Modules over a Ringed Space

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In these notes we collect some useful facts about sheaves of modules on a ringed space that are either left as exercises in [Har77] or omitted completely. Most of these observations can be found in [Gro60], which is of course always the best reference. Nonetheless it may be useful for the reader to have an english reference when these points arise in our other notes on algebraic geometry. These notes are definitely not intended as an introduction to the subject.

Contents

1 Modules over Ringed Spaces

Definition 1. Let X be a topological space and $\mathcal{C}$ the small category of open sets of X. In our Algebra in a Category notes we defined the following concepts: *presheaf of rings*,*sheaf of rings*, *presheaf of commutative rings*, *sheaf of commutative rings*. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a *commutative* sheaf of rings $\mathcal{O}_X$. Note that if say that $\mathscr{S}$ is a sheaf of rings, we do *not* require $\mathscr{S}$ to be commutative.

If R is a presheaf of rings on X, then we have the complete grothendieck abelian category $(P(\mathcal{C}); R)$**Mod** of presheaves of left R-modules (LC,Corollary 3), which we denote by $Mod(R)$ or $Mod(X)$ for a ringed space $(X, \mathcal{O}_X)$. If $\mathscr{S}$ is a sheaf of rings on X, the complete grothendieck abelian category $(Sh_J(\mathcal{C}); \mathscr{S})$**Mod** of sheaves of left $\mathscr{S}$-modules (LC,Corollary 4) is denoted $\mathfrak{Mod}(\mathscr{S})$ or $\mathfrak{Mod}(X)$ for a ringed space $(X, \mathcal{O}_X)$.

Throughout this section $(X, \mathcal{O}_X)$ denotes a ringed space, and all modules are $\mathcal{O}_X$-modules. For any module $\mathscr{F}$ you can form the conglomerate of subobjects $Sub\mathscr{F}$. Recall that a *subobject* is a monomorphism $\phi : \mathscr{G} \longrightarrow \mathscr{F}$ in $\mathfrak{Mod}(X)$, whereas a *submodule* is a subobject for which ϕ_U is the inclusion of a subset for all open U. Every subobject is equivalent to a submodule, and two submodules are equivalent iff. they are the same module with the same embedding in $\mathscr{F}$, so there is a bijection between $Sub\mathscr{F}$ and the set of submodules of $\mathscr{F}$. You can form the intersection and union of any set of subobjects.

Lemma 1. *Let $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves of modules. Then the subsheaf $Im\phi$ is defined by the following condition: given an open set U and $s \in \mathscr{G}(U)$ we have $s \in (Im\phi)(U)$ if and only if for every point $x \in U$ there is an open set V with $x \in V \subseteq U$ and $s|_V \in Im(\phi_V)$.*

Lemma 2. *Let G be a subpresheaf of modules of a sheaf of modules $\mathscr{F}$. Then the induced morphism of sheaves $\mathbf{a}G \longrightarrow \mathscr{F}$ is a monomorphism whose image $\mathscr{G}$ is defined by the following condition: given an open set U and $s \in \mathscr{F}(U)$ we have $s \in \mathscr{G}(U)$ if and only if for every point $x \in U$ there is an open set V with $x \in V \subseteq U$ and $s|_V \in G(V)$.*

Note that the submodule $\mathscr{G}$ of $\mathscr{F}$ above is the smallest submodule of $\mathscr{F}$ containing G. Let $\{\mathscr{G}_i\}_{i \in I}$ be a nonempty family of submodules of a sheaf of modules $\mathscr{F}$. Then the union $\sum_i \mathscr{G}_i$ is the image of the induced morphism $\bigoplus_i \mathscr{G}_i \longrightarrow \mathscr{F}$.

Lemma 3. *Let $\{\mathscr{G}_i\}_{i \in I}$ be a nonempty set of subsheaves of a sheaf of modules $\mathscr{F}$. Then the subsheaf $\sum_i \mathscr{G}_i$ is defined by the following condition: given an open set U and $s \in \mathscr{F}(U)$ we have $s \in (\sum_i \mathscr{G}_i)(U)$ if and only if for every point $x \in U$ there exists an open neighborhood V with $x \in V \subseteq U$ and elements $a_{i_1}, \dots, a_{i_n}$ with $a_{i_k} \in \mathscr{G}_{i_k}(V)$ such that $s|_V = a_{i_1} + \cdots + a_{i_n}$.*

Proof. Taking the union of subsheaves of modules is the same as taking the union of subsheaves of abelian groups, so the result follows from (SGR,Lemma 11). ☐

Lemma 4. *Let $\{\mathscr{G}_i\}_{i \in I}$ be a direct family of subsheaves of a sheaf of modules $\mathscr{F}$. Then for an open set U and $s \in \mathscr{F}(U)$ we have $s \in (\sum_i \mathscr{G}_i)(U)$ if and only if for every point $x \in U$ there exists an open neighborhood V with $x \in V \subseteq U$ and $k \in I$ such that $s|_V \in \mathscr{G}_k(V)$. If U is a quasi-compact open subset of X then any section $s \in (\sum_i \mathscr{G}_i)(U)$ belongs to $\mathscr{G}_i(U)$ for some $i \in I$.*

Proof. Since the family is direct, this follows easily from the explicit description of the union given above. ☐

Remark 1. Since $\mathfrak{Mod}(X)$ is grothendieck abelian, for any family of sheaves of modules $\{\mathscr{F}_i\}_{i \in I}$ the canonical morphism $\bigoplus_i \mathscr{F}_i \longrightarrow \prod \mathscr{F}_i$ is a monomorphism. In particular for every open set $U \subseteq X$ we have an inclusion

$$\Gamma(U, \bigoplus_i \mathscr{F}_i) \longrightarrow \prod_i \Gamma(U, \mathscr{F}_i)$$

which at first glance seems puzzling: the sheaf coproduct is not necessarily the presheaf (i.e. pointwise) coproduct, but nonetheless it consists of sequences. The difference is that $\Gamma(U, \bigoplus_i \mathscr{F}_i)$ can contain some genuinely infinite sequences that do not belong to the pointwise coproduct. In the case where X is quasi-noetherian and U quasi-compact there is sufficient finiteness about to stop any sequences "escaping" $\bigoplus_i \Gamma(U, \mathscr{F}_i)$ and becoming infinite (COS,Proposition 23), so the presheaf and sheaf coproduct agree.

Since $\mathfrak{Mod}(X)$ is a grothendieck abelian category it follows from Mitchell III 1.2 that if $\mathscr{F}_i$ are a direct family of submodules of a module $\mathscr{F}$ then the induced morphism $\varinjlim \mathscr{F}_i \longrightarrow \mathscr{F}$ is in fact a monomorphism, isomorphic as a subobject to the categorical union $\sum_i \mathscr{F}_i$. In the case where X is a noetherian space we can give a nice characterisation of this submodule.

Lemma 5. *Let $\{\alpha_i : \mathscr{F}_i \longrightarrow \mathscr{F}\}_{i \in I}$ be a nonempty family of submodules of a $\mathcal{O}_X$-module $\mathscr{F}$. Then the intersection is the following submodule*

$$\left(\bigcap_i \mathscr{F}_i \right)(U) = \bigcap_i \mathscr{F}_i(U)$$

If X is a noetherian topological space and the $\mathscr{F}_i$ are a direct family of submodules, then the union is the following submodule

$$\left(\sum_i \mathscr{F}_i\right)(U) = \bigcup_i \mathscr{F}_i(U)$$

Proof. Same proof as (SGR,Lemma 12). □

Lemma 6. *Let $\mathscr{F}_1, \dots, \mathscr{F}_n$ be submodules of a $\mathcal{O}_X$-module $\mathscr{F}$. Then for $x \in X$ we have*

$$(\mathscr{F}_1 \cap \dots \cap \mathscr{F}_n)_x = \mathscr{F}_{1,x} \cap \dots \cap \mathscr{F}_{n,x}$$

as $\mathcal{O}_{X,x}$-submodules of $\mathscr{F}_x$.

Lemma 7. *Let $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of $\mathcal{O}_X$-modules and suppose $\mathscr{H}$ is a submodule of $\mathscr{G}$. Then the inverse image $\psi^{-1}\mathscr{H}$ is the submodule defined by*

$$(\psi^{-1}\mathscr{H})(U) = \psi_U^{-1}(\mathscr{H}(U))$$

Proof. See the proof of (SGR,Lemma 14). □

Lemma 8. *Let $\mathscr{F}$ be a $\mathcal{O}_X$-module. The following conditions on two submodules $\mathscr{G}, \mathscr{H}$ of $\mathscr{F}$ are equivalent:*

*(i) $\mathscr{G}$ precedes $\mathscr{H}$ as a subobject in the category $\mathcal{O}_X$***Mod***;*

(ii) $\mathscr{G}(U) \subseteq \mathscr{H}(U)$ for all open U;

(iii) $\mathscr{G}_x \subseteq \mathscr{H}_x$ as submodules of $\mathscr{F}_x$ for all $x \in X$.

Proof. It is clear that $(i) \Leftrightarrow (ii)$ and $(ii) \Rightarrow (iii)$. If $\mathscr{G}_x \subseteq \mathscr{H}_x$ for all $x \in X$ and $s \in \mathscr{G}(U)$ then we can find an open neighborhood $x \in U_x \subseteq U$ and $h_x \in \mathscr{H}(U_x)$ such that $s|_{U_x} = h_x$ for all x. The fact that $\mathscr{H}$ is a sheaf implies that the amalgamation s must belong to $\mathscr{H}(U)$, as required. □

Definition 2. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules and $\{s_i\}_{i \in I}$ a nonempty set of sections $s_i \in \mathscr{F}(U_i)$ over open sets $U_i \subseteq X$. The *submodule of $\mathscr{F}$ generated by the set $\{s_i\}$* is the intersection of all submodules of $\mathscr{F}$ containing all the s_i. This is the smallest submodule containing the s_i, in the sense that it precedes any other such submodule. We say that $\mathscr{F}$ is *generated* by the set $\{s_i\}_{i \in I}$ if the submodule generated by the s_i is all of $\mathscr{F}$.

In particular, $\mathscr{F}$ is *generated by global sections* if there is a nonempty set of global sections $\{s_i\}_{i \in I}$ which generate $\mathscr{F}$. This property is stable under isomorphisms of modules and ringed spaces, and under restriction to open subsets.

Proposition 9. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a $\mathcal{O}_X$-module. Given sections $s_i \in \mathscr{F}(U_i)$ let $G_x \subseteq \mathscr{F}_x$ be the $\mathcal{O}_{X,x}$-submodule generated by the set $\{germ_x s_i \,|\, x \in U_i\}$. For $U \subseteq X$ let*

$$\mathscr{G}(U) = \{s \in \mathscr{F}(U) \,|\, germ_x s \in G_x \text{ for all } x \in U\}$$

Then $\mathscr{G}$ is the submodule generated by the set $\{s_i\}$.

Proof. It is clear that $\mathscr{G}(U)$ is a $\mathcal{O}_X(U)$-submodule of $\mathscr{F}(U)$, that these modules are closed under the restriction of $\mathscr{F}$, and the resulting presheaf of $\mathcal{O}_X$-modules is actually a sheaf. So $\mathscr{G}$ is in fact a submodule of $\mathscr{F}$, and clearly $s_i \in \mathscr{G}(U_i)$ for all i. If $\mathscr{H}$ is any other submodule of $\mathscr{F}$ with $s_i \in \mathscr{H}(U_i)$ for all i, we claim that $\mathscr{G}$ precedes $\mathscr{H}$ as a subobject of $\mathscr{F}$. Since it is clear from the definition that $\mathscr{G}_x = G_x \subseteq \mathscr{H}_x$ for all $x \in X$ this follows from the previous Lemma. □

Remark 2. It follows that if $(X, \mathcal{O}_X)$ is a ringed space, $\mathscr{F}$ a $\mathcal{O}_X$-module and $\{s_i\}_{i \in I}$ global sections, the s_i generate $\mathscr{F}$ if and only if $germ_x s_i$ generate $\mathscr{F}_x$ as a $\mathcal{O}_{X,x}$-module for every $x \in X$.

Lemma 10. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of $\mathcal{O}_X$-modules. The image of φ is the submodule of $\mathscr{G}$ generated by the elements $\{\varphi_U(s) \,|\, s \in \mathscr{F}(U)\}$.*

Proof. Earlier we defined the submodule $Im\varphi$ of $\mathscr{G}$ by $t \in (Im\varphi)(U)$ iff. every $x \in U$ has an open neighborhood $x \in V \subseteq U$ such that $t|_V \in Im(\varphi_V)$, which is iff. $germ_x t \in Im(\varphi_x)$. For $x \in X$ the $\mathcal{O}_{X,x}$-submodule $G_x \subseteq \mathscr{G}_x$ corresponding to the collection $\{\varphi_U(s) \,|\, s \in \mathscr{F}(U)\}$ is precisely $Im(\varphi_x)$, so the result is clear. □

Definition 3. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is *globally finitely presented* if there are integers $p, q > 0$ and an exact sequence

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathscr{F} \longrightarrow 0$$

We say that $\mathscr{F}$ is *locally finitely presented* if every point $x \in X$ has an open neighborhood $x \in U \subseteq X$ such that $\mathscr{F}|_U$ is a globally finitely presented sheaf of $\mathcal{O}_X|_U$-modules. Clearly a finitely free sheaf is globally finitely presented, and a locally finitely free sheaf is locally finitely presented.

Proposition 11. *Let $(X, \mathcal{O}_X)$ be a ringed space. There is a nonempty set $\mathcal{I}$ of locally finitely presented sheaves of modules on X which is* representative. *That is, every locally finitely presented sheaf of modules is isomorphic to some element of $\mathcal{I}$.*

Proof. The general idea is that up to isomorphism all the globally finitely presented sheaves can be obtained by taking $p, q > 0$ and the canonical cokernel of all morphisms $\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q$. To get all the *locally* finitely presented sheaves you get all the globally finitely presented sheaves over open covers and all the possible ways of glueing these sheaves. Let us now get into the specifics. Let C be the set of all nonempty open covers of X. For each open set $U \subseteq X$ we define the following union set

$$F_U = \bigcup_{p,q>0} Hom_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U^p, \mathcal{O}_X|_U^q)$$

For every pair of open sets $U, V \subseteq X$ we define the following union set

$$G_{U,V} = \bigcup_{(f,g) \in F_U \times F_V} Hom_{\mathcal{O}_X|_{U\cap V}}(Coker(f)|_{U\cap V}, Coker(g)|_{U\cap V})$$

A *glueing datum* D consists of the following data: (i) a nonempty open cover $\mathfrak{U}$ of X (ii) for each open set $U \in \mathfrak{U}$ an element f_U of F_U (iii) for each pair of open sets $U, V \in \mathfrak{U}$ an isomorphism $\varphi_{U,V} : Coker(f_U)|_{U\cap V} \longrightarrow Coker(f_V)|_{U\cap V}$ with the property that the sheaves $Coker(f_U)$ and isomorphisms $\varphi_{U,V}$ satisfy the hypothesis of (GS,Proposition 1). Using the sets $F_U, G_{U,V}$ it is clear how to formalise this in such a way that we have a set $\mathcal{G}$ of all glueing datums. We can glue any glueing datum D to produce a locally finitely presented sheaf of modules $\mathscr{F}_D$ on X. This defines a set of locally finitely presented sheaves $\{\mathscr{F}_D\}_{D\in\mathcal{G}}$.

Now let $\mathscr{G}$ be any locally finitely presented sheaf of modules on X and let $\mathfrak{U}$ be a nonempty open cover with $\mathscr{G}|_U$ globally finitely presented for every $U \in \mathfrak{U}$. That is, we can find $f_U \in F_U$ together with a canonical isomorphism $\psi_U : Coker(f_U) \cong \mathscr{G}|_U$ for each $U \in \mathfrak{U}$. For each pair of open sets $U, V \in \mathfrak{U}$ let $\varphi_{U,V}$ be $\psi_V^{-1}|_{U\cap V}\psi_U|_{U\cap V}$. This clearly defines a glueing datum D, and we have $\mathscr{F}_D \cong \mathscr{G}$ by construction, which completes the proof. □

Lemma 12. *Let X be a topological space, and let $\mathcal{Z}$ be the sheafification of the constant presheaf of rings $U \mapsto \mathbb{Z}$. Then there is a canonical isomorphism $\mathfrak{Mod}(\mathcal{Z}) \cong \mathfrak{Ab}(X)$.*

Proof. Let P be the presheaf of rings defined by $P(\emptyset) = \{\emptyset\}$ and $P(U) = \mathbb{Z}$ for nonempty open U. The restriction maps are the obvious ones, and we denote by $\mathcal{Z}$ the sheaf of rings associated to this presheaf. We claim that the canonical forgetful functor $F : \mathfrak{Mod}(\mathcal{Z}) \longrightarrow \mathfrak{Ab}(X)$ is an isomorphism.

Let $\mathscr{F}$ be a sheaf of abelian groups on X. Then $\mathscr{F}$ becomes a sheaf of P-modules in the obvious way, and therefore also a sheaf of $\mathcal{Z}$-modules. This defines a functor $G : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Mod}(\mathcal{Z})$. It is clear that $FG = 1$, and it is not difficult to check that $GF = 1$, which completes the proof. □

1.1 Stalks

Throughout this section let $(X, \mathcal{O}_X)$ be a fixed ringed space. All sheaves of modules will be over X. For any point $x \in X$ we have the additive stalk functor

$$(-)_x : \mathfrak{Mod}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Mod}$$

Given an $\mathcal{O}_{X,x}$-module M we define a sheaf of modules by

$$\Gamma(U, Sky_x(M)) = \begin{cases} M & x \in U \\ 0 & \text{otherwise} \end{cases}$$

where for $x \in U$ the abelian group $M = \Gamma(U, Sky_x(M))$ becomes an $\mathcal{O}_X(U)$-module via the ring morphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}$. One defines Sky_x on morphisms in the obvious way, so that we have an additive functor

$$Sky_x(-) : \mathcal{O}_{X,x}\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$$

For a sheaf of modules $\mathscr{F}$ and $\mathcal{O}_{X,x}$-module M we have natural morphisms

$$\eta : \mathscr{F} \longrightarrow Sky_x(\mathscr{F}_x), \quad \eta_U(m) = (U, m)$$
$$\varepsilon : (Sky_x M)_x \longrightarrow M, \quad \varepsilon(U, m) = m$$

In fact these are the unit and counit of an adjunction $(-)_x \dashv Sky_x$. It is clear that ε is a natural equivalence, from which we deduce that Sky_x is fully faithful (AC,Proposition 21). It is also distinct on objects, so Sky_x gives a full embedding of $\mathcal{O}_{X,x}\mathbf{Mod}$ into $\mathfrak{Mod}(X)$. Given a sheaf of modules $\mathscr{F}$ we define an abelian group

$$\mathscr{F}^x = Hom_{\mathcal{O}_X}(Sky_x(\mathcal{O}_{X,x}), \mathscr{F})$$

which becomes a $\mathcal{O}_{X,x}$-module with action $(r \cdot s)_V(t) = s_V(rt)$ for open sets V containing x. A morphism of sheaves of modules $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ determines a morphism of $\mathcal{O}_{X,x}$-modules ψ^x defined by $\psi^x(s) = \psi \circ s$. This defines an additive functor

$$(-)^x : \mathfrak{Mod}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Mod}$$

For an $\mathcal{O}_{X,x}$-module M, any element $m \in M$ corresponds to a morphism of modules $m : \mathcal{O}_{X,x} \longrightarrow M$ and therefore to a morphism of sheaves $Sky_x(m) \in Sky_x(M)^x$. Since $Sky_x(-)$ is fully faithful, this is a natural isomorphism of $\mathcal{O}_{X,x}$-modules

$$\eta : M \longrightarrow Sky_x(M)^x$$
$$\eta(m) = Sky_x(m)$$

In fact this is the unit of an adjunction $Sky_x(-) \dashv (-)^x$. To see this, suppose we are given a sheaf of modules $\mathscr{G}$ and morphism of $\mathcal{O}_{X,x}$-modules $\alpha : M \longrightarrow \mathscr{G}^x$. Then the following morphism of sheaves of modules

$$\phi : Sky_x(M) \longrightarrow \mathscr{G}$$
$$\phi_V(m) = \alpha(m)_V(1)$$

is unique with the property that $\phi^x \eta = \alpha$. This establishes the desired adjunction, so we have a triple of adjoints

$$(-)_x \dashv Sky_x(-) \dashv (-)^x$$

which in particular means that $Sky_x(-)$ is exact and preserves all limits and colimits.

Remark 3. You can't expect the stalk functor $(-)_x$ to preserve products in general, because a sequence in $\prod_i \mathscr{F}_{i,x}$ could contain germs over a strictly descending sequence of open sets. Then it makes sense that no open section of $\prod_i \mathscr{F}_i$ (which is a sequence of sections over a *fixed* open set) can induce all of these germs at once.

Remark 4. It is clear that for any sheaf of modules $\mathscr{F}$ the morphism of $\mathcal{O}_{X,x}$-modules $\eta_x : \mathscr{F}_x \longrightarrow Sky_x(\mathscr{F}_x)_x$ is an isomorphism. Suppose now that x is a *closed point* and that $Supp(\mathscr{F}) = \{x\}$. Then the stalk of $Sky_x(\mathscr{F}_x)$ is zero at every point $y \neq x$, so it is clear that η is an isomorphism of sheaves of modules. So if x is a closed point and $\mathscr{F}$ a sheaf of modules, $\mathscr{F}$ is a skyscraper sheaf at x if and only if $Supp(\mathscr{F}) \subseteq \{x\}$.

Lemma 13. *Let $x \in X$ be a point and I an injective $\mathcal{O}_{X,x}$-module. Then $Sky_x(I)$ is an injective object in $\mathfrak{Mod}(X)$.*

Proof. The functor $Sky_x(-)$ has an exact left adjoint $(-)_x$ and therefore preserves injectives (AC,Proposition 25). ☐

Lemma 14. *Let $x \in X$ be a point, $\mathscr{F}$ a sheaf of modules on X and M an $\mathcal{O}_{X,x}$-module. Then there is a canonical morphism of sheaves of modules natural in $\mathscr{F}$ and M*

$$\lambda : \mathscr{F} \otimes_{\mathcal{O}_X} Sky_x(M) \longrightarrow Sky_x(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} M)$$
$$a \dot{\otimes} b \mapsto (U, a) \otimes b$$

with the property that λ_x is an isomorphism. If x is a closed point then λ is an isomorphism.

Proof. Tensoring the counit $\varepsilon : (Sky_x M)_x \longrightarrow M$ with $\mathscr{F}_x$ and composing with the canonical isomorphism yields an isomorphism of $\mathcal{O}_{X,x}$-modules

$$(\mathscr{F} \otimes_{\mathcal{O}_X} Sky_x(M))_x \longrightarrow \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} M$$

By the adjunction this corresponds to a morphism λ with the required properties. This result should be compared with Lemma 80. ☐

Lemma 15. *Let $x \in X$ be a point and M, N two $\mathcal{O}_{X,x}$-modules. There is a canonical isomorphism of sheaves of modules natural in M, N*

$$Sky_x Hom_{\mathcal{O}_{X,x}}(M, N) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(Sky_x M, Sky_x N)$$
$$\alpha \mapsto Sky_x(\alpha)|_U$$

Lemma 16. *Let $x \in X$ be a point, M an $\mathcal{O}_{X,x}$-module and $\mathscr{F}$ a complex of sheaves of modules. There is a canonical isomorphism of sheaves of modules natural in $\mathscr{F}, M$*

$$Sky_x Hom_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, Sky_x M)$$

Proof. We compose the isomorphism of Lemma 15 with the unit of the adjunction between $Sky_x(-)$ and $(-)_x$ to obtain a canonical morphism of sheaves of modules

$$Sky_x Hom_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) \cong \mathscr{H}om_{\mathcal{O}_X}(Sky_x(\mathscr{F}_x), Sky_x M) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, Sky_x M)$$

which is clearly natural in both variables, and is easily checked to be an isomorphism. This result should be compared with Corollary 87. ☐

1.2 Inverse and Direct Image

Lemma 17. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{F}$ a sheaf of modules on Y. Then for open $U \subseteq X$, $s \in f^*\mathscr{F}(U)$ and $x \in U$ there exists open V with $x \in V \subseteq U$ such that $s|_V = s_1 + \cdots + s_n$ where each s_j has the form*

$$s_j = [T, c] \dot{\otimes} b$$

with $T \supseteq f(V)$, $c \in \mathscr{F}(T)$ and $b \in \mathcal{O}_X(V)$.

Lemma 18. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of sheaves of modules on Y. For open $U \subseteq X$, $T \supseteq f(U)$ and $c \in \mathscr{F}(T), b \in \mathcal{O}_X(U)$ we have*

$$(f^*\phi)_U([T, c] \dot{\otimes} b) = [T, \phi_T(c)] \dot{\otimes} b$$

Theorem 19. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. Then there is an adjunction $f^* \dashv f_*$. For a sheaf of modules $\mathscr{F}$ on X the unit is*

$$\eta : \mathscr{F} \longrightarrow f_* f^* \mathscr{F}$$
$$\eta_U(s) = [U, s] \dot{\otimes} 1$$

If $\phi : \mathscr{G} \longrightarrow f_*\mathscr{F}$ is a morphism of sheaves of modules on Y, the adjoint partner is

$$\psi : f^*\mathscr{G} \longrightarrow \mathscr{F}$$
$$\psi_U([V, t] \dot{\otimes} r) = r \cdot \phi_V(t)|_U$$

for open $V \supseteq f(U)$ and $t \in \mathscr{G}(V)$.

Remark 5. Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{G}$ a sheaf of modules on Y. For $x \in X$ the $\mathcal{O}_{X,x}$-module $(f^*\mathscr{G})_x$ is also a $(f^{-1}\mathcal{O}_Y)_x$-module as $f^*\mathscr{G}$ is the tensor product of $f^{-1}\mathcal{O}_Y$-modules $f^{-1}\mathscr{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. The ring isomorphism $(f^{-1}\mathcal{O}_Y)_x \cong \mathcal{O}_{Y,f(x)}$ therefore defines a $\mathcal{O}_{Y,f(x)}$-module structure on $(f^*\mathscr{G})_x$. It is straightforward to check this agrees with the structure obtained from the canonical ring morphism $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$. The canonical isomorphism of abelian groups $(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}$ maps the action of $(f^{-1}\mathcal{O}_Y)_x$ to the action of $\mathcal{O}_{Y,f(x)}$.

