irreducible subset of Y consisting of the whole space. This is obvious anyway, since any curve has the finite complement topology, so the only closed irreducible subsets are points and the whole space.

Let X be a nonsingular curve over k. Then if $x \in X$ is a closed point, $\mathcal{O}_{X,x}$ is a discrete valuation ring with quotient field $K(X)$, so there is a discrete valuation v_x on $K(X)$ with valuation ring $\mathcal{O}_{X,x}$. If $f \in K(X)^*$ then we can represent f by a section on the domain of definition U_f of f, and it is not difficult to see that $v_x(f) \geq 0$ if and only if $x \in U_f$.

Proposition 16. *Let X be a nonsingular curve over k with function field K. Then the following conditions are equivalent:*

(a) X is projective;

(b) X is complete;

(c) $X \cong t(C_K)$ as schemes over k, where C_K is the abstract nonsingular curve of $(I.6)$ and t is the functor from varieties to schemes of (2.6).

Proof. In our Varieties as Schemes notes we show that $t(C_K)$ is a nonsingular curve. $(a) \Rightarrow (b)$ Follows from (4.9). $(b) \Rightarrow (c)$ If $x \in X$ is a closed point then $dim\mathcal{O}_{X,x} = dimX = 1$ by Ex 3.20, so $\mathcal{O}_{X,x}$ is a discrete valuation ring. Considered as a subring of K, $\mathcal{O}_{X,x}$ determines a discrete valuation v_x of K/k. If v is a discrete valuation of K/k, then by Ex 4.5, v has a unique center $x \in X$. That is, the valuation ring R of v dominates $\mathcal{O}_{X,x}$. Since $R \subset K$, it follows from Lemma 15 that x must be a closed point. Therefore $\mathcal{O}_{X,x}$ is a discrete valuation ring, which is maximal under domination, so $R = \mathcal{O}_{X,x}$. Since the center of v is unique, this gives a bijection between the closed points of X and the discrete valuations of K/k, which are the points of C_K.

The points of $t(C_K)$ are the points of C_K plus a generic point, so by matching generic points we get a bijection $X \cong t(C_K)$. Since both spaces have the finite complement topology, this is trivially a homeomorphism. By Ex 3.6 there is an injective ring morphism $\mathcal{O}_X(U) \longrightarrow K$ for any nonempty open $U \subseteq X$. Clearly $\mathcal{O}_X(U) \subseteq \cap_{x \in U} \mathcal{O}_{X,x}$, and it is not hard to see this is an equality. So $X \cong t(C_K)$ is actually an isomorphism of schemes. ☐

This result shows that just as in (H, I.6), up to isomorphism there is only one projective nonsingular curve with a given function field.

Lemma 17. *If $f : X \longrightarrow Y$ is a finite morphism of curves over k, then f maps the generic point of X to the generic point of Y. The degree of the field extension $K(X)/K(Y)$ is finite.*

Proof. Let $\xi \in X, \eta \in Y$ be the generic points and suppose that $f(\xi) = z \neq \eta$. Then z is a closed point, so $f^{-1}(z)$ is closed, and therefore $f^{-1}(z) = X$. But f is finite, and a finite morphism is quasi-finite by Ex 3.5, so this is a contradiction. Using the canonical injective morphism of k-algebras $K(Y) \longrightarrow K(X)$ we consider $K(Y)$ as a subfield of $K(X)$. Since $K(X), K(Y)$ both have transcendence degree 1 over k, it follows that $K(X)$ is algebraic over $K(Y)$. Since $K(X)$ is finitely generated over k, it is finitely generated over $K(Y)$, so $K(X)/K(Y)$ is a finite extension. □

Definition 8. Let $f : X \longrightarrow Y$ be a finite morphism of curves over k. The *degree* of f is the degree of the field extension $[K(X) : K(Y)]$.

Lemma 18. *Let A be a normal domain, K its quotient field, F a finite algebraic extension of K, and A' the integral closure of A in F. Then F is the quotient field of A'.*

Proof. We used this several times in Section 6 of Chapter 1. The proof is part of Theorem 7, Section 4, Chapter V of Zariski & Samuel. See p.30 of our EFT notes (marked with a star). □

Lemma 19. *Let A be a Dedekind domain with quotient field K. Then the valuation rings of K containing A are precisely the subrings $A_\mathfrak{p}$ with $\mathfrak{p}$ a prime ideal of A.*

Proof. It is not hard to see that the subrings $A_\mathfrak{p}$ of A are all distinct, and since A is Dedekind they are all valuation rings of K (if $\mathfrak{p} \neq 0$ then $A_\mathfrak{p}$ is a discrete valuation ring of K). If $(V, \mathfrak{m})$ is a valuation ring of K containing A then $\mathfrak{m} \cap A = \mathfrak{p}$ is a prime ideal of A. If $\mathfrak{p} = 0$ then $V = K$. Otherwise V must dominate the valuation ring $A_\mathfrak{p}$, and hence $V = A_\mathfrak{p}$. □

Example 4. Take $A = k[x]$ for a field k. Then the nonzero primes of A are all of the form $\mathfrak{p} = (f)$ for an irreducible polynomial f. The valuation on $K = k(x)$ corresponding to $A_\mathfrak{p}$ takes a quotient g/h and spits out the power of f dividing g minus the power of f dividing h. Lemma 19 shows that these are the only valuations of K/A. This example shows intuitively why $A_\mathfrak{p}$ is a valuation ring of K, since any quotient g/h can be reduced until f divides at most one of g, h. So it is clear that one of $g/h, h/g$ must belong to $A_\mathfrak{p}$.

If A is not Dedekind then the local rings may not be valuation rings of K. Take for example $A = k[x, y]$ and $\mathfrak{p} = (x, y)$. Then $a = x/y$ is an element of the quotient field with $a \notin A_\mathfrak{p}, a^{-1} \notin A_\mathfrak{p}$.

Proposition 20. *Let X be a complete nonsingular curve over k, let Y be any nonsingular curve over k and let $f : X \longrightarrow Y$ be a morphism of schemes over k. Then either (1) $f(X) =$ a point, or (2) $f(X) = Y$. In case (2), $K(X)$ is a finite extension field of $K(Y)$, Y is complete and f is a finite morphism.*

Proof. It follows from Exercise 4.4 that f is proper, and hence $f(X)$ is closed. Since X is irreducible, so is $f(X)$. So it is clear that one of $(1), (2)$ must hold. Suppose that (2) holds and let $Z \longrightarrow Y$ be the image of f. The underlying topological space of Z is $f(X)$, so $Z \longrightarrow Y$ is a surjective closed immersion. Since Y is reduced, this means that $Z \longrightarrow Y$ is an isomorphism (see notes following our proof of (5.9)). By Exercise 4.4 the composite $Z \longrightarrow Y \longrightarrow S$ is proper, so Y is complete.

Since Y is infinite and f is surjective, f must map the generic point of X to the generic point of Y, which gives an injection of k-algebras $K(Y) \subseteq K(X)$. Using the argument of Lemma 17 we see that $K(X)/K(Y)$ is a finite field extension. For $x \in X$ taking the intersection $\mathcal{O}_{X,x} \cap K(Y)$ gives a valuation ring of $K(Y)$, which dominates $\mathcal{O}_{Y,f(x)}$ since the morphism $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is local. Therefore $\mathcal{O}_{X,x} \cap K(Y) = \mathcal{O}_{Y,f(x)}$, since valuation rings are maximal with respect to domination. Note that since $K(X)/K(Y)$ is algebraic, if x is not the generic point $\mathcal{O}_{X,x} \subset K(X)$ and therefore $\mathcal{O}_{Y,f(x)} \subset K(Y)$. That is, the fiber of f over the generic point of Y consists precisely of the generic point of X.

Now suppose that Y is nonsingular. Let $V \cong SpecB$ be an affine open subset of Y, with B a finitely generated k-domain. Using Ex 3.20e and the fact that Y is nonsingular, we see that B is a Dedekind domain. We can identify B with a subring of $K(Y)$ (which is then the quotient field of B) and hence with a subring of $K(X)$. Let A be the integral closure of B in $K(X)$. Then using $(I, 6.3A)$ and $(I, 3.9A)$ we see that A is a Dedekind finitely generated k-domain, which is also a finitely generated B-module. By Lemma 18, $K(X)$ is the quotient field of A.

Therefore we can find a nonsingular curve $Z \subseteq \mathbb{A}^n$ for some $n \geq 1$ (in the sense of Chapter 1) with coordinate ring $A(Z) \cong A$ as k-algebras, and function field $K(Z)$ k-isomorphic to $K(X)$. By $(I, 6.7)$ there is an isomorphism of "abstract nonsingular curves" $Z \cong U$, where U is an open subset of the abstract nonsingular curve $C_{K(Z)}$. Hence $t(Z) \cong t(U) \subseteq t(C_{K(Z)}) \cong t(C_{K(X)})$ as schemes over k. By (VS,Proposition 2) there is an isomorphism of schemes $t(Z) \cong SpecA$ over k, so finally $SpecA$ is isomorphic to an open subset of $t(C_{K(X)}) \cong X$. This isomorphism sends a nonzero prime $\mathfrak{p} \in SpecA$ to the discrete valuation ring $A_{\mathfrak{p}}$ of $K(X)$ and then to the corresponding point of X.

Since X, Y are both complete, for both curves there is a bijection between discrete valuation rings over k and closed points. The closed points $y \in V$ correspond to subrings $\mathcal{O}_{Y,y} \subseteq K(Y)$, which by Lemma 19 are precisely the discrete valuation rings of $K(Y)$ containing B. Therefore, if $x \in X$ we have $f(x) \in V$ if and only if $\mathcal{O}_{X,x} \supseteq B$, and since A is the integral closure of B, this is if and only if $\mathcal{O}_{X,x} \supseteq A$. Since A is the intersection of all the valuation rings of $K(X)$ containing A (see Corollary 5.22 A & M), we have

$$A = \bigcap_{x \in f^{-1}V} \mathcal{O}_{X,x} = \mathcal{O}_X(f^{-1}V)$$

We have constructed an isomorphism schemes over k of $SpecA$ with the open subset $f^{-1}V$ of X. By construction A is a finitely generated B-module, and we can cover Y with affine open sets of the form V, so this shows that f is finite. $\square$

Now we come to the study of divisors on curves. If X is a nonsingular curve, then X satisfies the condition $(*)$ used above, so we can talk about Weil divisors on X. A prime divisor is just a closed point, so an arbitrary divisor can be written $D = \sum n_i P$ where the P_i are closed points, and $n_i \in \mathbb{Z}$. We define the *degree* of D to be $\sum n_i$. Clearly this defines a morphism of abelian groups $deg : DivX \longrightarrow \mathbb{Z}$.

