Schemes via Noncommutative Localisation

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In this note we give an exposition of the well-known results of Gabriel, which show how to define affine schemes in terms of the theory of noncommutative localisation.

Contents

1 Noncommutative Rings

Throughout this section a "ring" means a not necessarily commutative ring. All modules in this section are right modules.

Definition 1. Let A be a ring, $\mathfrak{a}$ a right ideal and $h \in A$. Then the set $(\mathfrak{a} : h) = \{a \in A \,|\, ha \in \mathfrak{a}\}$ is also a right ideal. If $\mathfrak{a}$ is a left ideal then so is the set $(h : \mathfrak{a}) = \{a \in A \,|\, ah \in \mathfrak{a}\}$.

{definition_right

Definition 2. Let A be a ring. A set J of right ideals of A is a *right gabriel topology* if it satisfies the following conditions

(i) The improper ideal A belongs to J.

(ii) If $\mathfrak{a} \in J$ and $h \in A$ then $(\mathfrak{a} : h) \in J$.

(iii) If $\mathfrak{a} \in J$ and $\mathfrak{b}$ is any right ideal with $(\mathfrak{b} : h) \in J$ for every $h \in \mathfrak{a}$, then $\mathfrak{b} \in J$.

The set of all right gabriel topologies on A is partially ordered by inclusion. The sets J_0 and J_1 defined by $J_0 = \{A\}$ and $J_1 = \{\mathfrak{a} \,|\, \mathfrak{a} \text{ is a right ideal}\}$ are right gabriel topologies, and they are respectively the initial and terminal objects of the set of all right gabriel topologies.

{definition_leftg

Definition 3. Let A be a ring. A set J of left ideals of A is a *left gabriel topology* if it satisfies the following conditions

(i) The improper ideal A belongs to J.

(ii) If $\mathfrak{a} \in J$ and $h \in A$ then $(h : \mathfrak{a}) \in J$.

(iii) If $\mathfrak{a} \in J$ and $\mathfrak{b}$ is any left ideal with $(h : \mathfrak{b}) \in J$ for every $h \in \mathfrak{a}$, then $\mathfrak{b} \in J$.

The set of all left gabriel topologies on A is partially ordered by inclusion. The sets J_0 and J_1 defined above are also left gabriel topologies, and they are respectively the initial and terminal objects of the set of all left gabriel topologies.

Remark 1. Let A be a ring. A set of left ideals J is a left gabriel topology if and only if it is a right gabriel topology on A^{op}, so it suffices to talk about right gabriel topologies. Throughout the rest of this section, a *gabriel topology* is a right gabriel topology.

Remark 2. Let A be a ring with gabriel topology J. The following are trivial consequences of the definition

- If $\mathfrak{a}, \mathfrak{b} \in J$ then $\mathfrak{a} \cap \mathfrak{b} \in J$.
- If $\mathfrak{a} \in J$ and $\mathfrak{b} \supseteq \mathfrak{a}$ then $\mathfrak{b} \in J$.

{definition_plus}

Definition 4. Let A be a ring with gabriel topology J and let M be an A-module. The set J is a directed set under reverse inclusion and the abelian groups $\{Hom_A(\mathfrak{a}, M)\}_{\mathfrak{a} \in J}$ are a direct system over this directed set. We define the abelian group M^+ by

$$M^+ = \varinjlim_{\mathfrak{a} \in J} Hom_A(\mathfrak{a}, M)$$

Given $h \in A$ and a morphism of A-modules $\varphi : \mathfrak{a} \longrightarrow M$, we define a morphism of A-modules

$$\begin{aligned} \varphi \cdot h : (\mathfrak{a} : h) &\longrightarrow M \\ x &\mapsto \varphi(hx) \end{aligned}$$

It is not difficult to check that this makes M^+ into a well-defined A-module. If $\alpha : M \longrightarrow N$ is a morphism of A-modules then there is a well-defined morphism of A-modules

$$\begin{aligned} \alpha^+ : M^+ &\longrightarrow N^+ \\ \alpha^+(\mathfrak{a}, \varphi) &= (\mathfrak{a}, \alpha\varphi) \end{aligned}$$

This defines an additive functor $(-)^+ : \mathbf{Mod}A \longrightarrow \mathbf{Mod}A$. We make the following comments

- We call a morphism of A-modules $\varphi : \mathfrak{a} \longrightarrow M$ an *additive matching family* and sometimes represent it by the notation $\{x_f \,|\, f \in \mathfrak{a}\}$ where $x_f = \varphi(f)$. With this notation we have

$$\begin{aligned} \{x_f \,|\, f \in \mathfrak{a}\} + \{y_g \,|\, g \in \mathfrak{b}\} &= \{x_h + y_h \,|\, h \in \mathfrak{a} \cap \mathfrak{b}\} \\ \{x_f \,|\, f \in \mathfrak{a}\} \cdot h &= \{x_{hx} \,|\, x \in (\mathfrak{a} : h)\} \end{aligned}$$

- Given $m \in M$ let φ_m denote the morphism of A-modules $\varphi_m : A \longrightarrow M$ with $\varphi_m(1) = m$. Then $m \mapsto \varphi_m$ defines a morphism of A-modules $\mu : M \longrightarrow M^+$ which is natural in M.
- The construction of the A-module M^+ depends on the topology J, which is not reflected in the notation. Since we are only interested in M^+ as an intermediate step, this will have no chance to cause confusion.

{lemma_technical}

Lemma 1. *Let A be a ring with gabriel topology J and suppose $\mathfrak{a}, \mathfrak{b} \in J$. If $\alpha : \mathfrak{a} \longrightarrow A$ is a morphism of A-modules then $\alpha^{-1}\mathfrak{b} \in J$.*

Proof. Clearly $\alpha^{-1}\mathfrak{b}$ is a right ideal of A, and for $a \in \mathfrak{a}$ we have

$$(\alpha^{-1}\mathfrak{b} : a) = \{h \,|\, ah \in \mathfrak{a} \text{ and } \alpha(ah) \in \mathfrak{b}\} = (\mathfrak{a} : a) \cap (\mathfrak{b} : \alpha(a))$$

Using the second and third axioms of a topology, we see that $\alpha^{-1}\mathfrak{b} \in J$, as required. □

In terms of the additive matching family $\{a_f \,|\, f \in \mathfrak{a}\}$ corresponding to α (so $a_f = \alpha(f)$) the right ideal $\alpha^{-1}\mathfrak{b}$ is $\{f \in \mathfrak{a} \,|\, a_f \in \mathfrak{b}\}$.

Let A be a ring with gabriel topology J and fix an A-module M. We define a function $M^+ \times A^+ \longrightarrow M^+$ as follows. Given matching families $\{x_f \,|\, f \in \mathfrak{b}\}$ and $\{a_g \,|\, g \in \mathfrak{a}\}$ representing elements of M^+, A^+ respectively, let $\mathfrak{c} = \{h \in \mathfrak{a} \,|\, a_h \in \mathfrak{b}\}$. By the Lemma this belongs to J. For $h \in \mathfrak{c}$ we define $c_h = x_{a_h}$. This gives an additive matching family $\{c_h \,|\, h \in \mathfrak{c}\}$ which defines an element of M^+. Our first task is to show that this assignment is well-defined: suppose

$\{x_f \,|\, f \in \mathfrak{b}\} = \{x'_f \,|\, f \in \mathfrak{b}'\}$ in M^+ and $\{a_g \,|\, g \in \mathfrak{a}\} = \{a'_g \,|\, g \in \mathfrak{a}'\}$ in A^+. Let $\mathfrak{e} \subseteq \mathfrak{b} \cap \mathfrak{b}'$ and $\mathfrak{d} \subseteq \mathfrak{a} \cap \mathfrak{a}'$ be right ideals where the respective pairs of matching families agree. Let $\{c_h \,|\, h \in \mathfrak{c}\}$ and $\{c'_h \,|\, h \in \mathfrak{c}'\}$ be produced using the original and prime matching families respectively, so

$$\mathfrak{c} = \{h \in \mathfrak{a} \,|\, a_h \in \mathfrak{b}\}, \quad \mathfrak{c}' = \{h \in \mathfrak{a}' \,|\, a'_h \in \mathfrak{b}'\}$$

By Lemma 1 the right ideal $\mathfrak{t} = \{w \in \mathfrak{a} \,|\, a_w \in \mathfrak{e}\}$ belongs to J and hence so does $\mathfrak{d} \cap \mathfrak{t}$. It is easy to see that $c_w = c'_w$ for $w \in \mathfrak{d} \cap \mathfrak{t}$, as required. So there is a well-defined action of A^+ on M^+ given by choosing representatives and calculating:

$$\{x_f \,|\, f \in \mathfrak{b}\} \cdot \{a_g \,|\, g \in \mathfrak{a}\} = \{x_{a_h} \,|\, h \in \mathfrak{c}\} \quad \text{where } \mathfrak{c} = \{h \in \mathfrak{a} \,|\, a_h \in \mathfrak{b}\}$$

One checks easily that this action is linear in each variable. This makes A^+ into a ring with identity $\{x_f = f \,|\, f \in A\}$ and M^+ into a right A^+-module. The canonical morphism of A-modules $A \longrightarrow A^+$ defined by $a \mapsto \{af \,|\, f \in A\}$ is a morphism of rings, and it is clear that if M is an A-module then the A-module structure on M^+ induced by $A \longrightarrow A^+$ and the above A^+-module structure is just the canonical A-module structure.