Proposition 20. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{G}$ a sheaf of modules on Y. Then there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{G}$*

$$\tau : (f^*\mathscr{G})_x \longrightarrow \mathscr{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$
$$germ_x([V, s] \dot{\otimes} b) \mapsto germ_{f(x)} s \otimes germ_x b$$

Proof. Identifying the rings $\mathcal{O}_{Y,f(x)}$ and $(f^{-1}\mathcal{O}_Y)_x$ using the canonical isomorphism, we have an isomorphism of $\mathcal{O}_{Y,f(x)}$-modules

$$\begin{aligned}
(f^*\mathscr{G})_x &= (f^{-1}\mathscr{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)_x \\
&\cong (f^{-1}\mathscr{G})_x \otimes_{(f^{-1}\mathcal{O}_Y)_x} \mathcal{O}_{X,x} \\
&= (f^{-1}\mathscr{G})_x \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \\
&\cong \mathscr{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}
\end{aligned}$$

It is straightforward to check that this is an isomorphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{G}$. ☐

Lemma 21. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then as functors between the categories of sheaves of modules $(gf)_* = g_* f_*$. Also $1_* = 1$ and therefore if f is an isomorphism of ringed spaces, $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ is an isomorphism of categories.*

Remark 6. Let $(X, \mathcal{O}_X)$ be a ringed space and $U \subseteq X$ be an open subset with inclusion $i : U \longrightarrow X$. If $\mathscr{F}$ is a sheaf of $\mathcal{O}_X$ modules then the isomorphism of sheaves of abelian groups $\mathscr{F}|_U \longrightarrow i^{-1}\mathscr{F}$ is actually an isomorphism of sheaves of $\mathcal{O}_X|_U$-modules. The module structure on $i^{-1}\mathscr{F}$ comes from the natural $i^{-1}\mathcal{O}_X$-module structure together with the isomorphism $\mathcal{O}_X|_U \longrightarrow i^{-1}\mathcal{O}_X$.

Proposition 22. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_Z$-modules. There is a canonical isomorphism of sheaves of $(gf)^{-1}\mathcal{O}_Z$ modules natural in $\mathscr{F}$*

$$\psi : (gf)^{-1}\mathscr{F} \longrightarrow f^{-1}g^{-1}\mathscr{F}$$

Proof. The sheaf $g^{-1}\mathscr{F}$ is naturally a $g^{-1}\mathcal{O}_Z$-module, so $f^{-1}g^{-1}\mathscr{F}$ is naturally a $f^{-1}g^{-1}\mathcal{O}_Z$-module. We use the isomorphism $(gf)^{-1}\mathcal{O}_Z \longrightarrow f^{-1}g^{-1}\mathcal{O}_Z$ to make $f^{-1}g^{-1}\mathscr{F}$ into a $(gf)^{-1}\mathcal{O}_Z$-module.

We claim the isomorphism of sheaves of abelian groups $(gf)^{-1}\mathscr{F} \longrightarrow f^{-1}g^{-1}\mathscr{F}$ is actually an isomorphism of sheaves of $(gf)^{-1}\mathcal{O}_Z$-modules. First we set up our notation. Let Z', Z be the presheaves on X which sheafify to give $(gf)^{-1}\mathscr{F}$ and $f^{-1}g^{-1}\mathscr{F}$ respectively and let R', R be the presheaves of rings sheafifying to give $(gf)^{-1}\mathcal{O}_Z$ and $f^{-1}g^{-1}\mathcal{O}_Z$ respectively. Let $\phi' : Z' \longrightarrow Z$ and $\psi' : Z' \longrightarrow Z$ be the canonical morphisms of presheaves. For $x \in X$ and $a \in Z'_x$ and $b \in R'_x$ it is not difficult to check that

$$\phi'_x(b \cdot a) = \psi'_x(b) \cdot \phi'_x(a)$$

Let $\phi : (gf)^{-1}\mathscr{F} \longrightarrow f^{-1}g^{-1}\mathscr{F}$ and $\psi : (gf)^{-1}\mathcal{O}_Z \longrightarrow f^{-1}g^{-1}\mathcal{O}_Z$ be the sheafifications of ϕ, ψ respectively. If $U \subseteq X$ is open and $m \in (gf)^{-1}\mathscr{F}(U)$ and $r \in (gf)^{-1}\mathcal{O}_Z(U)$ then

$$\begin{aligned} \phi_U(r \cdot m)(x) &= \phi'_x((r \cdot m)(x)) \\ &= \phi'_x(r(x) \cdot m(x)) \\ &= \psi'_x(r(x)) \cdot \phi'_x(m(x)) \\ &= (\psi_U(r) \cdot \phi_U(m))(x) \end{aligned}$$

This shows that $\phi : (gf)^{-1}\mathscr{F} \longrightarrow f^{-1}g^{-1}\mathscr{F}$ is an isomorphism of $(gf)^{-1}\mathcal{O}_Z$-modules, as required. □

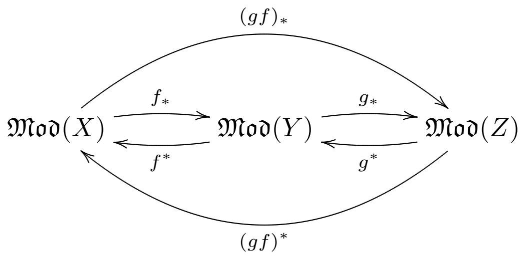
Proposition 23. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of ringed spaces. There is a canonical isomorphism of sheaves of modules on X natural in $\mathscr{F}$*

$$\alpha : (gf)^*\mathscr{F} \longrightarrow f^*g^*\mathscr{F}$$

which is unique making the following diagram commute

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\eta} & g_*g^*\mathscr{F} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle g_*\eta} \\ (gf)_*(gf)^*\mathscr{F} & \xrightarrow[(gf)_*\alpha]{} & (gf)_*f^*g^*\mathscr{F} \end{array} \qquad (1)$$

Proof. We have the following diagram of adjoint pairs



Let $\eta_f, \eta_g, \eta_{gf}$ be the units for the adjunctions $f^* \dashv f_*$, $g^* \dashv g_*$ and $(gf)^* \dashv (gf)_*$. By composition of adjoints, the functor f^*g^* is left adjoint to $g_*f_* = (gf)_*$ with unit $\nu = g_*\eta_f g^* \circ \eta_g$. We deduce a canonical natural equivalence $\alpha : (gf)^* \longrightarrow f^*g^*$ which is unique making the necessary diagram commute for each sheaf of modules $\mathscr{F}$ on Z. □

Remark 7. With the notation of Proposition 23 one checks that for a sheaf of modules $\mathscr{F}$ on X the following diagram commutes

$$\begin{array}{ccc} f^*g^*g_*f_*\mathscr{F} & \xrightarrow{f^*\varepsilon_{f_*\mathscr{F}}} & f^*f_*\mathscr{F} \\ {\scriptstyle\alpha^{-1}_{(gf)_*\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ (gf)^*(gf)_*\mathscr{F} & \xrightarrow[\varepsilon]{} & \mathscr{F} \end{array}$$

Remark 8. With the above notation, let $\mathscr{F}$ be a sheaf of modules on Z, $U \subseteq X$ and $T \supseteq (gf)(U)$ open sets and $c \in \mathscr{F}(T), b \in \mathcal{O}_X(U)$ sections. Then we have

$$\alpha : (gf)^*\mathscr{F} \longrightarrow f^*g^*\mathscr{F}$$
$$\alpha_U([T,c] \dot\otimes b) = [g^{-1}T, [T,c] \dot\otimes 1] \dot\otimes b$$

In checking this one can reduce to the case $b = 1$ and $U = (gf)^{-1}T$, which follows from commutativity of (1). Also for open $U \subseteq X, V \supseteq f(U), T \supseteq g(V)$ and $r \in \mathcal{O}_X(U), s \in \mathcal{O}_Y(V)$ and $c \in \mathscr{F}(T)$ we have

$$\alpha_U^{-1}([V,[T,c] \dot\otimes s] \dot\otimes r) = [T,c] \dot\otimes (rf_V^{\#}(s)|_U)$$

Checking this for $s = 1$ is trivial. We can reduce to this case in the following way

$$[V,[T,c] \dot\otimes s] \dot\otimes r = ([V,s] \cdot [V,[T,c] \dot\otimes 1]) \dot\otimes r = [V,[T,c] \dot\otimes 1] \dot\otimes (rf_V^{\#}(s)|_U)$$

since the tensor product is over $f^{-1}\mathcal{O}_Y(U)$.

1.3 Sheafification

Throughout this section $(X, \mathcal{O}_X)$ is a ringed space and $\mathbf{a} : Mod(X) \longrightarrow \mathfrak{Mod}(X)$ the canonical sheafification functor.

Lemma 24. *Let $U \subseteq X$ be an open subset. Then the following diagram of functors commutes up to canonical natural equivalence*

$$\begin{array}{ccc} Mod(X) & \xrightarrow{\mathbf{a}} & \mathfrak{Mod}(X) \\ {\scriptstyle |_U}\downarrow & & \downarrow{\scriptstyle |_U} \\ Mod(U) & \xrightarrow[\mathbf{a}]{} & \mathfrak{Mod}(U) \end{array}$$

Proof. The analogous result for presheaves of sets, groups or rings is easily checked (see *p.*7 of our Section 2.1 notes). Let F be a presheaf of modules on X, and let $\phi_x : (F|_U)_x \longrightarrow F_x$ be the canonical isomorphism of modules (compatible with the ring isomorphism $(\mathcal{O}_X|_U)_x \cong \mathcal{O}_{X,x}$) for $x \in U$. Then we define

$$\eta : \mathbf{a}(F|_U) \longrightarrow (\mathbf{a}F)|_U$$
$$\eta_W(s)(x) = \phi_x(s(x))$$

This is a morphism of sheaves of $\mathcal{O}_X$-modules, which is clearly an isomorphism. Naturality in F is also easily checked. ☐

Lemma 25. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then the following diagram of functors commutes up to canonical natural equivalence*

$$\begin{array}{ccc} Mod(X) & \xrightarrow{\mathbf{a}} & \mathfrak{Mod}(X) \\ {\scriptstyle f_*}\Downarrow & & \Downarrow{\scriptstyle f_*} \\ Mod(Y) & \xrightarrow[\mathbf{a}]{} & \mathfrak{Mod}(Y) \end{array}$$

Proof. We checked this for abelian groups in Section 2.5 p.33. The natural equivalence given there is easily checked to give a natural equivalence of functors in our present situation. ☐

1.4 Extension By Zero

Associated to any morphism of topological spaces is an adjoint pair of functors between the categories of sheaves of abelian groups: direct and inverse image. We saw in (SGR,Theorem 32) that in the special cases of open and closed embeddings, two extra adjoints make an appearance (an extra left adjoint for open embeddings, and an extra right adjoint for closed embeddings). The situation is similar for morphisms of ringed spaces and in this section we define the additional adjoint for open immersions. The adjoint for closed immersions will have to wait until Section 1.13.

Definition 4. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is an *open immersion* if it is an open embedding and the induced morphism of ringed spaces $(X, \mathcal{O}_X) \longrightarrow (f(X), \mathcal{O}_X|_{f(X)})$ is an isomorphism.

Definition 5. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a *closed immersion* if it is a closed embedding and the morphism $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is an epimorphism of sheaves of abelian groups. Equivalently, the induced map $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is surjective for every $x \in X$.

Definition 6. Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an open immersion of ringed spaces. If $\mathscr{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then we write $\mathscr{G}|_f$ (or more commonly just $\mathscr{G}|_X$) to denote the sheaf of $\mathcal{O}_X$-modules defined by $\Gamma(U, \mathscr{G}|_X) = \Gamma(f(U), \mathscr{G})$. This defines an additive functor $(-)|_X : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$.

Lemma 26. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an open immersion of ringed spaces. For any sheaf of $\mathcal{O}_Y$-modules $\mathscr{G}$ there is a canonical isomorphism of sheaves of modules natural in $\mathscr{G}$*

$$\theta : f^*\mathscr{G} \longrightarrow \mathscr{G}|_X$$
$$\theta_Q([T, c] \dot{\otimes} b) = b \cdot c|_{f(Q)}$$

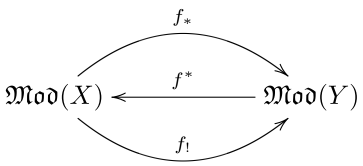
Proof. Follows from Proposition 111 Proposition 107. □

Definition 7. Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an open immersion of ringed spaces with open image U. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules and define a presheaf of $\mathcal{O}_Y$-modules by

$$\mathscr{F}_E(V) = \begin{cases} \mathscr{F}(f^{-1}V) & V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

Denote the sheafification $\mathbf{a}\mathscr{F}_E$ by $f_!\mathscr{F}$, which is a sheaf of $\mathcal{O}_Y$-modules. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of modules then the morphism ϕ_E of (SGR,Definition 16) is a morphism of presheaves of $\mathcal{O}_Y$-modules, so we have a morphism of sheaves of modules $f_!\phi = \mathbf{a}\phi_E : f_!\mathscr{F} \longrightarrow f_!\mathscr{F}'$. This defines an exact additive functor $f_!(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$.

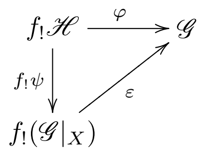
Proposition 27. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an open immersion of ringed spaces. Then we have a triple of adjoints*



$$f_! \dashv f^* \dashv f_*$$

The functors $f_!$ and f^* are exact.

Proof. We need only show that $f_!$ is left adjoint to f^*, and by Lemma 26 it is enough to show that $f_!$ is left adjoint to $(-)|_X$. Let $\mathscr{G}$ be a sheaf of $\mathcal{O}_Y$-modules. There is an obvious morphism of presheaves of $\mathcal{O}_Y$-modules $(\mathscr{G}|_X)_E \longrightarrow \mathscr{G}$ which is the identity for $V \subseteq U$ and zero otherwise. This induces a morphism of sheaves of $\mathcal{O}_Y$-modules $\varepsilon : f_!(\mathscr{G}|_X) \longrightarrow \mathscr{G}$, which is natural in $\mathscr{G}$. If $\mathscr{H}$ is a sheaf of $\mathcal{O}_X$-modules and $\varphi : f_!\mathscr{H} \longrightarrow \mathscr{G}$ a morphism of sheaves of $\mathcal{O}_Y$-modules, we define a morphism of $\psi : \mathscr{H} \longrightarrow \mathscr{G}|_X$ by $\psi_V(s) = \varphi_{f(V)}(\dot{s})$. This is the unique morphism of sheaves of $\mathcal{O}_X$-modules making the following diagram commute



which shows that $f_!$ is left adjoint to $(-)|_X$ with counit ε. □

Remark 9. Let $(X, \mathcal{O}_X)$ be a ringed space and $U \subseteq X$ an open subset with inclusion $j : U \longrightarrow X$. There is a canonical isomorphism of sheaves of modules $\eta : \mathscr{H} \longrightarrow (j_!\mathscr{H})|_U$ for any sheaf of modules $\mathscr{H}$ on U. It gives rise to an isomorphism of $\mathcal{O}_{X,x}$-modules $\mathscr{H}_x \longrightarrow (j_!\mathscr{H})_x$ for any $x \in U$. Clearly $(j_!\mathscr{H})_x = 0$ for any $x \notin U$.

Corollary 28. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an open immersion of ringed spaces. Then the functor $(-)|_X : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ preserves injectives.*

Proof. This follows from the fact that $(-)|_X$ has an exact left adjoint (AC,Proposition 25). □

There is an intuitive description of the sections of the extension by zero as those sections of the original sheaf which "fade out" in an appropriate way, and can therefore be extended by zero to a larger open set.

Lemma 29. *Let $(X, \mathcal{O}_X)$ be a ringed space and U an open subset with inclusion $j : U \longrightarrow X$. For any sheaf of modules $\mathscr{F}$ on U there is a canonical monomorphism of sheaves of modules natural in $\mathscr{F}$*

$$\psi : j_!\mathscr{F} \longrightarrow j_*\mathscr{F}$$

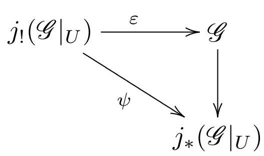
whose image is the submodule $\mathscr{G}$ defined by

$$\Gamma(V, \mathscr{G}) = \{s \in \Gamma(V \cap U, \mathscr{F}) \,|\, \textit{for every } x \in V \setminus U \textit{ there is an open set } x \in W \subseteq V \textit{ such that } s|_{W \cap U} = 0\}$$

Proof. Let $\mathscr{F}_E$ be the presheaf of modules on X defined by $\Gamma(V, \mathscr{F}_E) = \Gamma(V, \mathscr{F})$ for $V \subseteq U$ and zero otherwise. This is a subpresheaf of $j_*\mathscr{F}$ and the induced morphism of sheaves $\psi : j_!\mathscr{F} \longrightarrow j_*\mathscr{F}$ has an image $\mathscr{G}$ defined by the condition of Lemma 2. Examining this condition one checks that $\Gamma(V, \mathscr{G}) \subseteq \Gamma(V, j_*\mathscr{F})$ consists of the sections satisfying the given condition. □

Remark 10. With the notation of Lemma 29 let $\mathscr{G}$ be a sheaf of modules on X and set $\mathscr{F} = \mathscr{G}|_U$. Let $V \subseteq X$ be open and $s \in \Gamma(V, j_!(\mathscr{G}|_U))$ a section identified with a section of $\Gamma(V \cap U, \mathscr{G})$ with the "fading out" property. For every $x \in V \setminus U$ choose an open set $x \in W_x \subseteq V$ with $s|_{W_x \cap U} = 0$. Then these sets together with $V \cap U$ cover V, and s together with all the zero sections amalgamates to give a section $t \in \Gamma(V, \mathscr{G})$. This is the extension of s by zero.

Of course the counit $\varepsilon : j_!(\mathscr{G}|_U) \longrightarrow \mathscr{G}$ already knows all about this: it is precisely the map which extends every section by zero. In other words, $\varepsilon_V(s) = t$. This is the observation that the following diagram commutes



since ε_x is zero for $x \notin U$, so the image under ε_V of any section must have zero germs at every point of $x \in V \setminus U$.

Definition 8. Let $(X, \mathcal{O}_X)$ be a ringed space, $U \subseteq X$ an open subset and $\mathscr{G}$ a sheaf of modules on X. Let $V \subseteq X$ be open and suppose $s \in \Gamma(V \cap U, \mathscr{G})$ is a section with the property that for every $x \in V \setminus U$ there is an open set $x \in W \subseteq V$ such that $s|_{W \cap U} = 0$. Then there is a unique section $s|^V \in \Gamma(V, \mathscr{G})$ with the property that $(s|^V)|_{V \cap U} = s$ and $germ_x(s|^V) = 0$ for all $x \in V \setminus U$. We call this the *extension by zero* of s. Clearly $\varepsilon_V(s) = s|^V$.

1.5 Generators

Throughout this section $(X, \mathcal{O}_X)$ is a fixed ringed space. For any open subset $U \subseteq X$ with inclusion $i : U \longrightarrow X$ we have the functor $i_!(-) : \mathfrak{Mod}(U) \longrightarrow \mathfrak{Mod}(X)$ of Definition 7. We denote by $\mathcal{O}_U$ the sheaf of modules $i_!(\mathcal{O}_X|_U)$. For open sets $U \subseteq V$ there is a canonical monomorphism of sheaves of modules $\rho_{U,V} : \mathcal{O}_U \longrightarrow \mathcal{O}_V$. Clearly $\rho_{U,U} = 1$ and $\rho_{V,T}\rho_{U,V} = \rho_{U,T}$. We have $\mathcal{O}_\emptyset \cong 0$ and the module $\mathcal{O}_U$ for $U = X$ is isomorphic to the structure sheaf $\mathcal{O}_X$. The adjunction $i_! \dashv (-)|_U$ means that we can compute explicitly morphism sets of the form $Hom(\mathcal{O}_U, \mathscr{F})$.

Proposition 30. *For any open subset $U \subseteq X$ and $\mathcal{O}_X$-module $\mathscr{F}$ there is a canonical isomorphism of abelian groups*

$$Hom_{\mathcal{O}_X}(\mathcal{O}_U, \mathscr{F}) \longrightarrow \Gamma(U, \mathscr{F})$$

which is natural in U and $\mathscr{F}$ in the sense that for an inclusion $U \subseteq V$ and a morphism $\mathscr{F} \longrightarrow \mathscr{G}$ the following two diagrams commute

$$\begin{array}{ccc} Hom(\mathcal{O}_V, \mathscr{F}) & \longrightarrow & \Gamma(V, \mathscr{F}) \\ \downarrow & & \downarrow \\ Hom(\mathcal{O}_U, \mathscr{F}) & \longrightarrow & \Gamma(U, \mathscr{F}) \end{array} \qquad \begin{array}{ccc} Hom(\mathcal{O}_U, \mathscr{F}) & \longrightarrow & \Gamma(U, \mathscr{F}) \\ \downarrow & & \downarrow \\ Hom(\mathcal{O}_U, \mathscr{G}) & \longrightarrow & \Gamma(U, \mathscr{G}) \end{array}$$

Proof. Given $U \subseteq X$ let F_U be the presheaf sheafifying to give $\mathcal{O}_U$. For $s \in \mathscr{F}(U)$ we define $F_U \longrightarrow \mathscr{F}$ by sending $1 \in F_U(V)$ to $s|_V \in \mathscr{F}(V)$. Conversely a morphism $\phi : F_U \longrightarrow \mathscr{F}$ is mapped to $\phi_U(1)$. This gives an isomorphism of abelian groups $Hom(F_U, \mathscr{F}) \cong \mathscr{F}(U)$ which is natural in U and $\mathscr{F}$. Using the adjunction between inclusion and sheafification, it is straightforward to finish the proof. We note that the isomorphism $Hom(\mathcal{O}_U, \mathscr{F}) \longrightarrow \mathscr{F}(U)$ sends ϕ to $\phi_U(1)$ where 1 is the section mapping every $x \in U$ to $1 \in \mathcal{O}_{X,x}$. □

Remark 11. With the notation of Proposition 30 let $\mathcal{O}_U \longrightarrow \mathscr{F}$ correspond to a section $m \in \Gamma(U, \mathscr{F})$. Then it is easy to check that the composite $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x} \longrightarrow \mathscr{F}_x$ sends the identity to $germ_x m$.

Corollary 31. *Let $\{U_i\}_{i \in I}$ be a basis of open sets for X. Then the sheaves $\{\mathcal{O}_{U_i} \,|\, i \in I\}$ form a generating family for $\mathfrak{Mod}(X)$.*

Proof. The previous result shows that if $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules and $s \in \mathscr{F}(U)$ corresponds to $\alpha : \mathcal{O}_U \longrightarrow \mathscr{F}$, then the composite $\phi\alpha$ corresponds to $\phi_U(s) \in \mathscr{G}(U)$. So it is clear that if ϕ is nonzero, there is $i \in I$ and a morphism $\mathcal{O}_{U_i} \longrightarrow \mathscr{F}$ such that $\mathcal{O}_U \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}$ is nonzero. □

Lemma 32. *For any open set $U \subseteq X$ we have $Supp(\mathcal{O}_U) \subseteq U$ with equality if $(X, \mathcal{O}_X)$ is a locally ringed space.*

Proof. This is immediate from Remark 9. □

Lemma 33. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $V \subseteq Y$ an open subset. There is a canonical isomorphism of sheaves of modules natural in V*

$$\alpha : f^*\mathcal{O}_V \longrightarrow \mathcal{O}_{f^{-1}V}$$

Proof. The identity $1 \in \Gamma(V, f_*\mathcal{O}_{f^{-1}V})$ corresponds to a morphism $\mathcal{O}_V \longrightarrow f_*\mathcal{O}_{f^{-1}V}$ and by adjointness to a morphism $f^*\mathcal{O}_V \longrightarrow \mathcal{O}_{f^{-1}V}$. To check this is an isomorphism it suffices to check on stalks, which follows immediately from Proposition 20. By naturality in V we mean that for open sets $W \subseteq V$ of Y the following diagram commutes

$$\begin{array}{ccc} f^*\mathcal{O}_W & \longrightarrow & \mathcal{O}_{f^{-1}W} \\ \downarrow & & \downarrow \\ f^*\mathcal{O}_V & \longrightarrow & \mathcal{O}_{f^{-1}V} \end{array}$$

which is easily checked. □

Next we introduce the analogue of the sheaf $\mathcal{O}_U$ for *closed* subsets. Let $Z \subseteq X$ be a closed subset with open complement U and let $j : Z \longrightarrow X$ be the inclusion. Then $\mathcal{O}_Z = j_*(j^{-1}\mathcal{O}_X)$ is a sheaf of rings on X and there is a canonical morphism of sheaves of rings $\omega : \mathcal{O}_X \longrightarrow \mathcal{O}_Z$ which makes the latter sheaf into a sheaf of $\mathcal{O}_X$-modules. From (SGR,Lemma 26) we deduce a canonical short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0 \tag{2}$$

and in particular whenever $Z \subseteq Q$ are closed subsets of X we have an epimorphism of $\mathcal{O}_X$-modules $\mathcal{O}_Q \longrightarrow \mathcal{O}_Z$ compatible with the projections from $\mathcal{O}_X$. By passing to stalks one checks that $\mathcal{O}_{Z,x} = 0$ for $z \notin Z$, and $\omega_x : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Z,x}$ is an isomorphism for $x \in Z$.

