Relative Affine Schemes

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The fundamental $Spec(-)$ construction associates an affine scheme to any ring. In this note we study the relative version of this construction, which associates a scheme to any sheaf of algebras. The contents of this note are roughly EGA II §1.2, §1.3.

Contents

1 Affine Morphisms

Definition 1. Let $f : X \longrightarrow Y$ be a morphism of schemes. Then we say f is an *affine morphism* or that X *is affine over* Y, if there is a nonempty open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of Y by open affine subsets V_α such that for every α, $f^{-1}V_\alpha$ is also affine. If X is empty (in particular if Y is empty) then f is affine. Any morphism of affine schemes is affine. Any isomorphism is affine, and the affine property is stable under composition with isomorphisms on either end.

Example 1. Any closed immersion $X \longrightarrow Y$ is an affine morphism by our solution to (Ex 4.3).

Remark 1. A scheme X affine over S is not necessarily affine (for example $X = S$) and if an affine scheme X is an S-scheme, it is not necessarily affine over S. However, if S is separated then an S-scheme X which is affine is affine over S.

Lemma 1. *An affine morphism is quasi-compact and separated. Any finite morphism is affine.*

Proof. Let $f : X \longrightarrow Y$ be affine. Then f is separated since any morphism of affine schemes is separated, and the separatedness condition is local. Since any affine scheme is quasi-compact it is clear that f is quasi-compact. It follows from (Ex. 3.4) that a finite morphism is an affine morphism of a very special type. □

The next two result show that being affine is a property local on the base:

Lemma 2. *If $f : X \longrightarrow Y$ is affine and $V \subseteq Y$ is open then $f^{-1}V \longrightarrow V$ is also affine.*

Proof. Let $\{Y_\alpha\}_{\alpha \in \Lambda}$ be a nonempty affine open cover of Y such that $f^{-1}Y_\alpha$ is affine for every $\alpha \in \Lambda$. For every point $y \in V$ find α such that $y \in V \cap Y_\alpha$. If $Y_\alpha \cong SpecB_\alpha$ then there is $g \in B_\alpha$ with $y \in D(g) \subseteq V \cap Y_\alpha$. If $f^{-1}Y_\alpha \cong SpecA_\alpha$ and $\varphi : B_\alpha \longrightarrow A_\alpha$ corresponds to $f^{-1}Y_\alpha \longrightarrow Y_\alpha$ then $D(\varphi(g)) = f^{-1}D(g)$ is an affine open subset of $f^{-1}V$. □

Lemma 3. *If $f : X \longrightarrow Y$ is a morphism of schemes and $\{Y_\alpha\}_{\alpha \in \Lambda}$ is a nonempty open cover of Y such that $f^{-1}Y_\alpha \longrightarrow Y_\alpha$ is affine for every α, then f is affine.*

Proof. We can take a cover of Y consisting of affine open sets contained in some Y_α whose inverse image is affine. This shows that f is affine. □

Proposition 4. *A morphism $f : X \longrightarrow Y$ is affine if and only if for every open affine subset $U \subseteq Y$ the open subset $f^{-1}U$ is also affine.*

Proof. The condition is clearly sufficient, so suppose that f is affine. Using Lemma 2 we can reduce to the case where $Y = SpecA$ is affine and we only have to show that X is affine. Cover Y with open affines $Y_\alpha \cong SpecB_\alpha$ with $f^{-1}Y_\alpha$ affine. If $x \in Y_\alpha$ then there is $g \in A$ with $x \in D(g) \subseteq Y_\alpha$, and since $D(g)$ corresponds to $D(h)$ for some $h \in B_\alpha$ the open set $f^{-1}D(g)$ is affine. Therefore we can assume $Y_\alpha = D(g_\alpha)$ and further that there is only a finite number $D(g_1), \dots, D(g_n)$ with the g_i generating the unit ideal of A. Let t_i be the image of g_i under the canonical ring morphism $A \longrightarrow \mathcal{O}_X(X)$. Then the t_i generate the unit ideal of $\mathcal{O}_X(X)$ and the open sets $X_{t_i} = f^{-1}D(g_i)$ cover X, so by Ex. 2.17 the scheme X is affine. $\square$

Proposition 5. *Affine morphisms are stable under pullback. That is, suppose we have a pullback diagram*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ G & \xrightarrow[f]{} & H \end{array}$$

If f is an affine morphism then so is g.