Definition 9. If $f : X \longrightarrow Y$ is a finite morphism of nonsingular curves over k, we define a morphism of abelian groups $f^* : DivY \longrightarrow DivX$ as follows. For any closed point $Q \in Y$, let $t \in \mathcal{O}_{Y,Q}$ be a *local parameter* at Q, which is an element of $K(Y)$ with $v_Q(t) = 1$, where v_Q is the valuation corresponding to the discrete valuation ring $\mathcal{O}_{Y,Q}$. We define

$$f^*Q = \sum_{f(P)=Q} v_P(t) \cdot P$$

Since f is a finite morphism this is a finite sum (see Ex 3.5), so we get a divisor on X. Note that f^*Q is independent of the choice of the local parameter t. If t' is another local parameter, then $t' = ut$ where u is a unit in $\mathcal{O}_{Y,Q}$. For any point $P \in X$ with $f(P) = Q$, u will map to a unit in $\mathcal{O}_{X,P}$, so $v_P(t) = v_P(t')$. We extend the definition by linearity to all divisors on Y.

Let $P \in X, Q \in Y$ be closed points with $f(P) = Q$, and let t be a local parameter at Q. If $0 \neq g \in \mathcal{O}_{Y,Q}$ then $v_Q(g)$ is the largest integer $k \geq 0$ for which $g \in \mathfrak{m}_Q^k$. So it is clear that $v_P(g) \geq v_Q(g)$. We claim that $v_P(g) = v_P(t)v_Q(g)$. This is trivial if $v_Q(g) = 0$ since then $v_P(g) = 0$. If $v_Q(g) = k \geq 1$, then $g = ut^k$ where u is a unit in $\mathcal{O}_{Y,Q}$. Therefore u is also a unit in $\mathcal{O}_{P,X}$, so $v_P(g) = v_P(t^k) = v_P(t)k$, as required. It follows that the image under f^* of the principal divisor (g) is the principal divisor (g) obtained from $g \in K(X)$. Hence f^* induces a morphism of abelian groups $f^* : ClY \longrightarrow ClX$.

Corollary 21. *A principal divisor on a complete nonsingular curve X over k has degree zero. Consequently the degree function induces an epimorphism $deg : ClX \longrightarrow \mathbb{Z}$.*

Lemma 22. *Let X be a complete nonsingular curve over k. Then X is rational if and only if there are two distinct closed points $P, Q \in X$ with $P \sim Q$.*

rationality of a variety over k is defined in Section 2.8 notes.

3 Cartier Divisors

Now we want to extend the notion of divisor to an arbitrary scheme. It turns out that using the irreducible subvarieties of codimension one doesn't work very well. So instead, we take as our point of departure the idea that a divisor should be something which locally looks like the divisor of a rational function. This is not exactly a generalisation of the Weil divisors (as we will see), but it gives a good notion to use on arbitrary schemes.

Definition 10. Let X be a scheme. For each open set U let $S(U)$ denote the set of elements of $\mathcal{O}_X(U)$ which are regular in $\mathcal{O}_{X,x}$ for every $x \in U$. For nonempty U this is a multiplicatively closed subset, and we define $Q(U) = S(U)^{-1}\mathcal{O}_X(U)$. If $U \subseteq V$ then restriction maps $S(V)$ to $S(U)$, so there is a morphism of rings $Q(V) \longrightarrow Q(U)$ defined by $a/s \mapsto a|_V/s|_V$. Thus defined Q is a presheaf of commutative rings, whose associated sheaf of commutative rings $\mathscr{K}_X$ we call the *sheaf of total quotient rings of* X. On an arbitrary scheme, the sheaf $\mathscr{K}_X$ replaces the concept of function field of an integral scheme. We simply write $\mathscr{K}$ for $\mathscr{K}_X$ if there is no chance of confusion.

Lemma 23. *Let $(X, \mathcal{O}_X)$ be a ringed space and let $M(U)$ be the set of invertible elements of the ring $\mathcal{O}_X(U)$. Then M is a sheaf of (multiplicative) abelian groups.*

We denote by $\mathscr{K}^*$ the sheaf of (multiplicative abelian groups) of invertible elements in the sheaf of rings $\mathscr{K}$. Similarly $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}_X$. There are canonical morphisms of presheaves of rings $\mathcal{O}_X \longrightarrow Q$ and $Q \longrightarrow \mathscr{K}$. The composite $\mathcal{O}_X \longrightarrow \mathscr{K}$ makes $\mathscr{K}$ into a sheaf of $\mathcal{O}_X$-algebras, and gives rise to a morphism of sheaves of abelian groups $\mathcal{O}^* \longrightarrow \mathscr{K}^*$.

Lemma 24. *The morphism $\mathcal{O}_X \longrightarrow \mathscr{K}$ is a monomorphism of sheaves of commutative rings, and $\mathcal{O}^* \longrightarrow \mathscr{K}^*$ is a monomorphism of sheaves of abelian groups.*

Proof. The second claim follows immediately from the first. For $x \in X$ we need to show that the composite $\mathcal{O}_{X,x} \longrightarrow Q_x \longrightarrow \mathscr{K}_x$ is injective. The second morphism is an isomorphism, so we reduce to showing that $\mathcal{O}_X(U) \longrightarrow Q(U)$ is injective for all $U \subseteq X$. But if $a \in \mathcal{O}_X(U)$ and $0 = a/1 \in Q(U)$ then $sa = 0$ for some $s \in S(U)$. By definition $germ_x s$ is regular in $\mathcal{O}_{X,x}$ for all $x \in U$ and consequently $germ_x a = 0$ for all $x \in U$, which implies that $a = 0$, as required. $\square$

Recall that the cokernel $\mathscr{K}^*/\mathcal{O}^*$ of the subobject $\mathcal{O}^* \longrightarrow \mathscr{K}^*$ is the sheafification of the presheaf $U \mapsto \mathscr{K}^*(U)/\mathcal{O}^*(U)$ of abelian groups.

Proposition 25. *Let X be a scheme and $\phi : \mathcal{O}_X \longrightarrow \mathscr{B}$ a morphism of sheaves of commutative rings on X with the property that for every open $U \subseteq X$ the morphism $\phi_U : \mathcal{O}_X(U) \longrightarrow \mathscr{B}(U)$ sends $S(U)$ to units. Then there is a unique morphism $\psi : \mathscr{K} \longrightarrow \mathscr{B}$ of sheaves of rings making the following diagram commute*

$$\begin{tikzcd} \mathscr{K} \arrow[r, dotted, "\psi"] & \mathscr{B} \\ \mathcal{O}_X \arrow[u] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proof. For nonempty open $U \subseteq X$ we obtain in the usual way a morphism of commutative rings $\psi'_U : Q(U) \longrightarrow \mathscr{B}(U)$ defined by $a/s \mapsto \phi_U(a)\phi_U(s)^{-1}$ unique making the following diagram commute

$$\begin{tikzcd} Q(U) \arrow[r, "\psi'_U"] & \mathscr{B}(U) \\ \mathcal{O}_X(U) \arrow[u] \arrow[ur, "\phi_U"'] & \end{tikzcd}$$

This defines a morphism of presheaves of rings $\psi' : Q \longrightarrow \mathscr{B}$, which induces the required morphism $\psi : \mathscr{K} \longrightarrow \mathscr{B}$. $\square$

Lemma 26. *Let X be a scheme and $V \subseteq X$ an open subset. Then there is a canonical isomorphism $\mathscr{K}_V \longrightarrow \mathscr{K}_X|_V$ of sheaves of algebras on V.*

Proof. Let Q_X be the presheaf of rings on X sheafifiying to give $\mathscr{K}_X$ and Q_V the presheaf of rings on V sheafifying to give $\mathscr{K}_V$. It is clear that $Q_X|_V = Q_V$ and therefore that we have a canonical isomorphism of sheaves of rings $\mathscr{K}_V \longrightarrow \mathscr{K}_X|_V$ (see *p.*7 of our Section 2.1 notes). It is not difficult to check this is a morphism of $\mathcal{O}_X|_V$-algebras. $\square$

Lemma 27. *Let $f : X \longrightarrow Y$ be an isomorphism of schemes. There is a canonical isomorphism $\mathscr{K}_Y \longrightarrow f_*\mathscr{K}_X$ of sheaves of algebras on Y.*

Proof. There is an obvious isomorphism of presheaves of $\mathcal{O}_Y$-algebras $Q_Y \cong f_*Q_X$ which leads to an isomorphism of sheaves of algebras $\mathscr{K}_Y \cong \mathbf{a}(f_*Q_X) \cong f_*(\mathbf{a}Q_X) = f_*\mathscr{K}_X$. $\square$

Lemma 28. *Let A be an integral domain and set $X = SpecA$. Let S be the multiplicatively closed subset of all regular elements of A, and let $Q = S^{-1}A$. We claim there is a canonical isomorphism $\theta : Q^\sim \longrightarrow \mathscr{K}_X$ of sheaves of algebras on X.*