Proposition 34. *For any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ there is a canonical isomorphism of abelian groups $Hom_{\mathcal{O}_X}(\mathcal{O}_Z, \mathscr{F}) \cong \Gamma_Z(X, \mathscr{F})$ natural in $\mathscr{F}$ and Z.*

Proof. Recall that $\Gamma_Z(X, \mathscr{F})$ denotes the submodule consisting of sections $s \in \Gamma(X, \mathscr{F})$ satisfying $s|_U = 0$, where $U = X \setminus Z$. Given a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_Z \longrightarrow \mathscr{F}$ we can compose with the projection $\mathcal{O}_X \longrightarrow \mathcal{O}_Z$ to obtain a morphism $\mathcal{O}_X \longrightarrow \mathscr{F}$ which corresponds to an element $s \in \Gamma(X, \mathscr{F})$. We claim that this is the desired isomorphism. To see this, apply $Hom_{\mathcal{O}_X}(-, \mathscr{F})$ to the exact sequence (2) to obtain an exact sequence

$$0 \longrightarrow Hom_{\mathcal{O}_X}(\mathcal{O}_Z, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(U, \mathscr{F})$$

as required. Naturality in $\mathscr{F}$ is obvious. By naturality in Z we just mean that given closed sets $Z \subseteq Q$ and morphism $\mathcal{O}_Z \longrightarrow \mathscr{F}$ corresponding to $s \in \Gamma_Z(X, \mathscr{F}) \subseteq \Gamma_Q(X, \mathscr{F})$, the composite $\mathcal{O}_Q \longrightarrow \mathcal{O}_Z \longrightarrow \mathscr{F}$ also corresponds to s. ☐

Lemma 35. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $Z \subseteq Y$ a closed subset. There is a canonical isomorphism of sheaves of modules natural in Z*

$$\beta : f^*\mathcal{O}_Z \longrightarrow \mathcal{O}_{f^{-1}Z}$$

Proof. The global section $1 \in \Gamma(X, \mathcal{O}_{f^{-1}Z})$ is also a global section of $f_*\mathcal{O}_{f^{-1}Z}$ and it corresponds by the isomorphism of Proposition 34 to a morphism of sheaves of modules $\mathcal{O}_Z \longrightarrow f_*\mathcal{O}_{f^{-1}Z}$, and therefore by the adjunction to a morphism $\beta : f^*\mathcal{O}_Z \longrightarrow \mathcal{O}_{f^{-1}Z}$. One checks this is an isomorphism by passing to stalks. By naturality in Z we mean that for closed sets $Z \subseteq Q$ of Y the following diagram commutes

$$\begin{array}{ccc} f^*\mathcal{O}_Q & \longrightarrow & \mathcal{O}_{f^{-1}Q} \\ \downarrow & & \downarrow \\ f^*\mathcal{O}_Z & \longrightarrow & \mathcal{O}_{f^{-1}Z} \end{array}$$

which is easily checked. ☐

1.6 Cogenerators

Throughout this section $(X, \mathcal{O}_X)$ is a fixed ringed space. For each $x \in X$ let Λ_x be an injective cogenerator of $\mathcal{O}_{X,x}$**Mod** and let $\lambda_x = Sky_x(\Lambda_x)$ be the skyscraper sheaf. By adjointness of $(-)_x$ and $Sky_x(-)$ for any sheaf of modules $\mathscr{F}$ we have a canonical isomorphism of abelian groups natural in both variables

$$Hom_{\mathcal{O}_X}(\mathscr{F}, \lambda_x) \longrightarrow Hom_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \Lambda_x)$$

By Lemma 13 the sheaves λ_x are injective, and we claim they form a family of injective cogenerators for $\mathfrak{Mod}(X)$. In particular $\mathscr{F}$ is zero if and only if $Hom_{\mathcal{O}_X}(\mathscr{F}, \lambda_x) = 0$ for every $x \in X$.

Lemma 36. *The sheaves $\{\lambda_x \,|\, x \in X\}$ form a cogenerating family for $\mathfrak{Mod}(X)$.*

Proof. Suppose we are given a nonzero morphism of sheaves of modules $\phi : \mathscr{F} \longrightarrow \mathscr{G}$. Then ϕ_x is nonzero for some $x \in X$ and therefore admits a morphism $a : \mathscr{G}_x \longrightarrow \Lambda_x$ with $a\phi_x \neq 0$. By adjointness this corresponds to a morphism $\alpha : \mathscr{G} \longrightarrow \lambda_x$ with $\alpha\phi \neq 0$, completing the proof. ☐

1.7 Open Subsets

Proposition 37. *Let $(X, \mathcal{O}_X)$ be a ringed space and $U \subseteq X$ an open subset with inclusion $i : (U, \mathcal{O}_X|_U) \longrightarrow (X, \mathcal{O}_X)$. Then $-|_U \cong i^*$ and so we have a diagram of adjoints*

$$\mathfrak{Mod}(U) \underset{-|_U}{\overset{i_*}{\rightleftarrows}} \mathfrak{Mod}(X) \quad (\text{with } i^* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)) \qquad -|_U \cong i^* \dashv i_*$$

The functor $-|_U$ is exact and preserves all limits and colimits. The functor i_* is fully faithful and injective on objects.

Proof. We know from our Section 2.5 notes that i^* is left adjoint to i_* and $i^* \cong -|_U$, so it follows that $-|_U$ is also left adjoint to i_*. This shows that $-|_U$ preserves epimorphisms and all colimits. It is not hard to check that for a morphism $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ of sheaves of modules on X, $(Ker\psi)|_V = Ker\psi|_V$ for any open set V, so $-|_U$ also preserves kernels and is therefore exact. To prove $-|_U$ preserves all limits, we need only show it preserves products. But this is trivial since products in $\mathfrak{Mod}(X)$ are taken pointwise. The claims about i_* are easily checked. ☐

Note that the unit of the adjunction $-|_U \dashv i_*$ is the map $\mathscr{F} \longrightarrow i_*(\mathscr{F}|_U)$ given by restriction $\mathscr{F}(V) \longrightarrow \mathscr{F}(U \cap V)$ and the counit $(i_*\mathscr{G})|_U \longrightarrow \mathscr{G}$ is the identity. It follows from Proposition 37 that i_* gives an isomorphism of $\mathfrak{Mod}(U)$ with a full abelian subcategory of $\mathfrak{Mod}(X)$, so we can identify $\mathfrak{Mod}(U)$ with this subcategory. It also follows from Proposition 37 that $\mathfrak{Mod}(U)$ is a Giraud subcategory of $\mathfrak{Mod}(X)$. The reflection of a sheaf of modules $\mathscr{F}$ is the sheaf $i_*(\mathscr{F}|_U)$.

Lemma 38. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\{U_i\}_{i \in I}$ a nonempty open cover. Suppose we have a sequence of sheaves of modules on X*

$$\mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \tag{3}$$

This sequence is exact if and only if $\mathscr{F}'|_{U_i} \longrightarrow \mathscr{F}|_{U_i} \longrightarrow \mathscr{F}''|_{U_i}$ is an exact sequence of sheaves of modules on U_i for every $i \in I$.

Proof. The condition is clearly necessary, so assume the sequences on the U_i are all exact. We know the sequence (3) is exact iff. it is exact as a sequence of sheaves of abelian groups, which by (H, Ex. 1.2) is iff. the sequence of abelian groups $\mathscr{F}'_x \longrightarrow \mathscr{F}_x \longrightarrow \mathscr{F}''_x$ is exact for every $x \in X$. But if $x \in U_i$ we have a commutative diagram of abelian groups

$$\begin{array}{ccccc} \mathscr{F}'_x & \longrightarrow & \mathscr{F}_x & \longrightarrow & \mathscr{F}''_x \\ \Downarrow & & \Downarrow & & \Downarrow \\ (\mathscr{F}'|_{U_i})_x & \longrightarrow & (\mathscr{F}|_{U_i})_x & \longrightarrow & (\mathscr{F}''|_{U_i})_x \end{array}$$

By assumption the bottom row is exact, hence so is the top row, which completes the proof. ☐

Proposition 39. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\{U_\alpha\}_{\alpha \in \Lambda}$ a nonempty open cover. Let D be a diagram of sheaves of modules on X. Then a cone $\{H, \rho_i : H \longrightarrow D_i\}_{i \in D}$ is a limit if and only if $\{H|_{U_\alpha}, \rho_i|_{U_\alpha} : H|_{U_\alpha} \longrightarrow D_i|_{U_\alpha}\}_{i \in D}$ is a limit for the restricted diagram of sheaves of modules on U_α for every $\alpha \in \Lambda$.*

Proof. The categories $\mathfrak{Mod}(X), \mathfrak{Mod}(U_\alpha)$ are all complete and cocomplete, so let $\mu_i : Y \longrightarrow D_i$ be a limit for D in $\mathfrak{Mod}(X)$. There is an induced morphism $\eta : H \longrightarrow Y$ with $\mu_i \eta = \rho_i$, and the morphisms ρ_i are a limit iff. η is an isomorphism. But this is iff. $\eta|_{U_\alpha}$ is an isomorphism for all α, which is iff. the morphisms $\rho_i|_{U_\alpha}$ are a limit for all α. This completes the proof. ☐

Proposition 40. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\{U_\alpha\}_{\alpha \in \Lambda}$ a nonempty open cover. Let D be a diagram of sheaves of modules on X. Then a cocone $\{H, \rho_i : D_i \longrightarrow H\}_{i \in D}$ is a colimit if and only if $\{H|_{U_\alpha}, \rho_i|_{U_\alpha} : D_i|_{U_\alpha} \longrightarrow H|_{U_\alpha}\}_{i \in D}$ is a colimit for the restricted diagram of sheaves of modules on U_α for every $\alpha \in \Lambda$.*

1.8 Tensor products

Definition 9. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}_1, \ldots, \mathscr{F}_n$ sheaves of $\mathcal{O}_X$-modules for $n \geq 1$, and $\mathscr{G}$ a sheaf of abelian groups. A morphism of sheaves of sets $f : \mathscr{F}_1 \times \cdots \times \mathscr{F}_n \longrightarrow \mathscr{G}$ is *multilinear* if for every open set $U \subseteq X$ the function

$$f_U : \mathscr{F}_1(U) \times \cdots \times \mathscr{F}_n(U) \longrightarrow \mathscr{G}(U)$$

is multilinear, in the sense of (TES,Definition 2). If $\mathscr{G}$ is a sheaf of $\mathcal{O}_X$-modules and each f_U is a multilinear *form* then we say that f is a multilinear form. The canonical morphism of sheaves of sets

$$\gamma : \mathscr{F}_1 \times \cdots \times \mathscr{F}_n \longrightarrow \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n$$
$$(a_1, \ldots, a_n) \mapsto a_1 \dot{\otimes} \cdots \dot{\otimes} a_n$$

is clearly a multilinear form. If $n = 1$ then a multilinear map is just a morphism of sheaves of abelian groups $\mathscr{F}_1 \longrightarrow \mathscr{G}$, and a multilinear form is a morphism of sheaves of modules.

Proposition 41. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}_1, \ldots, \mathscr{F}_n, \mathscr{G}$ sheaves of $\mathcal{O}_X$-modules for $n \geq 1$. If $f : \mathscr{F}_1 \times \cdots \times \mathscr{F}_n \longrightarrow \mathscr{G}$ is a multilinear form then there is a unique morphism of sheaves of modules $\theta : \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n \longrightarrow \mathscr{G}$ making the following diagram commute*

$$\begin{array}{ccc} \mathscr{F}_1 \times \cdots \times \mathscr{F}_n & \xrightarrow{\gamma} & \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \theta} & \\ \mathscr{G} & & \end{array}$$

Proof. For $n = 2$ this follows from the definition of the tensor product. For $n > 2$ fix an open set $U \subseteq X$ and $a \in \mathscr{F}_1(U)$. Define a multilinear form

$$f' : \mathscr{F}_2|_U \times \cdots \times \mathscr{F}_n|_U \longrightarrow \mathscr{G}|_U$$
$$f'_V(a_2, \ldots, a_n) = f_V(a|_V, a_2, \ldots, a_n)$$

By the inductive hypothesis this induces a morphism of sheaves of modules $\theta'_a : (\mathscr{F}_2 \otimes \cdots \otimes \mathscr{F}_n)|_U \longrightarrow \mathscr{G}|_U$. Define a multilinear form

$$f'' : \mathscr{F}_1 \times (\mathscr{F}_2 \otimes \cdots \otimes \mathscr{F}_n) \longrightarrow \mathscr{G}$$
$$f''_U(a, t) = (\theta'_a)_U(t)$$

The induced morphism of sheaves of modules $\theta : \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n \longrightarrow \mathscr{G}$ is the one we require. □

Remark 12. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}_1, \ldots, \mathscr{F}_n$ sheaves of $\mathcal{O}_X$ for $n \geq 1$. Define a presheaf of $\mathcal{O}_X$-modules by

$$P(U) = \mathscr{F}_1(U) \otimes \cdots \otimes \mathscr{F}_n(U)$$

using the multiple tensor product of (TES,Definition 3). This sheafifies to give a sheaf $\mathscr{Q}$ of $\mathcal{O}_X$-modules together with a multilinear form $\mathscr{F}_1 \times \cdots \times \mathscr{F}_n \longrightarrow \mathscr{Q}$, which one checks has the universal property described in Proposition 41. We deduce a canonical isomorphism of sheaves of modules

$$\mathscr{Q} \longrightarrow \mathscr{F}_1 \otimes (\mathscr{F}_2 \otimes (\cdots \otimes (\mathscr{F}_{n-1} \otimes \mathscr{F}_n) \cdots) \qquad (4)$$

defined by $a_1 \dot{\otimes} \cdots \dot{\otimes} a_n \mapsto a_1 \dot{\otimes} (a_2 \dot{\otimes} \cdots)$. If we define another presheaf of $\mathcal{O}_X$-modules by

$$P'(U) = \mathscr{F}_1(U) \otimes (\mathscr{F}_2(U) \otimes (\cdots \otimes (\mathscr{F}_{n-1}(U) \otimes \mathscr{F}_n(U)) \cdots)$$

using the usual tensor product repeatedly, then there is by (TES,Lemma 2) a canonical isomorphism of presheaves $P \cong P'$ and therefore a canonical isomorphism of the sheafification of P' with the sheaves of (4). These isomorphisms are all natural in every variable.

Definition 10. Let $\phi : \mathscr{O} \longrightarrow \mathscr{O}'$ be a morphism of sheaves of commutative rings on a topological space X. Restriction of scalars defines an additive functor $F : \mathfrak{Mod}(\mathcal{O}') \longrightarrow \mathfrak{Mod}(\mathcal{O})$. If $\mathscr{F}$ is a sheaf of $\mathscr{O}$-modules then define the following presheaf of $\mathcal{O}'$-modules

$$P(U) = \mathscr{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}'(U)$$

Let $\mathscr{F} \otimes_{\mathcal{O}} \mathcal{O}'$ denote the sheaf of $\mathcal{O}'$-modules associated to P. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of $\mathcal{O}$-modules then define a morphism of presheaves of $\mathcal{O}'$-modules

$$\begin{gathered} \phi' : P \longrightarrow P' \\ \phi'_U : \mathscr{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}'(U) \longrightarrow \mathscr{F}'(U) \otimes_{\mathcal{O}(U)} \mathcal{O}'(U) \\ \phi'_U = \phi_U \otimes 1 \end{gathered}$$

Let $\phi \otimes_{\mathcal{O}} \mathcal{O}' : \mathscr{F} \otimes_{\mathcal{O}} \mathcal{O}' \longrightarrow \mathscr{F}' \otimes_{\mathcal{O}} \mathcal{O}'$ be the morphism of sheaves of $\mathcal{O}'$-modules associated to ϕ'. This defines an additive functor $- \otimes_{\mathcal{O}} \mathcal{O}' : \mathfrak{Mod}(\mathcal{O}) \longrightarrow \mathfrak{Mod}(\mathcal{O}')$.

Proposition 42. *Let $\phi : \mathcal{O} \longrightarrow \mathcal{O}'$ be a morphism of sheaves of commutative rings on a topological space X. Then we have a diagram of adjoints*

$$\mathfrak{Mod}(\mathcal{O}) \underset{F}{\overset{-\otimes_{\mathcal{O}}\mathcal{O}'}{\rightleftarrows}} \mathfrak{Mod}(\mathcal{O}') \qquad - \otimes_{\mathcal{O}} \mathcal{O}' \longrightarrow\!\!\!\!\!\dashv F$$

Proof. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}$-modules and define a morphism of sheaves of $\mathcal{O}$-modules natural in $\mathscr{F}$

$$\begin{gathered} \eta : \mathscr{F} \longrightarrow \mathscr{F} \otimes_{\mathcal{O}} \mathcal{O}' \\ \eta_U(s) = s \dot{\otimes} 1 \end{gathered}$$

So to complete the proof we need only show that the pair $(\mathscr{F} \otimes_{\mathcal{O}} \mathcal{O}', \eta)$ is a reflection of $\mathscr{F}$ along F. If $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ is another morphism of sheaves of $\mathcal{O}$-modules, where $\mathscr{G}$ is a sheaf of $\mathcal{O}'$-modules, then for each open U define a $\mathcal{O}_X(U)$-bilinear map

$$\begin{gathered} \varepsilon : \mathscr{F}(U) \times \mathcal{O}'(U) \longrightarrow \mathscr{G}(U) \\ (m, r) \mapsto r \cdot \alpha_U(m) \end{gathered}$$

This induces a morphism of sheaves of $\mathcal{O}'$-modules $\tau : \mathscr{F} \otimes_{\mathcal{O}} \mathcal{O}' \longrightarrow \mathscr{G}$ which is unique such that $\tau\eta = \alpha$, as required. □

1.9 Ideals

Definition 11. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of ideals* on X is a submodule $\mathscr{J}$ of $\mathcal{O}_X$. If $\mathscr{J}$ is a sheaf of ideals on X and $\mathscr{F}$ is a sheaf of modules on X, then $\mathscr{J}\mathscr{F}$ denotes the submodule of $\mathscr{F}$ given by the image of the following morphism

$$\mathscr{J} \otimes \mathscr{F} \longrightarrow \mathcal{O}_X \otimes \mathscr{F} \cong \mathscr{F}$$

It follows from the next result and Lemma 2 that $\mathscr{J}\mathscr{F}$ is the submodule of $\mathscr{F}$ given by sheafifying the presheaf $U \mapsto \mathscr{J}(U)\mathscr{F}(U)$. So $\mathscr{J}\mathscr{F}$ is the smallest submodule of $\mathscr{F}$ containing $\mathscr{J}(U)\mathscr{F}(U)$ for every open U. In particular if $\mathscr{J}_1, \mathscr{J}_2$ are two sheaves of ideals, then we have the product sheaf of ideals $\mathscr{J}_1\mathscr{J}_2$. If $\mathscr{J}$ is a sheaf of ideals then $\mathscr{J}^n$ denotes the n-fold product for $n \geq 1$. It is clear that if $U \subseteq X$ is open then $(\mathscr{J}\mathscr{F})|_U = \mathscr{J}|_U\mathscr{F}|_U$.

Fix a sheaf of ideals $\mathscr{J}$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules then there are induced morphisms of sheaves of modules $\mathscr{J}\phi : \mathscr{J}\mathscr{F} \longrightarrow \mathscr{J}\mathscr{G}$ and $\mathscr{F}/\mathscr{J}\mathscr{F} \longrightarrow \mathscr{G}/\mathscr{J}\mathscr{G}$ unique making the following diagram commute with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{J}\mathscr{F} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}/\mathscr{J}\mathscr{F} & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \mathscr{J}\phi} & & \downarrow {\scriptstyle \phi} & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{J}\mathscr{G} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{G}/\mathscr{J}\mathscr{G} & \longrightarrow & 0 \end{array}$$

In this way we define additive functors $\mathscr{J}- : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ and $(-)/\mathscr{J}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$. Note that if $\mathscr{J}, \mathscr{K}, \mathscr{L}$ are sheaves of ideals then $\mathscr{J}\mathscr{K} \subseteq \mathscr{L}$ if and only if for every open $U \subseteq X$ we have $\mathscr{J}(U)\mathscr{K}(U) \subseteq \mathscr{L}(U)$. This follows from the fact that $\mathscr{J}\mathscr{K}$ is the sheafification of $U \mapsto \mathscr{J}(U)\mathscr{K}(U)$ and Lemma 2.

Lemma 43. *Let $\mathscr{J}$ be a sheaf of ideals on X, $\mathscr{F}$ a sheaf of modules. For open $U \subseteq X$ and $s \in \mathscr{F}(U)$ we have $s \in (\mathscr{J}\mathscr{F})(U)$ if and only if for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $s|_V = s_1 + \cdots + s_n$ where each s_i is of the form*

$$s_i = r \cdot m$$

for some $r \in \mathscr{J}(V)$ and $m \in \mathscr{F}(V)$.

Proof. This follows immediately from Lemma 1. □

Lemma 44. *Let $\mathscr{J}$ be a sheaf of ideals on X and let $n \geq 1$. For open $U \subseteq X$ and $s \in \mathscr{F}(U)$ we have $s \in \mathscr{J}^n(U)$ if and only if for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $s|_V$ belongs to the ideal $\mathscr{J}(V)^n$ of $\mathcal{O}_X(V)$.*

Lemma 45. *Let $\mathscr{F}$ be a sheaf of modules on X. Then we have the following properties of ideal product*

(i) *If $\mathscr{J}, \mathscr{K}$ are sheaves of ideals then $\mathscr{J}\mathscr{K} = \mathscr{K}\mathscr{J} \subseteq \mathscr{J} \cap \mathscr{K}$.*

(ii) *If $\mathscr{J}_1, \mathscr{J}_2$ are sheaves of ideals then $\mathscr{J}_1(\mathscr{J}_2\mathscr{F}) = (\mathscr{J}_1\mathscr{J}_2)\mathscr{F}$. In particular the product of sheaves of ideals is associative.*

(iii) *If $\mathscr{J}$ is a sheaf of ideals then $(\mathscr{J}\mathscr{F})_x = \mathscr{J}_x\mathscr{F}_x$.*

(iv) *$\mathcal{O}_X\mathscr{F} = \mathscr{F}$.*

(v) *If $\mathscr{J}, \mathscr{K}, \mathscr{L}$ are sheaves of ideals with $\mathscr{J} \subseteq \mathscr{K}$ then $\mathscr{J}\mathscr{L} \subseteq \mathscr{K}\mathscr{L}$. In particular if $\mathscr{J} \subseteq \mathscr{K}$ then $\mathscr{J}^n \subseteq \mathscr{K}^n$ for all $n \geq 1$.*

Proof. (i) and (ii) follow from Lemma 43. For (iii) use the fact that the stalk functor $\mathfrak{Mod}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Mod}$ is exact to see that $(\mathscr{J}\mathscr{F})_x$ is the image of the following morphism

$$\mathscr{J}_x \otimes \mathscr{F}_x \longrightarrow \mathcal{O}_{X,x} \otimes \mathscr{F}_x \cong \mathscr{F}_x$$

which is clearly $\mathscr{J}_x\mathscr{F}_x$. (iv) and (v) are trivial. □

Definition 12. Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and let $\mathscr{J}$ be a sheaf of ideals on Y. The morphism of sheaves of modules $\mathscr{J} \longrightarrow \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ induces a morphism of sheaves of modules $f^*\mathscr{J} \longrightarrow \mathcal{O}_X$. Let $f^{-1}\mathscr{J} \cdot \mathcal{O}_X$ denote the sheaf of ideals on X given by the image of this morphism. We use the notation $\mathscr{J} \cdot \mathcal{O}_X$ if no confusion seems likely to result, and call $\mathscr{J} \cdot \mathcal{O}_X$ the *inverse image ideal sheaf*.

Lemma 46. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $\mathscr{J}$ a sheaf of ideals on Y, and let $U \subseteq X$ be open. If $s \in \mathcal{O}_X(U)$ then $s \in (\mathscr{J} \cdot \mathcal{O}_X)(U)$ if and only if for every point $x \in U$ there are open neighborhoods $x \in V \subseteq U$ and $f(V) \subseteq W$ such that $s|_V = s_1 + \cdots + s_n$ where each s_i is of the form*

$$s_i = r \cdot f^\#_W(t)|_V$$

for some $r \in \mathcal{O}_X(V)$ and $t \in \mathscr{J}(W)$.

Proof. Let $\psi : f^* \mathscr{J} \longrightarrow \mathcal{O}_X$ be the morphism of sheaves of modules in the Definition. For open $V \subseteq X, W \supseteq f(V)$ and $r \in \mathcal{O}_X(V), t \in \mathscr{J}(W)$ we have $\psi_V([W,t] \dot{\otimes} r) = r \cdot f^\#_W(t)|_V$ by Theorem 19. Suppose $s \in (\mathscr{J} \cdot \mathcal{O}_X)(U)$ and let $x \in U$ be given. Then there is an open neighborhood $x \in V \subseteq U$ such that $s|_V = \psi_V(m)$ for some $m \in (f^*\mathscr{J})(V)$ (by (SGR,Lemma 9)). By making V sufficiently small we can assume m is of the form $m = m_1 + \cdots + m_n$ with each m_i of the form $[W,t] \dot{\otimes} r$ for $W \supseteq f(V), t \in \mathscr{J}(W), r \in \mathcal{O}_X(V)$. We can assume W is constant for all i, and therefore $s|_V = \psi_V(m)$ has the desired form. The converse is obvious. □

Corollary 47. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $\mathscr{J}$ a sheaf of ideals on Y. Then $\mathscr{J} \cdot \mathcal{O}_X$ is the smallest submodule of $\mathcal{O}_X$ containing all the sections $f^\#_W(t)$ for open $W \subseteq Y$ and $t \in \mathscr{J}(W)$.*

Proof. This follows immediately from Lemma 46. □

Lemma 48. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $V \subseteq Y$ open and set $U = f^{-1}V$. If $\mathscr{J}$ is a sheaf of ideals on Y then $(\mathscr{J} \cdot \mathcal{O}_X)|_U = \mathscr{J}|_V \cdot \mathcal{O}_X|_U$ as submodules of $\mathcal{O}_X|_U$.*

Proof. To be a little more precise, let $g : U \longrightarrow V$ be induced by f. We claim that

$$(f^{-1}\mathscr{J} \cdot \mathcal{O}_X)|_U = g^{-1}(\mathscr{J}|_V) \cdot \mathcal{O}_X|_U$$

This follows immediately from the definition and Proposition 112. □

Lemma 49. *If $h : X \longrightarrow Y$ is an isomorphism of ringed spaces with inverse f, then there is a bijection between the sets of submodules of $\mathcal{O}_Y$ and $\mathcal{O}_X$*

$$\begin{gathered} Sub(\mathcal{O}_Y) \longrightarrow Sub(\mathcal{O}_X) \\ \mathscr{J} \mapsto \mathscr{J} \cdot \mathcal{O}_X \end{gathered}$$

For open $U \subseteq X$ and $s \in \mathcal{O}_X(U)$ we have $s \in (\mathscr{J} \cdot \mathcal{O}_X)(U)$ if and only if $f^\#_U(s) \in \mathscr{J}(h(U))$. In particular we have $(\mathscr{J} \cdot \mathcal{O}_X) \cdot \mathcal{O}_Y = \mathscr{J}$.