Proof. By Lemma 3 it suffices to find an open neighborhood V of each point $y \in Y$ such that $g^{-1}V \longrightarrow V$ is affine. Given $y \in Y$ let S be an affine open neighborhood of $q(y)$ and set $U = f^{-1}S, V = q^{-1}S$. By assumption U is affine and $g^{-1}V = p^{-1}U$ so we have a pullback diagram

$$\begin{array}{ccc} g^{-1}V & \xrightarrow{g_V} & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array}$$

If $W \subseteq V$ is affine then $g_V^{-1}W = U \times_S W$ is a pullback of affine schemes and therefore affine. Therefore g_V is an affine morphism, as required. $\square$

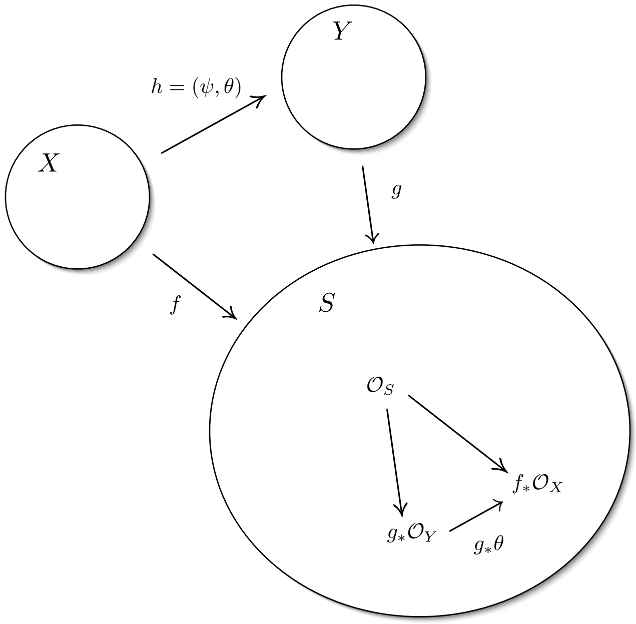
Lemma 6. *Let X be a scheme and $\mathscr{L}$ an invertible sheaf with global section $f \in \Gamma(X, \mathscr{L})$. Then the inclusion $X_f \longrightarrow X$ is affine.*

Proof. The open set X_f is defined in (MOS,Lemma 29), and it follows from Remark (MOS,Remark 2) that we can find an affine open cover of X whose intersections with X_f are all affine. This shows that the inclusion $X_f \longrightarrow X$ is an affine morphism of schemes. $\square$

If X is a scheme over S with structural morphism $f : X \longrightarrow S$ then we denote by $\mathscr{A}(X)$ the canonical $\mathcal{O}_S$-algebra $f_*\mathcal{O}_X$, when there is no chance of confusion. If X is an S-scheme affine over S then it follows from Lemma 1 and (5.18) that $\mathscr{A}(X)$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. If X, Y are two schemes over S with structural morphisms f, g then a morphism $h : X \longrightarrow Y$ of schemes over S consists of a continuous map $\psi : X \longrightarrow Y$ and a morphism of sheaves of rings $\theta : \mathcal{O}_Y \longrightarrow h_*\mathcal{O}_X$. The composite $g_*\mathcal{O}_Y \longrightarrow g_*h_*\mathcal{O}_X = f_*\mathcal{O}_X$ is a morphism of $\mathcal{O}_S$-algebras $\mathscr{A}(h) : \mathscr{A}(Y) \longrightarrow \mathscr{A}(X)$. This defines a contravariant functor

$$\mathscr{A}(-) : \mathbf{Sch}/S \longrightarrow \mathfrak{Alg}(S)$$

from the category of schemes over S to the category of sheaves of commutative $\mathcal{O}_S$-algebras.



We showed in (Ex 2.4) that for a ring B, a B-algebra A and B-scheme X there is a bijection $Hom_B(X, SpecA) \cong Hom_B(A, \mathcal{O}_X(X))$ between morphisms of schemes over $SpecB$ and morphisms of B-algebras. We now present a relative version of this result:

Proposition 7. *Let X, Y be S-schemes with Y affine over S. Then the map $h \mapsto \mathscr{A}(h)$ gives a bijection $Hom_S(X,Y) \cong Hom_{\mathfrak{Alg}(S)}(\mathscr{A}(Y), \mathscr{A}(X))$.*