Proof. For the moment let A be any nonzero commutative ring. By Proposition 25 it suffices to show that $Q^\sim$ has the same universal property as $\mathscr{K}_X$. Let $\phi : \mathcal{O}_X \longrightarrow \mathscr{B}$ be a morphism of sheaves of commutative rings on X which sends $S(U)$ to units for every open $U \subseteq X$. By (SOA,Proposition 5) the functor $\widetilde{-} : A\mathbf{Alg} \longrightarrow \mathfrak{Alg}(X)$ is left adjoint to the global sections functor $\Gamma(-) : \mathfrak{Alg}(X) \longrightarrow A\mathbf{Alg}$. So corresponding to ϕ there is a morphism of A-algebras $\Phi : A \longrightarrow \Gamma(\mathscr{B})$ which sends elements of S to units (it is not hard to see that the ring isomorphism $A \cong \Gamma(\mathcal{O}_X)$ identifies S with $S(X)$, since if A is a nonzero ring then $a \in A$ is regular in A iff. $a/1$ is regular in $A_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A). Therefore there is a unique morphism of rings $\Psi : Q \longrightarrow \Gamma(\mathscr{B})$ with $A \longrightarrow Q \longrightarrow \Gamma(\mathscr{B}) = \Phi$. Using the properties of the adjunction, it is not difficult to check that the corresponding morphism of sheaves of rings $\psi : Q^\sim \longrightarrow \mathscr{B}$ is unique making the following diagram commute

$$\begin{tikzcd} Q^\sim \arrow[r, "\psi"] & \mathscr{B} \\ \mathcal{O}_X \arrow[u] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Next we show that the morphism of sheaves of rings $\mathcal{O}_X \longrightarrow Q^\sim$ sends elements of $S(U)$ to units for every open $U \subseteq X$. It suffices to show that $\mathcal{O}_X(D(f)) \longrightarrow Q^\sim(D(f))$ sends $S(D(f))$ to units for every nonzero $f \in A$. But by the same argument as above, the ring isomorphism $\mathcal{O}_X(D(f)) \cong A_f$ identifies $S(D(f))$ with the set of regular elements of the ring A_f, so we have only to show that $A_f \longrightarrow Q_f$ sends regular elements to units (SOA,Proposition 4).

Now assume that A is an integral domain. If a/f^n is regular in A_f it follows that a is nonzero and therefore regular in A, so clearly a/f^n is a unit in Q_f, as required. Using the universal property we obtain a *unique* isomorphism $\theta : Q^\sim \longrightarrow \mathscr{K}_X$ of sheaves of algebras on X. $\square$

Proposition 29. *If X is an integral scheme then $\mathscr{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras.*

Proof. Combining Lemma 26 and Lemma 27 we reduce to the case $X = SpecA$ for an integral domain A, which is handled in Lemma 28. $\square$

Definition 11. A *Cartier divisor* on a scheme X is a global section of the sheaf $\mathscr{K}^*/\mathcal{O}^*$. A Cartier divisor is *principal* if it is in the image of the natural map $\mathscr{K}^*(X) \longrightarrow \mathscr{K}^*/\mathcal{O}^*(X)$. Two Cartier divisors are *linearly equivalent* if their difference is principal. Although the group operation on $\mathscr{K}^*/\mathcal{O}^*$ is multiplication, we will use the language of additive groups when speaking of Cartier divisors, so as to preserve the analogy with Weil divisors.

If X is the zero scheme, then the group of Cartier divisors is the zero group. Otherwise, giving a Cartier divisor on X is equivalent to giving a cover $\{U_i\}_{i \in I}$ of X by nonempty open sets, and for each $i \in I$ an element $f_i \in \mathscr{K}^*(U_i)$ such that for all $i, j \in I$, $f_i|_{U_i \cap U_j}/f_j|_{U_i \cap U_j}$ belongs to the image of $\mathcal{O}^*(U_i \cap U_j)$ in $\mathscr{K}^*(U_i \cap U_j)$. We represent this situation by saying that the divisor is represented by the pairs $\{(U_i, f_i)\}_{i \in I}$. This clashes slightly with our notation for germs, but it should be clear from context what we mean.

Lemma 30. *If X is an integral scheme then $\mathscr{K}$ is isomorphic as a sheaf of $\mathcal{O}_X$-algebras to the constant sheaf associated to the function field K of X.*

Proof. We begin by proving that for $x \in X$ there is an isomorphism of rings $Q_x \cong K$. First notice that for a nonempty open subset U the set $S(U)$ consists precisely of those $s \in \mathcal{O}_X(U)$ with $s_\xi \neq 0$ (where ξ is the generic point of X). So the ring morphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\xi} = K$ sends the elements of $S(U)$ to units and induces a morphism of rings $Q(U) \longrightarrow K$, and therefore a morphism of rings $\rho_x : Q_x \longrightarrow K$ for any $x \in K$, defined by $(U, a/s) \mapsto a_\xi(s_\xi)^{-1}$.

To show ρ_x injective for all x, it suffices to show that $Q(U) \longrightarrow K$ is injective for nonempty open U. But if $a, b \in \mathcal{O}_X(U)$ and $s, t \in S(U)$ are such that $a_\xi(s_\xi)^{-1} = b_\xi(t_\xi)^{-1}$ then $(at - bs)_\xi = 0$ and thus $at - bs = 0$ in $\mathcal{O}_X(U)$, showing that $a/s = b/t \in Q(U)$. The map ρ_x is surjective since K is the quotient field of $\mathcal{O}_{X,x}$ for any $x \in X$. Hence ρ_x is an isomorphism of rings.

For nonempty open U define $\mathscr{K}(U) \longrightarrow K$ by $a \mapsto \rho_\xi(a(\xi))$. Elements of $\mathscr{K}(U)$ map points $x \in U$ to germs $a(x) \in Q_x$, and since X is integral $\rho_x(a(x))$ will be constant for all $x \in U$. So we may as well use ξ. It is not difficult to see that this gives an isomorphism of sheaves of rings of $\mathscr{K}$ with the constant sheaf on X corresponding to the field K. Clearly the sheaf of abelian groups $\mathscr{K}^*$ is isomorphic to the constant sheaf on X corresponding to the multiplicative abelian group K^*. $\square$

Notice that the isomorphism of abelian groups $\mathscr{K}^*(U) \cong K^*$ for nonempty U fits into the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}^*(U) \arrow[r] \arrow[dr] & \mathscr{K}^*(U) \arrow[d, Rightarrow] \\ & K^* \end{tikzcd}$$

Lemma 31. *Set $X = \mathbb{P}^n_k$ for a field k and fix a nonzero homogenous polynomial $f \in S_1$. Identify the function field K with $S_{((0))}$. For $x \in X$ let i be an arbitrary integer with $x \in D_+(x_i)$ and define*

$$C_f(x) = (X, f/x_i + \mathcal{O}^*(X))$$

Then C_f is a Cartier divisor. Equivalently C_f is defined by the family of sections $\{(D_+(x_i), f/x_i + \Gamma(D_+(x_i), \mathcal{O}^*))\}_{0 \le i \le n}$. Given $0 \le \ell \le n$ we write C_ℓ for the Cartier divisor C_{x_ℓ}.

Proof. First we have to check the definition makes sense. Suppose we have $x \in D_+(x_i) \cap D_+(x_j)$. To show the germs $(X, f/x_i + \mathcal{O}^*(X)), (X, f/x_j + \mathcal{O}^*(X))$ agree, it suffices to show that $(f/x_i)/(f/x_j) \in S_{((0))}$ corresponds to an element $\mathcal{O}^*(U) \subseteq \mathscr{K}^*(U)$ for some open neighborhood U of x. But if we take $U = D_+(x_i) \cap D_+(x_j)$ then $x_j/x_i \in \mathcal{O}^*(U)$, as required.

Next we have to check that $C_\ell \in \Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$. But for every $y \in D_+(x_i)$ we have $C_\ell(y) = (D_+(x_i), x_\ell/x_i + \Gamma(D_+(x_i), \mathcal{O}^*))$, so this is obvious. $\square$

Lemma 32. *Let X be a normal scheme satisfying $(*)$ and let U be a nonempty open subset. If $f, g \in K^*$ are such that $v_Y(f) = v_Y(g)$ for all prime divisors Y with $Y \cap U \neq \emptyset$, then $f/g \in \mathcal{O}^*(U)$.*

Proof. Considering $\mathcal{O}_X(U)$ as a subring of K, it suffices to show that $f/g, g/f \in \mathcal{O}_X(U)$. By symmetry it suffices to show $f/g \in \mathcal{O}_X(U)$, and for this it suffices to produce an open cover $\{U_i\}$ of U with $f/g \in \mathcal{O}_X(U_i)$ for all i. Given $x \in U$ let $V \cong SpecA$ be an affine open neighborhood of x. Since X is normal, A is a normal noetherian domain. If $\mathfrak{p}$ is a prime ideal of height 1 in A then $V(\mathfrak{p})$ corresponds to a prime divisor of U, which is the restriction of a prime divisor Y of X (with generic point $\eta \in U$ corresponding to $\mathfrak{p}$). Let Q be the quotient field of A, and let $h \in Q$ be the image of f/g under the isomorphism $K \cong Q$. Then $h \in A_\mathfrak{p}$ since $v_Y(f/g) = 0$ implies $f/g \in \mathcal{O}_{X,\eta}$. By Proposition 5 it follows that $h \in A \subseteq K$ and hence $f/g \in \mathcal{O}_X(V)$, which gives the required open cover of U and completes the proof. $\square$

As a particular case this shows that if X is a normal scheme satisfying $(*)$ and U is a nonempty open subset, if $f \in K^*$ and $v_Y(f) = 0$ for all prime divisors Y with $Y \cap U \neq \emptyset$ then $f \in \mathcal{O}^*(U)$.

Definition 12. A scheme X is *locally factorial* if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a unique factorisation domain. Since a regular local ring is a unique factorisation domain (MAT,Theorem 158) any nonsingular scheme is locally factorial.

Proposition 33. *Let X be an integral, separated noetherian scheme which is locally factorial. Then the group $DivX$ of Weil divisors on X is isomorphic to the group of Cartier divisors $\Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Proof. First note that X is normal, hence satisfies $(*)$, since a UFD is normal. So it makes sense to talk about Weil divisors. In the case where X has no prime divisors, then $X \cong SpecK$ for a field K and the presheaf of abelian groups $U \mapsto \mathscr{K}^*(U)/\mathcal{O}^*(U)$ is the zero presheaf, so $DivX = \Gamma(X, \mathscr{K}^*/\mathcal{O}^*) = 0$, and this isomorphism clearly preserves principal divisors. So we can assume that X has at least one principal divisor.