Proof. Let $\mathscr{J}$ be a sheaf of ideals on Y. Then $\mathscr{J} \cdot \mathcal{O}_X$ is the image of the morphism $h^*\mathscr{J} \longrightarrow \mathcal{O}_X$, which is the top row in the following commutative diagram

$$\begin{array}{ccccc} h^*\mathscr{J} & \longrightarrow & h^*\mathcal{O}_Y & \xrightarrow{\varepsilon} & \mathcal{O}_X \\ \Uparrow & & \Uparrow & \nearrow\!\!\!\nearrow_{(f^\#)^{-1}} & \\ f_*\mathscr{J} & \longrightarrow & f_*\mathcal{O}_Y & & \end{array}$$

Where ε is the adjoint partner of $h^\# : \mathcal{O}_Y \longrightarrow h_*\mathcal{O}_X$. To check commutativity, use Proposition 107. Therefore $\mathscr{J} \cdot \mathcal{O}_X$ is the image of $f_*\mathscr{J} \longrightarrow f_*\mathcal{O}_Y \cong \mathcal{O}_X$, which shows that $s \in (\mathscr{J} \cdot \mathcal{O}_X)(U)$ iff. $f^\#_U(s) \in \mathscr{J}(h(U))$. Then it is clear that $(\mathscr{J} \cdot \mathcal{O}_X) \cdot \mathcal{O}_Y = \mathscr{J}$ and therefore we have a bijection, whose inverse is the map $Sub(\mathcal{O}_X) \longrightarrow Sub(\mathcal{O}_Y)$ induced by the isomorphism $f : Y \longrightarrow X$. □

Lemma 50. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of ringed spaces and $\mathscr{J}$ a sheaf of ideals on Z. Then $\mathscr{J} \cdot \mathcal{O}_X = (\mathscr{J} \cdot \mathcal{O}_Y) \cdot \mathcal{O}_X$.*

Proof. One inclusion is obvious from Corollary 47, and the other follows easily from Lemma 46. □

Lemma 51. *Suppose we have a commutative diagram of ringed spaces*

$$\begin{array}{ccc} X & \xRightarrow{\;h\;} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}$$

If $\mathscr{J}$ is a sheaf of ideals on Y then the canonical bijection between submodules of $\mathcal{O}_X$ and $\mathcal{O}_{X'}$ identifies $\mathscr{J} \cdot \mathcal{O}_X$ and $\mathscr{J} \cdot \mathcal{O}_{X'}$. In particular $h_*(\mathscr{J} \cdot \mathcal{O}_X) \cong \mathscr{J} \cdot \mathcal{O}_{X'}$. Now suppose we have a commutative diagram of ringed spaces

$$\begin{array}{ccc} & X & \\ {}^{g}\swarrow & & \searrow^{f} \\ Y' & \overset{}{\underset{h}{\Longrightarrow}} & Y \end{array}$$

If $\mathscr{J}$ is an ideal sheaf on Y which corresponds to the ideal sheaf $\mathscr{K}$ on Y' then $\mathscr{J} \cdot \mathcal{O}_X = \mathscr{K} \cdot \mathcal{O}_X$.

Proof. Immediate from Lemma 50. □

Proposition 52. *Let $f : X \longrightarrow Y$ be an isomorphism of ringed spaces, $\mathscr{J}$ a sheaf of ideals on X and $\mathscr{F}$ a sheaf of modules on X. Then*

$$f_*(\mathscr{J}\mathscr{F}) = (\mathscr{J} \cdot \mathcal{O}_Y) f_*\mathscr{F}$$
$$\mathscr{J}^n \cdot \mathcal{O}_Y = (\mathscr{J} \cdot \mathcal{O}_Y)^n$$

Proof. The first claim follows immediately from Lemma 49 and Lemma 43, while the second uses Lemma 44. □

Definition 13. Let $\mathscr{F}, \mathscr{G}$ be submodules of a sheaf of $\mathcal{O}_X$-modules $\mathscr{H}$. Then we define a sheaf of ideals $(\mathscr{F} : \mathscr{G})$ on X by

$$(\mathscr{F} : \mathscr{G})(U) = \{ r \in \mathcal{O}_X(U) \,|\, r|_V \mathscr{G}(V) \subseteq \mathscr{F}(V) \text{ for all open } V \subseteq U \}$$

It is clear that $(\mathscr{F} : \mathscr{G}) = \mathcal{O}_X$ if and only if $\mathscr{G} \subseteq \mathscr{F}$, and $(\mathscr{F} : \mathscr{G})\mathscr{G} \subseteq \mathscr{F}$. If $U \subseteq X$ is an open subset then $(\mathscr{F} : \mathscr{G})|_U = (\mathscr{F}|_U : \mathscr{G}|_U)$.

Lemma 53. *Let $f : X \longrightarrow Y$ be an isomorphism of ringed spaces. If $\mathscr{F}, \mathscr{G}$ are $\mathcal{O}_X$-submodules of a sheaf of $\mathcal{O}_X$-modules $\mathscr{H}$ then $(\mathscr{F} : \mathscr{G}) \cdot \mathcal{O}_Y = (f_*\mathscr{F} : f_*\mathscr{G})$.*

Lemma 54. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{J}$ a sheaf of ideals on X. Then there is a canonical isomorphism of sheaves of modules on X natural in $\mathscr{F}$*

$$\rho : \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathscr{J} \longrightarrow \mathscr{F}/\mathscr{J}\mathscr{F}$$
$$a \dot{\otimes} (b \dot{+} \Gamma(U, \mathscr{J})) \mapsto ba \dot{+} \Gamma(U, \mathscr{J}\mathscr{F})$$

Proof. By definition $\mathscr{J}\mathscr{F}$ is the image of the morphism $\mathscr{F} \otimes \mathscr{J} \longrightarrow \mathscr{F} \otimes \mathcal{O}_X \cong \mathscr{F}$. By tensoring $\mathscr{F}$ with the exact sequence $0 \to \mathscr{J} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathscr{J} \longrightarrow 0$ we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathscr{F} \otimes \mathscr{J} & \longrightarrow & \mathscr{F} \otimes \mathcal{O}_X & \longrightarrow & \mathscr{F} \otimes \mathcal{O}_X/\mathscr{J} & \longrightarrow & 0 \\ \downarrow & & \Downarrow & & \vdots & & \\ \mathscr{J}\mathscr{F} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}/\mathscr{J}\mathscr{F} & \longrightarrow & 0 \end{array}$$

where ρ is the unique morphism making the right hand square commute. This is clearly an isomorphism, and naturality in $\mathscr{F}$ is easily checked. □

1.10 Locally Free Sheaves

Definition 14. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is *free* if it is a coproduct of copies of $\mathcal{O}_X$. A sheaf of modules $\mathscr{F}$ is *locally free* if every point $x \in X$ has an open neighborhood U with $\mathscr{F}|_U$ a free $\mathcal{O}_X|_U$-module. We say that $\mathscr{F}$ is *invertible* if every point $x \in X$ has an open neighborhood U with $\mathscr{F}|_U \cong \mathcal{O}_X|_U$. We say that $\mathscr{F}$ is *locally finitely free* if every point $x \in X$ has an open neighborhood U with $\mathscr{F}|_U$ a coproduct of a finite set of copies of $\mathcal{O}_X$.

Remark 13. It is treacherous to talk about the rank of a free module over the zero ring, and it is similarly treacherous for free sheaves of modules. We say a ringed space $(X, \mathcal{O}_X)$ is *nontrivial* if X is nonempty and $\mathcal{O}_{X,x} \neq 0$ for every $x \in X$. A nonempty locally ringed space is nontrivial.

Definition 15. Let $(X, \mathcal{O}_X)$ be a nontrivial ringed space. If $\mathscr{F}$ is a free sheaf of $\mathcal{O}_X$-modules then for $x \in X$ the $\mathcal{O}_{X,x}$-module $\mathscr{F}_x$ is free and the ranks $rank_{\mathcal{O}_{X,x}} \mathscr{F}_x$ are all equal to the same element $n \in \{0, 1, 2, \ldots, \infty\}$, called the *rank* of $\mathscr{F}$ and written $rank_{\mathcal{O}_X} \mathscr{F}$. Clearly if $\mathscr{F}$ can be written as a coproduct of n copies of $\mathcal{O}_X$ then $rank_{\mathcal{O}_X} \mathscr{F} = n$.

If $\mathscr{F}$ is locally free then for $x \in X$ the $\mathcal{O}_{X,x}$-module $\mathscr{F}_x$ is free and we call $rank_{\mathcal{O}_{X,x}} \mathscr{F}_x$ the *rank of* $\mathscr{F}$ *at* x. If U is an open subset with $\mathscr{F}|_U$ free then $rank_{\mathcal{O}_X|_U} \mathscr{F}|_U = rank_{\mathcal{O}_{X,x}} \mathscr{F}_x$ for every $x \in U$. For $n \in \{0, 1, 2, \ldots, \infty\}$ we say that $\mathscr{F}$ is *locally free of rank n* if it is locally free and the rank of $\mathscr{F}$ at every point $x \in X$ is n. Equivalently, every point $x \in X$ has an open neighborhood U with $\mathscr{F}|_U$ free of rank n. A sheaf of modules $\mathscr{F}$ is locally free of rank 0 if and only if $\mathscr{F} = 0$. We say that a sheaf of modules $\mathscr{F}$ is *locally free of finite rank* if it is locally free of rank n for some finite $n \geq 0$.

Remark 14. Note that the concept of an invertible sheaf is defined for all ringed spaces, including those that are empty or have trivial local rings. But in the case where $(X, \mathcal{O}_X)$ is nontrivial, a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is invertible iff. it is locally free of rank 1.

Lemma 55. *Let $(X, \mathcal{O}_X)$ be a nontrivial ringed space and $U \subseteq X$ open. If $\mathscr{F}$ is a free module of rank n then $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module of the same rank. If $\mathscr{F}$ is locally free of rank n then $\mathscr{F}|_U$ is also locally free of the same rank.*

Proof. The morphism of ringed spaces $\mathcal{O}_X|_U \longrightarrow \mathcal{O}_X$ induces a pair of adjoint functors between the module categories, with restriction being left adjoint to direct image. Hence restriction preserves coproducts, and thus preserves free modules and their rank. If $\mathscr{F}$ is locally free of rank n and $x \in U$ let $x \in V \subseteq X$ be such that $\mathscr{F}|_V$ is free of rank n. Then $(\mathscr{F}|_U)|_{U \cap V} = (\mathscr{F}|_V)|_{U \cap V}$ which is free of rank n, completing the proof. □

Lemma 56. *Suppose $(X, \mathcal{O}_X)$ is a nontrivial ringed space and let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules. Then*

(i) If $\mathscr{F}, \mathscr{G}$ are free of finite ranks m, n then $\mathscr{F} \otimes \mathscr{G}$ is free of rank mn.

(ii) If $\mathscr{F}, \mathscr{G}$ are locally free of finite ranks m, n then $\mathscr{F} \otimes \mathscr{G}$ is locally free of rank mn.

(iii) If $\mathscr{F}, \mathscr{G}$ are locally free of finite ranks m, n then $\mathscr{F} \oplus \mathscr{G}$ is locally free of rank $m + n$.

(iv) For any ringed space, the tensor product of invertible sheaves is invertible.

Proof. Tensoring with a $\mathcal{O}_X$-module is an additive functor, hence preserves finite products and coproducts since $\mathcal{O}_X$**Mod** is abelian. So (i) is obvious. For (ii) let $x \in X$ be given and find open neighborhoods U, V of x such that $\mathscr{F}|_U$ is free of rank m and $\mathscr{G}|_V$ is free of rank n. Since restriction preserves coproducts the same can be said of $\mathscr{F}|_{U \cap V}$ and $\mathscr{G}|_{U \cap V}$ respectively, so by (i) the tensor product $\mathscr{F} \otimes \mathscr{G}$ restricts to $\mathscr{F}|_{U \cap V} \otimes \mathscr{G}|_{U \cap V}$ which is a free module of rank mn, as required. (iii) and (iv) are trivial. □

Lemma 57. *Let $(X, \mathcal{O}_X)$ be a ringed space and $f : \mathscr{L} \longrightarrow \mathscr{M}$ an epimorphism of invertible sheaves on X. Then f is an isomorphism.*

Proof. We reduce immediately to proving the following result of algebra: if $f : M \longrightarrow N$ is a surjective morphism of free modules of rank 1 over a commutative ring R, then f is an isomorphism. Let m be a basis for M and n a basis for N, and suppose $f(m) = \alpha \cdot n$ for $\alpha \in R$. Then since f is surjective, there is $k \in R$ with $f(k \cdot m) = n$ and therefore $n = k\alpha \cdot n$ and $1 = k\alpha$. Therefore α is a unit. If $f(l \cdot m) = 0$ then $l\alpha = 0$ and therefore $l = 0$, so f is injective and therefore an isomorphism, as required. □

1.11 Exponential Tensor Products

Throughout this section $(X, \mathcal{O}_X)$ is a ringed space. Let $-^{\otimes 0} : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ be the functor mapping every object to $\mathcal{O}_X$ and every morphism to the identity. Define $-^{\otimes 1} : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ to be the identity functor, and define inductively for $n > 1$

$$\begin{aligned} -^{\otimes n} : \mathfrak{Mod}(X) &\longrightarrow \mathfrak{Mod}(X) \\ \mathscr{F}^{\otimes n} &= \mathscr{F} \otimes \mathscr{F}^{\otimes (n-1)} \\ \phi^{\otimes n} &= \phi \otimes \phi^{\otimes (n-1)} \end{aligned}$$

Where $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of $\mathcal{O}_X$-modules. In other words, $\mathscr{F}^{\otimes n}$ is the tensor product $\mathscr{F} \otimes (\mathscr{F} \otimes \cdots (\mathscr{F} \otimes \mathscr{F}) \ldots)$ with n copies of $\mathscr{F}$ involved. We prove by induction on n that $-^{\otimes n}$ is a functor

$$(1_{\mathscr{F}})^{\otimes n} = 1_{\mathscr{F}} \otimes (1_{\mathscr{F}})^{\otimes (n-1)} = 1_{\mathscr{F}} \otimes 1_{\mathscr{F}^{\otimes (n-1)}} = 1_{\mathscr{F}^{\otimes n}}$$

And

$$\begin{aligned} \phi^{\otimes n} \psi^{\otimes n} &= (\phi \otimes \phi^{\otimes (n-1)})(\psi \otimes \psi^{\otimes (n-1)}) \\ &= (\phi\psi) \otimes (\phi^{\otimes (n-1)} \psi^{\otimes (n-1)}) \\ &= (\phi\psi) \otimes (\phi\psi)^{\otimes (n-1)} \\ &= (\phi\psi)^{\otimes n} \end{aligned}$$

It is not generally true that $-^{\otimes n}$ is additive. Given $U \subseteq X$ and $f \in \mathscr{F}(U)$, we define $f^{\otimes 0} = 1, f^{\otimes 1} = f$ and for $n > 1$, $f^{\otimes n}$ denotes the element $f \dot{\otimes} f^{\otimes (n-1)}$ of $\mathscr{F}^{\otimes n}(U)$. More generally for $n > 1$ and $f_1, \ldots, f_n \in \mathscr{F}(U)$ write $f_1 \dot{\otimes} f_2 \dot{\otimes} \cdots \dot{\otimes} f_n$ for the section $f_1 \dot{\otimes} (f_2 \dot{\otimes} \cdots (f_{n-1} \dot{\otimes} f_n) \cdots)$ of $\mathscr{F}^{\otimes n}(U)$. With this notation, $f^{\otimes n} = f \dot{\otimes} \cdots \dot{\otimes} f$ with n factors. We sometimes write f^n for $f^{\otimes n}$. If $V \subseteq U$ is open then

$$(f_1 \dot{\otimes} \cdots \dot{\otimes} f_n)|_V = (f_1|_V) \dot{\otimes} \cdots \dot{\otimes} (f_n|_V)$$

Lemma 58. *Let $U \subseteq X$ be open, $n > 1$ and $t \in \mathscr{F}^{\otimes n}(U)$. For any $x \in U$ there is an open set V with $x \in V \subseteq U$ such that $t|_V = \sum_{i=1}^m f_{i1} \dot{\otimes} \cdots \dot{\otimes} f_{in}$ for some $f_{ij} \in \mathscr{F}(V)$.*

Lemma 59. *For $n > 0$ there is a canonical morphism of sheaves of sets $\eta^n : \mathscr{F} \longrightarrow \mathscr{F}^{\otimes n}$ which is natural in $\mathscr{F}$, defined by $\eta^n_U(f) = f^{\otimes n}$.*

Proof. The proof is by induction on n. For $n = 1$ the result is trivial since $\eta = 1_{\mathscr{F}}$. So assume $n > 1$. Then

$$\begin{aligned} \eta_U(f)|_V &= f^{\otimes n}|_V = (f \dot{\otimes} f^{\otimes (n-1)})|_V \\ &= f|_V \dot{\otimes} (f^{\otimes (n-1)})|_V = f|_V \dot{\otimes} (f|_V)^{\otimes (n-1)} \\ &= \eta_V(f|_V) \end{aligned}$$

Once again η is not necessarily additive. To see that η is natural, let $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ be given. Then

$$\begin{aligned} \eta_U \phi_U(f) &= (\phi_U(f))^{\otimes n} \\ &= \phi_U(f) \dot{\otimes} \phi_U(f)^{\otimes (n-1)} \\ &= \phi_U(f) \dot{\otimes} \phi_U^{\otimes (n-1)}(f^{\otimes (n-1)}) \\ &= (\phi \otimes \phi^{\otimes (n-1)})_U (f \dot{\otimes} f^{\otimes (n-1)}) \\ &= \phi_U^{\otimes n} \eta_U(f) \end{aligned}$$

So the following diagram commutes

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\phi} & \mathscr{F}' \\ \big\downarrow & & \big\downarrow \\ \mathscr{F}^{\otimes n} & \xrightarrow[\phi^{\otimes n}]{} & \mathscr{F}'^{\otimes n} \end{array}$$

□

Proposition 60. *For any open subset $U \subseteq X$, integer $n \geq 1$ and $\mathcal{O}_X$-module $\mathscr{F}$ there is a canonical isomorphism of modules natural in $\mathscr{F}$*

$$\xi : (\mathscr{F}^{\otimes n})|_U \longrightarrow (\mathscr{F}|_U)^{\otimes n}$$

For open $V \subseteq U$ and $s_1, \dots, s_n \in \mathscr{F}(V)$ we have $s_1 \dot{\otimes} \cdots \dot{\otimes} s_n \mapsto s_1 \dot{\otimes} \cdots \dot{\otimes} s_n$.

Proof. The proof is by induction on n. If $n = 1$ then $-^{\otimes n}$ is the identity functor, so the result is trivial. So assume $n > 1$ and that we have an isomorphism $\Phi_{\mathscr{F}} : (\mathscr{F}^{\otimes(n-1)})|_U \longrightarrow (\mathscr{F}|_U)^{\otimes(n-1)}$ with the necessary property and natural in $\mathscr{F}$. Let $\Psi_{\mathscr{F}}$ be the composite

$$\begin{aligned}(\mathscr{F}^{\otimes n})|_U = (\mathscr{F} \otimes \mathscr{F}^{\otimes(n-1)})|_U &\cong \mathscr{F}|_U \otimes (\mathscr{F}^{\otimes(n-1)})|_U \\ &\cong \mathscr{F}|_U \otimes (\mathscr{F}|_U)^{\otimes(n-1)} = (\mathscr{F}|_U)^{\otimes n}\end{aligned}$$

Naturality of this isomorphism in $\mathscr{F}$ follows from the naturality of the individual isomorphisms. □

Lemma 61. *For $n > 0$ there is a canonical isomorphism of modules $\mathcal{O}_X^{\otimes n} \longrightarrow \mathcal{O}_X$ which maps $f_1 \dot{\otimes} \cdots \dot{\otimes} f_n$ to $f_1 \cdots f_n$. In particular $f^{\otimes n}$ is mapped to f^n.*

Proof. By induction on n. For $n = 1$ take the identity morphism. Let $n > 1$ and let $\theta^{n-1} : \mathcal{O}_X^{\otimes(n-1)} \longrightarrow \mathcal{O}_X$ be the isomorphism constructed in stage $n-1$. Let θ^n be the following composite

$$\mathcal{O}_X^{\otimes n} = \mathcal{O}_X \otimes \mathcal{O}_X^{\otimes(n-1)} \cong \mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$$

It is clear that $f_1 \dot{\otimes} \cdots \dot{\otimes} f_n$ maps to $f_1 \cdots f_n$ □

Lemma 62. *For an $\mathcal{O}_X$-module $\mathscr{F}$ and $n > 0$ there is a canonical isomorphism of modules natural in $\mathscr{F}$*

$$\kappa : \mathscr{F} \otimes (\mathcal{O}_X)^{\otimes n} \longrightarrow \mathscr{F}$$

For $s \in \mathscr{F}(U)$ and $f_1, \dots, f_n \in \mathcal{O}_X(U)$ we have $s \dot{\otimes} (f_1 \dot{\otimes} \cdots \dot{\otimes} f_n) \mapsto (f_1 \cdots f_n) \cdot s$.

Proof. Let κ be the $1 \otimes \theta^n$ followed by the isomorphism $\mathscr{F} \otimes \mathcal{O}_X \longrightarrow \mathscr{F}$. Naturality in $\mathscr{F}$ and the other claims are easily checked. □

Proposition 63. *For an $\mathcal{O}_X$-module $\mathscr{F}$ and $i, j \geq 0$ there is a canonical isomorphism of modules natural in $\mathscr{F}$*

$$\lambda^{i,j} : \mathscr{F}^{\otimes i} \otimes \mathscr{F}^{\otimes j} \cong \mathscr{F}^{\otimes(i+j)}$$

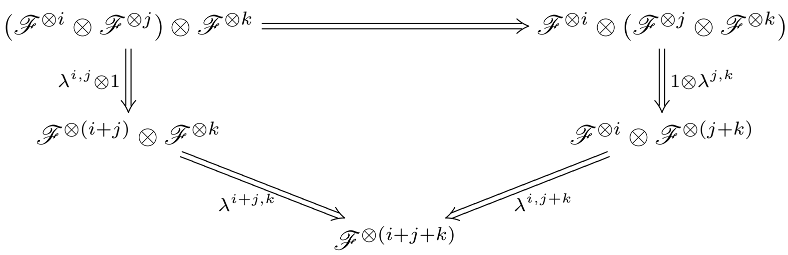
For $s \in \mathscr{F}(U)$ we have $s^{\otimes i} \dot{\otimes} s^{\otimes j} \mapsto s^{\otimes(i+j)}$. For $i, j > 0$ and $s_1, \dots, s_i, s_{i+1}, \dots, s_{i+j} \in \mathscr{F}(U)$ we have $(s_1 \dot{\otimes} \cdots \dot{\otimes} s_i) \dot{\otimes} (s_{i+1} \dot{\otimes} \cdots \dot{\otimes} s_{i+j}) \mapsto s_1 \dot{\otimes} \cdots \dot{\otimes} s_{i+j}$.

Proof. For $i = 0$ or $j = 0$ we use the canonical isomorphisms $\mathscr{F}^{\otimes i} \otimes \mathcal{O}_X \cong \mathscr{F}^{\otimes i}$ and $\mathcal{O}_X \otimes \mathscr{F}^{\otimes j} \cong \mathscr{F}^{\otimes j}$. The other cases we handle by induction on $m = i + j$. If $m = 2$ then $i = j = 1$ and this is obvious. Assume $m > 2$ and the result is true for $m-1$. If $i = 1$ then $j = m-1$ and the result is trivial. Otherwise let $\lambda^{i,j}$ be the composite

$$\begin{aligned} \mathscr{F}^{\otimes i} \otimes \mathscr{F}^{\otimes j} &= (\mathscr{F} \otimes \mathscr{F}^{\otimes (i-1)}) \otimes \mathscr{F}^{\otimes j} \\ &\cong \mathscr{F} \otimes (\mathscr{F}^{\otimes (i-1)} \otimes \mathscr{F}^{\otimes j}) \\ &\cong \mathscr{F} \otimes \mathscr{F}^{\otimes (i+j-1)} \\ &= \mathscr{F}^{\otimes m} \end{aligned}$$

By the inductive hypothesis it is clear that $\lambda^{i,j}$ has the required properties. Naturality in $\mathscr{F}$ is easily checked. □

Proposition 64. *For an $\mathcal{O}_X$-module $\mathscr{F}$ and $i, j, k \geq 0$ the following diagram commutes*



Proof. In the cases where one of i, j, k is zero, this is easily checked. If $i, j, k > 0$ then by Lemma 58 it suffices to check commutativity on an element of the form

$$\left((s_1 \dot\otimes \cdots \dot\otimes s_i) \dot\otimes (t_1 \dot\otimes \cdots \dot\otimes t_j)\right) \dot\otimes (q_1 \dot\otimes \cdots \dot\otimes q_k)$$

which is not difficult. □

Proposition 65. *For an $\mathcal{O}_X$-module $\mathscr{F}$ and $i, j \geq 0$ there is a canonical isomorphism of modules*

$$\mu^{i,j} : (\mathscr{F}^{\otimes i})^{\otimes j} \cong \mathscr{F}^{\otimes (ij)}$$

For $s \in \mathscr{F}(U)$ we have $(s^{\otimes i})^{\otimes j} \mapsto s^{\otimes (ij)}$. For $i, j > 0$ and $s_1, \ldots, s_i \in \mathscr{F}(U)$, $(s_1 \dot\otimes \cdots \dot\otimes s_i)^{\otimes j}$ is the element $s_1 \dot\otimes \cdots \dot\otimes s_i$ repeated j times.