{definition_torsi

Definition 5. Let A be a ring with gabriel topology J. The *J-torsion* submodule of an A-module M is the submodule $t_J(M) = \{x \in M \,|\, Ann(x) \in J\}$. Any morphism of A-modules $\psi : M \longrightarrow N$ restricts to a morphism of A-modules $t_J(M) \longrightarrow t_J(N)$, and this defines an additive functor $t_J(-) : \mathbf{Mod}A \longrightarrow \mathbf{Mod}A$. We say that M is *J-torsion* if $t_J(M) = M$ and *J-torsion-free* if $t_J(M) = 0$. Where there is no chance of confusion we will often drop J from the notation. It is clear that the properties of being J-torsion and J-torsion-free are stable under isomorphism of A-modules. If J, K are two gabriel topologies on A with $J \subseteq K$ then it is clear that $t_J(M) \subseteq t_K(M)$. In particular, K-torsion-free implies J-torsion-free.

Remark 3. With the notation of Definition 5 it is clear that the A-module $t_J(M)$ is J-torsion.

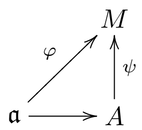
{lemma_quotientis

Lemma 2. *Let A be a ring with gabriel topology J. If M is an A-module, then $M/t_J(M)$ is J-torsion-free.*

Proof. If $m + t_J(M)$ were a J-torsion element, say $\mathfrak{a} \in J$ with $m \cdot \mathfrak{a} \subseteq t_J(M)$. For $f \in \mathfrak{a}$ let $\mathfrak{a}_f \in J$ be such that $(m \cdot f) \cdot \mathfrak{a}_f = 0$. Then $\mathfrak{c} = \sum_f f\mathfrak{a}_f \in J$ and clearly $m \cdot \mathfrak{c} = 0$, so m is J-torsion and thus $m + t_J(M) = 0$, as required. □

{definition_injec

Definition 6. Let A be a ring with gabriel topology J. An A-module M is *J-injective* if for every $\mathfrak{a} \in J$ and morphism of A-modules $\varphi : \mathfrak{a} \longrightarrow M$ there is a morphism $\psi : A \longrightarrow M$ making the following diagram commute



In other words, the canonical morphism of abelian groups

$$Hom_A(A, M) \longrightarrow Hom_A(\mathfrak{a}, M) \qquad (1)$$ {eq_injectivehom}

is surjective. The property of being J-injective is stable under isomorphism of A-modules. It is well-known that M is injective in the usual sense if and only if (1) is surjective for *every* right ideal $\mathfrak{a}$. That is, injectivity and J_1-injectivity are equivalent.

If J, K are two gabriel topologies on A with $J \subseteq K$ then it is clear that K-injectivity implies J-injectivity. In particular, if M is injective then it is J-injective for any gabriel topology J. At the other extreme, *every* A-module M is J_0-injective.

Remark 4. Let A be a ring with gabriel topology J. It is not hard to see that an A-module M is J-torsion-free if and only if the map (1) is injective for all $\mathfrak{a} \in J$.

{definition_close

Definition 7. Let A be a ring with gabriel topology J. An A-module M is J-*closed* if it is J-torsion-free and J-injective. Equivalently, for every $\mathfrak{a} \in J$ and morphism of A-modules $\varphi : \mathfrak{a} \longrightarrow M$ there is a *unique* morphism $\psi : A \longrightarrow M$ making the following diagram commute

$$\begin{array}{ccc} & & M \\ & \overset{\varphi}{\nearrow} & \uparrow{\scriptstyle \psi} \\ \mathfrak{a} & \longrightarrow & A \end{array}$$

The property of being J-closed is stable under isomorphism of A-modules. If J, K are two gabriel topologies on A with $J \subseteq K$ then it is clear that K-closed implies J-closed.

{definition_local

Definition 8. Let A be a ring with gabriel topology J. We denote by $\mathbf{Mod}(A, J)$ the preadditive subcategory of $\mathbf{Mod}A$ consisting of all J-closed A-modules. Observe that the zero module is J-closed for any gabriel topology J. If J, K are two gabriel topologies with $J \subseteq K$ then $\mathbf{Mod}(A, K) \subseteq \mathbf{Mod}(A, J)$.

1.1 Localisation

{definition_first

Definition 9. Let A be a ring with gabriel topology J, and let M be an A-module. We denote the A-module $(M/t_J(M))^+$ by M_J, and call it the *localisation of* M *with respect to* J. If $\alpha : M \longrightarrow N$ is a morphism of A-modules there is an induced morphism of A-modules $\alpha' : M/t_J(M) \longrightarrow N/t_J(N)$, so we have a morphism of A-modules $\alpha_J = (\alpha')^+$ which is defined by

$$\alpha_J : M_J \longrightarrow N_J$$
$$\alpha_J(\mathfrak{a}, \varphi) = (\mathfrak{a}, \alpha'\varphi)$$

This defines an additive functor $(-)_J : \mathbf{Mod}A \longrightarrow \mathbf{Mod}A$. There is a canonical morphism of A-modules $M \longrightarrow M_J$ natural in M, given by the composite $M \longrightarrow M/t_J(M) \longrightarrow (M/t_J(M))^+$. By (LOR,Section 2.1) there is a canonical isomorphism of A-modules $M_J \cong (M^+)^+$ natural in M.

{prop_localisatio

Proposition 3. *Let A be a ring with gabriel topology J. If M is an A-module then M_J is J-closed.*

Proof. Since $M/t_J(M)$ is J-torsion-free the result follows from (LOR,Proposition 9). ☐

{prop_localisatio

Proposition 4. *Let A be a ring with gabriel topology J, and let M be an A-module. Let N be a J-closed A-module and suppose we have a morphism of A-modules $\theta : M \longrightarrow N$. Then there is a* unique *morphism of A-modules $\psi : M_J \longrightarrow N$ making the following diagram commute*

$$\begin{array}{ccc} M & \xrightarrow{\theta} & N \\ \downarrow & \overset{\psi}{\nearrow} & \\ M_J & & \end{array} \tag{2}$$ {eq_localisationu

Proof. By assumption N is J-closed, so θ sends J-torsion elements of M to zero, and we have an induced morphism $\theta' : M/t_J(M) \longrightarrow N$. By (LOR,Proposition 7) there is a unique morphism $\psi : M_J \longrightarrow N$ making (2) commute, as required. For $(\mathfrak{a}, \varphi) \in M_J$, the element $\psi(\mathfrak{a}, \varphi) \in N$ is unique with the property that for all $h \in \mathfrak{a}$ we have $\psi(\mathfrak{a}, \varphi) \cdot h = \theta'\varphi(h)$. ☐

Definition 10. Let A be a ring with gabriel topology J. It follows from Proposition 3 that localisation defines an additive functor $(-)_J : \mathbf{Mod}A \longrightarrow \mathbf{Mod}(A, J)$. By Proposition 4 this functor is left adjoint to the inclusion $\mathbf{Mod}(A, J) \longrightarrow \mathbf{Mod}A$, with unit given by the canonical morphisms $M \longrightarrow M_J$. In fact $\mathbf{Mod}(A, J)$ is a grothendieck abelian category (LOR,Corollary 17), and the functor $(-)_J : \mathbf{Mod}A \longrightarrow \mathbf{Mod}(A, J)$ is exact (LOR,Proposition 16).