Let a Cartier divisor C be given, and let Y be a prime divisor of X with generic point η. Let $C(\eta) = (U, f + \mathcal{O}^*(U))$ for some open neighborhood U of η and $f \in \mathscr{K}^*(U)$. Take the coefficient C_Y of Y to be the integer $v_Y(f)$, where we use the isomorphism of groups $\mathscr{K}^*(U) \cong K^*$ to identify f with an element of K^*. To be precise, this integer is $v_Y(\rho_\eta(f(\eta)))$. This is independent of the choice of element f used to represent the germ $C(\eta)$, since if $g \in \mathscr{K}^*(V)$ gives the same germ then there is an open neighborhood $\eta \in W \subseteq U \cap V$ with $f|_W/g|_W \in \mathcal{O}^*(W) \subseteq \mathscr{K}^*(W)$. For any element $a \in \mathcal{O}_X(W)$ the germ $a_\xi \in K^*$ is assigned a non-negative value by v_Y since η belongs to the domain of definition of a_ξ. And by assumption $f|_W/g|_W$ is the image in $\mathscr{K}^*(W)$ of a *unit* in $\mathcal{O}_X(W)$, whose value under v_Y must be zero. Consequently $v_Y(f/g) = 0$ and so $v_Y(f) = v_Y(g)$, as required.

We claim that $C_Y \neq 0$ for only finite many prime divisors Y. Since X is noetherian we can cover it in a finite number of nonempty open sets U_i together with $f_i \in \mathscr{K}^*(U_i)$ such that for all $y \in U_i, C(y) = (U_i, f_i + \mathcal{O}^*(U_i))$. By Lemma 3, $v_Y(f_i) \neq 0$ for only finitely many Y, so it follows that $C_Y \neq 0$ for only finitely many Y. Thus we obtain a well-defined Weil divisor $D = \sum C_Y \cdot Y$ on X.

Conversely, if D is a Weil divisor on X, let $x \in X$ be any point other than the generic point ξ. By Lemma 2 there is at least one prime divisor passing through x, and if $f : Spec(\mathcal{O}_{X,x}) \longrightarrow X$ is canonical then $Y \mapsto f^{-1}Y$ gives an injective map from the set of the divisors passing through x to the prime divisors of $Spec(\mathcal{O}_{X,x})$. So D induces a Weil divisor D_x on $Spec(\mathcal{O}_{X,x})$. The divisor D_x is principal by Proposition 4, so $D_x = (f_x)$ for some $f_x \in Q^*$, where Q is the function field of $Spec(\mathcal{O}_{X,x})$. Let $h_x \in K^*$ be the corresponding element under the isomorphism $K \cong Q$. Then the principal divisor (h_x) has the same values as D on any prime divisor meeting x. Since there are only finitely many prime divisors which do not contain x on which either D or (h_x) has a nonzero value, there is an open neighborhood U_x of x such that D and (h_x) have the same restriction to U_x. It follows from Lemma 32 that $h_x/h_y \in \mathcal{O}^*(U_x \cap U_y)$ for any $x, y \in X$. So the elements of $\mathscr{K}^*(U_x)$ corresponding to the h_x patch together to give a Cartier divisor on X.

It is clear that given a Weil divisor D and corresponding Cartier divisor C that the Weil divisor produced $\sum_Y C_Y \cdot Y$ produced from C is just D. In particular the construction in the previous paragraph is independent of the chosen $f_x \in Q^*$ with $(f_x) = D$ (which was the only choice involved). In the other direction, given a Cartier divisor C and corresponding Weil divisor $D = \sum C_Y \cdot Y$ pick for each $x \in X$ an open set U_x and $h_x \in \mathscr{K}^*(U_x)$ such that $C(y) = (U_x, h_x + \mathcal{O}^*(U_x))$ for all $y \in U_x$. Let $f_x \in Q^*$ correspond to h_x. Then $(f_x) = D_x$ and C is the Cartier divisor constructed from D, as required.

We have established a bijection $DivX \longrightarrow \Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$, which is easily seen to be a morphism of abelian groups. If $f \in \mathscr{K}^*(X)$ then the Weil divisor corresponding to f is just the principal divisor (f) (identifying $\mathscr{K}^*(X)$ with K^*), so the isomorphism identifies principal Weil divisors with principal Cartier divisors. □

Definition 13. Let X be a scheme. The *Cartier divisor class group* of X, denoted $CaClX$, is the group of Cartier divisors modulo the principal Cartier divisors.

Proposition 33 implies that for an integral, separated noetherian locally factorial scheme X we have isomorphisms of abelian groups $DivX \cong \Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$ and $ClX \cong CaClX$. In particular this is true when X is a nonsingular variety over a field k.

Remark 1. Let $X = \mathbb{P}^n_k$ where k is a field and $n \geq 1$. We have already observed that X is a variety over k (TPC,Corollary 5). In fact, it is a nonsingular variety since it is covered by affine open subsets isomorphic to $Speck[x_1, \dots, x_n]$, and the local rings of $k[x_1, \dots, x_n]$ are all regular (MAT,Corollary 119). Since a regular local ring is a UFD (MAT,Theorem 158), X is locally factorial and we can apply Proposition 33.

Lemma 34. *Let $X = \mathbb{P}^n_k$ where k is a field and $n \geq 1$. Under the isomorphism $DivX \cong \Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$ the hyperplane $V(g)$ corresponds to the Cartier divisor C_g for any nonzero homogenous polynomial $g \in S_1$.*

Proof. See Lemma 31 for the definition of the Cartier divisor C_g. Let $Y = V(f)$ be a prime divisor of X with generic point η. Then $C_g(\eta) = (X, g/x_i + \mathcal{O}^*(X))$ where $\eta \in D_+(x_i)$. In any case, $f \neq x_i$ and so clearly $v_Y(g/x_i) = 0$ by Proposition 11, unless f is an associate of g in which case $C_\ell(\eta) = 1$. So Proposition 33 associates C_g with $V(g)$. $\square$

4 Invertible Sheaves

We will see now that invertible sheaves on a scheme are closely related to divisor classes modulo linear equivalence. For some background material necessary in this section, the reader should consult (MRS,Definition 2) and related results.

Proposition 35. *If $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves on a ringed space X, so is $\mathscr{L} \otimes \mathscr{M}$. If $\mathscr{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathscr{L}^{-1}$ on X such that $\mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathcal{O}_X$.*

Proof. See (MRS,Lemma 83) for the proof. The inverse $\mathscr{L}^{-1}$ is the dual $\mathscr{H}om(\mathscr{L}, \mathcal{O}_X)$. $\square$

Definition 14. Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$ as above. Let $\mathscr{L}(D)$ be the submodule of the $\mathcal{O}_X$-module $\mathscr{K}$ generated by the set $\{f_i^{-1}\}$. We call $\mathscr{L}(D)$ the *sheaf associated to* D. Clearly $\mathscr{L}(0) = \mathcal{O}_X$.

Lemma 36. *Let D be a Cartier divisor on a scheme X. The $\mathcal{O}_X$-module $\mathscr{L}(D)$ is independent of the matching family $\{(U_i, f_i)\}$ chosen to represent D, and for $U \subseteq U_i$, $\Gamma(U, \mathscr{L}(D))$ is the $\mathcal{O}_X(U)$-submodule generated by $f_i^{-1}|_U$.*

Proof. Let D be a Cartier divisor represented by $\{(U_i, f_i)\}$. Then for $x \in U_i$ the submodule $G_x \subseteq \mathscr{K}_x$ generated by the set $\{germ_x f_i^{-1} \,|\, x \in U_i\}$ is in fact $G_x = (germ_x f_i^{-1})$. To see this, note that by definition $f_i|_{U_i \cap U_j}/f_j|_{U_i \cap U_j} \in \mathcal{O}^*(U_i \cap U_j)$ so in the $\mathcal{O}_X(U_i \cap U_j)$-module $\mathscr{K}(U_i \cap U_j)$ we have $f_i^{-1}|_{U_i \cap U_j} = u \cdot f_j^{-1}|_{U_i \cap U_j}$ for a unit u. So for all $x \in U_i \cap U_j$ we have $germ_x f_j^{-1} \in (germ_x f_i^{-1})$. This proves that $G_x = (germ_x f_i^{-1})$ for $x \in U_i$.

If $\{(V_j, g_j)\}$ is another cover amalgamating to give D and $x \in X$ then say $x \in U_i \cap V_j$. Then in Q_x we have $(V_j, g_j + \mathcal{O}^*(V_j)) = D(x) = (U_i, f_i + \mathcal{O}^*(U_i))$. As before this shows that $germ_x g_j^{-1}$ and $germ_x f_i^{-1}$ generate the same submodule of $\mathscr{K}_x$, and so by the previous paragraph the two covers determine the same submodule $\mathscr{L}(D)$ of $\mathscr{K}$.

For the second claim suppose $a \in \Gamma(U, \mathscr{L}(D))$ for a nonempty open subset U. The fact that $germ_x a \in (germ_x f_i^{-1})$ for all $x \in U$ means we can find a cover of U by nonempty open W_α with $r_\alpha \in \mathcal{O}_X(W_\alpha)$ such that $a|_{W_\alpha} = r_\alpha \cdot f_i^{-1}|_{W_\alpha}$. If $\varphi : \mathcal{O}_X \longrightarrow \mathscr{K}$ is the canonical monomorphism of sheaves of rings (which gives $\mathscr{K}$ its $\mathcal{O}_X$-module structure) then $r_\alpha \cdot f_i^{-1}|_{W_\alpha} = \varphi_{W_\alpha}(r_\alpha) f_i^{-1}|_{W_\alpha}$. Since f_i is a unit and φ is a monomorphism it follows that the r_α form a matching family, which yields $r \in \mathcal{O}_X(U)$ with $r|_{W_\alpha} = r_\alpha$ for all α. Clearly $a = r \cdot f_i^{-1}|_U$, as required. $\square$

Before giving the next result we need to study more closely which sections of $\mathscr{K}$ are units.