Proof. For $i = 0$ and $j > 0$ we use the isomorphism of Lemma 61. For $j = 0$ there is an equality. We handle the other cases by induction on $m = ij$. If $m = 1$ then $i = j = 1$ and this is obvious. Assume $m > 1$ and that the result is true for all smaller m. If $j = 1$ then $i = m$ and the result is trivial. Otherwise let $\mu^{i,j}$ be the composite

$$\begin{aligned} (\mathscr{F}^{\otimes i})^{\otimes j} &= \mathscr{F}^{\otimes i} \otimes (\mathscr{F}^{\otimes i})^{\otimes (j-1)} \\ &\cong \mathscr{F}^{\otimes i} \otimes \mathscr{F}^{\otimes ij-i} \\ &\cong \mathscr{F}^{\otimes (ij)} \end{aligned}$$

By the inductive hypothesis it is clear that $\mu^{i,j}$ has the required property. □

Proposition 66. *Let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. For open $U \subseteq X$, $s_1, \ldots, s_n \in \mathscr{L}(U)$ $(n > 1)$ and any permutation σ on n letters we have*

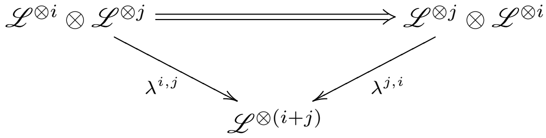
$$s_1 \dot\otimes s_2 \dot\otimes \cdots \dot\otimes s_{n-1} \dot\otimes s_n = s_{\sigma(1)} \dot\otimes s_{\sigma(2)} \dot\otimes \cdots \dot\otimes s_{\sigma(n-1)} \dot\otimes s_{\sigma(n)}$$

Proof. That is, the sections of $\mathscr{L}^{\otimes n}$ are order independent. For $x \in U$ let $x \in V \subseteq U$ be such that $\mathscr{L}|_V \cong \mathcal{O}_X|_V$. Let $\eta \in \mathscr{L}(V)$ be a $\mathcal{O}_X(V)$-basis, and say $r_i \in \mathcal{O}_X(V)$ are such that $s_i|_V = r_i \cdot \eta$ for $1 \leq i \leq n$. Then

$$(s_1 \dot\otimes \cdots \dot\otimes s_n)|_V = (r_1 \cdot \eta) \dot\otimes \cdots \dot\otimes (r_n \cdot \eta) = (r_1 \cdots r_n) \cdot (\eta \dot\otimes \cdots \dot\otimes \eta)$$

Since $\mathcal{O}_X(V)$ is commutative, it is clear that any permutation of the s_i will restrict to the same section on V, which completes the proof. □

Corollary 67. *Let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. Then for $i, j \geq 0$ the following diagram commutes*



Proof. As in Proposition 64 it suffices to check commutativity on sections of the form

$$(s_1 \dot{\otimes} \cdots \dot{\otimes} s_i) \dot{\otimes} (t_1 \dot{\otimes} \cdots \dot{\otimes} t_j)$$

Using Proposition 66 this is not difficult. $\square$

Lemma 68. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{F}$ a sheaf of modules on X. For $d > 0$ there is a canonical morphism of sheaves of modules on Y natural in $\mathscr{F}$*

$$\alpha : f_*(\mathscr{F})^{\otimes d} \longrightarrow f_*(\mathscr{F}^{\otimes d})$$
$$f_1 \dot{\otimes} \cdots \dot{\otimes} f_d \mapsto f_1 \dot{\otimes} \cdots \dot{\otimes} f_d$$

If f is an isomorphism of ringed spaces, then α is also an isomorphism.

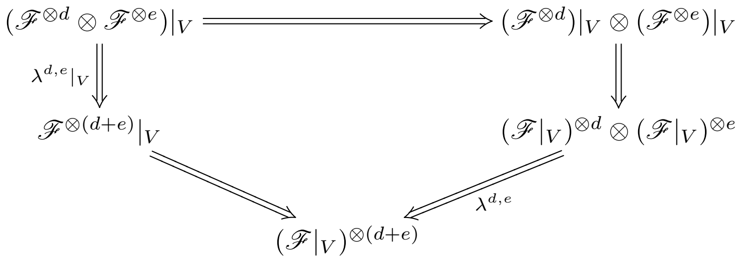
Proof. The case $n = 2$ is done in our Section 2.5 notes and the rest follows by induction. $\square$

Lemma 69. *Let $\mathscr{F}$ be a sheaf of modules on X. Then for $x \in X$ and $n \geq 1$ we have a canonical isomorphism of $\mathcal{O}_{X,x}$-modules $(\mathscr{F}^{\otimes n})_x \longrightarrow (\mathscr{F}_x)^{\otimes n}$ with the property that*

$$germ_x(f_1 \dot{\otimes} \cdots \dot{\otimes} f_n) \mapsto germ_x f_1 \otimes \cdots \otimes germ_x f_n$$

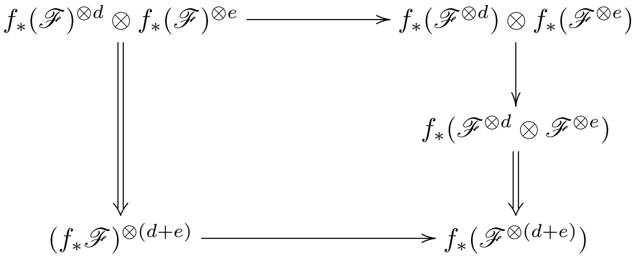
Proof. For $n = 1$ this is trivial, and for $n \geq 2$ it follows by a simple induction. $\square$

Lemma 70. *Let $\mathscr{F}$ be a sheaf of modules on X, $V \subseteq X$ open and $d, e \geq 1$. Then the following diagram commutes*



Proof. We can reduce to checking commutativity on sections of the form $(s_1 \dot{\otimes} \cdots \dot{\otimes} s_d) \dot{\otimes} (t_1 \dot{\otimes} \cdots \dot{\otimes} t_e)$ which is straightforward. $\square$

Lemma 71. *Let $f : X \longrightarrow Y$ be an morphism of ringed spaces and $\mathscr{F}$ a sheaf of modules on X. Then for $d, e \geq 1$ the following diagram commutes*



Proof. By reduction to sections of the form $(s_1 \dot{\otimes} \cdots \dot{\otimes} s_d) \dot{\otimes} (t_1 \dot{\otimes} \cdots \dot{\otimes} t_e)$. $\square$

1.12 Sheaf Hom

Definition 16. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules. We define a sheaf of $\mathcal{O}_X$-modules $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ by

$$\Gamma(U, \mathscr{H}om(\mathscr{F}, \mathscr{G})) = Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

Addition and restriction are defined in the obvious way, and for $r \in \mathcal{O}_X(U)$ and $\phi \in \mathscr{H}om(\mathscr{F}, \mathscr{G})(U)$ we define $(r \cdot \phi)_V(s) = r|_V \cdot \phi_V(s)$. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of $\mathcal{O}_X$-modules then we define a morphism of $\mathcal{O}_X$-modules

$$\mathscr{H}om(\phi, \mathscr{G}) : \mathscr{H}om(\mathscr{F}', \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})$$
$$\mathscr{H}om(\phi, \mathscr{G})_U(\psi) = \psi\phi|_U$$

This defines a contravariant additive functor $\mathscr{H}om(-, \mathscr{G}) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$. Similarly if $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of $\mathcal{O}_X$-modules we define a morphism of $\mathcal{O}_X$-modules

$$\mathscr{H}om(\mathscr{F}, \phi) : \mathscr{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}')$$
$$\mathscr{H}om(\mathscr{F}, \phi)_U(\psi) = \phi|_U\psi$$

This defines a covariant additive functor $\mathscr{H}om(\mathscr{F}, -) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$. It is easy to check that for morphisms of $\mathcal{O}_X$-modules $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ and $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ that we have

$$\mathscr{H}om(\mathscr{F}, \psi)\mathscr{H}om(\phi, \mathscr{G}) = \mathscr{H}om(\phi, \mathscr{G}')\mathscr{H}om(\mathscr{F}', \psi) \tag{5}$$

It follows that $\mathscr{H}om(-, -)$ defines a bifunctor $\mathfrak{Mod}(X)^{\text{op}} \times \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ with $\mathscr{H}om(\phi, \psi) : \mathscr{H}om(\mathscr{F}', \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}')$ given by the equivalent expressions in (5). The partial functors are the functors $\mathscr{H}om(\mathscr{F}, -)$ and $\mathscr{H}om(-, \mathscr{G})$ given above.

Throughout this section let $(X, \mathcal{O}_X)$ be a ringed space, and assume that all sheaves of modules are sheaves of $\mathcal{O}_X$-modules.

Lemma 72. *If $\mathscr{F}, \mathscr{G}$ are sheaves of modules then*

(i) For any open $U \subseteq X$ there is an equality of modules $\mathscr{H}om(\mathscr{F}, \mathscr{G})|_U = \mathscr{H}om(\mathscr{F}|_U, \mathscr{G}|_U)$ natural in $\mathscr{F}, \mathscr{G}$.

(ii) There is an isomorphism of modules $\mathscr{H}om(\mathcal{O}_X, \mathscr{F}) \cong \mathscr{F}$ natural in $\mathscr{F}$.

(iii) The functor $\mathscr{H}om(\mathscr{F}, -)$ preserves monomorphisms and limits.

(iv) The functor $\mathscr{H}om(-, \mathscr{G})$ maps epimorphisms to monomorphisms and maps colimits to limits.

Proof. (i) The two modules are equal by definition, and the equality is clearly natural in both variables. (ii) For any ringed space there is an isomorphism of abelian groups $Hom(\mathcal{O}_X, \mathscr{F}) \cong \mathscr{F}(X)$ defined by $\phi \mapsto \phi_X(1)$ and $m \mapsto \phi_m$ where $(\phi_m)_U(1) = m|_U$. Define

$$\eta : \mathscr{H}om(\mathcal{O}_X, \mathscr{F}) \longrightarrow \mathscr{F}$$
$$\eta_U(\phi) = \phi_U(1)$$

It is easily checked that this is an isomorphism of modules natural in $\mathscr{F}$, with inverse $\eta^{-1} : \mathscr{F} \longrightarrow \mathscr{H}om(\mathcal{O}_X, \mathscr{F})$ defined by $\eta_U^{-1}(m)_V(1) = m|_V$.

(iii) Let $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ be a monomorphism. If $t : \mathscr{T} \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a morphism with $\mathscr{H}om(\mathscr{F}, \phi)t = 0$ then for all $U \subseteq X$ and $s \in \mathscr{T}(U)$ we have $\phi|_U t_U(s) = 0$. Since $\phi|_U$ is a monomorphism, it follows that $t_U(s) = 0$. Hence $t = 0$, and $\mathscr{H}om(\mathscr{F}, t)$ is a monomorphism.

Next we show that $\mathscr{H}om(\mathscr{F}, -)$ preserves limits. This functor preserves zero objects since it is additive, so we can restrict our attention to nonempty diagrams D. Let the morphisms $p_i : \mathscr{L} \longrightarrow \mathscr{D}_i$ form a limit in $\mathfrak{Mod}(X)$. It suffices to show that for all open $U \subseteq X$ the morphisms

$$\mathscr{H}om(\mathscr{F}, p_i)_U : \mathscr{H}om(\mathscr{F}, \mathscr{L})(U) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{D}_i)(U) \tag{6}$$

form a limit of $\mathcal{O}_X(U)$-modules. But the functor $-|_U$ preserves limits, as does the additive functor $Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, -) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Ab}$, and by definition $\mathscr{H}om(\mathscr{F}, \phi)_U = Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \phi|_U)$ for any morphism of modules ϕ, so it follows that the morphisms in (6) are a limit of groups and thus of modules, as required.

(iv) Let $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ be an epimorphism. If $t : \mathscr{T} \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a morphism with $\mathscr{H}om(\phi, \mathscr{G})t = 0$ then for all $U \subseteq X$ and $s \in \mathscr{T}(U)$ we have $t_U(s)\phi|_U = 0$. Since $\phi|_U$ is an epimorphism, it follows that $t_U(s) = 0$. Hence $t = 0$ and $\mathscr{H}om(\phi, \mathscr{G})$ is a monomorphism.

A functor between abelian categories is right exact (= cokernel preserving) iff. it is finite colimit preserving. The dual of an abelian category is abelian, so to show that $\mathscr{H}om(-, \mathscr{G})$ maps finite colimits to limits it suffices to show that it maps cokernels to kernels. Suppose we are given an exact sequence of $\mathcal{O}_X$-modules

$$\mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \longrightarrow 0$$

We have to prove that for any module $\mathscr{G}$ the following sequence is also exact:

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}'', \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}', \mathscr{G})$$

We already know that $\mathscr{H}om(\psi, \mathscr{G})$ is a monomorphism, so it suffices to show that any morphism $t : \mathscr{T} \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})$ with $\mathscr{H}om(\phi, \mathscr{G})t = 0$ factors through $\mathscr{H}om(\psi, \mathscr{G})$. For $U \subseteq X$ and $s \in \mathscr{T}(U)$ we have by assumption $t_U(s)\phi|_U = 0$. Since the functor $-|_U$ is exact we see that $\psi|_U$ is the cokernel of $\phi|_U$, so there is a unique morphism of modules $\gamma_U(s) : \mathscr{F}''|_U \longrightarrow \mathscr{G}|_U$ making the following diagram commute:

$$\begin{array}{ccccccc}
\mathscr{F}'|_U & \xrightarrow{\phi|_U} & \mathscr{F}|_U & \xrightarrow{\psi|_U} & \mathscr{F}''|_U & \longrightarrow & 0 \\
 & & {\scriptstyle t_U(s)}\downarrow & \swarrow{\scriptstyle \gamma_U(s)} & & & \\
 & & \mathscr{G}|_U & & & &
\end{array}$$

Using the fact that t is a morphism of $\mathcal{O}_X$-modules one checks that this defines a morphism of $\mathcal{O}_X$-modules $\gamma : \mathscr{T} \longrightarrow \mathscr{H}om(\mathscr{F}'', \mathscr{G})$. Clearly $\mathscr{H}om(\psi, \mathscr{G})\gamma = t$, which completes the proof that $\mathscr{H}om(-, \mathscr{G})$ sends finite colimits to limits. It only remains to show that it sends arbitrary coproducts to products. This is straightforward to check, because restriction preserves coproducts and $Hom_{\mathcal{O}_X|_U}(-, \mathscr{G}|_U)$ sends coproducts to products. $\square$

In particular for a finite index set I, modules $\mathscr{F}_i, i \in I$ and a module $\mathscr{G}$ there is an isomorphism

$$\xi : \mathscr{H}om\left(\prod_{i \in I} \mathscr{F}_i, \mathscr{G}\right) \longrightarrow \prod_{i \in I} \mathscr{H}om(\mathscr{F}_i, \mathscr{G})$$

Here we use the fact that in an abelian category finite products and coproducts coincide. Let $p_i : \prod_{i \in I} \mathscr{F}_i \longrightarrow \mathscr{F}_i$ be the projections out of the canonical pointwise product. Then the induced morphisms $u_i : \mathscr{F}_i \longrightarrow \prod_{i \in I} \mathscr{F}_i$ are a coproduct, where $(u_i)_V$ is the canonical injection of $\mathscr{F}(V)$ in $\prod_{i \in I} \mathscr{F}(V)$. By the lemma the morphisms

$$\mathscr{H}om(u_i, \mathscr{G}) : \mathscr{H}om\left(\prod_{i \in I} \mathscr{F}_i, \mathscr{G}\right) \longrightarrow \mathscr{H}om(\mathscr{F}_i, \mathscr{G})$$

are a product. Since $\mathscr{H}om(-, \mathscr{G})$ is additive the morphisms $\mathscr{H}om(p_i, \mathscr{G})$ are also a coproduct. The morphisms $\mathscr{H}om(u_i, \mathscr{G})$ induce the desired isomorphism ξ, which has the following form:

$$\xi_U : Hom_{\mathcal{O}_X|_U}\left(\prod_{i \in I} \mathscr{F}_i|_U, \mathscr{G}|_U\right) \longrightarrow \prod_{i \in I} Hom_{\mathcal{O}_X|_U}(\mathscr{F}_i|_U, \mathscr{G}|_U)$$
$$\xi_U(\phi)_i = \mathscr{H}om(u_i, \mathscr{G})_U(\phi) = \phi \circ u_i|_U$$

Given a sequence $(\psi_i)_{i \in I}$ with $\psi_i : \mathscr{F}_i|_U \longrightarrow \mathscr{G}|_U$ let $\phi : \prod_{i \in I} \mathscr{F}_i|_U \longrightarrow \mathscr{G}|_U$ be induced out of the coproduct. Then $\xi_U(\phi) = (\psi_i)_{i \in I}$.

Definition 17. Let A be a ring and M an A-module. The A-module $Hom_A(M, A)$ is called the *dual* of M and is denoted $M^\vee$. The functor $(-)^\vee = Hom_A(-, A)$ is a contravariant additive functor $A\mathbf{Mod} \longrightarrow A\mathbf{Mod}$. Given an exact sequence of A-modules

$$L \longrightarrow M \longrightarrow N \longrightarrow 0$$

the following sequence of the duals is exact

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow L^\vee$$

In general the operation of taking duals is not exact, but in some important cases it is.

Lemma 73. *Let A be a ring and suppose we have a short exact sequence of A-modules with N projective $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$. Then the corresponding sequence of duals is exact*

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow L^\vee \longrightarrow 0 \tag{7}$$

Proof. We have a long exact sequence of Ext groups

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow L^\vee \longrightarrow Ext^1(N, A) \longrightarrow Ext^1(M, A) \longrightarrow \cdots$$

and since N is projective, $Ext^1(N, A) = 0$ which shows that (7) is exact. ☐

Definition 18. Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathscr{F}$ is a $\mathcal{O}_X$-module then we define the *dual* of $\mathscr{F}$ to be the module $\mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$, which we denote by $\mathscr{F}^\vee$. Taking duals defines a contravariant additive functor $(-)^\vee = \mathscr{H}om(-, \mathcal{O}_X)$. Dualising commutes with restriction in the sense that $\mathscr{F}^\vee|_U = (\mathscr{F}|_U)^\vee$. More generally the following diagram of functors commutes

$$\begin{array}{ccc} \mathfrak{Mod}(X) & \xrightarrow{(-)^\vee} & \mathfrak{Mod}(X) \\ {\scriptstyle -|_U}\downarrow & & \downarrow{\scriptstyle -|_U} \\ \mathfrak{Mod}(U) & \xrightarrow[(-)^\vee]{} & \mathfrak{Mod}(U) \end{array}$$

Taking duals is left exact in the sense that given a short exact sequence of sheaves of modules $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$ the sequence $0 \longrightarrow \mathscr{H}^\vee \longrightarrow \mathscr{G}^\vee \longrightarrow \mathscr{F}^\vee$ is exact.

Proposition 74. *For modules $\mathscr{F}, \mathscr{E}$ there is a canonical morphism of modules natural in $\mathscr{F}$*

$$\tau : \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{H}om(\mathscr{F}, \mathscr{E}), \mathscr{E})$$
$$\tau_U(t)_V(\phi) = \phi_V(t|_V)$$

If $\mathscr{F}$ is locally finitely free and $\mathscr{E} = \mathcal{O}_X$, this is an isomorphism $\mathscr{F} \longrightarrow (\mathscr{F}^\vee)^\vee$.

Proof. Let $U \subseteq X$ be open. We define

$$\tau_U : \mathscr{F}(U) \longrightarrow Hom_{\mathcal{O}_X|_U}(\mathscr{H}om(\mathscr{F}|_U, \mathscr{E}|_U), \mathscr{E}|_U)$$

as follows. Given $t \in \mathscr{F}(U)$ we define $\Phi_t : \mathscr{H}om(\mathscr{F}|_U, \mathscr{E}|_U) \longrightarrow \mathscr{E}|_U$ by $(\Phi_t)_V(\phi) = \phi_V(t|_V)$. One checks that $\tau_U(t) = \Phi_t$ is a well-defined morphism of $\mathcal{O}_X(U)$-modules natural in U. This defines the required morphism of modules τ, and naturality in $\mathscr{F}$ is easily checked.

Now suppose $\mathscr{F}$ is locally finitely free. To show that τ is an isomorphism it suffices to show $\tau|_U$ is an isomorphism for all U in an open cover of X. So we can reduce to the case where $\mathscr{F}$ is a coproduct of some finite number of copies of $\mathcal{O}_X$, and since τ is natural we can assume that $\mathscr{F} = \prod_{i=1}^n \mathcal{O}_X$ is the canonical pointwise product for some $n \geq 1$ (if $\mathscr{F} = 0$ the result is trivial). Let u_i, p_i for $1 \le i \le n$ be the canonical injections and projections. Consider the chain of isomorphisms

$$(\mathscr{F}^\vee)^\vee = \mathscr{H}om(\mathscr{H}om(\prod_{i=1}^n \mathcal{O}_X, \mathcal{O}_X), \mathcal{O}_X) \cong \mathscr{H}om(\prod_{i=1}^n \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X), \mathcal{O}_X)$$
$$\cong \mathscr{H}om(\prod_{i=1}^n \mathcal{O}_X, \mathcal{O}_X) \cong \prod_{i=1}^n \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X) \cong \prod_{i=1}^n \mathcal{O}_X = \mathscr{F}$$

It only remains to show that this composite is actually τ^{-1}. Let $U \subseteq X$ be open, $(r_1, \dots, r_n)_{i \in I}, r_i \in \mathcal{O}_X(U)$ a sequence in $\mathscr{F}(U)$. We chase this element up the chain and show that it ends up being mapped to $\tau_U(r_1, \dots, r_n)$. Given $a \in \mathcal{O}_X(U)$ we denote by θ_a the corresponding morphism $\mathcal{O}_X|_U \longrightarrow \mathcal{O}_X|_U$ defined by $(\theta_a)_V(1) = a|_V$. We follow $(r_1, \dots, r_n)$ up the rows: In row 4 we have the sequence $(\theta_{r_1}, \dots, \theta_{r_n})$. In row 3 this becomes the morphism $\theta : \prod_{i=1}^n \mathcal{O}_X|_U \longrightarrow \mathcal{O}_X|_U$ with $\theta \circ u_i|_U = \theta_{r_i}$. Let κ be the canonical isomorphism:

$$\mathscr{H}om(\prod_{i=1}^n \mathcal{O}_X, \mathcal{O}_X) \cong \prod_{i=1}^n \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X) \cong \prod_{i=1}^n \mathcal{O}_X$$

Then in row 1 the element we have is $\theta\kappa|_U \in (\mathscr{F}^\vee)^\vee(U)$. We must show that for $V \subseteq U$ and $\phi : \mathscr{F}|_V \longrightarrow \mathcal{O}_X|_V$ there is an equality of $(\theta\kappa|_U)_V(\phi)$ with $\phi_V(r_1|_V, \dots, r_n|_V)$. But

$$\begin{aligned}(\theta\kappa|_U)_V(\phi) = \theta_V(\kappa_V(\phi)) &= \sum_{i=1}^n (\theta_{r_i})_V\,(\phi_V((u_i)_V(1)))\\ &= \sum_{i=1}^n \phi_V((u_i)_V(1)) r_i|_V = \sum_{i=1}^n \phi_V(r_i|_V \cdot (u_i)_V(1))\\ &= \phi_V(r_1|_V, \dots, r_n|_V)\end{aligned}$$

as required. This completes the proof that τ is an isomorphism. ☐

Proposition 75. *For modules $\mathscr{F}, \mathscr{E}$ there is a canonical morphism of modules natural in both variables:*

$$\omega : \mathscr{E}^\vee \otimes \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{F})$$
$$\omega_U(\nu \dot{\otimes} t)_V(e) = \nu_V(e) \cdot t|_V$$

If $\mathscr{E}$ is locally finitely free, this is an isomorphism.