{lemma_secondusef

Lemma 5. *Let A be a ring with gabriel topology J and suppose $\mathfrak{a}, \mathfrak{b} \in J$. If $\alpha : \mathfrak{a} \longrightarrow A/t_J(A)$ is a morphism of A-modules then $\mathfrak{d}_{\alpha,\mathfrak{b}} = \alpha^{-1}\big((\mathfrak{b} + t_J(A))/t_J(A)\big) \in J$.*

Proof. Clearly $\mathfrak{d}_{\alpha,\mathfrak{b}}$ is a right ideal of A. For $a \in \mathfrak{a}$ choose $c \in A$ with $\alpha(a) = c + t_J(A)$. Then

$$(\mathfrak{d}_{\alpha,\mathfrak{b}} : a) = \{h \,|\, ah \in \mathfrak{a} \text{ and } \alpha(ah) \in (\mathfrak{b} + t_J(A))/t_J(A)\} = (\mathfrak{a} : a) \cap (\mathfrak{b} + t_J(A) : c)$$

Using the second and third axioms of a topology, we see that $\mathfrak{d}_{\alpha,\mathfrak{b}} \in J$, as required. □

Let A be a ring with gabriel topology J and fix an A-module M. Let $\mathfrak{a}, \mathfrak{b} \in J$ and suppose we are given morphisms of A-modules

$$\alpha : \mathfrak{a} \longrightarrow A/t_J(A)$$
$$\varphi : \mathfrak{b} \longrightarrow M/t_J(M)$$

Since φ maps $t_J(\mathfrak{b}) = t_J(A) \cap \mathfrak{b}$ into $t_J(M)$, there is an induced morphism of A-modules

$$\varphi' : (\mathfrak{b} + t_J(A))/t_J(A) \cong \mathfrak{b}/t_J(\mathfrak{b}) \longrightarrow M/t_J(M)$$

Let $\mathfrak{d}_{\alpha,\mathfrak{b}}$ be the right ideal of A defined in Lemma 5. Then α induces a morphism of A-modules

$$\alpha' : \mathfrak{d}_{\alpha,\mathfrak{b}} \longrightarrow (\mathfrak{b} + t_J(A))/t_J(A)$$

The composite $\varphi'\alpha' : \mathfrak{d}_{\alpha,\mathfrak{b}} \longrightarrow M/t_J(M)$ is a morphism of A-modules which we denote by $\varphi \cdot \alpha$. Given $x \in \mathfrak{d}_{\alpha,\mathfrak{b}}$ we calculate $(\varphi \cdot \alpha)(x)$ by choosing $b \in \mathfrak{b}$ with $\alpha(x) = b + t_J(A)$. Then $(\varphi \cdot \alpha)(x) = \varphi(b)$. We define a map

$$M_J \times A_J \longrightarrow M_J$$
$$((\mathfrak{b}, \varphi), (\mathfrak{a}, \alpha)) \mapsto (\mathfrak{d}_{\alpha,\mathfrak{b}}, \varphi \cdot \alpha)$$

One checks this is well-defined and additive in each variable, and we write $(\mathfrak{b}, \varphi) \cdot (\mathfrak{a}, \alpha)$ for $(\mathfrak{d}_{\alpha,\mathfrak{b}}, \varphi \cdot \alpha)$. Let $1 \in A_J$ denote the equivalence class of the canonical epimorphism of A-modules $A \longrightarrow A/t_J(A)$. Then $(\mathfrak{b}, \varphi) \cdot 1 = (\mathfrak{b}, \varphi)$. In particular we have a map $A_J \times A_J \longrightarrow A_J$ which makes A_J into a ring, and then $M_J \times A_J \longrightarrow M_J$ makes M_J into an A-module.

Definition 11. Let A be a ring with gabriel topology J and let M be an A-module. Then we have a ring A_J and a canonical A_J-module structure on M_J. If $\theta : M \longrightarrow N$ is a morphism of A-modules then $\theta_J : M_J \longrightarrow N_J$ defined in Definition 9 is a morphism of A_J-modules. This defines an additive functor

$$(-)_J : \mathbf{Mod}A \longrightarrow \mathbf{Mod}A_J$$

The canonical morphism of A-modules $A \longrightarrow A_J$ is clearly a morphism of rings, and for an A-module M the canonical A-module structure on M_J agrees with the structure obtained by restriction of scalars.

{prop_localisatio

Proposition 6. *Let A be a ring with gabriel topology J, and let $\theta : A \longrightarrow R$ be a ring morphism with R a J-closed A-module. Then there is a* unique *morphism of A-algebras $\psi : A_J \longrightarrow R$ making the following diagram commute*

$$\begin{array}{ccc} A & \xrightarrow{\theta} & R \\ \downarrow & \nearrow_{\psi} & \\ A_J & & \end{array} \tag{3}$$

{eq_localisationu

Proof. We need only show the morphism ψ of Proposition 4 is a morphism of rings, which is straightforward using the unique property of $\psi(\mathfrak{a}, \varphi)$. □

{lemma_morphismde

Lemma 7. *Let A be a ring with gabriel topology J, $\mathfrak{b} \subseteq \mathfrak{a}$ right ideals in J and M a J-torsion-free A-module. If $f, g : \mathfrak{a} \longrightarrow M$ are morphisms of A-modules which agree on $\mathfrak{b}$, then $f = g$.*

Proof. It suffices to prove that if $h : \mathfrak{a} \longrightarrow M$ is a morphism of A-modules with $h = 0$ on $\mathfrak{b}$, then $h = 0$. Let $a \in \mathfrak{a}$. Then $(\mathfrak{b} : a) \in J$ and everything in $(\mathfrak{b} : a)$ kills $h(a)$, so $Ann(h(a)) \supseteq (\mathfrak{b} : a)$ and hence $h(a)$ is J-torsion, which implies that $h(a) = 0$ since M is J-torsion-free. □

{remark_equalityc

Remark 5. Let A be a ring with gabriel topology J and let M be an A-module. By Lemma 2 the A-module $M/t_J(M)$ is J-torsion-free, so Lemma 7 implies that two pairs $(\mathfrak{b}, \varphi), (\mathfrak{b}', \varphi')$ determine the same equivalence class of M_J if and only if $\varphi|_{\mathfrak{b} \cap \mathfrak{b}'} = \varphi'|_{\mathfrak{b} \cap \mathfrak{b}'}$. In particular, $(\mathfrak{b}, \varphi) = 0$ in M_J if and only if $\varphi = 0$. This shows that the canonical morphism of abelian groups

$$Hom_A(\mathfrak{a}, M/t_J(M)) \longrightarrow M_J$$

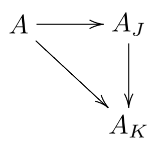
is injective for every $\mathfrak{a} \in J$.

{definition_gamma

Definition 12. Let A be a ring with gabriel topologies $J \subseteq K$ and let M be an A-module. Then $t_J(M) \subseteq t_K(M)$ so there is a canonical morphism of A-modules $\theta : M/t_J(M) \longrightarrow M/t_K(M)$ and therefore a canonical morphism of A-modules

$$\varphi_{J,K} : M_J \longrightarrow M_K$$
$$(\mathfrak{a}, \varphi) \mapsto (\mathfrak{a}, \theta\varphi)$$

One checks easily that $\varphi_{J,J} = 1$ and for three gabriel topologies $J \subseteq K \subseteq Q$ we have $\varphi_{K,Q}\varphi_{J,K} = \varphi_{J,Q}$. In the special case $M = A$ the morphism $\varphi_{J,K} : A_J \longrightarrow A_K$ is a morphism of rings which makes the following diagram commute



In fact the morphism $M_J \longrightarrow M_K$ sends the action of A_J to the action of A_K in a way compatible with the ring morphism $A_J \longrightarrow A_K$.