Lemma 37. *Let X be a scheme and $U \subseteq X$ an open subset. Then $f \in \mathscr{K}(U)$ is a unit if and only if $germ_x f \in \mathscr{K}_x$ is $\mathcal{O}_{X,x}$-torsion-free for all $x \in U$.*

Proof. This is trivial for $U = \emptyset$, so we may assume U is nonempty. Suppose $f(x) \in Q_x$ is a unit for all $x \in U$ and define $g(x) = f(x)^{-1}$. Then g is a well-defined element of $\mathscr{K}(U)$ since if $x \in U$ is given we can find $x \in V \subseteq U$ and $a \in Q(V)$ with $f(y) = (V, a)$ for all $y \in V$. If $(W, b) = (V, a)^{-1}$ in Q_x then $(W, b) = (V, a)^{-1}$ in Q_y for all y in some open $x \in Q \subseteq V \cap W$ and consequently $g(y) = (Q, b|_Q)$ for all $y \in Q$. So f is a unit in $\mathscr{K}(U)$ iff. $f(x)$ is a unit in Q_x for all $x \in U$.

Since there is an isomorphism of rings $Q_x \cong \mathscr{K}_x$ we have reduced to showing that $f(x)$ is a unit in Q_x for all $x \in U$ iff. $f(x)$ is $\mathcal{O}_{X,x}$-torsion-free for all $x \in U$. One implication is clear. For the other, suppose $x \in U$ is given, and let $x \in V \subseteq U$ be such that $f(y) = (V, a/s) \in Q_y$ for all $y \in V$, where $a \in \mathcal{O}_X(V)$ and $s \in S(V)$. By assumption $(V, a/s)$ is torsion-free in Q_y for all $y \in V$, from which it follows that (V, a) is torsion-free and therefore regular in $\mathcal{O}_{X,y}$ for all $y \in V$. That is, $a \in S(V)$, from which it follows that $(V, a/s)$ is a unit in Q_x. So $f(x)$ is a unit for all $x \in U$, as required. □

Corollary 38. *Let X be a scheme and $\mathscr{L}$ an invertible submodule of $\mathscr{K}$. Then for any $x \in X$, $\mathscr{L}_x$ is generated as an $\mathcal{O}_{X,x}$-submodule of $\mathscr{K}_x$ by a unit.*

Lemma 39. *Let A be a commutative ring and B a commutative A-algebra. Let M, N be A-submodules of B with one of M, N generated as an A-module by a regular element of B. Then there is a canonical isomorphism of A-modules $M \otimes_A N \longrightarrow M \cdot N$ given by $m \otimes n \mapsto mn$.*

Proof. We assume without loss of generality that M is generated as an A-module by a regular element $a \in B$. The map $M \times N \longrightarrow M \cdot N$ given by $(m, n) \mapsto mn$ is clearly A-bilinear, and induces a surjective morphism of A-modules $M \otimes_A N \longrightarrow M \cdot N$. Suppose that $\sum_i (a_i a \otimes b_i)$ is mapped to zero in B. Then $(\sum_i a_i b_i) a = 0$ and therefore $\sum_i a_i b_i = 0$. But then

$$\sum_i (a_i a \otimes b_i) = a \otimes (\sum_i a_i b_i) = 0$$

therefore $M \otimes_A N \longrightarrow M \cdot N$ is an isomorphism. □

Definition 15. Let X be a scheme and $\mathscr{F}, \mathscr{G}$ submodules of the commutative $\mathcal{O}_X$-algebra $\mathscr{K}$. In (SOA,Section 2.3) we defined the product $\mathscr{M}\mathscr{L}$, which is a submodule of $\mathscr{K}$. If we identify $\mathcal{O}_X$ with a submodule of $\mathscr{K}$ then it is clear that $\mathcal{O}_X \mathscr{F} = \mathscr{F}$ for any submodule $\mathscr{F}$ of $\mathscr{K}$.

Proposition 40. *Let X be a scheme and $\mathscr{M}, \mathscr{L}$ submodules of $\mathscr{K}$ with one of $\mathscr{M}, \mathscr{L}$ invertible. Then there is a canonical isomorphism $\mathscr{M} \otimes \mathscr{L} \longrightarrow \mathscr{M}\mathscr{L}$ of sheaves of modules on X.*

Proof. By definition we have an epimorphism of sheaves of modules $\mathscr{M} \otimes \mathscr{L} \longrightarrow \mathscr{M}\mathscr{L}$, so it suffices to show that the map $\mathscr{M}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{L}_x \longrightarrow \mathscr{K}_x$ is injective for every $x \in X$. This follows immediately from Lemma 39 and Corollary 38. □

Proposition 41. *Let X be a scheme. Then*

(a) For any Cartier divisor D, $\mathscr{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathscr{L}(D)$ gives a bijection between Cartier divisors on X and invertible submodules of $\mathscr{K}$.

(b) $\mathscr{L}(D_1 + D_2) = \mathscr{L}(D_1)\mathscr{L}(D_2)$ as submodules of $\mathscr{K}$. In particular there is a canonical isomorphism $\mathscr{L}(D_1 + D_2) \cong \mathscr{L}(D_1) \otimes \mathscr{L}(D_2)$ of $\mathcal{O}_X$-modules.

(c) If $\mathscr{F}$ is an invertible submodule of $\mathscr{K}$ then so is $(\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})$, and $\mathscr{F}(\mathcal{O}_X :_{\mathscr{K}} \mathscr{F}) = \mathcal{O}_X$. There is a canonical isomorphism $\mathscr{H}om(\mathscr{F}, \mathcal{O}_X) \cong (\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})$ of $\mathcal{O}_X$-modules.

(d) The invertible submodules of $\mathscr{K}$ form an abelian group $Inv\mathscr{K}$ under multiplication, and $D \mapsto \mathscr{L}(D)$ defines an isomorphism of abelian groups.

(e) If $u \in \mathscr{K}^*(X)$ then (u) is an invertible submodule of $\mathscr{K}$. The map $u \mapsto (u)$ is a morphism of abelian groups $\mathscr{K}^*(X) \longrightarrow Inv\mathscr{K}$ with kernel $\mathcal{O}^*(X)$. An invertible submodule of $\mathscr{K}$ is principal *if it is in the image of this morphism, and given invertible submodules $\mathscr{F}, \mathscr{G}$ of $\mathscr{K}$ we write $\mathscr{F} \sim \mathscr{G}$ if $\mathscr{F}\mathscr{G}^{-1}$ is principal.*

(f) If $\mathscr{F}, \mathscr{G}$ are invertible submodules of $\mathscr{K}$ then $\mathscr{F} \sim \mathscr{G}$ if and only if $\mathscr{F} \cong \mathscr{G}$ as $\mathcal{O}_X$-modules.

(g) $D_1 \sim D_2$ if and only if $\mathscr{L}(D_1) \sim \mathscr{L}(D_2)$ if and only if $\mathscr{L}(D_1) \cong \mathscr{L}(D_2)$ as $\mathcal{O}_X$-modules.

Proof. (a) Let D be represented by $\{(U_i, f_i)\}$. For each i the morphism of $\mathcal{O}_X|_{U_i}$-modules $\mathcal{O}_X|_{U_i} \longrightarrow \mathscr{L}(D)|_{U_i}$ corresponding to f_i^{-1} is an isomorphism by the previous Lemma, so $\mathscr{L}(D)$ is invertible. Now suppose $\mathscr{L}$ is an invertible submodule of $\mathscr{K}$ and let U_i be a cover over X by nonempty open sets with $\mathscr{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ for all i. Let $f_i \in \mathscr{L}(U_i) \subseteq \mathscr{K}(U_i)$ correspond to the identity under this isomorphism. We have to show that f_i is a unit, for which it suffices to show that $germ_x f_i$ is $\mathcal{O}_{X,x}$-torsion-free for all $x \in U_i$. But this follows from the fact that $germ_x f_i$ is torsion-free in $\mathscr{L}_x$ for all $x \in U_i$.

We claim that the $f_i^{-1} \in \mathscr{K}^*(U_i)$ give rise to a Cartier divisor. Given indices i, j for which the intersection $U_i \cap U_j$ is nonempty, the isomorphisms $\mathscr{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ and $\mathscr{L}|_{U_j} \cong \mathcal{O}_X|_{U_j}$ show that $f_i|_{U_i \cap U_j}$ and $f_j|_{U_i \cap U_j}$ both give a basis for the $\mathcal{O}_X(U_i \cap U_j)$-module $\mathscr{L}(U_i \cap U_j)$. So there is a unit $u \in \mathcal{O}_X(U_i \cap U_j)$ with $f_i|_{U_i \cap U_j} = u \cdot f_j|_{U_i \cap U_j}$ from which we conclude that $f_i^{-1}|_{U_i \cap U_j}/f_j^{-1}|_{U_i \cap U_j} \in \mathcal{O}^*(U_i \cap U_j)$ which implies that the f_i^{-1} determine a Cartier divisor D with $D(x) = (U_i, f_i^{-1} + \mathcal{O}^*(U_i))$ for all $x \in U_i$. The divisor D is independent of the cover U_i and isomorphisms $\mathscr{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ chosen - suppose V_j is another cover of X by nonempty open subsets together with isomorphisms $\mathscr{L}|_{V_j} \cong \mathcal{O}_X|_{V_j}$ and let $g_j \in \mathscr{L}(V_j)$ correspond to the identities. Given $x \in X$ find i, j such that $x \in U_i \cap V_j$. Both $f_i|_{U_i \cap V_j}$ and $g_j|_{U_i \cap V_j}$ are a basis for the $\mathcal{O}_X(U_i \cap V_j)$-module $\mathscr{L}(U_i \cap V_j)$, so there is a unit u with $f_i^{-1}|_{U_i \cap V_j}/g_j^{-1}|_{U_i \cap V_j} = u$, which shows that the two divisors agree at x. Since x was arbitrary, this shows that the construction of D is independent of the chosen local isomorphisms.