Proof. We begin with the case where $Y = SpecA$ and $S = SpecB$ are affine for a B-algebra A. We know from (Ex 2.4) that there is a bijection $Hom_S(X,Y) \cong Hom_B(A, \mathcal{O}_X(X))$. Let $f : Y \longrightarrow S$ be the structural morphism, so that $\mathscr{A}(Y) = f_*\mathcal{O}_Y$. Then by (5.2) we have $\mathscr{A}(Y) \cong \widetilde{A}$ as sheaves of algebras (see our Modules over a Ringed Space notes for the properties of the functor $\widetilde{-} : B\mathbf{Alg} \longrightarrow \mathfrak{Alg}(S)$). Therefore the adjunction $\widetilde{-} \longrightarrow\!\!\!\!\dashv \Gamma$ (these are functors between $B\mathbf{Alg}$ and $\mathfrak{Alg}(S)$) gives us a bijection

$$Hom_{\mathfrak{Alg}(S)}(\mathscr{A}(Y), \mathscr{A}(X)) \cong Hom_{\mathfrak{Alg}(S)}(\widetilde{A}, \mathscr{A}(X)) \cong Hom_B(A, \mathcal{O}_X(X))$$

It is not difficult to check that this has the desired form, so we have the desired bijection in this special case. One extends easily to the case where $Y \cong SpecA$ and $S \cong SpecB$.

In the general case, cover S with open affines S_α with $g^{-1}S_\alpha$ an affine open subset of Y (by Proposition 4 we may as well assume $\{S_\alpha\}_{\alpha\in\Lambda}$ is the set of *all* affine subsets of S). Let $f_\alpha : f^{-1}S_\alpha \longrightarrow S_\alpha, g_\alpha : g^{-1}S_\alpha \longrightarrow S_\alpha$ be induced by f, g respectively. Then by the special case already treated, the following maps are bijections

$$Hom_{S_\alpha}(f^{-1}S_\alpha, g^{-1}S_\alpha) \longrightarrow Hom_{\mathfrak{Alg}(S_\alpha)}(\mathscr{A}(Y)|_{S_\alpha}, \mathscr{A}(X)|_{S_\alpha}) \qquad (1)$$

Suppose that $h, h' : X \longrightarrow Y$ are morphisms of schemes over S with $\mathscr{A}(h) = \mathscr{A}(h')$. Then $\mathscr{A}(h)|_{S_\alpha} = \mathscr{A}(h')|_{S_\alpha}$ and therefore $h_\alpha = h'_\alpha$. Since the $g^{-1}S_\alpha$ cover Y it follows that $h = h'$. This shows that the map $Hom_S(X,Y) \longrightarrow Hom_{\mathfrak{Alg}(S)}(\mathscr{A}(Y), \mathscr{A}(X))$ is injective. To see that it is surjective, let $\phi : \mathscr{A}(Y) \longrightarrow \mathscr{A}(X)$ be given, and let $h_\alpha : f^{-1}S_\alpha \longrightarrow g^{-1}S_\alpha$ correspond to $\phi|_{S_\alpha}$ via (1). Since every affine open subset of S occurs among the S_α, we can use injectivity of (1) to see that the h_α can be glued to give $h : X \longrightarrow Y$ with $\mathscr{A}(h) = \phi$, which completes the proof. $\square$

Let $\mathbf{Sch}/_A S$ denote the full subcategory of $\mathbf{Sch}/S$ consisting of all schemes affine over S. Then there is an induced functor $\mathscr{A}(-) : \mathbf{Sch}/_A S \longrightarrow \mathfrak{QcoAlg}(S)$ into the full subcategory of all *quasi-coherent* sheaves of commutative $\mathcal{O}_S$-algebras.

Corollary 8. *The functor $\mathscr{A}(-) : \mathbf{Sch}/_A S \longrightarrow \mathfrak{QcoAlg}(S)$ is fully faithful. In particular, if X, Y are S-schemes affine over S then an S-morphism $h : X \longrightarrow Y$ is an isomorphism if and only if $\mathscr{A}(h) : \mathscr{A}(Y) \longrightarrow \mathscr{A}(X)$ is an isomorphism.*

2 The Spec Construction

In this section X is a scheme and $\mathscr{B}$ is a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. Then for an affine open subset $U \subseteq X$ the scheme $Spec\mathscr{B}(U)$ is canonically a scheme over U, via the morphism $\pi_U : Spec\mathscr{B}(U) \longrightarrow Spec\mathcal{O}_X(U) \cong U$. It is clear that π_U is affine, and given another affine open set V we denote $\pi_U^{-1}(U \cap V)$ by $X_{U,V}$. If $U \subseteq V$ are affine open subsets, then let $\rho_{U,V} : Spec\mathscr{B}(U) \longrightarrow Spec\mathscr{B}(V)$ be induced by the restriction $\mathscr{B}(V) \longrightarrow \mathscr{B}(U)$. Our aim in this section is to glue the schemes $Spec\mathscr{B}(U)$ to define a "canonical" scheme affine over X associated to $\mathscr{B}$. We begin with an important technical Lemma (which will also be used in our construction of **Proj**).