{prop_transitivel

Proposition 8. *Let A be a ring with gabriel topologies $J \subseteq K$, and let M be an A-module. Then there is a canonical isomorphism of A-modules $(M_J)_K \cong M_K$ natural in M.*

Proof. This follows immediately from (LOR,Corollary 23). □

2 Commutative rings

Throughout this section a "ring" means a not necessarily commutative ring. All modules in this section are right modules unless there is some indication to the contrary. If A is a commutative ring, then an A-*algebra* is a (not necessarily commutative) ring B together with a ring morphism $A \longrightarrow B$ with image contained in the center of B. See (TES,Definition 1) for more details. We only consider algebras over commutative rings A.

Let A be a commutative ring. A set of ideals J is a right gabriel topology if and only if it is a left gabriel topology, and we simply call J a *gabriel topology*. Summarising the results of the previous section, for every A-module M we have an A-module M_J defined as the following direct limit *of A-modules*

$$M_J = \varinjlim_{\mathfrak{a} \in J} Hom_A(\mathfrak{a}, M/t_J(M))$$

In particular we have a ring A_J, which together with the canonical ring morphism $A \longrightarrow A_J$ is an A-algebra. There is a canonical A_J-module structure on M_J which restricts to the above A-module structure, and we have an additive functor $(-)_J : \mathbf{Mod}A \longrightarrow \mathbf{Mod}A_J$.

Remark 6. In many cases the ring A_J is actually commutative, but I doubt this is true in general. Elements of A_J are essentially morphisms $\mathfrak{a} \longrightarrow A/t_J(A)$. The product of two such morphisms α, β is essentially the composite $\alpha \circ \beta$, and composition of morphisms is certainly not commutative (although when α, β are something like $x \mapsto x \cdot r, x \mapsto x \cdot s$ then they can commute past each other if r, s can, which is the source of commutativity of A_J in most of our examples).

Remark 7. If A is a commutative ring then axiom (b) of a gabriel topology follows from axiom (c). That is, a set of ideals J is a gabriel topology if and only if $(a) A \in J$ and (c) If $\mathfrak{a} \in J$ and $\mathfrak{b}$ is an ideal with $(\mathfrak{b} : h) \in J$ for every $h \in \mathfrak{a}$, then $\mathfrak{b} \in J$.

{definition_topol

Definition 13. Let A be a commutative ring and let S be a multiplicatively closed subset of A (so $1 \in S$ and $st \in S$ for any $s, t \in S$). It is easy to check that $J(S) = \{\mathfrak{a} \,|\, \mathfrak{a} \cap S \neq \emptyset\}$ is a gabriel topology. If M is an A-module then $x \in M$ is $J(S)$-torsion if and only if there exists $s \in S$ with $m \cdot s = 0$.

{prop_generalises

Proposition 9. *Let A be a commutative ring, $S \subseteq A$ a multiplicatively closed subset and M an A-module. Then there is a canonical isomorphism of A-modules natural in M*

$$\phi : S^{-1}M \longrightarrow M_{J(S)}$$
$$\phi\left(\frac{x}{s}\right)(sa) = xa + t_{J(S)}(M)$$

Proof. Let $x \in M, s \in S$ be given. Then (s) is an ideal belonging to $J(S)$ and we define a morphism of A-modules

$$\varphi_{x,s} : (s) \longrightarrow M/t_{J(S)}(M)$$
$$\varphi_{x,s}(sa) = xa + t_{J(S)}(M)$$

To see this is well-defined, suppose that $sa = sb$. Then $s(a-b) = 0$ and it follows that the element $x(a-b)$ of M is $J(S)$-torsion. Therefore $xa + t_{J(S)}(M) = xb + t_{J(S)}(M)$, so $\varphi_{x,s}$ is well-defined. It is not difficult to check it is a morphism of A-modules. We define a morphism of A-modules

$$\phi : S^{-1}M \longrightarrow M_{J(S)}$$
$$\phi(x/s) = ((s), \varphi_{x,s})$$

To see this is well-defined, suppose that $x/s = y/t$ in $S^{-1}M$. Then there is $q \in S$ with $(xt - ys)q = 0$. The morphisms $\varphi_{x,s}, \varphi_{y,t}$ therefore agree on the ideal $(stq) \in J(S)$, so $((s), \varphi_{x,s}) = ((t), \varphi_{y,t})$ in $M_{J(S)}$. Therefore ϕ is well-defined, and it is easy to see it is a morphism of A-modules natural in M.

The map ϕ is injective since if $\phi(x/s) = 0$ then $\varphi_{x,s} = 0$ by Remark 5. But this implies $x \in t_{J(S)}(M)$, which is another way of saying $x/s = 0$ in $S^{-1}M$. Any element of $M_{J(S)}$ can be represented by a morphism of A-modules $\varphi : (s) \longrightarrow M/t_{J(S)}(M)$ for some $s \in S$. Choose $m \in M$ with $\varphi(s) = m + t_{J(S)}(M)$. Then it is not hard to check that $\phi(m/s) = ((s), \varphi)$, which completes the proof. □

Remark 8. With the notation of Proposition **??** suppose that $M = A$. Then it is not difficult to check that $\phi : S^{-1}A \longrightarrow A_{J(S)}$ is actually an isomorphism of A-algebras. In particular, the ring $A_{J(S)}$ is commutative.

Corollary 10. *Let $\theta : A \longrightarrow B$ be a morphism of commutative rings and $S \subseteq A$ a multiplicatively closed set. Then B is $J(S)$-closed as an A-module if and only if θ sends the elements of S to units.*

Proof. Suppose that B is $J(S)$-closed. Then by Proposition 9 and Proposition 6 there is a morphism of A-algebras $S^{-1}A \longrightarrow B$ which implies the the images of the elements of S in B are all units. Conversely, suppose that the elements of $\theta(S)$ are all units. Then it is clear that B is $J(S)$-torsion-free. To see that it is $J(S)$-injective it suffices by Lemma 7 to show that any morphism of A-modules $\varphi : (s) \longrightarrow B$ for $s \in S$ can be extended to all of A. If $\varphi(s) = q$ then let $\psi : A \longrightarrow B$ be the morphism of A-modules $1 \mapsto q\theta(s)^{-1}$. It is easy to check that $\psi|_{(s)} = \varphi$, as required. □

{lemma_commgabtop

Lemma 11. *Let A be a commutative ring with gabriel topology J. Then*

(i) If $\mathfrak{a}, \mathfrak{b} \in J$ then $\mathfrak{a}\mathfrak{b} \in J$.

(ii) An ideal $\mathfrak{a}$ maximal with respect to $\mathfrak{a} \notin J$ is prime.

(iii) If every ideal in J contains a finitely generated ideal in J, then for any ideal $\mathfrak{a} \notin J$ there exists $\mathfrak{p} \in V(\mathfrak{a})$ with $\mathfrak{p} \notin J$.

Proof. (i) For any $a \in \mathfrak{a}$ we have $(\mathfrak{ab} : a) \supseteq \mathfrak{b}$ so it is clear that $\mathfrak{ab} \in J$. (ii) Suppose $a, b \notin \mathfrak{a}$ with $ab \in \mathfrak{a}$. Then by maximality $(a) + \mathfrak{a}, (b) + \mathfrak{a} \in J$ and so by (i) $((a) + \mathfrak{a})((b) + \mathfrak{a}) \in J$. But this ideal is contained in $\mathfrak{a}$, which contradicts the fact that $\mathfrak{a} \notin J$. Hence $\mathfrak{a}$ is prime. (iii) The idea is that any ideal not in J can be expanded to a prime not in J. Suppose $\mathfrak{a} \notin J$ and let $\mathcal{O}$ be the set of all ideals $\mathfrak{b} \supseteq \mathfrak{a}$ not in J. The fact that chains in $\mathcal{O}$ have upper bounds follows from the assumption, since if $\{\mathfrak{b}_i\}$ is a chain and $\bigcup \mathfrak{b}_i \in J$ then there is a finitely generated ideal $\mathfrak{c} \in J$ with $\mathfrak{c} \subseteq \bigcup \mathfrak{b}_i$. Hence $\mathfrak{c} \subseteq \mathfrak{b}_j$ for some j, which contradicts the fact that $\mathfrak{b}_j \notin J$. Hence we can use Zorn's Lemma to find an ideal $\mathfrak{p} \supseteq \mathfrak{a}$ maximal with respect to $\mathfrak{p} \notin J$, which is prime by (ii). □

{prop_topologyfro

Proposition 12. *Let A be a commutative ring and let $\mathcal{P}$ be a set of prime ideals of A. Then the following defines a gabriel topology*

$$J_{\mathcal{P}} = \{\mathfrak{a} \,|\, V(\mathfrak{a}) \cap \mathcal{P} = \emptyset\}$$

Conversely associated to any gabriel topology J is the following subset of $SpecA$

$$D(J) = \{\mathfrak{p} \in SpecA \,|\, \mathfrak{p} \notin J\}$$

The following conditions on a gabriel topology J are equivalent

(a) $J = J_{\mathcal{P}}$ for some $\mathcal{P} \subseteq SpecA$.