Proof. Let P be the presheaf of $\mathcal{O}_X$-modules defined by

$$P(U) = \mathscr{E}^\vee(U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U) = Hom_{\mathcal{O}_X|_U}(\mathscr{E}|_U, \mathcal{O}_X|_U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U)$$

We define a $\mathcal{O}_X(U)$-bilinear map

$$\beta : Hom_{\mathcal{O}_X|_U}(\mathscr{E}|_U, \mathcal{O}_X|_U) \times \mathscr{F}(U) \longrightarrow Hom_{\mathcal{O}_X|_U}(\mathscr{E}|_U, \mathscr{F}|_U)$$
$$\beta(u,t)_V(e) = u_V(e) \cdot t|_V$$

One checks that $\beta(u,t)$ is actually a morphism of modules and that β is bilinear. This induces a morphism of $\mathcal{O}_X(U)$-modules

$$\omega'_U : P(U) \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{F})(U)$$
$$\omega'_U(u \otimes t)_V(e) = u_V(e) \cdot t|_V$$

Together these give a morphism of presheaves of $\mathcal{O}_X$-modules $P \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{F})$ which sheafifies to give the desired morphism ω. Explicitly:

$$\omega : \mathscr{E}^\vee \otimes \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{F})$$
$$germ_x \omega_U(s) = \omega'_x(s(x))$$

Naturality of this morphism in both variables is easily checked. The morphism also commutes with restriction, in the sense that if $U \subseteq X$ is open and $\sigma : \mathscr{E}|_U^\vee \otimes \mathscr{F}|_U \longrightarrow \mathscr{H}om(\mathscr{E}|_U, \mathscr{F}|_U)$ is defined for the $\mathcal{O}_X|_U$-modules $\mathscr{E}|_U, \mathscr{F}|_U$ as above, then the following diagram commutes:

$$\begin{array}{ccc} (\mathscr{E}^\vee \otimes \mathscr{F})|_U & \xrightarrow{\ \omega|_U\ } & \mathscr{H}om(\mathscr{E}, \mathscr{F})|_U \\ \Downarrow & & \Downarrow \\ \mathscr{E}|_U^\vee \otimes \mathscr{F}|_U & \xrightarrow[\ \sigma\ ]{} & \mathscr{H}om(\mathscr{E}|_U, \mathscr{F}|_U) \end{array} \tag{8}$$

By virtue of (8), to show ω is an isomorphism in the case where $\mathscr{E}$ is locally finitely free, it suffices to prove it in the case where ω is *free*. In fact, since ω is natural in $\mathscr{E}$ we can assume that $\mathscr{E} = \prod_{i=1}^n \mathcal{O}_X$ is the canonical pointwise product for some $n \geq 1$ (the case $\mathscr{E} = 0$ being trivial). Consider the following chain of isomorphisms

$$\begin{aligned} \mathscr{E}^\vee \otimes \mathscr{F} &= \mathscr{H}om\left(\prod_{i=1}^n \mathcal{O}_X, \mathcal{O}_X\right) \otimes \mathscr{F} \cong \left(\prod_{i=1}^n \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X)\right) \otimes \mathscr{F} \\ &\cong \left(\prod_{i=1}^n \mathcal{O}_X\right) \otimes \mathscr{F} \cong \prod_{i=1}^n (\mathcal{O}_X \otimes \mathscr{F}) \cong \prod_{i=1}^n \mathscr{F} \cong \prod_{i=1}^n \mathscr{H}om(\mathcal{O}_X, \mathscr{F}) \\ &\cong \mathscr{H}om(\prod_{i=1}^n \mathcal{O}_X, \mathscr{F}) = \mathscr{H}om(\mathscr{E}, \mathscr{F}) \end{aligned}$$

It only remains to show that this composite is actually ω. Let $U \subseteq X$ be open and let $u \in \mathscr{H}om(\mathscr{E}, \mathcal{O}_X)(U), t \in \mathscr{F}(U)$ be given. We keep the notation of the previous Proposition, so we denote the injections into $\prod_{i=1}^n \mathcal{O}_X$ by u_i and the morphism $\mathcal{O}_X|_U \longrightarrow \mathscr{F}|_U$ corresponding to $a \in \mathscr{F}(U)$ by θ_a. The element $u \dot{\otimes} t$ in the first row becomes $(u \circ u_1|_U, \ldots, u \circ u_n|_U) \dot{\otimes} t$ in the second row and then $(u_U(u_1)_U(1) \cdot t, \ldots, u_U(u_n)_U(1) \cdot t)$ in the fifth row. In the seventh row we are have the morphism $\kappa : \prod_{i=1}^n \mathcal{O}_X|_U \longrightarrow \mathscr{F}|_U$ defined by $\kappa \circ (u_i)|_U = \theta_{u_U(u_i)_U(1) \cdot t}$. We need to show that κ agrees with $\omega_U(u \dot{\otimes} t)$. Given $V \subseteq U$ and $e = (e_1, \ldots, e_n) \in \prod_{i=1}^n \mathcal{O}_X(V)$ we have $\omega_U(u \dot{\otimes} t)_V(e) = u_V(e) \cdot t|_V$ while

$$\begin{aligned} \kappa_V(e_1, \ldots, e_n) &= \sum_{i=1}^n (\theta_{u_U(u_i)_U(1) \cdot t})_V(e_i) = \sum_{i=1}^n e_i \cdot (u_U(u_i)_U(1)|_V \cdot t|_V) \\ &= \sum_{i=1}^n u_V(e_i \cdot (u_i)_V(1)) \cdot t|_V = u_V(e_1, \ldots, e_n) \cdot t|_V \end{aligned}$$

as required. This completes the proof that ω is an isomorphism. □

Proposition 76. *For modules $\mathscr{F}, \mathscr{E}, \mathscr{G}$ there is a canonical isomorphism of abelian groups natural in all three variables*

$$\zeta : Hom_{\mathcal{O}_X}(\mathscr{F} \otimes \mathscr{E}, \mathscr{G}) \longrightarrow Hom_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om(\mathscr{E}, \mathscr{G}))$$

Proof. Let $\phi : \mathscr{F} \otimes \mathscr{E} \longrightarrow \mathscr{G}$, $U \subseteq X$ and $s \in \mathscr{F}(U)$ be given. We define $\zeta(\phi)_U(s) : \mathscr{E}|_U \longrightarrow \mathscr{G}|_U$ by

$$\zeta(\phi)_U(s)_V(e) = \phi_V(s|_V \dot{\otimes} e)$$

It is not difficult to check that $\zeta(\phi)_U(s)$ is a morphism of $\mathcal{O}_X|_U$ modules, and that $\zeta(\phi) : \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{G})$ is a morphism of $\mathcal{O}_X$-modules. So our map ζ is a well-defined morphism of abelian groups. Naturality in all three variables is not difficult to check. Next we show that ζ is an isomorphism (we place no finiteness restrictions on any of the modules). It is clear that ζ is injective. To see that it is surjective, let $\alpha : \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{G})$ be given. For $U \subseteq X$ define a $\mathcal{O}_X(U)$-bilinear map

$$\begin{aligned} \mathscr{F}(U) \times \mathscr{E}(U) &\longrightarrow \mathscr{G}(U) \\ (s, e) &\mapsto \alpha_U(s)_U(e) \end{aligned}$$

This induces a morphism of presheaves of $\mathcal{O}_X$-modules $P \longrightarrow \mathscr{G}$ where P sheafifies to give $\mathscr{F} \otimes \mathscr{E}$. So finally we induce a morphism of $\mathcal{O}_X$-modules:

$$\phi : \mathscr{F} \otimes \mathscr{E} \longrightarrow \mathscr{G}$$
$$\phi_U(s \dot{\otimes} e) = \alpha_U(s)_U(e)$$

It is easy to check that $\zeta(\phi) = \alpha$, which completes the proof that ζ is an isomorphism. □

Remark 15. The adjunction isomorphism of Proposition 76 determines a unit morphism

$$\eta : \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{F} \otimes \mathscr{E})$$
$$\eta_U(s)_V(e) = s|_V \dot{\otimes} e$$

and counit morphism

$$\varepsilon : \mathscr{H}om(\mathscr{E}, \mathscr{G}) \otimes \mathscr{E} \longrightarrow \mathscr{G}$$
$$\varepsilon_U(\varphi \dot{\otimes} e) = \varphi_U(e)$$

The counit determines a morphism $\mathscr{E} \otimes \mathscr{H}om(\mathscr{E}, \mathscr{G}) \longrightarrow \mathscr{G}$ which corresponds under the adjunction of Proposition 76 to the morphism $\tau : \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{H}om(\mathscr{F}, \mathscr{E}), \mathscr{E})$ of Proposition 74.

Proposition 77. *For modules $\mathscr{F}, \mathscr{E}, \mathscr{G}$ there is a canonical isomorphism of modules natural in all three variables*

$$\nu : \mathscr{H}om(\mathscr{F} \otimes \mathscr{E}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{H}om(\mathscr{E}, \mathscr{G}))$$

Proof. It is not difficult to check that the isomorphism ζ of Proposition 76 is actually a morphism of $\Gamma(X, \mathcal{O}_X)$-modules. So for every open subset $U \subseteq X$ we have an isomorphism of $\Gamma(U, \mathcal{O}_X)$-modules

$$\begin{aligned} \nu_U : Hom_{\mathcal{O}_X|_U}((\mathscr{F} \otimes \mathscr{E})|_U, \mathscr{G}|_U) &\cong Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U \otimes \mathscr{E}|_U, \mathscr{G}|_U) \\ &\cong Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{H}om(\mathscr{E}|_U, \mathscr{G}|_U)) \\ &= Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{H}om(\mathscr{E}, \mathscr{G})|_U) \end{aligned}$$

This defines the required isomorphism of modules ν, which is natural in all three variables. □

Corollary 78. *For a ringed space $(X, \mathcal{O}_X)$ and any $\mathcal{O}_X$-module $\mathscr{E}$ there is an adjunction of covariant additive functors:*

$$\mathfrak{Mod}(X) \underset{\mathscr{H}om(\mathscr{E}, -)}{\overset{-\otimes\mathscr{E}}{\rightleftarrows}} \mathfrak{Mod}(X) \qquad - \otimes \mathscr{E} \dashv \mathscr{H}om(\mathscr{E}, -)$$

In particular $- \otimes \mathscr{E}$ and $\mathscr{E} \otimes -$ preserve all colimits.

Proof. The adjunction follows from Proposition 76 and since $- \otimes \mathscr{E}$ is naturally equivalent to $\mathscr{E} \otimes -$ if one has a right adjoint so does the other. □

Lemma 79. *For modules $\mathscr{F}, \mathscr{E}$ there is a canonical morphism of modules natural in both variables*

$$\mathscr{E}^\vee \otimes \mathscr{F}^\vee \longrightarrow (\mathscr{E} \otimes \mathscr{F})^\vee$$

If $\mathscr{E}$ is locally finitely free, this is an isomorphism.

Proof. Combining Proposition 77 and Proposition 75 we have a canonical morphism natural in both variables

$$\mathscr{E}^\vee \otimes \mathscr{F}^\vee \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)) \cong \mathscr{H}om(\mathscr{E} \otimes \mathscr{F}, \mathcal{O}_X) = (\mathscr{E} \otimes \mathscr{F})^\vee$$

which is an isomorphism if $\mathscr{E}$ is locally finitely free. □

Let $F : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ be an additive functor. Then F preserves finite products and coproducts (take for example $\mathscr{E} \otimes -$ or $\mathscr{H}om(\mathscr{E}, -)$). For $n \geq 1$ let $\prod_{i=1}^n \mathscr{F}_i$ be the canonical pointwise product with projections p_i. The isomorphism

$$\mu : F(\prod_{i=1}^n \mathscr{F}_i) \longrightarrow \prod_{i=1}^n F(\mathscr{F}_i)$$

is defined by $\mu_U(s) = ((p_1)_U(s), \ldots, (p_n)_U(s))$.

Lemma 80 (Projection Formula). *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For any $\mathcal{O}_Y$-module $\mathscr{E}$ and $\mathcal{O}_X$-module $\mathscr{F}$ there is a canonical morphism of modules natural in both variables*

$$\pi : f_*\mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{E} \longrightarrow f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E})$$

If $\mathscr{E}$ is locally finitely free, this is an isomorphism.

Proof. The desired morphism is the composite

$$f_*\mathscr{F} \otimes \mathscr{E} \xrightarrow{1\otimes\eta} f_*\mathscr{F} \otimes f_*f^*\mathscr{E} \xrightarrow{\alpha} f_*(\mathscr{F} \otimes f^*\mathscr{E})$$

Where $\eta : \mathscr{E} \longrightarrow f_*f^*\mathscr{E}$ is canonical and α is defined earlier in our notes. It is readily checked that this morphism is natural in $\mathscr{F}, \mathscr{E}$. Next we must show that this morphism commutes with restriction, in the sense that if $V \subseteq Y$ is open, $U = f^{-1}V$ and $g : U \longrightarrow V$ induced from f, there is a commutative diagram:

$$\begin{array}{ccc} (f_*\mathscr{F} \otimes \mathscr{E})|_V & \xrightarrow{\pi|_V} & f_*(\mathscr{F} \otimes f^*\mathscr{E})|_V \\ \Downarrow & & \Downarrow \\ g_*(\mathscr{F}|_U) \otimes \mathscr{E}|_V & \longrightarrow & g_*(\mathscr{F}|_U \otimes g^*(\mathscr{E}|_V)) \end{array}$$

Where the bottom morphism is π defined for the morphism of ringed spaces g, and modules $\mathscr{F}|_U, \mathscr{E}|_V$. We piece this diagram together from two commutative squares: one for $1 \otimes \eta$ and one for α. First we consider $1 \otimes \eta$.

We claim that for any $\mathcal{O}_Y$-module $\mathscr{E}$ the unit $\eta' : \mathscr{E}|_V \longrightarrow g_*g^*(\mathscr{E}|_V)$ of the adjunction $g^* \dashv g_*$ fits into the following commutative diagram:

$$\begin{array}{ccc} \mathscr{E}|_V & \xrightarrow{\eta'} & g_*g^*(\mathscr{E}|_V) \\ & {\scriptstyle \eta|_V} \searrow \quad \swarrow {\scriptstyle g_*\rho} & \\ & (f_*f^*\mathscr{E})|_V & \end{array} \tag{9}$$

Where $\rho : g^*(\mathscr{E}|_V) \longrightarrow (f^*\mathscr{E})|_U$ is the canonical isomorphism of modules, which is described explicitly in our Section 5 notes. If $W \subseteq V$ is open and $s \in \mathscr{F}(W)$ then $\eta'_W(s) = (\dot{W, s})\dot{\otimes}1$ and from our notes on ρ we see that $\rho_{g^{-1}W}(\eta'_W(s))$ is the element $(\dot{W, s})\dot{\otimes}1$ of $f^*\mathscr{F}(g^{-1}W)$, which is $\eta_W(s)$. Therefore the above diagram commutes, and using the isomorphisms $g_*(-|_U) = (-|_V)f_*$ and $g^*(-|_V) \cong (-|_U)f^*$ one builds a commutative diagram:

$$\begin{array}{ccc} (f_*\mathscr{F} \otimes \mathscr{E})|_V & \xrightarrow{(1\otimes\eta)|_V} & (f_*\mathscr{F} \otimes f_*f^*\mathscr{E})|_V \\ \Downarrow & & \Downarrow \\ g_*(\mathscr{F}|_U) \otimes \mathscr{E}|_V & \xrightarrow[1\otimes\eta']{} & g_*(\mathscr{F}|_U) \otimes g_*g^*(\mathscr{E}|_V) \end{array} \tag{10}$$

Similarly by chasing an element $a\dot{\otimes}b$ with $a \in \mathscr{F}(g^{-1}W)$ and $b \in g^*(\mathscr{E}|_V)(g^{-1}W)$ around the following diagram we see that it too is commutative:

$$
\begin{array}{ccc}
(f_*\mathscr{F} \otimes f_*f^*\mathscr{E})|_V & \xrightarrow{\alpha|_V} & f_*(\mathscr{F} \otimes f^*\mathscr{E})|_V \\
\Uparrow & & \Uparrow \\
(f_*\mathscr{F})|_V \otimes (f_*f^*\mathscr{E})|_V & & g_*((\mathscr{F} \otimes f^*\mathscr{E})|_U) \\
\Uparrow & & \Uparrow \\
g_*(\mathscr{F}|_U) \otimes g_*g^*(\mathscr{E}|_V) & \longrightarrow & g_*(\mathscr{F}|_U \otimes g^*(\mathscr{E}|_V))
\end{array}
\tag{11}
$$

By turning around the isomorphisms and pasting (11) and (10) we get the desired commutative diagram in (9). Suppose now that $\mathscr{E}$ is locally finitely free. Using (9) we can reduce to the case where $\mathscr{E} = \prod_{i=1}^n \mathcal{O}_Y$ is the canonical pointwise product for $n \geq 1$ (the case $\mathscr{E} = 0$ is trivial).

It is then sufficient to show commutativity of the following diagram:

$$
\begin{array}{ccc}
f_*(\mathscr{F}) \otimes \prod_{i=1}^n \mathcal{O}_Y & \xrightarrow{\pi} & f_*(\mathscr{F} \otimes f^* \prod_{i=1}^n \mathcal{O}_Y) \\
\Big\Downarrow & & \Downarrow \\
 & & f_*(\mathscr{F} \otimes \prod_{i=1}^n f^*\mathcal{O}_Y) \\
 & & \Downarrow \\
 & & f_*(\mathscr{F} \otimes \prod_{i=1}^n \mathcal{O}_X) \\
 & & \Downarrow \\
 & & f_*(\prod_{i=1}^n \mathscr{F} \otimes \mathcal{O}_X) \\
\Big\Downarrow & & \Downarrow \\
\prod_{i=1}^n f_*\mathscr{F} & = & f_*(\prod_{i=1}^n \mathscr{F})
\end{array}
$$

Where f_*, f^* are additive so they preserve finite products, and we use the isomorphism of $\mathcal{O}_X$-modules $f^*\mathcal{O}_Y \cong \mathcal{O}_X$. It suffices to show that for $V \subseteq Y$, $s \in \mathscr{F}(f^{-1}V)$ and $r_1, \ldots, r_n \in \mathcal{O}_Y(V)$ that both legs of the diagram agree on $s\dot{\otimes}(r_1, \ldots, r_n)$. This is easily checked, provided we observe that $f^\#_V(r_i) = \xi_{f^{-1}V}((\dot{V, r_i}))$ where $\xi : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is the adjoint partner of $f^\#$. Hence π is an isomorphism and the proof is complete. $\square$

Remark 16. With the notation of Lemma 80 let $V \subseteq Y$ be an open subset and $g : U \longrightarrow V$ the induced morphism of ringed spaces, where $U = f^{-1}V$. Then one checks that the projection morphism is local, by which we mean that the following diagram commutes

$$
\begin{array}{ccc}
(f_*\mathscr{F} \otimes \mathscr{E})|_V & \xrightarrow{\pi|_V} & f_*(\mathscr{F} \otimes f^*\mathscr{E})|_V \\
\Downarrow & & \Downarrow \\
g_*(\mathscr{F}|_U) \otimes \mathscr{E}|_V & \xrightarrow[\pi]{} & g_*(\mathscr{F}|_U \otimes g^*(\mathscr{E}|_U))
\end{array}
$$

Lemma 81. *For any module $\mathscr{F}$ there is a canonical morphism of modules*

$$
\gamma : \mathcal{O}_X \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{F})
$$
$$
\gamma_U(r)_V(m) = r|_V \cdot m
$$

If $\mathscr{F}$ is invertible this is an isomorphism.

Proof. The morphism γ is defined as follows:

$$\gamma_U : \mathcal{O}_X(U) \longrightarrow Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{F}|_U)$$
$$\gamma_U(r)_V(m) = r|_V \cdot m$$

One checks easily that this defines a morphism of $\mathcal{O}_X$-modules. Now suppose $\mathscr{F}$ is invertible. Given $x \in X$ let U be an open neighborhood of x with $\mathscr{F}|_U \cong \mathcal{O}_X|_U$. Then $\mathscr{H}om(\mathscr{F}|_U, \mathscr{F}|_U) \cong \mathscr{H}om(\mathcal{O}_X|_U, \mathscr{F}|_U)$ which is isomorphic to $\mathcal{O}_X|_U$. One checks that this isomorphism is in fact $\gamma|_U$, which completes the proof. Observe that γ is just the unit of Remark 15. □

Remark 17. Let $\mathscr{F}$ be any module. Under the bijection of Proposition 76 the morphism $\gamma : \mathcal{O}_X \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{F})$ corresponds to the canonical isomorphism $\mathcal{O}_X \otimes \mathscr{F} \longrightarrow \mathscr{F}$.

Lemma 82. *Let $(X, \mathcal{O}_X)$ be a nontrivial ringed space and $\mathscr{L}$ a locally free sheaf of finite rank $n \geq 0$. Then $\mathscr{L}^\vee$ is a locally free sheaf of the same rank. In particular for any ringed space if $\mathscr{L}$ is invertible then so is $\mathscr{L}^\vee$.*

Proof. If $n = 0$ this is trivial, so assume $n \geq 1$. If $x \in X$ let U be an open neighborhood with $\mathscr{L}|_U$ free of rank n. Then using Lemma 72 we have an isomorphism of sheaves of modules

$$\mathscr{H}om(\mathscr{L}, \mathcal{O}_X)|_U = \mathscr{H}om(\mathscr{L}|_U, \mathcal{O}_X|_U) \cong \bigoplus_{i=1}^{n} \mathscr{H}om(\mathcal{O}_X|_U, \mathcal{O}_X|_U) = \bigoplus_{i=1}^{n} \mathcal{O}_X|_U$$

which is what we wanted to show. □

Lemma 83. *For any module $\mathscr{L}$ there is a canonical morphism of modules*

$$\delta : \mathscr{L}^\vee \otimes \mathscr{L} \longrightarrow \mathcal{O}_X$$
$$\delta_U(\nu \,\dot{\otimes}\, a) = \nu_U(a)$$

If $\mathscr{L}$ is invertible this is an isomorphism.

Proof. One proves the existence of δ in the usual way. To show that δ is an isomorphism, we show that it agrees with the composite $\mathscr{L}^\vee \otimes \mathscr{L} \cong \mathscr{H}om(\mathscr{L}, \mathscr{L}) \cong \mathcal{O}_X$ of the isomorphisms of Proposition 75 and Lemma 81. In doing this, we can clearly reduce to the case where $\mathscr{L} = \mathcal{O}_X$, which is easily checked. □

Remark 18. Let $\mathscr{L}$ be any module. Under the bijection of Proposition 76 the morphism $\delta : \mathscr{L}^\vee \otimes \mathscr{L} \longrightarrow \mathcal{O}_X$ corresponds to the identity $\mathscr{L}^\vee \longrightarrow \mathscr{L}^\vee$. In other words δ is the counit of Remark 15.

Remark 19. The next definition involves some set-theoretical issues. We refer to (FCT,Section 4) and in particular (FCT,Remark 4) for the relevant background.

Definition 19. For a ringed space $(X, \mathcal{O}_X)$ let Pic$X$ denote the conglomerate of of isomorphism classes of invertible sheaves on X. Then $- \otimes -$ gives a well-defined, associative, commutative binary operation with identity $\mathcal{O}_X$ and inverses given by duals. This makes Pic$X$ into an abelian group (whose underlying conglomerate need not be a set), called the *Picard group* of X.

Remark 20. We will see later (H, III Ex.4.5) that $PicX$ can be expressed as the cohomology group $H^1(X, \mathcal{O}_X^*)$.

Just using the properties of the group $PicX$ we can see that for $m > 0$ and an invertible sheaf $\mathscr{L}$ there is an isomorphism $(\mathscr{L}^{\otimes m})^\vee \cong (\mathscr{L}^\vee)^{\otimes m}$. However, we can produce a canonical *natural* isomorphism of this form with a little work.

Lemma 84. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{L}$ a locally finitely free sheaf of modules. Then for $n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of modules $(\mathscr{L}^\vee)^{\otimes n} \cong (\mathscr{L}^{\otimes n})^\vee$. For $n \geq 0$ this is natural in $\mathscr{L}$ and defined by*

$$\varrho : (\mathscr{L}^\vee)^{\otimes n} \longrightarrow (\mathscr{L}^{\otimes n})^\vee$$
$$\varrho_U(\nu_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, \nu_n)_V(a_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, a_n) = \nu_1(a_1) \cdots \nu_n(a_n)$$

Proof. It suffices to prove the result for $n \geq 0$, which we do by recursion. For $n = 0$ take the canonical isomorphism $\mathcal{O}_X \cong \mathcal{O}_X^\vee$ of Lemma 81 and for $n = 1$ take the identity. For $n \geq 2$ we define the isomorphism as follows

$$\begin{aligned}(\mathscr{L}^{\otimes n})^\vee &= \mathscr{H}om(\mathscr{L}^{\otimes n}, \mathcal{O}_X) = \mathscr{H}om(\mathscr{L} \otimes \mathscr{L}^{\otimes (n-1)}, \mathcal{O}_X)\\ &\cong \mathscr{H}om(\mathscr{L}, \mathscr{H}om(\mathscr{L}^{\otimes (n-1)}, \mathcal{O}_X)) = \mathscr{H}om(\mathscr{L}, (\mathscr{L}^{\otimes (n-1)})^\vee)\\ &\cong \mathscr{H}om(\mathscr{L}, (\mathscr{L}^\vee)^{\otimes (n-1)}) \cong \mathscr{L}^\vee \otimes (\mathscr{L}^\vee)^{\otimes (n-1)} = (\mathscr{L}^\vee)^{\otimes n}\end{aligned}$$

Naturality is easily checked. ☐

Proposition 85. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{L}$ an invertible $\mathcal{O}_X$-module. For $n \in \mathbb{Z}$ we define*

$$\mathscr{L}^{\otimes n} = \begin{cases} \mathscr{L}^{\otimes n} & n > 0 \\ \mathcal{O}_X & n = 0 \\ (\mathscr{L}^\vee)^{\otimes (-n)} & n < 0 \end{cases}$$

Then for all $n \in \mathbb{Z}$ the module $\mathscr{L}^{\otimes n}$ is invertible, and for $m, n \in \mathbb{Z}$ we have canonical isomorphisms of sheaves of modules

$$\mathscr{L}^{\otimes m} \otimes \mathscr{L}^{\otimes n} \cong \mathscr{L}^{\otimes (m+n)} \tag{12}$$

$$(\mathscr{L}^{\otimes m})^{\otimes n} \cong \mathscr{L}^{\otimes (mn)} \tag{13}$$

Proof. The sheaf $\mathscr{L}^\vee$ is invertible so by a simple induction $\mathscr{L}^{\otimes n}$ is invertible for any $n \in \mathbb{Z}$. For $m, n > 0$ or $m, n < 0$ the equations (12), (13) follow from Proposition 63 and Proposition 65. The cases where m or n are zero are trivial. The remaining cases follow in the multiplicative case from Lemma 84. In the additive case we can assume without loss of generality that $m < 0$ and $n > 0$, where we obtain an isomorphism by recursively pairing off $\mathscr{L}^\vee$ and $\mathscr{L}$. ☐

Remark 21. It is clear what the action of the isomorphism (12) of Proposition 85 is in the cases where $m, n \geq 0$ or $m, n < 0$, but it is worthwhile writing down exactly what happens in the case $m < 0$ and $n > 0$. Equivalently, taking $m, n > 0$ we have a canonical isomorphism

$$\alpha : \mathscr{L}^{\otimes (-m)} \otimes \mathscr{L}^{\otimes n} \longrightarrow \mathscr{L}^{\otimes (n-m)}$$

which is defined by

$$\alpha_U(\nu_1 \dot{\otimes} \cdots \dot{\otimes} \nu_m \dot{\otimes} a_1 \dot{\otimes} \cdots \dot{\otimes} a_n) = \begin{cases} \nu_1(a_1) \cdots \nu_n(a_n) & m = n \\ \nu_1(a_1) \cdots \nu_n(a_n) \cdot (\nu_{n+1} \dot{\otimes} \cdots \dot{\otimes} \nu_m) & m > n \\ \nu_1(a_1) \cdots \nu_m(a_m) \cdot (a_{m+1} \dot{\otimes} \cdots \dot{\otimes} a_n) & n > m \end{cases}$$

Proposition 86. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$ there is a canonical morphism of $\mathcal{O}_Y$-modules natural in $\mathscr{F}, \mathscr{G}$*

$$\kappa : f_* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathscr{F}, f_*\mathscr{G})$$

$$\kappa_V(\psi)_W = \psi_{f^{-1}W}$$

If f is an isomorphism of ringed spaces then κ is an isomorphism of $\mathcal{O}_Y$-modules.