Lemma 9. *Let X be a scheme and $U \subseteq V$ affine open subsets. If $\mathscr{F}$ is a quasi-coherent sheaf of modules on X then the following morphism of $\mathcal{O}_X(U)$-modules is an isomorphism*

$$\begin{gathered} \mathscr{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \longrightarrow \mathscr{F}(U) \\ a \otimes b \mapsto b \cdot a|_U \end{gathered}$$

Proof. We reduce immediately to the case where $X = V$, and then to the case where $X = SpecA$ is actually an affine scheme. So there is a ring B and an open immersion $i : Y = SpecB \longrightarrow X$ whose open image in X is U. Factor $U \longrightarrow X$ as an isomorphism $j : U \cong Y$ followed by i. Let $\mathscr{F}$ be a quasi-coherent sheaf of modules on X and induce a morphism of $\mathcal{O}_X(U)$-modules $\mathscr{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U) \longrightarrow \mathscr{F}(U)$ with $a \otimes b \mapsto b \cdot a|_U$. We have to show this is an isomorphism. Using various isomorphisms defined in our Section 2.5 notes we have an isomorphism of abelian groups

$$\begin{aligned} \mathscr{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U) &\cong \mathscr{F}(X) \otimes_A B \cong (\mathscr{F}(X) \otimes_A B)^{\sim}(Y) \\ &\cong i^*\widetilde{\mathscr{F}(X)}(Y) \cong i^*\mathscr{F}(Y) \\ &\cong (j_*\mathscr{F}|_U)(Y) \cong \mathscr{F}(U) \end{aligned}$$

Using the explicit calculations of of these isomorphisms, one can check that it sends $a \otimes b$ to $b \cdot a|_U$, which proves our claim. ∎

Proposition 10. *Let X be a scheme and $U \subseteq V$ affine open subsets. If $\mathscr{B}$ is a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras then the following diagram is a pullback*

$$\begin{array}{ccc} Spec\mathscr{B}(U) & \xrightarrow{\rho_{U,V}} & Spec\mathscr{B}(V) \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_V} \\ U & \longrightarrow & V \end{array}$$

In particular $\rho_{U,V}$ is an open immersion inducing an isomorphism of $Spec\mathscr{B}(U)$ with $\pi_V^{-1}U$.

Proof. It follows from Lemma 9 that the following diagram is a pushout of rings

$$\begin{array}{ccc} \mathcal{O}_X(V) & \longrightarrow & \mathcal{O}_X(U) \\ \downarrow & & \downarrow \\ \mathscr{B}(V) & \longrightarrow & \mathscr{B}(U) \end{array}$$

That is, the morphism $\mathscr{B}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \longrightarrow \mathscr{B}(U)$ given by $a \otimes b \mapsto b \cdot a|_U$ is an isomorphism of $\mathcal{O}_X(V)$-algebras (and $\mathcal{O}_X(U)$-algebras). Applying $Spec$ gives the desired pullback. Since open immersions are stable under pullback, we see that $\rho_{U,V}$ is an open immersion. Since $\pi_V^{-1}U$ is another candidate for the pullback $U \times_V Spec\mathscr{B}(V)$ it follows that the open image of $\rho_{U,V}$ is $\pi_V^{-1}U$. □

For open affine $U \subseteq X$ we denote by $\widetilde{\mathscr{B}(U)}$ the sheaf of algebras on U obtained by taking the direct image along $Spec\mathcal{O}_X(U) \cong U$ of the sheaf of algebras on $Spec\mathcal{O}_X(U)$ corresponding to the $\mathcal{O}_X(U)$-algebra $\mathscr{B}(U)$. There is an isomorphism of sheaves of algebras on U

$$\delta_U : \mathscr{A}(Spec\mathscr{B}(U)) \cong \widetilde{\mathscr{B}(U)}$$

defined as the direct image of the isomorphism of sheaves on $Spec\mathcal{O}_X(U)$ obtained from the algebra analogue of $(5.2d)$ (see our Modules over a Ringed Space notes). The sheaves of algebras are compatible with restriction in the following sense

Lemma 11. *Let X be a scheme, $W \subseteq U$ affine open subsets and $\mathscr{B}$ a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. There is a canonical isomorphism of sheaves of algebras on W*

$$\begin{gathered} \varepsilon_{W,U} : \widetilde{\mathscr{B}(U)}|_W \longrightarrow \widetilde{\mathscr{B}(W)} \\ a\dot{/}s \mapsto a|_W\dot{/}s|_W \end{gathered}$$