(b) $J = J_{D(J)}$.

(c) For every ideal $\mathfrak{a} \notin J$ there is $\mathfrak{p} \in V(\mathfrak{a})$ with $\mathfrak{p} \notin J$.

Proof. Let $\mathcal{P}$ be any set of prime ideals. It is clear that $A \in J_{\mathcal{P}}$. Given $\mathfrak{a} \in J_{\mathcal{P}}$ and $b \in A$ we must show that $(\mathfrak{a} : b)$ is not contained in any of the primes in $\mathcal{P}$. But if $\mathfrak{p} \in \mathcal{P}$ then $\mathfrak{a} \nsubseteq \mathfrak{p}$, say $a \in \mathfrak{a} \setminus \mathfrak{p}$. Then $a \in (\mathfrak{a} : b) \setminus \mathfrak{p}$, as required.

To show transitivity let ideals $\mathfrak{a}, \mathfrak{b}$ be given with $\mathfrak{a} \in J_{\mathcal{P}}$, so $\mathfrak{a} \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{P}$. Suppose $(\mathfrak{b} : a) \in J_{\mathcal{P}}$ for all $a \in \mathfrak{a}$ and suppose for a contradiction that $\mathfrak{b} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathcal{P}$. Since $\mathfrak{a} \nsubseteq \mathfrak{p}$ there is $a \in \mathfrak{a} \setminus \mathfrak{p}$, and the fact that $(\mathfrak{b} : a) \nsubseteq \mathfrak{p}$ means there exists c with $c \notin \mathfrak{p}$ and $ca \in \mathfrak{b}$, which contradicts the fact that $\mathfrak{p}$ is prime. Hence $J_{\mathcal{P}}$ is a topology.

Now we show that the three conditions $(a), (b), (c)$ are equivalent. First observe that $J_{D(J)} = \{\mathfrak{a} \,|\, V(\mathfrak{a}) \subseteq J\}$ for any topology J and consequently $J_{D(J)} \supseteq J$. Similarly $\mathcal{P} \subseteq D(J_{\mathcal{P}})$ for any subset $\mathcal{P} \subseteq SpecA$. $(a) \Rightarrow (b)$ Suppose $J = J_{\mathcal{P}}$. To show $J = J_{D(J)}$ it suffices to show $J_{D(J)} \subseteq J$. So suppose $V(\mathfrak{a}) \subseteq J = J_{\mathcal{P}}$. If $\mathfrak{q} \in V(\mathfrak{a}) \cap \mathcal{P}$ then $\mathfrak{q} \in J_{\mathcal{P}}$ implies that $\mathfrak{q} \notin \mathcal{P}$, a contradiction. Hence $V(\mathfrak{a}) \cap \mathcal{P} = \emptyset$ and $\mathfrak{a} \in J$. $(b) \Rightarrow (c)$ Suppose $J = J_{D(J)}$. If $\mathfrak{a} \notin J$ then $\mathfrak{a} \notin J_{D(J)}$ and hence $V(\mathfrak{a}) \nsubseteq J$ as required. $(c) \Rightarrow (a)$ We claim that $J = J_{D(J)}$. If $\mathfrak{a} \in J_{D(J)}$ then $V(\mathfrak{a}) \subseteq J$. By (c) this implies $\mathfrak{a} \in J$. □

Remark 9. Let A be a commutative ring. Note that $J_{\emptyset}$ is the topology consisting of all ideals, whereas $J_{SpecA} = \{A\}$. If $\mathcal{P}, \mathcal{Q}$ are two subsets of $SpecA$ with $\mathcal{P} \subseteq \mathcal{Q}$ then it is clear that $J_{\mathcal{Q}} \subseteq J_{\mathcal{P}}$.

The topology $J_{\{\mathfrak{p}\}}$ consists of all $\mathfrak{a}$ with $\mathfrak{a} \nsubseteq \mathfrak{p}$. Therefore $J_{\{\mathfrak{p}\}}$ is the topology $J(A \setminus \mathfrak{p})$ of Definition 13. It follows from Proposition 9 that there is a canonical isomorphism of A-algebras $A_{\mathfrak{p}} \longrightarrow A_{J_{\{\mathfrak{p}\}}}$.

Lemma 13. *Let A be a commutative ring and $S \subseteq A$ a multiplicatively closed set. Set $\mathcal{P} = \{\mathfrak{p} \,|\, \mathfrak{p} \cap S = \emptyset\}$. Then $J(S) = J_{\mathcal{P}}$.*

Proof. The inclusion $J(S) \subseteq J_{\mathcal{P}}$ is trivial. In the other direction, use the fact that any ideal not meeting S can be extended to a prime ideal with this property. □

{definition_gabri

Definition 14. Let A be a commutative ring. We say that two subsets $\mathcal{P}, \mathcal{Q}$ of $SpecA$ are *gabriel equivalent* and write $\mathcal{P} \sim \mathcal{Q}$ if we have $J_{\mathcal{P}} = J_{\mathcal{Q}}$. This is clearly an equivalence relation. We make the following comments

- It follows from Proposition 12 that for any set of primes $\mathcal{P}$ we have $\mathcal{P} \sim d(\mathcal{P})$ where $d(\mathcal{P}) = D(J_{\mathcal{P}})$. In fact it is easy to see that $\mathcal{P} \sim \mathcal{Q}$ if and only if $d(\mathcal{P}) = d(\mathcal{Q})$.
- By definition we can associate with every equivalence class E of $\sim$ a gabriel topology J with $J = J_{\mathcal{P}}$ for every representative $\mathcal{P}$ of E. In that case $D(J)$ belongs to E, and it contains every other representative of E.
- Clearly $D(J_{\emptyset}) = \emptyset$ and $D(J_{SpecA}) = SpecA$.
- For a subset $\mathcal{P}$ we have $\mathcal{P} \sim \emptyset$ iff. $\mathcal{P} = \emptyset$, so $\emptyset$ is the only set in its equivalence class.
- Let us consider the equivalence class of a singleton $\{\mathfrak{p}\}$. It is clear that $\{\mathfrak{p}\} \sim \{\mathfrak{q}\}$ iff. $\mathfrak{p} = \mathfrak{q}$, so each prime lives in a distinct equivalence class. More generally $\{\mathfrak{p}\} \sim \mathcal{P}$ iff. $\mathfrak{p} \in \mathcal{P}$ and every element of $\mathcal{P}$ is contained in $\mathfrak{p}$.

Given a subset $\mathcal{P}$ let $m(\mathcal{P})$ denote the primes in $\mathcal{P}$ not properly contained in any other prime of $\mathcal{P}$. These are precisely the closed points in the subspace topology on $\mathcal{P}$. Observe that

$$m(\mathcal{P}) = \{\mathfrak{p} \,|\, V(\mathfrak{p}) \cap \mathcal{P} = \{\mathfrak{p}\}\}$$
$$d(\mathcal{P}) = \{\mathfrak{p} \,|\, V(\mathfrak{p}) \cap \mathcal{P} \neq \emptyset\}$$

and by definition $m(\mathcal{P}) \subseteq \mathcal{P} \subseteq d(\mathcal{P})$.