By definition $\mathscr{L}(D)$ is the submodule generated by the set $\{f_i\}$, and $\mathscr{L}(D)_x = (germ_x f_i)$ for $x \in U_i$. By construction $\mathscr{L}_x = (germ_x f_i)$ for $x \in U_i$, and it follows that $\mathscr{L} = \mathscr{L}(D)$. From the construction it is apparent that the divisor produced from $\mathscr{L}(D)$ is D, so we have the desired bijection.

(b) Let Cartier divisors D_1, D_2 be given. We can choose a cover of X by nonempty open sets $\{U_i\}_{i \in I}$, such that D_1 is represented by $\{(U_i, f_i)\}_{i \in I}$ and D_2 by $\{(U_i, g_i)\}_{i \in I}$ for $f_i, g_i \in \mathscr{K}^*(U_i)$. Since we can represent the divisor $D_1 + D_2$ by $\{(U_i, f_i g_i)\}_{i \in I}$, we have $\mathscr{L}(D_1 + D_2)_x = (germ_x f_i^{-1} g_i^{-1})$ for $x \in U_i$. Since $\mathscr{L}(D_1)_x = (germ_x f_i^{-1}), \mathscr{L}(D_2)_x = (germ_x g_i^{-1})$ for $x \in U_i$ it follows from (SOA,Lemma 24)(iii) that $\mathscr{L}(D_1 + D_2) = \mathscr{L}(D_1)\mathscr{L}(D_2)$ as submodules of $\mathscr{K}$. Proposition 40 now implies that there is a canonical isomorphism $\mathscr{L}(D_1 + D_2) \cong \mathscr{L}(D_1) \otimes \mathscr{L}(D_2)$ of sheaves of $\mathcal{O}_X$-modules.

(c) See (SOA,Definition 8) for the definition of the submodule $(\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})$ of $\mathscr{K}$. Let $\mathscr{F}$ be an invertible submodule of $\mathscr{K}$ and suppose that there exists an invertible submodule $\mathscr{G}$ of $\mathscr{K}$ with $\mathscr{F}\mathscr{G} = \mathcal{O}_X$. Then we have

$$\mathscr{G} \subseteq (\mathcal{O}_X :_{\mathscr{K}} \mathscr{F}) = (\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})\mathscr{F}\mathscr{G} \subseteq \mathcal{O}_X \mathscr{G} = \mathscr{G}$$

We know from $(a), (b)$ that $\mathscr{G}$ exists, so this shows that $(\mathcal{O}_X :_{\mathscr{K}} \mathscr{F}) = \mathscr{G}$ is an invertible submodule of $\mathscr{K}$ with $\mathscr{F}(\mathcal{O}_X :_{\mathscr{K}} \mathscr{F}) = \mathcal{O}_X$. In fact we have shown that it is unique with this property. Using (MRS,Lemma 83) and Proposition 40 we have an isomorphism of $\mathcal{O}_X$-modules

$$\mathscr{F}^\vee \cong \mathscr{F}^\vee \otimes (\mathscr{F} \otimes (\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})) \cong (\mathscr{F}^\vee \otimes \mathscr{F}) \otimes (\mathcal{O}_X :_{\mathscr{K}} \mathscr{F}) \cong (\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})$$

Using this isomorphism we can freely identify $\mathscr{F}^\vee = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$ with a submodule of $\mathscr{K}$.

(d) It follows from Proposition 40 and (MRS,Lemma 56) that the set of invertible submodules of $\mathscr{K}$ is closed under the product of (SOA,Section 2.3). It follows from (c) that this is an abelian group with identity $\mathcal{O}_X$ and inverse of $\mathscr{F}$ given by $(\mathcal{O}_X :_{\mathscr{K}} \mathscr{F})$. We have already shown in (b) that the map $D \mapsto \mathscr{L}(D)$ between the group of Cartier divisors $\Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$ and the group of invertible submodules of $\mathscr{K}$ is an isomorphism of abelian groups. An immediate consequence is that for a Cartier divisor D we have a canonical isomorphism $\mathscr{L}(-D) = (\mathcal{O}_X :_{\mathscr{K}} \mathscr{L}(D)) \cong \mathscr{L}(D)^\vee$.

(e) Given $u \in \mathscr{K}^*(X)$ let D be principal Cartier divisor corresponding to u^{-1}. By definition $\mathscr{L}(D)$ is (u), the smallest submodule of $\mathscr{K}$ containing the global section u. Therefore (u) is invertible. Clearly $(u)(v) = (uv)$ for elements $u, v \in \mathscr{K}^*(X)$ so we have a morphism of abelian groups $\mathscr{K}^*(X) \longrightarrow Inv\mathscr{K}$. To see that the kernel of this morphism is $\mathcal{O}^*(X)$ it suffices to observe that if $(u) = \mathcal{O}_X$ then also $(u^{-1}) = \mathcal{O}_X$, which shows that $u, u^{-1} \in \mathcal{O}_X(X)$ and therefore $u \in \mathcal{O}^*(X)$.

(f) Using Proposition 40 and the fact that $\mathscr{G}^{-1}$ is isomorphic to the dual $\mathscr{H}om(\mathscr{G}, \mathcal{O}_X)$ it suffices to show that an invertible submodule $\mathscr{F}$ is principal if and only if $\mathscr{F} \cong \mathcal{O}_X$ as $\mathcal{O}_X$-modules. If $u \in \mathscr{K}^*(X)$ then (u) is the image of the unique morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \longrightarrow \mathscr{K}$ with $1 \mapsto u$. Since by assumption u is a unit, this is a monomorphism and therefore $(u) \cong \mathcal{O}_X$ as $\mathcal{O}_X$-modules. Conversely, suppose that $\mathscr{L}$ is an invertible submodule of $\mathscr{K}$ and that there is an isomorphism $\mathcal{O}_X \cong \mathscr{L}$ of $\mathcal{O}_X$-modules. Let v be the corresponding global section of $\mathscr{L}$. It follows from Lemma 37 that $v \in \mathscr{K}^*(X)$, so $\mathscr{L} = (v)$ is principal.

(g) If D is a principal Cartier divisor then $\mathscr{L}(D)$ is clearly principal. The converse follows from the construction given in (a), and the other claims then follow from (f). □

Corollary 42. *Let X be a scheme. Then*

(i) The map $D \mapsto \mathscr{L}(D)$ induces an isomorphism of abelian groups between $CaClX$ and $Inv\mathscr{K}/P$, where P denotes the subgroup of principal invertible submodules of $\mathscr{K}$.

(ii) The map $D \mapsto \mathscr{L}(D)$ gives an injective morphism of abelian groups $CaClX \longrightarrow PicX$.

Remark 2. The map $CaClX \longrightarrow PicX$ may not be surjective, because there may be invertible sheaves on X which are not isomorphic to any invertible subsheaf of $\mathscr{K}$. But it can be proved to be surjective under special circumstances.

Lemma 43. *Let X be an integral scheme and $\mathscr{L}$ an invertible sheaf. If ξ is the generic point then the canonical morphism of abelian groups $\mathscr{L}_x \longrightarrow \mathscr{L}_\xi$ is injective for all $x \in X$. Consequently $\mathscr{L}(U) \longrightarrow \mathscr{L}_\xi$ is injective for any open $U \subseteq X$.*

Proof. The morphism is defined by $(U, a) \mapsto (U, a)$. Find an open neighborhood U of x (which must contain ξ) with $\mathscr{L}|_U \cong \mathcal{O}_X|_U$ and reduce to the case where $\mathscr{L} = \mathcal{O}_X$, which is easily checked (see Section 3 Exercises). □

Proposition 44. *If X is an integral scheme, the morphism $CaClX \longrightarrow PicX$ is an isomorphism.*

Proof. We have only to show that every invertible sheaf is isomorphic to a submodule of $\mathscr{K}$, which in this case is isomorphic to the constant sheaf K, where K is the function field of X. So given an invertible sheaf $\mathscr{L}$ it suffices to show there is a monomorphism $\psi : \mathscr{L} \longrightarrow K$. We choose an isomorphism $\mathscr{L}_\xi \cong K$ and define ψ_U to be the composite $\mathscr{L}(U) \longrightarrow \mathscr{L}_\xi \cong K$. This is easily checked to be a monomorphism of $\mathcal{O}_X$-modules, which completes the proof. In particular this result shows that if X is integral, $PicX$ is small (that is, bijective to an element of our universe). □

Corollary 45. *If X is a noetherian, integral, separated, locally factorial scheme, then there is a canonical isomorphism of abelian groups $ClX \cong PicX$.*

Proof. This follows from Proposition 44 and Proposition 33. □

Example 5. If k is an algebraically closed field and X is a nonsingular variety over k, then since a regular local ring is a UFD, X satisfies the conditions of the Corollary and $ClX \cong PicX$.

Let $S = k[x_0, \dots, x_n]$ be a graded ring with $n \geq 1$ and $\mathfrak{p}$ a nonzero homogenous prime ideal. Then there is a morphism of rings $S_{(\mathfrak{p})} \longrightarrow S_{((0))}$ defined by $a/q \mapsto a/q$. If we set $X = ProjS$ then this morphism fits into the following commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X,\mathfrak{p}} \arrow[r] \arrow[d, Rightarrow] & \mathcal{O}_{X,0} \arrow[d, Rightarrow] \\ S_{(\mathfrak{p})} \arrow[r] & S_{((0))} \end{tikzcd}$$

Remark 3. Set $X = \mathbb{P}^n_k$ for some field k and $n \geq 1$ and let K be the function field of X. Denote also by K the corresponding constant sheaf. We make the following remarks

(i) For $d > 0$, a homogenous prime ideal $\mathfrak{p}$ of $S = k[x_0, \dots, x_n]$ and $f \in S_1$ with $f \notin \mathfrak{p}$ there is a canonical isomorphism of $S_{(\mathfrak{p})}$-modules $S(d)_{(\mathfrak{p})} \longrightarrow S_{(\mathfrak{p})}$ defined by $a/q \mapsto a/f^d q$. In particular if we fix $0 \leq \ell \leq n$ then there is a canonical isomorphism of $S_{((0))}$-modules $S(d)_{((0))} \longrightarrow S_{((0))}$ defined by $a/q \mapsto a/x_\ell^d q$.