Proof. For open $V \subseteq Y$ let $f_V : f^{-1}V \longrightarrow V$ be the induced morphism of ringed spaces. Given a morphism of $\mathcal{O}_X|_{f^{-1}V}$-modules $\psi : \mathscr{F}|_{f^{-1}V} \longrightarrow \mathscr{G}|_{f^{-1}V}$ let $\kappa_V(\psi)$ be the morphism $(f_V)_*\psi$. It is easy to see that this is a morphism of sheaves of $\mathcal{O}_Y$-modules. Naturality is easily checked, and clearly if f is an isomorphism then so is κ. ☐

Corollary 87. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For an $\mathcal{O}_X$-module $\mathscr{F}$ and $\mathcal{O}_Y$-module $\mathscr{G}$ there is a canonical isomorphism of $\mathcal{O}_Y$-modules natural in $\mathscr{F}, \mathscr{G}$*

$$f_* \mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{G}, \mathscr{F}) \longrightarrow \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

Proof. The morphism is defined to be the composite

$$f_* \mathscr{H}om_{\mathcal{O}_Y}(f^*\mathscr{G}, \mathscr{F}) \xrightarrow{\kappa} \mathscr{H}om_{\mathcal{O}_X}(f_* f^*\mathscr{G}, f_*\mathscr{F}) \xrightarrow{\mathscr{H}om(\eta, f_*\mathscr{F})} \mathscr{H}om_{\mathcal{O}_X}(\mathscr{G}, f_*\mathscr{F})$$

which is clearly natural in both variables. To check that it is an isomorphism, let open $V \subseteq Y$ be given and set $U = f^{-1}V$. Let $g : U \longrightarrow V$ be the induced morphism of ringed spaces. Then one checks that the following diagram commutes

$$\begin{array}{ccc} Hom_{\mathcal{O}_X|_U}((f^*\mathscr{G})|_U, \mathscr{F}|_U) & \longrightarrow & Hom_{\mathcal{O}_Y|_V}(\mathscr{G}|_V, (f_*\mathscr{F})|_V) \\ \Downarrow & & \Downarrow \\ Hom_{\mathcal{O}_X|_U}(g^*(\mathscr{G}|_V), \mathscr{F}|_U) & \Longrightarrow & Hom_{\mathcal{O}_Y|_V}(\mathscr{G}|_V, g_*(\mathscr{F}|_U)) \end{array}$$

where the bottom row is the usual adjunction isomorphism. It is therefore clear that our morphism is an isomorphism of sheaves of modules. □

Proposition 88. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For $\mathcal{O}_Y$-modules $\mathscr{F}, \mathscr{G}$ there is a canonical morphism of $\mathcal{O}_X$-modules natural in $\mathscr{F}, \mathscr{G}$*

$$\mu : f^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{F}, f^*\mathscr{G})$$
$$\mu_U([T, \psi] \dot{\otimes} b)_W([S, c] \dot{\otimes} d) = [S, \psi_S(c)] \dot{\otimes} b|_W d$$

If f is an isomorphism of ringed spaces then κ is an isomorphism of $\mathcal{O}_Y$-modules.

Proof. First we define a morphism of $\mathcal{O}_Y$-modules $\tau : \mathscr{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow f_* \mathscr{H}om(f^*\mathscr{F}, f^*\mathscr{G})$. Given an open subset $V \subseteq Y$ and a morphism of $\mathcal{O}_Y|_V$-modules $\psi : \mathscr{F}|_V \longrightarrow \mathscr{G}|_V$ let $f_V : f^{-1}V \longrightarrow V$ be the induced morphism of ringed spaces and $\tau_V(\psi)$ the composite of $(f_V)^*\psi$ with the canonical isomorphisms $(f_V)^*(\mathscr{F}|_V) \cong f^*(\mathscr{F})|_{f^{-1}V}$ and $(f_V)^*(\mathscr{G}|_V) \cong f^*(\mathscr{G})|_{f^{-1}V}$. In a more compact notation

$$\tau : \mathscr{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow f_* \mathscr{H}om(f^*\mathscr{F}, f^*\mathscr{G})$$
$$\tau_V(\psi)_U([T, c] \dot{\otimes} b) = [T, \psi_T(c)] \dot{\otimes} b$$

for open $T \subseteq V, U \subseteq f^{-1}T$ and $c \in \mathscr{F}(T), b \in \mathcal{O}_X(U)$. It is not hard to check that this is a morphism of $\mathcal{O}_Y$-modules. The adjoint partner μ of τ has the right action and is natural in $\mathscr{F}, \mathscr{G}$. Suppose that f is an isomorphism of ringed spaces with inverse g. Then it is not hard to check that μ agrees with the following isomorphism

$$f^* \mathscr{H}om(\mathscr{F}, \mathscr{G}) \cong g_* \mathscr{H}om(\mathscr{F}, \mathscr{G}) \cong \mathscr{H}om(g_*\mathscr{F}, g_*\mathscr{G}) \cong \mathscr{H}om(f^*\mathscr{F}, f^*\mathscr{G})$$

which completes the proof. □

Proposition 89. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ sheaves of $\mathcal{O}_X$-modules. Then for $x \in X$ there is a canonical morphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{F}, \mathscr{G}$*

$$\lambda : \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_x \longrightarrow Hom_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$
$$\lambda(U, \phi)(V, s) = (U \cap V, \phi_{U \cap V}(s|_{U \cap V}))$$

If $\mathscr{F}$ is locally finitely presented then this is an isomorphism.

Proof. Let $\phi : \mathscr{F}|_U \longrightarrow \mathscr{G}|_U$ be a morphism of $\mathcal{O}_X|_U$-modules for an open neighborhood U of x. Then $\phi_x : (\mathscr{F}|_U)_x \longrightarrow (\mathscr{G}|_U)_x$ together with the isomorphisms $(\mathscr{F}|_U)_x \cong \mathscr{F}_x$ and $(\mathscr{G}|_U)_x \cong \mathscr{G}_x$ give a morphism of $\mathcal{O}_{X,x}$-modules $\mathscr{F}_x \longrightarrow \mathscr{G}_x$ defined by $(V, s) \mapsto (U \cap V, \phi_{U \cap V}(s|_{U \cap V}))$. Thus defined, it is not hard to check that λ is a well-defined morphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{F}, \mathscr{G}$.

For $p > 0$ let $\mathscr{F}$ be the canonical coproduct $\mathcal{O}_X^p$. Since the contravariant functor $\mathscr{H}om(-, \mathscr{G}) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ maps finite colimits to limits by Lemma 72 and taking stalks $(-)_x$:

$\mathfrak{Mod}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Mod}$ preserves finite limits and all colimits, it is straightforward to check that in this case the morphism

$$\lambda : \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathscr{G})_x \longrightarrow Hom_{\mathcal{O}_{X,x}}((\mathcal{O}_X^p)_x, \mathscr{G}_x)$$

is an isomorphism. Next we prove λ is an isomorphism for $\mathscr{F}$ globally finitely presented, so that we have integers $p, q > 0$ and an exact sequence

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathscr{F} \longrightarrow 0$$

The functor $\mathscr{H}om(-, \mathscr{G})$ sends cokernels to kernels by Lemma 72 so the following is an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathscr{G})$$

Applying the exact functor $(-)_x$ and using naturality we get a commutative diagram with exact rows in $\mathcal{O}_{X,x}\mathbf{Mod}$

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_x & \longrightarrow & \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathscr{G})_x & \longrightarrow & \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathscr{G})_x \\
& \downarrow & & \Downarrow & & \Downarrow \\
0 \longrightarrow & Hom_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) & \longrightarrow & Hom_{\mathcal{O}_{X,x}}((\mathcal{O}_X^q)_x, \mathscr{G}_x) & \longrightarrow & Hom_{\mathcal{O}_{X,x}}((\mathcal{O}_X^p)_x, \mathscr{G}_x)
\end{array}$$

It follows that the left vertical morphism is also an isomorphism, as required. Finally assume that $\mathscr{F}$ is *locally* finitely presented. Since $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})|_U = \mathscr{H}om_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ in showing that λ is an isomorphism we can reduce easily to the case where $\mathscr{F}$ is globally finitely presented, so the proof is complete. □

Corollary 90. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ locally finitely presented sheaves of $\mathcal{O}_X$-modules. For $x \in X$ if there is an isomorphism of $\mathcal{O}_{X,x}$-modules $\mathscr{F}_x \cong \mathscr{G}_x$ then there is an open neighborhood U of x and an isomorphism of $\mathcal{O}_X|_U$-modules $\mathscr{F}|_U \cong \mathscr{G}|_U$.*

Proof. If $\varphi : \mathscr{F}_x \longrightarrow \mathscr{G}_x$ is an isomorphism of $\mathcal{O}_{X,x}$-modules with inverse $\psi : \mathscr{G}_x \longrightarrow \mathscr{F}_x$ then by Proposition 89 there is an open neighborhood V of x and morphisms $u : \mathscr{F}|_V \longrightarrow \mathscr{G}|_V$ and $v : \mathscr{G}|_V \longrightarrow \mathscr{F}|_V$ such that $u_x = \varphi$ and $v_x = \psi$. The fact that $\varphi\psi = 1$ and $\psi\varphi = 1$ means that we can find an open neighborhood $x \in U \subseteq V$ such that $(u \circ v)|_U = 1$ and $(v \circ u)|_U = 1$, which gives us the desired isomorphism. □

Corollary 91. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of locally finitely presented sheaves of $\mathcal{O}_X$-modules. If $\phi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x$ is an isomorphism of $\mathcal{O}_{X,x}$-modules then $\phi|_U$ is an isomorphism of $\mathcal{O}_X|_U$-modules for some open neighborhood U of x.*

Corollary 92. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{F}$ a locally finitely presented sheaf of $\mathcal{O}_X$-modules. Then for $x \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{F}$*

$$\lambda : (\mathscr{F}^\vee)_x \longrightarrow (\mathscr{F}_x)^\vee$$
$$\lambda(U, \nu)(V, s) = (U \cap V, \nu_{U \cap V}(s|_{U \cap V}))$$

Proposition 93. *Let $(X, \mathcal{O}_X)$ be a ringed space, and suppose we have a short exact sequence $0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{M} \longrightarrow \mathscr{N} \longrightarrow 0$ of locally finitely free sheaves of modules. Then the sequence of duals is also exact*

$$0 \longrightarrow \mathscr{N}^\vee \longrightarrow \mathscr{M}^\vee \longrightarrow \mathscr{L}^\vee \longrightarrow 0 \qquad (14)$$

Proof. For each $x \in X$ we have by Corollary 92 a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & (\mathscr{N}_x)^\vee & \longrightarrow & (\mathscr{M}_x)^\vee & \longrightarrow & (\mathscr{L}_x)^\vee & \longrightarrow 0 \\
& \Downarrow & & \Downarrow & & \Downarrow & \\
0 \longrightarrow & (\mathscr{N}^\vee)_x & \longrightarrow & (\mathscr{M}^\vee)_x & \longrightarrow & (\mathscr{L}^\vee)_x & \longrightarrow 0
\end{array}$$

Since $\mathscr{N}_x$ is free and therefore projective, Lemma 73 implies that the top row is exact. We deduce that the bottom row is also exact, and therefore (14) is exact. □

Lemma 94. *Let $\alpha : A \longrightarrow B$ be a ring morphism and M an A-module. Then there is a canonical morphism of B-modules natural in M*

$$\theta : M^\vee \otimes_A B \longrightarrow (M \otimes_A B)^\vee$$
$$\theta(\nu \otimes b)(m \otimes c) = bc \cdot \alpha\nu(m)$$

If M is free of finite rank, this is an isomorphism.

Proposition 95. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For an $\mathcal{O}_Y$-module $\mathscr{F}$ there is a canonical morphism of $\mathcal{O}_X$-modules natural in $\mathscr{F}$*

$$\theta : f^*(\mathscr{F}^\vee) \longrightarrow (f^*\mathscr{F})^\vee$$
$$\theta_U([V, \phi] \dot{\otimes} r)_W([T, s] \dot{\otimes} b) = r|_W \cdot b \cdot f_T^\#(\phi_T(s))|_W$$

If $\mathscr{F}$ is locally finitely free, this is an isomorphism.

Proof. Given an open set $V \subseteq Y$ let $f_V : f^{-1}V \longrightarrow V$ denote the induced morphism of ringed spaces. Given a morphism of sheaves of modules $\phi : \mathscr{F}|_V \longrightarrow \mathcal{O}_Y|_V$ we have the following composite, which we denote by $\theta'_V(\phi)$

$$(f^*\mathscr{F})|_{f^{-1}V} \Longrightarrow (f_V)^*(\mathscr{F}|_V) \xrightarrow{(f_V)^*\phi} (f_V)^*(\mathcal{O}_Y|_V) \Longrightarrow \mathcal{O}_X|_{f^{-1}V}$$

This defines a morphism of $\mathcal{O}_Y$-modules $\mathscr{F}^\vee \longrightarrow f_*(f^*\mathscr{F})^\vee$. By adjointness this corresponds to a morphism of $\mathcal{O}_X$-modules $\theta : f^*(\mathscr{F}^\vee) \longrightarrow (f^*\mathscr{F})^\vee$ with $\theta_U([V, \phi] \dot{\otimes} r) = r \cdot \theta'_V(\phi)|_U$. Naturality in $\mathscr{F}$ is easily checked. Now suppose that $\mathscr{F}$ is locally finitely free. Let $x \in X$ be given. Then using Lemma 94 and Corollary 92 we have a commutative diagram

$$\begin{array}{ccc} f^*(\mathscr{F}^\vee)_x & \xrightarrow{\theta_x} & ((f^*\mathscr{F})^\vee)_x \\ \Downarrow & & \Downarrow \\ (\mathscr{F}_{f(x)})^\vee \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} & \Longrightarrow & (\mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x})^\vee \end{array}$$

It follows that θ_x is an isomorphism, which completes the proof. □

1.13 Coinverse Image

Now that we have defined the sheaf Hom, we can continue the discussion of Section 1.4. Just as we constructed the functor f^* from f^{-1} and the tensor product, we can use $f^?$ and the sheaf Hom to define a right adjoint to the direct image functor for closed immersions.

Definition 20. Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a closed embedding of ringed spaces, and set $U = Y \setminus f(X)$. Let $\mathscr{G}$ be a sheaf of $\mathcal{O}_Y$-modules and define a sheaf of abelian groups on X by

$$f^!\mathscr{G} = f^? \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G})$$
$$\Gamma(V, f^!\mathscr{G}) = \{s \in \Gamma(f(V) \cup U, \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G})) \,|\, s|_U = 0\}$$
$$= \Gamma(f(V) \cup U, \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G}))$$

where $f^? : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$ is the additive functor defined in (SGR,Definition 17). We make $f^!\mathscr{G}$ into a sheaf of $\mathcal{O}_X$-modules by defining for $V \subseteq X$ and $r \in \Gamma(V, \mathcal{O}_X), s \in \Gamma(V, f^!\mathscr{G})$ by $(r \cdot s)_W(t) = s_W(r|_{f^{-1}W} t)$. If $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of sheaves of $\mathcal{O}_Y$-modules then $f^!\phi = f^? \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \phi)$ is a morphism of sheaves of $\mathcal{O}_X$-modules.

$$f^!\phi : f^!\mathscr{G} \longrightarrow f^!\mathscr{G}'$$
$$(f^!\phi)_V(s) = \phi|_{f(V) \cup U} \circ s$$

This defines an additive functor $f^!(-) : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$.

As we shall show in the next result, we could have used the functor f^{-1} instead of $f^?$ in this definition. While this would avoid the introduction of yet another new functor, it would obscure the true simplicity of the construction.

Proposition 96. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a closed embedding of ringed spaces and $\mathscr{G}$ a sheaf of $\mathcal{O}_Y$-modules. If we set $Z = f(X), U = Y \setminus Z$ then*

(i) There is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{G}$

$$f^!\mathscr{G} \longrightarrow f^{-1}\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G})$$
$$s \mapsto [f(V) \cup U, s]$$

(ii) If $f_*\mathcal{O}_X$ is locally finitely presented, then for $x \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{G}$

$$\lambda : f^!(\mathscr{G})_x \longrightarrow Hom_{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x}, \mathscr{G}_{f(x)})$$
$$\lambda(V, s)(T, r) = (Q, s_Q(r|_{T \cap V}))$$

where $Q = f(T \cap V) \cup U$.

Proof. (i) Set $Z = f(X)$ and $U = Y \setminus Z$. We have $\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G})|_U = \mathscr{H}om_{\mathcal{O}_Y|_U}(0, \mathscr{G}|_U) = 0$, so by (SGR,Lemma 35) there is an isomorphism of sheaves of abelian groups natural in $\mathscr{G}$

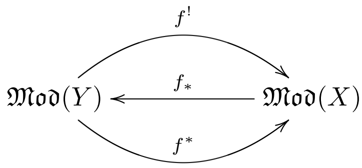
$$f^!\mathscr{G} = f^?\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G}) \longrightarrow f^{-1}\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G})$$

(ii) If $f_*\mathcal{O}_X$ is locally finitely presented, then we have by (i), Proposition 89 and (SGR,Proposition 21) an isomorphism of abelian groups

$$\begin{aligned} f^!(\mathscr{G})_x &\cong (f^{-1}\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G}))_x \\ &\cong \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{G})_{f(x)} \\ &\cong Hom_{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x}, \mathscr{G}_{f(x)}) \end{aligned}$$

One checks this is an isomorphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{G}$, where the Hom set acquires its natural module structure. ☐

Proposition 97. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a closed embedding of ringed spaces. Then we have a triple of adjoints*



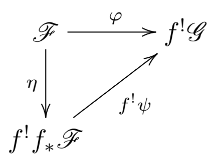
$$f^* \dashv f_* \dashv f^!$$

In particular, the functor f_* is exact.

Proof. We need only show that $f^!$ is right adjoint to f_*. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define a morphism of sheaves of $\mathcal{O}_X$-modules natural in $\mathscr{F}$ by

$$\eta : \mathscr{F} \longrightarrow f^!f_*\mathscr{F}$$
$$\eta_V(s)_W(t) = t \cdot s|_{f^{-1}W}$$

Suppose $\varphi : \mathscr{F} \longrightarrow f^!\mathscr{G}$ is a morphism of sheaves of $\mathcal{O}_X$-modules for a sheaf of $\mathcal{O}_Y$-modules $\mathscr{G}$. We define a morphism of sheaves of $\mathcal{O}_Y$-modules $\psi : f_*\mathscr{F} \longrightarrow \mathscr{G}$ by $\psi_T(s) = \varphi_{f^{-1}T}(s)_T(1)$. This is the unique morphism of $\mathcal{O}_Y$-modules making the following diagram commute



This shows that $f^!$ is right adjoint to f_* with unit η. ☐

Remark 22. Let $f : X \longrightarrow Y$ be a closed embedding of ringed spaces and $\mathscr{F}, \mathscr{G}$ two sheaves of modules on Y. It is not difficult to check that the morphism of abelian groups $f^! : Hom_Y(\mathscr{F}, \mathscr{G}) \longrightarrow Hom_X(f^!\mathscr{F}, f^!\mathscr{G})$ actually sends the action of $\mathcal{O}_Y(Y)$ to the action of $\mathcal{O}_X(X)$, so it is a morphism of modules.

2 Modules over Graded Ringed Spaces

Definition 21. Let X be a topological space. A *sheaf of graded rings* on X is a sheaf of rings $\mathscr{S}$ together with a set of subsheaves of abelian groups $\mathscr{S}_d, d \geq 0$ such that the morphisms $\mathscr{S}_d \longrightarrow \mathscr{S}$ induce an isomorphism of sheaves of abelian groups $\bigoplus_{d \geq 0} \mathscr{S}_d \cong \mathscr{S}$, and for $d, e \geq 0$, open $U \subseteq X$ and $s \in \mathscr{S}_d(U), t \in \mathscr{S}_e(U)$ we must have $st \in \mathscr{S}_{d+e}(U)$. We also require that $1 \in \mathscr{S}_0(X)$. This definition is a special case of the one given in our Algebra in a Category notes.

A *morphism* of graded rings $\mathscr{S}, \mathscr{S}'$ on X is a morphism of sheaves of rings $\mathscr{S} \longrightarrow \mathscr{S}'$ which carries $\mathscr{S}_d$ into $\mathscr{S}'_d$ for all $d \geq 0$. That is, for all $d \geq 0$ there is a commutative diagram of sheaves of abelian groups

$$\begin{array}{ccc} \mathscr{S} & \longrightarrow & \mathscr{S}' \\ \uparrow & & \uparrow \\ \mathscr{S}_d & \longrightarrow & \mathscr{S}'_d \end{array}$$

This makes the sheaves of graded rings on X into a category. If $\mathscr{S}$ is a sheaf of rings on X then we can consider it is as a sheaf of graded rings with $\mathscr{S}_0 = \mathscr{S}$ and $\mathscr{S}_d = 0$ for $d > 0$. A *graded ringed space* is a pair $(X, \mathscr{O}_X)$ consisting of a space X and a sheaf of *commutative* graded rings $\mathcal{O}_X$.

The sheaf of abelian groups $\bigoplus_{d \geq 0} \mathscr{S}_d$ denotes the coproduct in $\mathfrak{Ab}(X)$, which is the sheafification of the presheaf $P : U \mapsto \bigoplus_{d \geq 0} \mathscr{S}_d(U)$ together with the injections $\mathscr{S}_d \longrightarrow P \longrightarrow \bigoplus_{d \geq 0} \mathscr{S}_d$. Given a sheaf of abelian groups $\mathscr{F}$ and a nonempty collection of subsheaves $\{\mathscr{G}_i\}_{i \in I}$ we denote by $\sum_i \mathscr{G}_i$ the image of the induced morphism $\bigoplus_i \mathscr{G}_i \longrightarrow \mathscr{F}$. This is the union of subobjects in the category $\mathfrak{Ab}(X)$. In this case we can describe the subsheaf $\sum_i \mathscr{G}_i$ explicitly as follows

> For $U \subseteq X$ the section $s \in \mathscr{F}(U)$ belongs to $(\sum_i \mathscr{G}_i)(U)$ if and only if for every point $x \in U$ there exists an open neighborhood V with $x \in V \subseteq U$ and elements $a_{i_1}, \ldots, a_{i_n}$ with $a_{i_k} \in \mathscr{G}_{i_k}(V)$ such that $s|_V = a_{i_1} + \cdots + a_{i_n}$.

In particular if $\mathscr{S}$ is a sheaf of graded rings and $s \in \mathscr{S}(U)$ then for every point $x \in U$ we can find $x \in V \subseteq U$ with $s|_V \in \sum_{d \geq 0} \mathscr{S}_d(V)$.

Definition 22. Let X be a topological space and $\mathscr{S}$ a sheaf of graded rings. A *sheaf of graded $\mathscr{S}$-modules* is a sheaf of $\mathscr{S}$-modules $\mathscr{F}$ together with a set of subsheaves of abelian groups $\{\mathscr{F}_n\}_{n \in \mathbb{Z}}$ such that the morphisms $\mathscr{F}_n \longrightarrow \mathscr{F}$ induce an isomorphism of sheaves of abelian groups $\bigoplus_{n \in \mathbb{Z}} \mathscr{F}_n \cong \mathscr{F}$, and such that for open U, $d \geq 0, n \in \mathbb{Z}$ and $s \in \mathscr{S}_d(U), m \in \mathscr{F}_n(U)$ we have $s \cdot m \in \mathscr{F}_{d+n}(U)$. A *morphism* of sheaves of graded $\mathscr{S}$-modules is a morphism of sheaves of $\mathscr{S}$-modules $\mathscr{F} \longrightarrow \mathscr{G}$ which carries $\mathscr{F}_n$ into $\mathscr{G}_n$ for all $n \in \mathbb{Z}$. That is, for all $n \in \mathbb{Z}$ there is a commutative diagram of sheaves of abelian groups

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\phi} & \mathscr{G} \\ \uparrow & & \uparrow \\ \mathscr{F}_n & \xrightarrow[\phi_n]{} & \mathscr{G}_n \end{array}$$

This makes the sheaves of graded $\mathscr{S}$-modules into a preadditive category, denoted $\mathfrak{GrMod}(\mathscr{S})$ or if there is no chance of confusion simply by $\mathfrak{GrMod}(X)$. A *graded submodule* is a monomorphism $\phi : \mathscr{G} \longrightarrow \mathscr{F}$ in $\mathfrak{GrMod}(\mathscr{S})$ with the property that $\phi_U : \mathscr{G}(U) \longrightarrow \mathscr{F}(U)$ is the inclusion of a subset for every open $U \subseteq X$. Every subobject of $\mathscr{F}$ in $\mathfrak{GrMod}(\mathscr{S})$ is equivalent as a subobject to some graded submodule. A *sheaf of homogenous $\mathscr{S}$-ideals* is a graded submodule of $\mathscr{S}$.

If $\mathscr{S}$ is just a sheaf of rings, then we can consider it is a sheaf of graded rings and therefore talk about sheaves of graded $\mathscr{S}$-modules, the category of which we still denote $\mathfrak{GrMod}(\mathscr{S})$.