{prop_psimclosed}

Proposition 14. *Let A be a commutative noetherian ring and $\mathcal{P}$ a subset of $SpecA$. Then we have $\mathcal{P} \sim m(\mathcal{P})$.*

Proof. Since $m(\mathcal{P}) \subseteq \mathcal{P}$ it suffices to prove that $J_{m(\mathcal{P})} \subseteq J_{\mathcal{P}}$. Suppose to the contrary that there exists an ideal $\mathfrak{a}$ not contained in any closed point of $\mathcal{P}$ but which is contained in some $\mathfrak{p} \in \mathcal{P}$. Then $\mathfrak{p}$ is not closed, so there is $\mathfrak{p}_1 \in \mathcal{P}$ with $\mathfrak{p} \subset \mathfrak{p}_1$. For the same reason $\mathfrak{p}_1$ cannot be closed, so we produce in this way a strictly ascending chain $\mathfrak{p} \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots$ which is impossible since A is noetherian. This contradiction shows that $J_{m(\mathcal{P})} \subseteq J_{\mathcal{P}}$ and completes the proof. □

{lemma_topclosedp

Lemma 15. *Let A be a commutative noetherian ring. The following conditions on a pair of subsets $\mathcal{P}, \mathcal{P}' \subseteq SpecA$ are equivalent*

(i) $\mathcal{P} \sim \mathcal{P}'$.

(ii) *For every $\mathfrak{p} \in \mathcal{P}$ there is a prime in $\mathcal{P}'$ containing $\mathfrak{p}$, and for every $\mathfrak{q} \in \mathcal{P}'$ there is a prime in $\mathcal{P}$ containing $\mathfrak{q}$.*

(iii) $m(\mathcal{P}) = m(\mathcal{P}')$.

(iv) $d(\mathcal{P}) = d(\mathcal{P}')$.

Proof. $(i) \Rightarrow (ii)$ Suppose $\mathcal{P} \sim \mathcal{P}'$. By symmetry it suffices to show that every prime $\mathfrak{p} \in \mathcal{P}$ is contained in a prime of $\mathcal{P}'$. But $\mathfrak{p} \in \mathcal{P}$ means that $\mathfrak{p} \notin J_{\mathcal{P}}$ and hence $\mathfrak{p} \notin J_{\mathcal{P}'}$, so this is trivial. $(ii) \Rightarrow (i)$ By symmetry it suffices to show $J_{\mathcal{P}} \subseteq J_{\mathcal{P}'}$. Suppose $\mathfrak{a} \notin J_{\mathcal{P}'}$ so $\mathfrak{a} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathcal{P}'$. Then by (ii) there is $\mathfrak{p} \in \mathcal{P}$ with $\mathfrak{q} \subseteq \mathfrak{p}$ and hence $\mathfrak{a} \notin J_{\mathcal{P}}$, as required. $(ii) \Rightarrow (iii)$ By symmetry it suffices to show $m(\mathcal{P}) \subseteq m(\mathcal{P}')$. But if $\mathfrak{p} \in m(\mathcal{P})$ then there is $\mathfrak{q} \in \mathcal{P}'$ with $\mathfrak{q} \supseteq \mathfrak{p}$. Applying (ii) again we find $\mathfrak{p}' \in \mathcal{P}$ with $\mathfrak{p}' \supseteq \mathfrak{q} \supseteq \mathfrak{p}$. Maximality of $\mathfrak{p}$ implies that $\mathfrak{p} = \mathfrak{q}$, so at least $\mathfrak{p} \in \mathcal{P}'$. The same argument shows that $\mathfrak{p} \in m(\mathcal{P}')$. The equivalence $(i) \Leftrightarrow (iv)$ is trivial.

We have shown $(iv) \Leftrightarrow (i) \Leftrightarrow (ii) \Rightarrow (iii)$ without using the noetherian hypothesis. But we use it to prove $(iii) \Rightarrow (i)$, which is an immediate consequence of Proposition 14. □

Remark 10. Let A be a commutative noetherian ring. Then for any subset $\mathcal{P} \subseteq SpecA$ we have $m(\mathcal{P}) \sim \mathcal{P} \sim d(\mathcal{P})$ by Proposition 14. In fact it is not difficult to see that for another subset $\mathcal{Q}$ we have $\mathcal{P} \sim \mathcal{Q}$ if and only if $m(\mathcal{P}) \subseteq \mathcal{Q} \subseteq d(\mathcal{P})$.

{remark_generisat

Remark 11. Let A be a commutative noetherian ring. Another way of stating condition (ii) of Lemma 15 is that every point of $\mathcal{P}$ is a generisation of a point of $\mathcal{P}'$, and every point of $\mathcal{P}'$ is a generisation of a point of $\mathcal{P}$. This has the following consequences

- Open subsets are stable under generisation, so if U, V are two open subsets of $SpecA$ we have $U \sim V$ if and only if $U = V$.
- It is easy to check that $d(\mathcal{P})$ is stable under generisation for any subset $\mathcal{P}$ of $SpecA$.
- If $\mathcal{P} \subseteq \mathcal{Q}$ are subsets of $SpecA$ then $\mathcal{P} \sim \mathcal{Q}$ if and only if every point of $\mathcal{Q}$ is a generisation of a point of $\mathcal{P}$. In particular, every point of $\mathcal{P}$ is a generisation of a point of $m(\mathcal{P})$ and $d(\mathcal{P})$ is the set of *all* generisations of the points of $m(\mathcal{P})$.
- If $\mathcal{Q}$ is a subset of $SpecA$ stable under generisation then we must have $\mathcal{Q} = d(\mathcal{Q})$.
- Taking $\mathcal{P} = \{\mathfrak{p}\}$ we see that $d(\{\mathfrak{p}\})$ is the set of all generisations of the point $\mathfrak{p}$. In other words, $d(\{\mathfrak{p}\}) = \{\mathfrak{q} \,|\, \mathfrak{q} \subseteq \mathfrak{p}\}$.

Definition 15. Let A be a commutative ring, $\mathcal{P}$ a subset of $SpecA$ and M an A-module. To avoid excessive subscripts, we denote the A-submodule $t_{J_{\mathcal{P}}}(M)$ by $t_{\mathcal{P}}(M)$.

{lemma_intersectt

Lemma 16. *Let A be a commutative ring. For a nonempty family of subsets $\{\mathcal{P}_i\}_{i \in I}$ of $SpecA$ we have $J_{\cup_i \mathcal{P}_i} = \bigcap_i J_{\mathcal{P}_i}$. Consequently for an A-module M we have*

$$t_{\cup_i \mathcal{P}_i}(M) = \bigcap_i t_{\mathcal{P}_i}(M)$$

Proof. Both claims follow directly from the definitions. □

{prop_classifygab

Proposition 17. *Let A be a commutative noetherian ring. Then every gabriel topology J is of the form $J_{\mathcal{P}}$ for some subset $\mathcal{P} \subseteq SpecA$. In fact we have $J = J_{D(J)}$.*

Proof. If A is noetherian then it follows from Lemma 11(iii) that for any ideal $\mathfrak{a} \notin J$ there exists a prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$ with $\mathfrak{p} \notin J$. Then Proposition 12 implies that $J = J_{D(J)}$, as required. □

Example 1. Let A be a commutative noetherian ring, set $X = SpecA$ and let $U \subseteq X$ be an open subset. Let $\mathfrak{a}$ be the radical ideal with $U = X \setminus V(\mathfrak{a})$. Then we have

$$\begin{aligned} J_U &= \{\mathfrak{b} \,|\, V(\mathfrak{b}) \cap U = \emptyset\} \\ &= \{\mathfrak{b} \,|\, V(\mathfrak{b}) \subseteq V(\mathfrak{a})\} \\ &= \{\mathfrak{b} \,|\, \mathfrak{a} \subseteq \sqrt{\mathfrak{b}}\} \\ &= \{\mathfrak{b} \,|\, \mathfrak{a}^n \subseteq \mathfrak{b} \text{ for some } n \geq 1\} \end{aligned}$$

where we have used the fact that in a noetherian ring every ideal contains a power of its radical. This shows that J_U is *precisely the set of open ideals in the $\mathfrak{a}$-adic topology* on A.