Proof. We have the following commutative diagram

$$\begin{array}{ccc} Spec\mathcal{O}_X(W) & \xrightarrow{j} & Spec\mathcal{O}_X(U) \\ {\scriptstyle k}\Downarrow & & \Downarrow{\scriptstyle l} \\ W & \xrightarrow[i]{} & U \end{array}$$

Since j is an open immersion it induces an isomorphism $Spec\mathcal{O}_X(W) \cong Imj$. Let $g : Imj \longrightarrow Spec\mathcal{O}_X(W)$ be the inverse of this isomorphism, and let $s : Imj \longrightarrow W$ be the morphism induced by l. Let $\varepsilon_{W,U}$ be the following isomorphism of sheaves of algebras (using some results from our Modules over Ringed space notes)

$$\begin{aligned} \varepsilon_{W,U} : \widetilde{\mathscr{B}(U)}|_W &= s_*(\widetilde{\mathscr{B}(U)}|_{Imj}) = k_* g_*(\widetilde{\mathscr{B}(U)}|_{Imj}) \\ &\cong k_* j^*(\widetilde{\mathscr{B}(U)}) \cong k_*(\mathscr{B}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(W))^{\sim} \\ &\cong k_* \widetilde{\mathscr{B}(W)} = \widetilde{\mathscr{B}(W)} \end{aligned}$$

Suppose we are given an open set $T \subseteq W$, $a \in \mathscr{B}(U)$ and $s \in \mathcal{O}_X(U)$ with $l^{-1}T \subseteq D(s)$. Then using the explicit forms of the above isomorphisms, one checks that the section $a\dot{/}s \in \widetilde{\mathscr{B}(U)}(T)$ maps to the section $a|_W\dot{/}s|_W$ of $\widetilde{\mathscr{B}(W)}(T)$. □

Since $\mathscr{B}$ is quasi-coherent, there is an isomorphism $\mu_U : \widetilde{\mathscr{B}(U)} \cong \mathscr{B}|_U$ of sheaves of algebras on U. Together with δ_U this gives an isomorphism of sheaves of algebras on U

$$\beta_U = \mu_U \delta_U : \mathscr{A}(Spec\mathscr{B}(U)) \cong \widetilde{\mathscr{B}(U)} \cong \mathscr{B}|_U$$

In particular for affine $W \subseteq U$ and $b \in \mathscr{B}(U)$ we have the following action of β_U:

$$\begin{aligned} (\beta_U)_W : \mathcal{O}_{Spec\mathscr{B}(U)}(X_{U,W}) &\longrightarrow \mathscr{B}(W) \\ b\dot{/}1 &\mapsto b|_W \end{aligned} \tag{2}$$

The schemes $X_{U,V}$ and $X_{V,U}$ are both affine over $U \cap V$ by Lemma 2. And on $U \cap V$ we have an isomorphism of sheaves of algebras

$$\mathscr{A}(X_{U,V}) = \mathscr{A}(Spec\mathscr{B}(U))|_{U \cap V} \cong \mathscr{B}|_{U \cap V} \cong \mathscr{A}(Spec\mathscr{B}(V))|_{U \cap V} = \mathscr{A}(X_{V,U})$$

Therefore by Corollary 8 there is an isomorphism $\theta_{U,V} : X_{U,V} \longrightarrow X_{V,U}$ of schemes over $U \cap V$ with $\mathscr{A}(\theta_{U,V}) = (\beta_U)|_{U\cap V}^{-1}(\beta_V)|_{U\cap V}$. If $U \subseteq V$ then $X_{U,V} = Spec\mathscr{B}(U)$ and we claim that the following diagram commutes

$$\begin{array}{ccc} Spec\mathscr{B}(U) & \overset{\theta_{U,V}}{\Longrightarrow} & X_{V,U} \\ & {\scriptstyle \rho_{U,V}} \searrow & \downarrow \\ & & Spec\mathscr{B}(V) \end{array}$$

The two legs agree on global sections of $Spec\mathscr{B}(V)$ by (2), and they are therefore equal. It is clear that $\theta_{U,V} = \theta_{V,U}^{-1}$ and for affine opens $U, V, W \subseteq X$ we have $\theta_{U,V}(X_{U,V} \cap X_{U,W}) = X_{V,U} \cap X_{V,W}$ since $\theta_{U,V}$ is a morphism of schemes over $U \cap V$. So to glue the $Spec\mathscr{B}(U)$ it only remains to check that $\theta_{U,W} = \theta_{V,W} \circ \theta_{U,V}$ on $X_{U,V} \cap X_{U,W}$. But these are all morphisms of schemes affine over $U \cap V \cap W$, so by using the injectivity of $\mathscr{A}(-)$ the verification is straightforward.