Proposition 98. *Let X be a topological space and $\mathscr{S}$ a sheaf of graded rings. Then $\mathfrak{GrMod}(\mathscr{S})$ is a complete grothendieck abelian category. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of graded $\mathscr{S}$-modules then ϕ is a monomorphism, epimorphism or isomorphism if and only if it has this property as a morphism of sheaves of $\mathscr{S}$-modules. In particular*

- *ϕ is a monomorphism $\Leftrightarrow \phi_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is injective for all open $U \subseteq X$.*
- *ϕ is an epimorphism $\Leftarrow \phi_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is surjective for all open $U \subseteq X$.*
- *ϕ is an isomorphism $\Leftrightarrow \phi_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is bijective for all open $U \subseteq X$.*

Proof. This is a special case of (LC,Corollary 10) where $(\mathcal{C}, J)$ is the small site of open subsets of X with the open cover topology. □

Lemma 99. *Let X be a topological space and $\mathscr{S}$ a sheaf of graded rings. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of graded $\mathscr{S}$-modules then*

- *ϕ is a monomorphism $\Leftrightarrow \phi_n : M_n \longrightarrow N_n$ is a monomorphism for all $n \in \mathbb{Z}$.*
- *ϕ is an epimorphism $\Leftrightarrow \phi_n : M_n \longrightarrow N_n$ is a epimorphism for all $n \in \mathbb{Z}$.*
- *ϕ is an isomorphism $\Leftrightarrow \phi_n : M_n \longrightarrow N_n$ is a isomorphism for all $n \in \mathbb{Z}$.*

Proof. This is a special case of (LC,Lemma 12). □

Definition 23. Let X be a topological space and $\mathscr{S}$ a sheaf of graded rings on X. If $\mathscr{F}$ is a sheaf of graded $\mathscr{S}$-modules and $d \in \mathbb{Z}$ then let $\mathscr{F}(d)$ denote the sheaf of graded $\mathscr{S}$-modules that is $\mathscr{F}$ with the modified grading $\mathscr{F}(d)_n = \mathscr{F}_{n+d}$. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of graded $\mathscr{S}$-modules then the same morphism of sheaves of modules $\phi(d) : \mathscr{F}(d) \longrightarrow \mathscr{F}'(d)$ is a morphism of sheaves of graded $\mathscr{S}$-modules, and this defines an additive functor $-(d) : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$. In particular we have the sheaf of graded $\mathscr{S}$-modules $\mathscr{S}(d)$ for every $d \in \mathbb{Z}$.

Let $(X, \mathscr{S})$ be a graded ringed space with open subset $U \subseteq X$. Then $\mathscr{S}|_U$ together with the subsheaves of abelian groups $\mathscr{S}_d|_U$ is a graded ringed space, and if $\mathscr{F}$ is a sheaf of graded $\mathscr{S}$-modules then $\mathscr{F}|_U$ together with the submodules $\mathscr{F}_n|_U$ is a sheaf of graded $\mathscr{S}|_U$-modules. Therefore restriction defines an additive functor $\mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S}|_U)$. This is the identity for $U = X$ and the composite of the restriction functors for $V \subseteq U$ and $U \subseteq X$ is the functor for $V \subseteq X$.

2.1 Quasi-Structures

Throughout this section let X be a topological space and $\mathscr{S}$ a sheaf of graded rings on X.

Definition 24. If $\mathscr{F}$ is a sheaf of graded $\mathscr{S}$-modules and $d \in \mathbb{Z}$ then let $\mathscr{F}\{d\}$ denote the submodule of $\mathscr{F}$ given by the internal direct sum $\bigoplus_{n \geq d} \mathscr{F}_n$ together with the grading $\mathscr{F}\{d\}_n = \mathscr{F}_{n+d}$ for $n \geq 0$ and $\mathscr{F}\{d\}_n = 0$ for $n < 0$. This is the sheaf of graded $\mathscr{S}$-modules obtained from $\mathscr{F}(d)$ by removing all negative grades. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of graded $\mathscr{S}$-modules then ϕ restricts to give a morphism of sheaves of graded $\mathscr{S}$-modules $\phi\{d\} = \oplus_{n \geq d} \phi_n : \mathscr{F}\{d\} \longrightarrow \mathscr{G}\{d\}$. This defines an additive exact functor $-\{d\} : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$ (for exactness use Lemma 99, (LC,Corollary 10) and exactness of coproducts in $\mathfrak{Ab}(X)$). If $e \geq 0$ and $d \in \mathbb{Z}$ then it is clear that $(-\{e\}) \circ (-\{d\}) = -\{d + e\}$. In particular if $d \geq 0$ then $(-\{d\}) \circ (-\{-d\}) = -\{0\}$.

We say two sheaves of graded $\mathscr{S}$-modules $\mathscr{F}, \mathscr{G}$ are *quasi-isomorphic* and write $\mathscr{F} \sim \mathscr{G}$ if there exists an integer $d \geq 0$ such that $\mathscr{F}\{d\} \cong \mathscr{G}\{d\}$ as sheaves of graded $\mathscr{S}$-modules. This is an equivalence relation on the class of sheaves of graded $\mathscr{S}$-modules, and it is clear that if $\mathscr{F}, \mathscr{G}$ are isomorphic as sheaves of graded $\mathscr{S}$-modules then they are quasi-isomorphic. Clearly $\mathscr{F} \sim 0$ if and only if there exists $d \geq 0$ such that $\mathscr{F}_n = 0$ for all $n \geq d$.

Let $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves of graded $\mathscr{S}$-modules. We say ϕ is a *quasi-isomorphism* if $\phi\{d\}$ is an isomorphism in $\mathfrak{GrMod}(\mathscr{S})$ for some $d \geq 0$, and similarly we define *quasi-monomorphisms* and *quasi-epimorphisms*. It follows from Lemma 99 that ϕ is a quasi-isomorphism iff. there exists $d \geq 0$ such that $\phi_n : M_n \longrightarrow N_n$ is an isomorphism for all $n \geq d$, and similarly for quasi-monomorphisms and quasi-epimorphisms. Note that $\mathscr{F} \sim \mathscr{G}$ does *not* necessarily mean there exists a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{G}$, although the converse is obviously true.

Lemma 100. *Let $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves of graded $\mathscr{S}$-modules. Then*

(i) ϕ is a quasi-monomorphism $\Leftrightarrow$ $Ker\phi \sim 0$.

(ii) ϕ is a quasi-epimorphism $\Leftrightarrow$ $Coker\phi \sim 0$.

(iii) ϕ is a quasi-isomorphism $\Leftrightarrow$ $Ker\phi \sim 0$ and $Coker\phi \sim 0$.

Proof. All three statements follow immediately from the fact that the functors $-\{d\}$ are exact. □

2.2 Stalks

Throughout this section let $(X, \mathscr{S})$ be a graded ringed spaced space, so $\mathscr{S}$ is a sheaf of commutative graded rings. Since the stalk functor is exact and preserves all colimits, for $x \in X$ the ring $\mathscr{S}_x$ together with the images of the monomorphisms $\mathscr{S}_{d,x} \longrightarrow \mathscr{S}_x$ for $d \geq 0$ is a graded ring.

If $\mathscr{F}$ is a sheaf of graded $\mathscr{S}$-modules then for every $x \in X$, $\mathscr{F}_x$ is a graded $\mathscr{S}_x$-module with the degree n subgroup given by the image of the monomorphism $\mathscr{F}_{n,x} \longrightarrow \mathscr{F}_x$. If $\phi : \mathscr{G} \longrightarrow \mathscr{G}$ is a morphism of sheaves of graded $\mathscr{S}$-modules then $\phi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x$ is a morphism of graded $\mathscr{S}_x$-modules. So taking stalks defines an additive functor $(-)_x : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathscr{S}_x\mathbf{GrMod}$.

2.3 Tensor Products

Throughout this section let $(X, \mathscr{S})$ be a graded ringed space, so $\mathscr{S}$ is a sheaf of commutative graded rings. If $\mathscr{F}, \mathscr{G}$ are sheaves of graded $\mathscr{S}$-modules then we have already defined their tensor product $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}$ as sheaves of $\mathscr{S}$-modules. We want to make this module into a sheaf of graded $\mathscr{S}$-modules in a canonical way. The technique is a generalisation of (GRM,Section 6).

Let Q be the presheaf of abelian groups $Q(U) = \mathscr{F}(U) \otimes_{\mathbb{Z}} \mathscr{G}(U)$ and Z the presheaf of $\mathscr{S}$-modules $Z(U) = \mathscr{F}(U) \otimes_{\mathscr{S}(U)} \mathscr{G}(U)$. There is a canonical morphism of presheaves of abelian groups $a : Q \longrightarrow Z$ given by $a_U(x \otimes y) = x \otimes y$. Let $\alpha : \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G} \longrightarrow \mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}$ the sheafification of a. Since the tensor product of sheaves of abelian groups preserves all colimits by (SGR,Corollary 19) the morphisms $\mathscr{F}_n \otimes_{\mathbb{Z}} \mathscr{G}_m \longrightarrow \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G}$ for $m, n \in \mathbb{Z}$ are a coproduct in $\mathfrak{Ab}(X)$. For $n \in \mathbb{Z}$ we define

$$(\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G})_n = \bigoplus_{p+q=n} \mathscr{F}_p \otimes_{\mathbb{Z}} \mathscr{G}_q$$

Then $\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G} = \bigoplus_n (\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G})_n$. Let $(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_n$ denote the subsheaf of abelian groups of $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}$ given by the image of the composite $(\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G})_n \longrightarrow \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G} \longrightarrow \mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}$. Then using (SGR,Lemma 9) we have

> Given open U and $s \in (\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})(U)$ we have $s \in (\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_n(U)$ if and only if for every point $x \in U$ there is an open set V with $x \in V \subseteq U$ and such that $s|_V = s_1 + \cdots + s_t$ where each s_i is of the form $a \dot{\otimes} b$ for integers $p + q = n$ and $a \in \mathscr{F}_p(V), b \in \mathscr{G}_q(V)$.

This makes it clear that $\mathscr{S}_d(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_n \subseteq (\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_{n+d}$, so to show that these subsheaves make $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}$ into a sheaf of graded $\mathscr{S}$-modules it suffices to show that the morphisms $(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_n \longrightarrow \mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}$ are a coproduct of sheaves of abelian groups. By (SGR,Proposition 16) we reduce to showing that $(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_{n,x} \longrightarrow (\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_x$ is a coproduct for every $x \in X$. Using (SGR,Proposition 20) and the fact that taking stalks is exact and preserves all colimits, we see that $(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_{n,x}$ is the image of the subgroup $\bigoplus_{p+q=n} \mathscr{F}_{p,x} \otimes_{\mathbb{Z}} \mathscr{G}_{q,x}$ under the canonical morphism $\mathscr{F}_x \otimes_{\mathbb{Z}} \mathscr{G}_x \longrightarrow \mathscr{F}_x \otimes_{\mathscr{S}_x} \mathscr{G}_x$. Since $\mathscr{F}_x, \mathscr{G}_x$ are graded $\mathscr{S}_x$-modules, it follows from the construction of the grading of $\mathscr{F}_x \otimes_{\mathscr{S}_x} \mathscr{G}_x$ in (GRM,Section 6) that $\mathscr{F}_x \otimes_{\mathscr{S}_x} \mathscr{G}_x$ is the internal direct sum of these images, which is what we wanted to show. Therefore $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}$ is a sheaf of graded $\mathscr{S}$-modules.

If $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of sheaves of graded $\mathscr{S}$-modules then the morphism of sheaves of $\mathscr{S}$-modules $\mathscr{F} \otimes_{\mathscr{S}} \psi : \mathscr{F} \otimes_{\mathscr{S}} \mathscr{G} \longrightarrow \mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}'$ preserves grade, so we have an additive functor

$$\mathscr{F} \otimes_{\mathscr{S}} - : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$$

Similarly if $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of graded $\mathscr{S}$-modules then so is $\phi \otimes_{\mathscr{S}} \mathscr{G}$, so we have an additive functor

$$- \otimes_{\mathscr{S}} \mathscr{G} : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$$

Clearly the isomorphisms of sheaves of $\mathscr{S}$-modules $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G} \cong \mathscr{G} \otimes_{\mathscr{S}} \mathscr{G}$ and $\mathscr{S} \otimes_{\mathscr{S}} \mathscr{F} \cong \mathscr{F}$ are also isomorphisms of sheaves of graded $\mathscr{S}$-modules. If $U \subseteq X$ is open then the isomorphism of sheaves of $\mathscr{S}|_U$-modules $(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})|_U \cong \mathscr{F}|_U \otimes_{\mathscr{S}|_U} \mathscr{G}|_U$ is also an isomorphism of sheaves of graded $\mathscr{S}|_U$-modules. For $x \in X$ the isomorphism of $\mathscr{S}_x$-modules

$$(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G})_x \cong \mathscr{F}_x \otimes_{\mathscr{S}_x} \mathscr{G}_x$$

is also an isomorphism of graded $\mathscr{S}_x$-modules, with the canonical gradings. Finally, for sheaves of graded $\mathscr{S}$-modules $\mathscr{F}, \mathscr{G}, \mathscr{H}$ the canonical isomorphism of sheaves of $\mathscr{S}$-modules

$$(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G}) \otimes_{\mathscr{S}} \mathscr{H} \cong \mathscr{F} \otimes_{\mathscr{S}} (\mathscr{G} \otimes_{\mathscr{S}} \mathscr{H})$$

is an isomorphism of sheaves of graded $\mathscr{S}$-modules.

Lemma 101. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ sheaves of graded $\mathcal{O}_X$-modules. Then for $p \in \mathbb{Z}$ there is a canonical isomorphism of $\mathcal{O}_X$-modules*

$$(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})_p \cong \bigoplus_{m+n=p} \mathscr{F}_m \otimes_{\mathcal{O}_X} \mathscr{G}_n$$

Proof. For each pair $m, n \in \mathbb{Z}$ with $m + n = p$ we have a morphism of sheaves of $\mathcal{O}_X$-modules $u_{m,n} : \mathscr{F}_m \otimes_{\mathcal{O}_X} \mathscr{G}_n \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ induced by the inclusions. It follows from Corollary 78 that the tensor product preserves colimits in both variables, so the morphisms $u_{m,n}$ are a coproduct. Rearranging the summands, we have

$$\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G} = \bigoplus_p \bigoplus_{m+n=p} \mathscr{F}_m \otimes_{\mathcal{O}_X} \mathscr{G}_n$$

It is easy to check that the image of the monomorphism $\bigoplus_{m+n=p} \mathscr{F}_m \otimes_{\mathcal{O}_X} \mathscr{G}_n \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is $(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})_p$, which yields the desired isomorphism. ☐

Lemma 102. *For any integers $m, n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of graded $\mathscr{S}$-modules*

$$t^{m,n} : \mathscr{S}(m) \otimes_{\mathscr{S}} \mathscr{S}(n) \longrightarrow \mathscr{S}(m+n)$$
$$a \dot{\otimes} b \mapsto ab$$

Proof. As sheaves of $\mathscr{S}$-modules $\mathscr{S}(m), \mathscr{S}(n), \mathscr{S}(m+n)$ are all $\mathscr{S}$, so there is certainly an isomorphism of sheaves of $\mathscr{S}$-modules $t^{m,n}$ mapping $a \dot{\otimes} b$ to ab for open $U \subseteq X$ and $a \in \mathscr{S}(m)(U), b \in \mathscr{S}(n)(U)$. It is not hard to check that this is a morphism of sheaves of graded $\mathscr{S}$-modules. ☐

Lemma 103. *For integers $n, e, d \in \mathbb{Z}$ the following diagram of sheaves of graded $\mathscr{S}$-modules commutes*

$$\begin{array}{ccc}
(\mathscr{S}(n) \otimes \mathscr{S}(e)) \otimes \mathscr{S}(d) & \xRightarrow{\ \lambda\ } & \mathscr{S}(n) \otimes (\mathscr{S}(e) \otimes \mathscr{S}(d)) \\
\Big\Downarrow {\scriptstyle t^{n,e} \otimes 1} & & \Big\Downarrow {\scriptstyle 1 \otimes t^{e,d}} \\
\mathscr{S}(n+e) \otimes \mathscr{S}(d) & & \mathscr{S}(n) \otimes \mathscr{S}(e+d) \\
& {\scriptstyle t^{n+e,d}} \searrow \quad \swarrow {\scriptstyle t^{n,e+d}} & \\
& \mathscr{S}(n+e+d) &
\end{array}$$

Lemma 104. *For any sheaf of graded $\mathscr{S}$-modules $\mathscr{F}$ and integer $n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of graded $\mathscr{S}$-modules natural in $\mathscr{F}$*

$$\mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}(n) \cong \mathscr{F}(n)$$

Proof. The canonical isomorphism of sheaves of graded $\mathscr{S}$-modules $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{S} \cong \mathscr{F}$ is easily seen to map the grading of $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}(n)$ into the grading of $\mathscr{F}(n)$. Naturality in $\mathscr{F}$ is trivial. □

Definition 25. Let $\phi : \mathscr{S} \longrightarrow \mathscr{S}'$ be a morphism of sheaves of commutative graded rings on a topological space X. Restriction of scalars defines an additive functor $F : \mathfrak{GrMod}(\mathscr{S}') \longrightarrow \mathfrak{GrMod}(\mathscr{S})$. We can consider $\mathscr{S}'$ as a sheaf of graded $\mathscr{S}$-modules, so if $\mathscr{F}$ is a sheaf of graded $\mathscr{S}$-modules then $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}'$ is a sheaf of graded $\mathscr{S}$-modules. This grading together with the canonical structure of a sheaf of $\mathscr{S}'$-modules makes $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}'$ into a sheaf of graded $\mathscr{S}'$-modules.

If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of graded $\mathscr{S}$-modules then the canonical morphism of sheaves of $\mathscr{S}'$-modules $\phi \otimes_{\mathscr{S}} \mathscr{S}' : \mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}' \longrightarrow \mathscr{F}' \otimes_{\mathscr{S}} \mathscr{S}'$ is a morphism of sheaves of graded $\mathscr{S}'$-modules. This defines an additive functor $- \otimes_{\mathscr{S}} \mathscr{S}' : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S}')$.

Proposition 105. *Let $\phi : \mathscr{S} \longrightarrow \mathscr{S}'$ be a morphism of sheaves of commutative graded rings on a topological space X. Then we have a diagram of adjoints*

$$\mathfrak{GrMod}(\mathscr{S}) \underset{F}{\overset{-\otimes_{\mathscr{S}}\mathscr{S}'}{\rightleftarrows}} \mathfrak{GrMod}(\mathscr{S}') \qquad -\otimes_{\mathscr{S}}\mathscr{S}' \dashv F$$

Proof. Let $\mathscr{F}$ be a sheaf of graded $\mathscr{S}$-modules and define a morphism of sheaves of graded $\mathscr{S}$-modules natural in $\mathscr{F}$

$$\eta : \mathscr{F} \longrightarrow \mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}'$$
$$\eta_U(s) = s \dot{\otimes} 1$$

So to complete the proof we need only show that the pair $(\mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}', \eta)$ is a reflection of $\mathscr{F}$ along F. If $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ is another morphism of sheaves of graded $\mathscr{S}$-modules, where $\mathscr{G}$ is a sheaf of graded $\mathscr{S}'$-modules, then for each open U define a $\mathscr{S}(U)$-bilinear map

$$\varepsilon : \mathscr{F}(U) \times \mathscr{S}'(U) \longrightarrow \mathscr{G}(U)$$
$$(m, r) \mapsto r \cdot \alpha_U(m)$$

This induces a morphism of sheaves of graded $\mathscr{S}'$-modules $\tau : \mathscr{F} \otimes_{\mathscr{S}} \mathscr{S}' \longrightarrow \mathscr{G}$ which is unique such that $\tau\eta = \alpha$, as required. □

Lemma 106. *Let $\phi : \mathscr{S} \longrightarrow \mathscr{S}'$ be a morphism of sheaves of commutative graded rings on a topological space X. For $n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of graded $\mathscr{S}'$-modules*

$$\mathscr{S}(n) \otimes_{\mathscr{S}} \mathscr{S}' \cong \mathscr{S}'(n)$$

Proof. The canonical isomorphism of sheaves of $\mathscr{S}$-modules $\mathscr{S} \otimes_{\mathscr{S}} \mathscr{S}' \cong \mathscr{S}'$ is a morphism of sheaves of $\mathscr{S}'$-modules which maps the grading of $\mathscr{S}(n) \otimes_{\mathscr{S}} \mathscr{S}'$ into the grading of $\mathscr{S}'(n)$. □

3 Properties of Inverse and Direct Image

Proposition 107. *Let $f : X \longrightarrow Y$ be an isomorphism of ringed spaces with inverse g. There is a canonical natural equivalence $g_* \longrightarrow f^*$, with the property that for a sheaf of modules $\mathscr{F}$ on Y*

$$\kappa : g_*\mathscr{F} \longrightarrow f^*\mathscr{F}$$
$$\kappa_V(a) = [f(V), a] \dot{\otimes} 1$$

for open $V \subseteq X$ and $a \in \mathscr{F}(f(V))$.

Proof. This follows from our Section 2.5 notes. ◻

Proposition 108. *Suppose there is a commutative diagram of morphisms of ringed spaces*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle k}\Big\Downarrow & & \Big\Downarrow{\scriptstyle h} \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

Then there is a canonical natural equivalence of functors $\mu : k_* f^* \cong g^* h_*$ with the property that for a sheaf of modules $\mathscr{F}$ on Y, open $U \subseteq X$ and $T \supseteq f(U)$, $c \in \mathscr{F}(T)$ and $b \in \mathcal{O}_X(U)$ we have

$$\mu_{k(U)}([T, c] \dot{\otimes} b) = [h(T), c] \dot{\otimes} b'$$

where b' is the image of b under the isomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X'}(k(U))$.

Proof. The canonical natural equivalence is the following composite

$$\begin{aligned} k_* f^* &\cong (k^{-1})^* f^* \cong (fk^{-1})^* \\ &= (h^{-1}g)^* \cong g^*(h^{-1})^* \\ &\cong g^* h_* \end{aligned}$$

Using the explicit isomorphisms given above, it is not difficult to check this has the desired action on the special sections in the statement. ◻

Proposition 109. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. There is a canonical isomorphism of sheaves of modules $\xi : f^*\mathcal{O}_Y \cong \mathcal{O}_X$ with the property that for open $U \subseteq X$, $T \supseteq f(U)$ and $a \in \mathcal{O}_Y(T), b \in \mathcal{O}_X(U)$ we have*

$$\xi_U([T, a] \dot{\otimes} b) = b f^\#_T(a)|_U$$

Proof. We constructed this isomorphism in Section 2.5. ◻

Proposition 110. *Let $(X, \mathcal{O}_X)$ be a ringed space and $U \subseteq X$ an open subset with inclusion $f : U \longrightarrow X$. There is a canonical natural equivalence $(-|_U) \cong f^*$ which for a sheaf of modules $\mathscr{F}$ on X, open $V \subseteq U$ and $s \in \mathscr{F}(V)$ has the form*

$$\begin{aligned} \gamma &: \mathscr{F}|_U \longrightarrow f^*\mathscr{F} \\ \gamma_V(s) &= [V, s] \dot{\otimes} 1 \end{aligned}$$

and also

$$\gamma_V^{-1}([T, a] \dot{\otimes} b) = b \cdot a|_V$$

for open $T \supseteq V$ and $a \in \mathscr{F}(T), b \in \mathcal{O}_X(V)$.

Proposition 111. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $U \subseteq X$ open and suppose $V \subseteq Y$ is open with $f(U) \subseteq V$. Let $g : U \longrightarrow V$ be the unique morphism of ringed spaces making the following diagram commute*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\uparrow & & \Big\uparrow \\ U & \xrightarrow[g]{} & V \end{array}$$

Then there is a canonical natural equivalence $(-|_U)f^* \cong g^*(-|_V)$. For a sheaf of modules $\mathscr{F}$ on Y and open $Q \subseteq U, T \supseteq f(Q)$ and $c \in \mathscr{F}(T), b \in \mathcal{O}_X(Q)$ the isomorphism has the form

$$\begin{gathered} \theta : (f^*\mathscr{F})|_U \longrightarrow g^*(\mathscr{F}|_V) \\ \theta_Q([T, c] \dot{\otimes} b) = [T \cap V, c|_{T \cap V}] \dot{\otimes} b \end{gathered}$$

Proof. Use the isomorphism defined in our Section 2.5 notes and explicit formulae of Proposition 110 and Remark 8. ∎

The adjunction of Theorem 19 is natural in open subsets of Y, in the following sense

Proposition 112. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $V \subseteq Y$ an open subset. Let $g : f^{-1}V \longrightarrow V$ be induced by f and let $\mathscr{F}, \mathscr{G}$ be sheaves of modules on X, Y respectively. If $U = f^{-1}V$ and $\theta : g^*(\mathscr{G}|_V) \cong (f^*\mathscr{G})|_U$ is the canonical isomorphism, we claim that the following diagram commutes*

$$\begin{array}{ccc} Hom_{\mathfrak{Mod}(Y)}(\mathscr{G}, f_*\mathscr{F}) & \Longrightarrow & Hom_{\mathfrak{Mod}(X)}(f^*\mathscr{G}, \mathscr{F}) \\ & & \downarrow \\ \downarrow & & Hom_{\mathfrak{Mod}(U)}((f^*\mathscr{G})|_U, \mathscr{F}|_U) \\ & & \Downarrow {\scriptstyle -\circ\theta} \\ Hom_{\mathfrak{Mod}(V)}(\mathscr{G}|_V, g_*(\mathscr{F}|_U)) & \Longrightarrow & Hom_{\mathfrak{Mod}(U)}(g^*(\mathscr{G}|_V), \mathscr{F}|_U) \end{array}$$

Proof. Let a morphism of sheaves of modules $\phi : \mathscr{G} \longrightarrow f_*\mathscr{F}$ be given. It suffices to check both images $g^*(\mathscr{G}|_V) \longrightarrow \mathscr{F}|_U$ have the same action on special sections on open $W \subseteq U$ of the form $[T, c] \dot{\otimes} b$ for open $f(W) \subseteq T \subseteq V$, $c \in \mathscr{G}(T)$ and $b \in \mathcal{O}_X(W)$. Since we know θ explicitly, this is easy to check. ∎

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