{prop_setsclosedg

Proposition 18. *Let A be a commutative noetherian ring. Then $J \mapsto D(J)$ defines a bijection between the set of gabriel topologies on A and the set of subsets of $SpecA$ stable under generisation.*

Proof. Let J be a gabriel topology on A. Then $D(J) = \{\mathfrak{p} \,|\, \mathfrak{p} \notin J\}$ is clearly stable under generisation. Suppose $\mathcal{Q}$ is a subset of $SpecA$ closed under generisation. Then it follows from the comments of Remark 11 that $\mathcal{Q} = D(J_{\mathcal{Q}})$, which shows that the map $J \mapsto D(J)$ is a bijection. □

3 Algebraic Geometry

{lemma_sectionsto

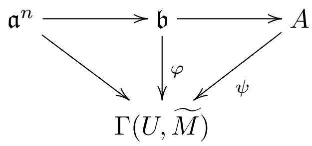
Lemma 19. *Let A be a commutative noetherian ring, set $X = SpecA$ and let M be an A-module. For any open $U \subseteq X$ the A-module $\Gamma(U, M^{\sim})$ is J_U-torsion-free.*

Proof. Let $\mathfrak{a}$ be the radical ideal with $U = X \setminus V(\mathfrak{a})$ and suppose that $x \in \Gamma(U, M^{\sim})$ is J_U-torsion. That is, $\mathfrak{a}^n x = 0$ for some $n \geq 1$. If $\mathfrak{a}$ is generated by $f_1, \dots, f_n$ then U is covered by the $D(f_i)$ and since $f_i^n x|_{D(f_i)} = 0$ we use the canonical isomorphism $\Gamma(D(f_i), M^{\sim}) \cong M_{f_i}$ to see that $x|_{D(f_i)} = 0$ for each i. This shows that $x = 0$, as required. $\square$

{prop_sectioninje

Proposition 20. *Let A be a commutative noetherian ring, set $X = SpecA$ and let M be an A-module. For any open $U \subseteq X$ the A-module $\Gamma(U, M^{\sim})$ is J_U-closed.*

Proof. By Lemma 19 it suffices to show that $\Gamma(U, M^{\sim})$ is J_U-injective. Suppose we are given an ideal $\mathfrak{b} \in J_U$ and a morphism of A-modules $\varphi : \mathfrak{b} \longrightarrow \Gamma(U, M^{\sim})$. There exists $n \geq 1$ with $\mathfrak{a}^n \subseteq \mathfrak{b}$. Suppose that the restriction of φ to $\mathfrak{a}^n$ admits an extension to a morphism $\psi : A \longrightarrow \Gamma(U, M^{\sim})$, as in the following diagram



By Lemma 19 the module $\Gamma(U, \widetilde{M})$ is J_U-torsion-free, so it follows from Lemma 7 that the above diagram must commute, and we can reduce to the case $\mathfrak{b} = \mathfrak{a}^n$ for some $n \geq 1$. Suppose that $\mathfrak{b} = (f_1, \dots, f_n)$, in which case $U = X \setminus V(\mathfrak{b})$ is covered by the open sets $D(f_i)$. For each i we can write $\varphi(f_i)|_{D(f_i)} = m_i \dot{/} f_i^N$ for some $N \geq 1$ independent of i. Let x_i be the section $m_i \dot{/} f_i^{N+1}$ of $\Gamma(D(f_i), M^{\sim})$. Then we have

$$\begin{aligned} f_i f_j \cdot x_i|_{D(f_i f_j)} &= f_i f_j m_i \dot{/} f_i^{N+1} \\ &= f_j \cdot \varphi(f_i)|_{D(f_i f_j)} \\ &= \varphi(f_i f_j)|_{D(f_i f_j)} \\ &= f_i f_j \cdot x_j|_{D(f_i f_j)} \end{aligned}$$

which shows that $x_i|_{D(f_i f_j)} = x_j|_{D(f_i f_j)}$. Since the $D(f_i)$ cover U, there exists a unique element $x \in \Gamma(U, M^{\sim})$ with $x|_{D(f_i)} = x_i$. Let $\psi : A \longrightarrow \Gamma(U, M^{\sim})$ be the morphism of A-modules corresponding to x. To show that $\psi|_{\mathfrak{b}} = \varphi$ it suffices to show that they agree on each f_i. Equivalently, we have to show that $f_i \cdot x_j = \varphi(f_i)|_{D(f_j)}$ for each pair i, j. We have

$$\begin{aligned} f_j \cdot (f_i \cdot x_j) &= f_j f_i m_j \dot{/} f_j^{N+1} \\ &= f_i \cdot \varphi(f_j)|_{D(f_j)} \\ &= \varphi(f_i f_j)|_{D(f_j)} \\ &= f_j \cdot \varphi(f_i)|_{D(f_j)} \end{aligned}$$

and therefore $f_i \cdot x_j = \varphi(f_i)|_{D(f_j)}$, as required. This shows that ψ extends φ and completes the proof. $\square$

{lemma_kernel}

Lemma 21. *Let A be a commutative noetherian ring, set $X = SpecA$ and let M be an A-module. For any open $U \subseteq X$ the kernel of the canonical morphism of A-modules $\theta : M \longrightarrow \Gamma(U, M^{\sim})$ is $t_U(M)$.*

Proof. By Lemma 19 the A-module $\Gamma(U, M^{\sim})$ is J_U-torsion-free, so the inclusion $t_U(M) \subseteq Ker\theta$ is trivial. As above, let $\mathfrak{a}$ be the radical ideal with $U = X \setminus V(\mathfrak{a})$ and write $\mathfrak{a} = (f_1, \dots, f_n)$. If $\theta(m) = 0$ then we have $m = 0$ in each M_{f_i}, and therefore $f_i^N m = 0$ for some $N \geq 1$ independent of i. We can therefore find $M \geq 1$ so large that $\mathfrak{a}^M m = 0$, which shows that m is J_U-torsion and completes the proof. $\square$

{lemma_image}

Lemma 22. *Let A be a commutative noetherian ring, set $X = SpecA$ and let M be an A-module. For any open $U \subseteq X$ and $s \in \Gamma(U, M^\sim)$ there exists an ideal $\mathfrak{b} \in J_U$ with $\mathfrak{b} \cdot s \subseteq Im\theta$.*

Proof. That is, we claim that there exists an ideal $\mathfrak{b} \in J_U$ such that $b \cdot s$ is of the form $\dot{m}/1$ for every $b \in \mathfrak{b}$. The first part of the proof uses a standard technique (see (H, II 2.2) for example). Since X is noetherian U is quasi-compact, and we can cover U with a finite number of open sets $D(h_1), \dots, D(h_n)$ such that $s|_{D(h_i)} = m_i/h_i$ for each i. Since $m_i/h_i = m_j/h_j$ on $D(h_i h_j)$ we deduce that there is $r \geq 1$ with

$$(h_i h_j)^r (h_j m_i - h_i m_j) = 0$$

where we may take r large enough to work for every pair i, j. Replacing h_i by h_i^{r+1} and m_i by $h_i^r m_i$ we can assume that $h_j m_i = h_i m_j$ in M for every pair i, j.

Let $\mathfrak{b}$ be the ideal $(h_1, \dots, h_n)$. Clearly $V(\mathfrak{b}) \cap U = \emptyset$, so $\mathfrak{b} \in J_U$. We define a map

$$\psi : \mathfrak{b} \longrightarrow \Gamma(U, \widetilde{M})$$
$$a_1 h_1 + \cdots + a_n h_n \mapsto \theta(a_1 m_1 + \cdots + a_n m_n)$$

To see that this map is well-defined, it suffices to show that if $a_1 h_1 + \cdots + a_n h_n = 0$ then $\theta(a_1 m_1 + \cdots + a_n m_n) = 0$. We have

$$\begin{aligned} 0 &= (a_1 h_1 + \cdots + a_n h_n) m_i = a_1 h_1 m_i + \cdots + a_n h_n m_i \\ &= a_1 h_i m_1 + \cdots + a_n h_i m_n = h_i (a_1 m_1 + \cdots + a_n m_n) \end{aligned}$$

Therefore $Ann(\sum_i a_i m_i)$ contains $\mathfrak{b}$, and thus itself belongs to J_U. This shows that $\sum_i a_i m_i$ is J_U-torsion, so Lemma 21 implies that $\theta(a_1 m_1 + \cdots + a_n m_n) = 0$, as required. It is easy to check that ψ is a morphism of A-modules.