Thus our family of schemes and patches satisfies the conditions of the Glueing Lemma (Ex. 2.12), and we have a scheme $Spec(\mathscr{B})$ together with open immersions $\psi_U : Spec\mathscr{B}(U) \longrightarrow Spec(\mathscr{B})$ for each affine open subset $U \subseteq X$. These morphisms have the following properties:

(a) The open sets $Im\psi_U$ cover $Spec(\mathscr{B})$.

(b) For affine open subsets $U, V \subseteq X$ we have $\psi_U(X_{U,V}) = Im\psi_U \cap Im\psi_V$ and $\psi_V|_{X_{V,U}}\theta_{U,V} = \psi_U|_{X_{U,V}}$.

In particular for affine open subsets $U \subseteq V$ we have a commutative diagram

$$\begin{array}{ccc} Spec\mathscr{B}(V) & \xrightarrow{\psi_V} & Spec(\mathscr{B}) \\ {\scriptstyle \rho_{U,V}} \uparrow & \nearrow {\scriptstyle \psi_U} & \\ Spec\mathscr{B}(U) & & \end{array} \tag{3}$$

The open sets $Im\psi_U$ are a nonempty open cover of $Spec(\mathscr{B})$, and it is a consequence of (b) above and the fact that $\theta_{U,V}$ is a morphism of schemes over $U \cap V$ that the morphisms $Im\psi_U \cong Spec\mathscr{B}(U) \longrightarrow U \longrightarrow X$ can be glued (that is, for open affines U, V the corresponding morphisms agree on $Im\psi_U \cap Im\psi_V$). Therefore there is a unique morphism of schemes $\pi : Spec(\mathscr{B}) \longrightarrow X$ with the property that for every affine open subset $U \subseteq X$ the following diagram commutes

$$\begin{array}{ccc} Spec\mathscr{B}(U) & \xrightarrow{\psi_U} & Spec(\mathscr{B}) \\ {\scriptstyle \pi_U} \downarrow & & \downarrow {\scriptstyle \pi} \\ U & \longrightarrow & X \end{array} \tag{4}$$

In fact it is easy to see that $\pi^{-1}U = Im\psi_U$, the above diagram is also a pullback. Moreover π is affine by Lemma 3 since all the morphisms π_U are affine.

Our next task is to show that $\mathscr{A}(Spec(\mathscr{B})) \cong \mathscr{B}$. For open affine $U \subseteq X$, let $j_U : Im\psi_U \longrightarrow U$ be the morphism induced by π and $\psi'_U : Spec\mathscr{B}(U) \cong Im\psi_U$ the isomorphism induced by ψ_U, so that $j_U\psi'_U = \pi_U$. Then there is an isomorphism of sheaves of algebras on U

$$\begin{aligned} \omega_U : \mathscr{A}(Spec(\mathscr{B}))|_U &= (j_U)_*(\mathcal{O}_{Spec(\mathscr{B})}|_{Im\psi_U}) \\ &\cong (j_U)_*((\psi'_U)_*\mathcal{O}_{Spec\mathscr{B}(U)}) \\ &= (\pi_U)_*\mathcal{O}_{Spec\mathscr{B}(U)} \\ &= \mathscr{A}(Spec\mathscr{B}(U)) \end{aligned}$$

For open affines $W \subseteq U$ let $\pi_U|_{X_{U,W}} : X_{U,W} \longrightarrow W$ be induced from π_U. Then there is an isomorphism of sheaves of algebras on W

$$\begin{aligned} \zeta_{W,U} : \mathscr{A}(Spec\mathscr{B}(U))|_W &= (\pi_U|_{X_{U,W}})_*(\mathcal{O}_{Spec\mathscr{B}(U)}|_{X_{U,W}}) \\ &\cong (\pi_U|_{X_{U,W}})_*((\theta_{W,U})_*\mathcal{O}_{Spec\mathscr{B}(W)}) \\ &= \mathscr{A}(Spec\mathscr{B}(W)) \end{aligned}$$

Despite the complicated notation, if one draws a picture it is straightforward to check that the following diagram commutes

$$\begin{array}{ccc} \mathscr{A}(Spec(\mathscr{B}))|_W & \xRightarrow{\omega_U|_W} & \mathscr{A}(Spec\mathscr{B}(U))|_W \\ & {\scriptstyle\omega_W}\searrow \quad \swarrow{\scriptstyle\zeta_{W,U}} & \\ & \mathscr{A}(Spec\mathscr{B}(W)) & \end{array} \tag{5}$$

We claim that the isomorphism $\zeta_{W,U}$ is compatible with the isomorphism $\varepsilon_{W,U}$ defined earlier.