(ii) For $d > 0$ and $0 \leq \ell \leq n$ there is a canonical monomorphism $\psi_\ell : \mathcal{O}(d) \longrightarrow \mathscr{K}$ which is defined as follows: we have an isomorphism of K-modules $\mathcal{O}(d)_0 \cong S(d)_{((0))} \cong S_{((0))} \cong K$ which induces a monomorphism of $\mathcal{O}_X$-modules $\mathcal{O}(d) \longrightarrow K$ as in Proposition 44. Composing with the canonical isomorphism $K \cong \mathscr{K}$ we have the desired monomorphism ψ_ℓ, and we denote the corresponding submodule of $\mathscr{K}$ by $\mathcal{O}_\ell(d)$. Given $\mathfrak{p} \in X$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}(d)_{\mathfrak{p}} \arrow[r, "\psi_{\ell,\mathfrak{p}}"] \arrow[d] & K_{\mathfrak{p}} \arrow[d, Rightarrow] \\ \mathcal{O}(d)_0 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & K \arrow[d, Rightarrow] \\ S(d)_{((0))} \arrow[r, Rightarrow] & S_{((0))} \end{tikzcd}$$

If $x_i \notin \mathfrak{p}$ then the stalk $\mathcal{O}_\ell(d)_{\mathfrak{p}}$ is the $\mathcal{O}_{X,\mathfrak{p}}$-submodule of $\mathscr{K}_{\mathfrak{p}}$ generated by x_i^d/x_ℓ^d (or more precisely, the image of this quotient under $S_{((0))} \cong K \cong K_{\mathfrak{p}} \cong \mathscr{K}_{\mathfrak{p}}$). It follows easily that for $d, e > 0$ we have $\mathcal{O}_\ell(d)\mathcal{O}_\ell(e) = \mathcal{O}_\ell(d+e)$ as submodules of $\mathscr{K}$.

Lemma 46. *If $X = \mathbb{P}^n_k$ for some field k and $n \geq 1$, then for $0 \leq \ell \leq n$ we have $\mathscr{L}(C_\ell) = \mathcal{O}_\ell(1)$ as submodules of $\mathscr{K}$. There is a canonical isomorphism of sheaves of modules $\mathcal{O}(1) \longrightarrow \mathscr{L}(C_\ell)$.*

Proof. Here C_ℓ is the Cartier divisor defined in Lemma 31. Let K be the function field of X and denote also by K the corresponding constant sheaf. We have by Remark 3(ii) a canonical monomorphism $\mathcal{O}(1) \longrightarrow \mathscr{K}$ with image $\mathcal{O}_\ell(1)$. We have to show that this agrees with the submodule $\mathscr{L}(C_\ell)$.

It suffices to show they agree on stalks. Let $\mathfrak{p} \in X$ be given and find x_i with $\mathfrak{p} \in D_+(x_i)$. Then we observed in Remark 3(ii) that $\mathcal{O}_\ell(1)_{\mathfrak{p}}$ is generated by x_i/x_ℓ. It is clear from the beginning of the proof of Lemma 36 and the definition of $\mathscr{L}(C_\ell)$ that this coincides with $\mathscr{L}(C_\ell)_{\mathfrak{p}}$, completing the proof. Observe that taking powers of both sides we have $\mathscr{L}(d \cdot C_\ell) = \mathcal{O}_\ell(d)$ for any $d > 0$, and in particular deduce a canonical isomorphism $\mathcal{O}(d) \longrightarrow \mathscr{L}(d \cdot C_\ell)$. $\square$

Corollary 47. *If $X = \mathbb{P}^n_k$ for some field k and $n \geq 1$, then every invertible sheaf on X is isomorphic to $\mathcal{O}(m)$ for some $m \in \mathbb{Z}$.*

Proof. We observed just before Lemma 34 that X has the properties necessary to apply Corollary 45 and Proposition 12. Hence $PicX \cong ClX \cong \mathbb{Z}$. Combining Lemma 34 and Lemma 46 we see that for any $0 \leq \ell \leq n$ this isomorphism maps the prime divisor $V(x_\ell)$ to the isomorphism class of $\mathcal{O}(1)$. Since $V(x_0)$ is identified with $1 \in \mathbb{Z}$ it follows that $PicX$ is a free abelian group generated by $\mathcal{O}(1)$. So every invertible sheaf is isomorphic to $\mathcal{O}(m)$ for some $m \in \mathbb{Z}$, and moreover all these modules are in distinct isomorphism classes. $\square$

We conclude this section with some remarks about closed subschemes of codimension one of a scheme X.

Lemma 48. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then for $x\in X$ there is a canonical isomorphism of abelian groups $(\mathcal{O}^*)_x\cong(\mathcal{O}_{X,x})^*$ where $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}_X$.*

Proof. To be clear, $(\mathcal{O}_{X,x})^*$ denotes the multiplicative group of all units in $\mathcal{O}_{X,x}$. It is not hard to check that the map $(\mathcal{O}^*)_x\longrightarrow(\mathcal{O}_{X,x})^*$ defined by $(U,s)\mapsto(U,s)$ is a well-defined isomorphism of abelian groups. □

Definition 16. Let X be a scheme with sheaf of total quotient rings $\mathscr{K}$. For $x\in X$ we can identify $\mathcal{O}_{X,x}$ with a subring of $\mathscr{K}_x$ and $(\mathcal{O}_{X,x})^*$ with a subgroup of $(\mathscr{K}_x)^*$. Let E_x denote the subset of $(\mathscr{K}_x)^*$ given by the intersection $(\mathscr{K}_x)^*\cap\mathcal{O}_{X,x}$ (i.e. elements of $\mathcal{O}_{X,x}$ which become invertible in $\mathscr{K}_x$). We have (possibly strict) inclusions $(\mathcal{O}_{X,x})^*\subseteq E_x\subseteq(\mathscr{K}_x)^*$. There is an isomorphism of abelian groups $(\mathscr{K}^*/\mathcal{O}^*)_x\cong(\mathscr{K}_x)^*/(\mathcal{O}_{X,x})^*$ and in the latter group we say that a coset is *effective* if it contains an element of E_x. We say that a Cartier divisor C on X is *effective* if $germ_xC\in(\mathscr{K}^*/\mathcal{O}^*)_x$ is effective for every $x\in X$. Clearly the zero divisor is effective and the sum of effective divisors is effective. A divisor C is zero if and only if both $C,-C$ are effective.

Lemma 49. *Let X be a nonempty scheme and C a Cartier divisor on X. Then C is effective if and only if it can be represented by a family $\{(U_i,f_i)\}_{i\in I}$ with each $f_i\in\Gamma(U_i,\mathcal{O}_X)$.*

Proof. Suppose that C is effective and let $x\in X$ be given. Let $C(x)=(U,f+\mathcal{O}^*(U))$ where U is an open neighborhood of x and $f\in\mathscr{K}^*(U)$. Since C is effective the element $germ_xf+(\mathcal{O}_{X,x})^*$ of $(\mathscr{K}_x)^*/(\mathcal{O}_{X,x})^*$ is effective, and therefore $germ_xf\in E_x$. It follows that there is an open neighborhood $x\in W\subseteq U$ with $f|_W\in\mathcal{O}_X(W)$. Since x was arbitrary, this shows that C can be represented by a family of the required type. The converse is easily checked. □

Definition 17. Let X be a scheme. If D,C are Cartier divisors then we write $D\geq C$ if $D-C$ is an effective Cartier divisor. So a Cartier divisor D is effective if and only if $D\geq 0$. This makes the group of Cartier divisors into a partially ordered abelian group (this is not necessarily a total order).

Proposition 50. *Let X be a scheme and D,C Cartier divisors on X. Then $D\geq C$ if and only if $\mathscr{L}(D)\supseteq\mathscr{L}(C)$. In particular C is effective if and only if $\mathscr{L}(C)\supseteq\mathcal{O}_X$.*

Proof. By Proposition 41 it suffices to show that a Cartier divisor C is effective if and only if $\mathscr{L}(C)\supseteq\mathcal{O}_X$, or equivalently $\mathscr{L}(-C)\subseteq\mathcal{O}_X$. Suppose that C is effective and is represented by the family $\{(U_i,f_i)\}_{i\in I}$. Then the Cartier divisor $-C$ is represented by $\{(U_i,f_i^{-1})\}_{i\in I}$ and therefore $\mathscr{L}(-C)$ is the submodule of $\mathscr{K}$ generated by the set $\{f_i\}_{i\in I}$. By assumption $f_i\in\Gamma(U_i,\mathcal{O}_X)$ for every $i\in I$ so this is clearly a submodule of $\mathcal{O}_X$.

Now suppose that $\mathscr{L}(-C)\subseteq\mathcal{O}_X$. The construction of Proposition 41(a) shows that $-C$ can be represented by a family $\{(U_i,f_i^{-1})\}_{i\in I}$ where $f_i\in\Gamma(U_i,\mathscr{L}(-C))\subseteq\Gamma(U_i,\mathcal{O}_X)$. Therefore C is represented by $\{(U_i,f_i)\}_{i\in I}$, which shows that C is effective. □

Proposition 51. *Let X be an integral, separated noetherian scheme which is locally factorial. Then the canonical isomorphism of abelian groups $DivX\cong\Gamma(X,\mathscr{K}^*/\mathcal{O}^*)$ identifies the effective Weil divisors with the effective Cartier divisors.*

Proof. Let C be an effective Cartier divisor on X. To show that the corresponding Weil divisor is effective we need to show that $C_Y\geq 0$ for every prime divisor $Y\subseteq X$ (notation of Proposition 33). Let η be the generic point of Y, and write $C(\eta)=(U,f+\mathcal{O}^*(U))$ where $f\in\mathcal{O}_X(U)$ and U is an open neighborhood of η. Therefore $germ_\xi f$ belongs to the subring $\mathcal{O}_{X,\eta}$ of the function field $K=\mathcal{O}_{X,\xi}$. By definition this is the valuation ring of v_Y, so $C_Y=v_Y(f)\geq 0$ as required.