By Proposition 20 there exists a unique element $t \in \Gamma(U, M^\sim)$ with $ht = \psi(h)$ for every $h \in \mathfrak{b}$. In particular we have $h_i t = \theta(m_i)$ for every i, from which we infer that

$$(h_i t)|_{D(h_i)} = \theta(m_i)|_{D(h_i)} = (h_i s)|_{D(h_i)}$$

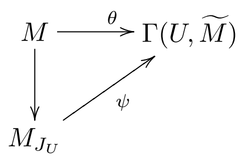
Or written differently, $h_i \cdot (t|_{D(h_i)} - s|_{D(h_i)}) = 0$. It follows that $t|_{D(h_i)} = s|_{D(h_i)}$ for each i and therefore $t = s$. By construction we have $hs = \psi(h) \in Im\theta$ for every $h \in \mathfrak{b}$, so $\mathfrak{b} \cdot s \subseteq Im\theta$, as required. □

{theorem_deligne}

Theorem 23. *Let A be a commutative noetherian ring and set $X = SpecA$. For every A-module M and open set $U \subseteq X$ there is a canonical isomorphism of A-modules natural in M and U*

$$\psi : M_{J_U} \longrightarrow \Gamma(U, \widetilde{M})$$

Proof. Let $\theta : M \longrightarrow \Gamma(U, M^\sim)$ be the canonical morphism of A-modules. By Proposition 20 the A-module $\Gamma(U, M^\sim)$ is J_U-closed, so by Proposition 4 there is a unique morphism of A-modules $\psi : M_{J_U} \longrightarrow \Gamma(U, M^\sim)$ making the following diagram commute



Since $\Gamma(U, M^\sim)$ is J_U-torsion-free, there is an induced morphism $\theta' : M/t_U(M) \longrightarrow \Gamma(U, M^\sim)$ which by Lemma 21 is injective. For $(\mathfrak{b}, \varphi) \in M_{J_U}$ the element $\psi(\mathfrak{b}, \varphi) \in \Gamma(U, M^\sim)$ is unique with the property that $h \cdot \psi(\mathfrak{b}, \varphi) = \theta' \varphi(h)$ for all $h \in \mathfrak{b}$.

Injectivity of ψ follows from injectivity of θ', so it only remains to show that ψ it is surjective. Let $s \in \Gamma(U, M^\sim)$ be given, and define the following ideal of A

$$\mathfrak{b} = \{h \in A \,|\, h \cdot s \in Im\theta\}$$

It follows from Lemma 22 that $\mathfrak{b}$ contains an ideal of J_U, and therefore belongs itself to J_U. For each $h \in \mathfrak{b}$ there is a unique element $\xi \in M/t_U(M)$ mapping to $h \cdot s$, and we define $\varphi(h) = \xi$. This defines a morphism of A-modules $\varphi : \mathfrak{b} \longrightarrow M/t_U(M)$. It is easy to check that $\psi(\mathfrak{b}, \varphi) = s$, which shows that ψ is an isomorphism. Naturality in M and U is straightforward, so the proof is complete. □

{corollary_delign

Corollary 24. *Let A be a commutative noetherian ring and set $X = SpecA$. For every open set $U \subseteq X$ there is a canonical isomorphism of A-algebras natural in U*

$$\psi : A_{J_U} \longrightarrow \mathcal{O}_X(U)$$

In particular, the ring A_{J_U} is commutative.

Proof. We need only check that the isomorphism of Theorem 23 is a morphism of rings, which is straightforward. □

Definition 16. Let A be a commutative ring and set $X = SpecA$. Let M be an A-module. For every open subset $U \subseteq X$ we have the gabriel topology J_U and an A-module $\mathcal{G}_M(U) = M_{J_U}$. For an inclusion $U \subseteq V$ we have $J_V \subseteq J_U$ and therefore a morphism of A-modules $M_{J_V} \longrightarrow M_{J_U}$ as in Definition 12. This defines the presheaf of abelian groups $\mathcal{G}_M$ on X.

In particular we have a presheaf of rings $\mathcal{G}_A$ on X. For each open set U there is a canonical $\mathcal{G}_A(U)$-module structure on the abelian group $\mathcal{G}_M(U)$, and this makes $\mathcal{G}_M$ into a presheaf of right modules over $\mathcal{G}_A$ (the only fact that needs checking is that restriction on $\mathcal{G}_M$ commutes with the action of $\mathcal{G}_A$, but we observed this in Definition 12). If $\alpha : M \longrightarrow N$ is a morphism of A-modules, the morphisms of A-modules $M_{J_U} \longrightarrow N_{J_U}$ of Definition 9 define a morphism of presheaves of right modules $\mathcal{G}_\alpha : \mathcal{G}_M \longrightarrow \mathcal{G}_N$. This defines an additive functor

$$\mathcal{G}_{(-)} : \mathbf{Mod}A \longrightarrow Mod\mathcal{G}_A$$
$$(\mathcal{G}_\alpha)_U = \alpha_{J_U}$$

{remark_sheavesof

Remark 12. Let A be a commutative noetherian ring and set $X = SpecA$. By Corollary 24 there is a canonical isomorphism of presheaves of rings $\psi : \mathcal{G}_A \longrightarrow \mathcal{O}_X$. In particular, $\mathcal{G}_A$ is a sheaf of commutative rings and the pair $(X, \mathcal{G}_A)$ is a scheme canonically isomorphic to X. If M is an A-module then by Theorem 23 there is a canonical isomorphism of presheaves of abelian groups $\mathcal{G}_M \longrightarrow M^\sim$, so $\mathcal{G}_M$ is a sheaf of left modules on the scheme $(X, \mathcal{G}_A)$. This defines an additive functor

$$\mathcal{G}_{(-)} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X, \mathcal{G}_A)$$
$$(\mathcal{G}_\alpha)_U = \alpha_{J_U}$$

{prop_isoofmodule

Proposition 25. *Let A be a commutative noetherian ring and set $X = SpecA$. For every A-module M there is a canonical isomorphism of sheaves of modules natural in M*

$$\psi : \mathcal{G}_M \longrightarrow \widetilde{M}$$

Proof. By Theorem 23 there is a canonical isomorphism of sheaves of abelian groups $\psi : \mathcal{G}_M \longrightarrow M^\sim$. Here $\mathcal{G}_M$ is a sheaf of modules on $(X, \mathcal{G}_A)$ while $M^\sim$ is a sheaf of modules on $(X, \mathcal{O}_X)$. By saying that ψ is an isomorphism of sheaves of modules, we mean that it sends the action of $\mathcal{G}_A(U)$ to the action of $\mathcal{O}_X(U)$ for every open set U. This is easy to check, and we already know that ψ is natural in M. □

{remark_universal

Remark 13. Let A be a commutative noetherian ring and set $X = SpecA$. Fix an open set $U \subseteq X$ and let $\theta : M \longrightarrow \Gamma(U, M^\sim)$ be the canonical morphism of A-modules. The A-module $\Gamma(U, M^\sim)$ is J_U-closed by Proposition 20, which means that given $\mathfrak{b} \in J_U$ and a morphism of A-modules

$\varphi : \mathfrak{b} \longrightarrow \Gamma(U, M^{\sim})$ there is a *unique* morphism of A-modules ψ making the following diagram commute

$$\begin{array}{ccc} \mathfrak{b} & \xrightarrow{\varphi} & \Gamma(U, \widetilde{M}) \\ \downarrow & \nearrow_{\psi} & \\ A & & \end{array}$$

Moreover by Theorem 23 the A-module $\Gamma(U, M^{\sim})$ is *universal with this property.* That is, given another J_U-closed A-module N and a morphism of A-modules $\alpha : M \longrightarrow N$ there is a *unique* morphism of A-modules κ making the following diagram commute

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle\theta}\downarrow & \nearrow_{\kappa} & \\ \Gamma(U, \widetilde{M}) & & \end{array}$$

{remark_universal

Remark 14. With the notation of Remark 13 the commutative ring $\Gamma(U, \mathcal{O}_X)$ is the *universal J_U-closed A-algebra.* That is, given another J_U-closed A-algebra R there is a *unique* morphism of A-algebras $\Gamma(U, \mathcal{O}_X) \longrightarrow R$. In other words, the morphism κ of Remark 14 is a morphism of rings.

Remark 15. Let A be a commutative noetherian ring. It follows from Corollary 24 that if A is a domain then so is A_{J_U} for any nonempty open $U \subseteq X$. On the other hand, it is not necessarily true that A_{J_U} is noetherian (since the ring $\mathcal{O}_X(U)$ is not always noetherian).