Lemma 12. *Let X be a scheme, $W \subseteq U$ affine open subsets and $\mathscr{B}$ a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. Then the following diagram of sheaves of algebras on W commutes*

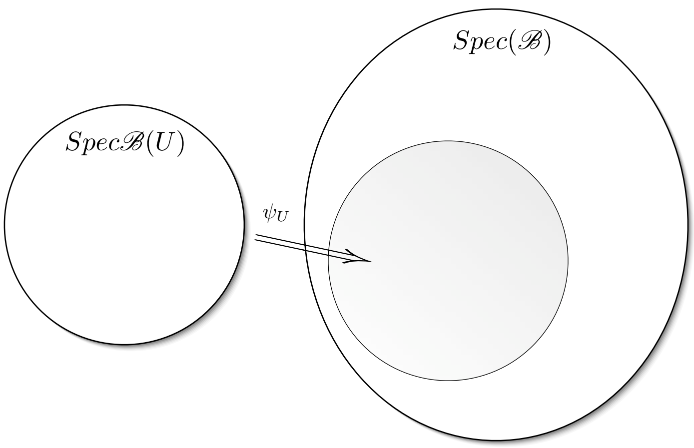
$$\begin{array}{ccccc} \mathscr{A}(Spec\mathscr{B}(U))|_W & \xRightarrow{\delta_U|_W} & \widetilde{\mathscr{B}(U)}|_W & & \\ {\scriptstyle\zeta_{W,U}}\Downarrow & & {\scriptstyle\varepsilon_{W,U}}\Downarrow & \searrow{\scriptstyle\mu_U|_W} & \\ \mathscr{A}(Spec\mathscr{B}(W)) & \xRightarrow[\delta_W]{} & \widetilde{\mathscr{B}(W)} & \xRightarrow[\mu_W]{} & \mathscr{B}|_W \end{array}$$

Proof. First we check that the square on the left commutes by beginning at $\widetilde{\mathscr{B}(U)}|_W$ and showing at the two morphisms to $\mathscr{A}(Spec\mathscr{B}(W))$ agree. For this, we need only check they agree on sections of the form $a\dot{/}s$, and this is straightforward. We can use the same trick to check commutativity of the triangle on the right. □

For an affine open subset $U \subseteq X$ let $\phi_U : \mathscr{A}(Spec(\mathscr{B}))|_U \longrightarrow \mathscr{B}|_U$ be the isomorphism of sheaves of algebras on U given by the composite $\phi_U = \beta_U \omega_U$. Lemma 12 and (5) show that $\phi_U|_W = \phi_W$ for open affines $W \subseteq U$, so together the ϕ_U give an isomorphism of sheaves of algebras $\phi : \mathscr{A}(Spec(\mathscr{B})) \longrightarrow \mathscr{B}$ with $\phi|_U = \phi_U$.

In summary:

Definition 2. Let X be a scheme and $\mathscr{B}$ a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Then we can canonically associate to $\mathscr{B}$ a scheme $\pi : Spec(\mathscr{B}) \longrightarrow X$ affine over X. For every open affine subset $U \subseteq X$ there is an open immersion $\psi_U : Spec\mathscr{B}(U) \longrightarrow Spec(\mathscr{B})$ with the property that the diagram (4) is a pullback and the diagram (3) commutes for any open affines $U \subseteq V$. There is also a canonical isomorphism of sheaves of algebras $\mathscr{A}(Spec(\mathscr{B})) \cong \mathscr{B}$.



Corollary 13. *Let S be a scheme. The functor $\mathscr{A}(-) : \mathbf{Sch}/_A S \longrightarrow \mathfrak{QcoAlg}(S)$ is an equivalence. In particular schemes X, Y affine over S are S-isomorphic if and only if $\mathscr{A}(X) \cong \mathscr{A}(Y)$.*

Proof. We know from Corollary 8 that the functor is fully faithful, and the above construction together with the fact that $\mathscr{A}(Spec(\mathscr{B})) \cong \mathscr{B}$ shows that it is representative. Therefore it is an equivalence. □

If A is a commutative ring, then the composite $\Gamma(-)\mathscr{A}(-)$ gives an equivalence $\mathbf{Sch}/_A SpecA \longrightarrow A\mathbf{Alg}$. Any quasi-coherent sheaf of commutative algebras on $SpecA$ is isomorphic to $\widetilde{B}$ for some commutative A-algebra B. The morphism $SpecB \longrightarrow SpecA$ is affine and $\mathscr{A}(SpecB) \cong \widetilde{B}$, so it follows that $Spec(\widetilde{B}) \cong SpecB$ as schemes over A. In particular any scheme X affine over $SpecA$ is A-isomorphic to $SpecB$ for some commutative A-algebra B. Therefore