In the other direction, let D be an effective Weil divisor on X. With the notation of the proof of Proposition 33, for every $x\in X$ the divisor (f_x) is an effective principal divisor on $Spec(\mathcal{O}_{X,x})$. It follows from Corollary 6 that $h_x\in\mathcal{O}_{X,x}$ and therefore the Cartier divisor corresponding to D is effective. □

5 Examples

Let A be a regular noetherian domain, so that $X = SpecA$ satisfies the conditions of Proposition 33. The map $\mathfrak{p} \mapsto V(\mathfrak{p})$ defines a bijection between the prime Weil divisors on X and the prime ideals $\mathfrak{p}$ of height 1. What is the Cartier divisor corresponding to $V(\mathfrak{p})$? To find out, we have to run through the proof of Proposition 33.

Fix a prime ideal $\mathfrak{p}$ of height 1 and set $Y = V(\mathfrak{p})$ and $U = X \setminus Y$. Let $\mathfrak{q}$ be a nonzero prime. If $\mathfrak{q} \notin Y$ then $\mathfrak{q} \not\supseteq \mathfrak{p}$ and we set $U_{\mathfrak{q}} = U, h_{\mathfrak{q}} = 1$, where $h_{\mathfrak{q}} \in \mathscr{K}^*(U_{\mathfrak{q}})$.

On the other hand, if $\mathfrak{q} \in Y$ then $\mathfrak{q} \supseteq \mathfrak{p}$. Since $A_{\mathfrak{q}}$ is a unique factorisation domain the ideal $\mathfrak{p}A_{\mathfrak{q}}$ is principal, generated by $f_{\mathfrak{q}}/1$ for some $f_{\mathfrak{q}} \in \mathfrak{p}$. Let Z be the set of all prime ideals $\mathfrak{r}$ of A with $ht.\mathfrak{r} = 1, \mathfrak{q} \not\supseteq \mathfrak{r}$ and $f_{\mathfrak{q}} \in \mathfrak{r}$. This is a finite set (possibly empty) and we set $U_{\mathfrak{q}}$ equal to $\bigcap_{\mathfrak{r}\in Z} D(\mathfrak{r})$ (which is all of X if Z is empty). This is an open neighborhood of $\mathfrak{q}$ and we set $h_{\mathfrak{q}} = f_{\mathfrak{q}}$, considered as an element of $\mathscr{K}^*(U_{\mathfrak{q}})$.

For every nonzero prime $\mathfrak{q}$ we have defined an open neighborhood $U_{\mathfrak{q}}$ of $\mathfrak{q}$ and an element $h_{\mathfrak{q}} \in \mathscr{K}^*(U_{\mathfrak{q}})$. The family $\{(U_{\mathfrak{q}}, h_{\mathfrak{q}})\}_{\mathfrak{q}\neq 0}$ is the effective Cartier divisor C corresponding to the Weil divisor Y.

The invertible submodule $\mathscr{L}(C)$ of $\mathscr{K}$ corresponding to the Cartier divisor C is the submodule generated by the set $\{h_{\mathfrak{q}}^{-1}\}_{\mathfrak{q}\neq 0}$. Let Q be the quotient field of A, so that by Lemma 28 there is a canonical isomorphism $Q^{\sim} \cong \mathscr{K}$ of sheaves of algebras on X. Under this isomorphism the submodule $\mathscr{L}(C) \subseteq \mathscr{K}$ corresponds to the submodule of $Q^{\sim}$ generated by the sections $\dot{1/f_{\mathfrak{q}}} \in \Gamma(U_{\mathfrak{q}}, Q^{\sim})$. The Cartier divisor $-C$ can be represented by $\{(U_{\mathfrak{q}}, h_{\mathfrak{q}}^{-1})\}_{\mathfrak{q}\neq 0}$ and therefore $\mathscr{L}(-C) \subseteq \mathscr{K}$ is the submodule generated by the set $\{h_{\mathfrak{q}}\}_{\mathfrak{q}\neq 0}$. This corresponds to the submodule $\mathscr{M}$ of $Q^{\sim}$ generated by the sections $\dot{f_{\mathfrak{q}}/1} \in \Gamma(U_{\mathfrak{q}}, Q^{\sim})$. For a nonzero prime $\mathfrak{q}$ the canonical isomorphism of $A_{\mathfrak{q}}$-modules $(Q^{\sim})_{\mathfrak{q}} \cong Q_{\mathfrak{q}} \cong Q$ identifies the submodule $\mathscr{M}_{\mathfrak{q}}$ with the submodule $\mathfrak{p}A_{\mathfrak{q}}$. We have monomorphisms of sheaves of modules

$$\widetilde{\mathfrak{p}} \longrightarrow \widetilde{A} \longrightarrow \widetilde{Q}$$

and we consider $\mathfrak{p}^{\sim}$ as a submodule of $Q^{\sim}$ in this way. By looking at stalks we see that $\mathscr{M} = \mathfrak{p}^{\sim}$ as submodules of $Q^{\sim}$. But by (SIAS,Lemma 5) the sheaf of ideals $\mathfrak{p}^{\sim}$ is the ideal sheaf of the closed set Y. So we have proved

Proposition 52. *Let A be a regular noetherian domain and set $X = SpecA$. For a prime ideal $\mathfrak{p}$ of height 1 let C be the Cartier divisor corresponding to the prime Weil divisor $V(\mathfrak{p})$. Then there is a canonical isomorphism of sheaves of modules $\mathscr{L}(-C) \cong \mathfrak{p}^{\sim}$.*

Remark 4. For the next result we recall some notation introduced in (SPM,Definition 3), where we associated to any points x, y of a scheme X with $y \in Y = \overline{\{x\}}$ a prime ideal $\mathfrak{p}_{y,x}$ of $\mathcal{O}_{X,y}$ with the property that $ht.\mathfrak{p}_{y,x} = codim(Y, X)$ (SPM,Lemma 7). Observe that if $\mathscr{J}_Y$ is the ideal sheaf of the closed set Y then (SI,Lemma 3) implies that for $y \in X$ we have

$$\mathscr{J}_{Y,y} = \begin{cases} \mathcal{O}_{X,y} & y \notin Y \\ \mathfrak{p}_{y,x} & y \in Y \end{cases}$$

Proposition 53. *Let X be a scheme satisfying the conditions of Proposition 33 and let $Y \subseteq X$ be a prime divisor with corresponding Cartier divisor C. If $\mathscr{J}_Y$ is the ideal sheaf of Y then there are canonical isomorphisms of sheaves of modules $\mathscr{L}(-C) \cong \mathscr{J}_Y$ and $\mathscr{L}(C) \cong \mathscr{J}_Y^{\vee}$.*

Proof. First let us calculate the Cartier divisor C. Let η be the generic point of Y and let $y \in X$ be any point other than the generic point ξ of X. If $y \notin Y$ set $U_y = X \setminus Y$ and $h_y = 1 \in \mathscr{K}^*(U_y)$. If $y \in Y$ then $\mathfrak{p}_{y,\eta}$ is a prime ideal of height 1 in the unique factorisation domain $\mathcal{O}_{X,y}$, so it can be generated as an ideal by a single element $f_y \in \mathfrak{p}_{y,\eta}$. With K the quotient field of X let U_y be a sufficiently small open neighborhood of y and h_y the image of $f_y \in \mathfrak{p}_{y,\eta} \subseteq \mathcal{O}_{X,y} \subseteq K$ under the canonical isomorphism $K^* \cong \mathscr{K}^*(U_y)$. Then the Cartier divisor C corresponding to Y under the bijection of Proposition 33 is represented by $\{(U_y, h_y)\}_{y\neq\xi}$. The invertible submodule

$\mathscr{L}(-C) \subseteq \mathscr{K}$ is generated by the sections $h_y \in \Gamma(U_y, \mathscr{K})$. By construction we have for $y \in X$ the following equality of $\mathcal{O}_{X,y}$-submodules of $\mathscr{K}_y$

$$\mathscr{L}(-C)_y = \begin{cases} \mathcal{O}_{X,y} & y \notin Y \\ \mathfrak{p}_{y,\eta} & y \in Y \end{cases}$$

By comparing stalks, it is now easy to see that $\mathscr{L}(-C)$ is the image of the monomorphism $\mathscr{J}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}$, which yields the required canonical isomorphism $\mathscr{L}(-C) \cong \mathscr{J}_Y$. Taking duals of both sides and using Proposition 41(c) we have $\mathscr{L}(C) \cong \mathscr{L}(-C)^\vee \cong \mathscr{J}_Y^\vee$. □

Remark 5. An immediate consequence of Proposition 53 is that if X is a scheme satisfying the conditions of Proposition 33 and $Y \subseteq X$ a prime divisor, then the ideal sheaf $\mathscr{J}_Y$ is invertible.

Corollary 54. *Let X be a scheme satisfying the conditions of Proposition 33 and let $\sum_{i=1}^n n_i \cdot Y_i$ be an effective Weil divisor with corresponding Cartier divisor C. Then there is a canonical isomorphism of sheaves of modules*

$$\mathscr{L}(-C) \cong \prod_{i=1}^n \mathscr{J}_{Y_i}^{n_i}$$

Proof. We have by Proposition 33 and Proposition 41 canonical isomorphisms of abelian groups

$$DivX \cong \Gamma(X, \mathscr{K}^*/\mathcal{O}^*) \cong Inv\mathscr{K}$$

Let $D = \sum_{i=1}^n n_i \cdot Y_i$ be a Weil divisor and set $D_i = 1 \cdot Y_i$. Then $-C = \sum_{i=1}^n n_i \cdot (-D_i)$ and therefore $\mathscr{L}(-C) = \prod_{i=1}^n \mathscr{L}(-D_i)^{n_i} \cong \prod_{i=1}^n \mathscr{J}_{Y_i}^{n_i}$, as required. □