Lemma 14. *Let S be an affine scheme. Then an S-scheme X is affine over S if and only if X is an affine scheme.*

3 The Sheaf Associated to a Module

Let X be a scheme and $\mathscr{B}$ a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Let $\mathfrak{Mod}(\mathscr{B})$ denote the category of all sheaves of $\mathscr{B}$-modules and $\mathfrak{Qco}(\mathscr{B})$ the full subcategory of quasi-coherent sheaves of $\mathscr{B}$-modules (SOA,Definition 1). Note that these are precisely the sheaves of $\mathscr{B}$-modules that are quasi-coherent as sheaves of $\mathcal{O}_X$-modules, so in this section there is no harm in simply calling these sheaves "quasi-coherent" (SOA,Proposition 19). In this section we define for every finite morphism $f : X \longrightarrow Y$ a functor

$$\widetilde{-} : \mathfrak{Qco}(\mathscr{A}(X)) \longrightarrow \mathfrak{Mod}(X)$$

which is the relative version of the functor $A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(SpecA)$ for a ring A. Note that $\mathfrak{Qco}(\mathscr{A}(X))$ is an abelian category (SOA,Corollary 20).

Lemma 15. *Let X be a scheme and $U \subseteq V$ affine open subsets. If $\mathscr{B}$ is a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras, and $\mathscr{M}$ a quasi-coherent sheaf of $\mathscr{B}$-modules, then the following morphism of $\mathscr{B}(U)$-modules is an isomorphism*

$$\mathscr{M}(V) \otimes_{\mathscr{B}(V)} \mathscr{B}(U) \longrightarrow \mathscr{M}(U) \qquad (6)$$
$$a \otimes b \mapsto b \cdot a|_U \qquad (7)$$

Proof. Such a morphism of $\mathscr{B}(U)$-modules certainly exists. We know from Lemma 9 that there are isomorphisms of $\mathcal{O}_X(U)$-modules

$$\mathscr{B}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \cong \mathscr{B}(U)$$
$$\mathscr{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \cong \mathscr{M}(U)$$

So at least we have an isomorphism of abelian groups

$$\begin{aligned}\mathscr{M}(V) \otimes_{\mathscr{B}(V)} \mathscr{B}(U) &\cong \mathscr{M}(V) \otimes_{\mathscr{B}(V)} (\mathscr{B}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U)) \\ &\cong (\mathscr{M}(V) \otimes_{\mathscr{B}(V)} \mathscr{B}(V)) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \\ &\cong \mathscr{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \\ &\cong \mathscr{M}(U)\end{aligned}$$

It is easily checked that this map agrees with (6), which is therefore an isomorphism. □

Throughout the remainder of this section, $f : X \longrightarrow Y$ is a finite morphism and $\mathscr{A}(X) = f_*\mathcal{O}_X$ the corresponding commutative quasi-coherent sheaf of $\mathcal{O}_Y$-algebras.

Proposition 16. *Let $\mathscr{M}$ a quasi-coherent sheaf of $\mathscr{A}(X)$-modules. There is a canonical quasi-coherent sheaf of modules $\widetilde{\mathscr{M}}$ on X with the property that for every affine open subset $V \subseteq Y$ there is an isomorphism*

$$\mu_V : \widetilde{\mathscr{M}}|_{f^{-1}V} \longrightarrow (\psi_V)_* \mathscr{M}(V)^{\sim}$$

where $\psi_V : Spec\mathcal{O}_X(f^{-1}V) \longrightarrow f^{-1}V$ is the canonical isomorphism.

Proof. Let $\mathfrak{U}$ be the set of all open affine subsets of Y, so that $\mathfrak{U}' = \{f^{-1}V\}_{V\in\mathfrak{U}}$ is an indexed affine open cover of X. For each $V \in \mathfrak{U}$ we have the $\Gamma(V, \mathscr{A}(X)) = \mathcal{O}_X(f^{-1}V)$-module $\mathscr{M}(V)$ and therefore a sheaf of modules $\mathscr{M}(V)^{\sim}$ on $X_V = Spec\mathcal{O}_X(f^{-1}V)$. Taking the direct image along ψ_V we have a sheaf of modules $\mathscr{M}_V = (\psi_V)_*\mathscr{M}(V)^{\sim}$ on $f^{-1}V$. We want to glue the sheaves $\mathscr{M}_V$.

Let $W \subseteq V$ be affine open susbets of Y and $\rho_{W,V} : X_W \longrightarrow X_V$ the canonical open immersion. Using Lemma 15 we have an isomorphism of sheaves of modules on X_W

$$\alpha_{W,V} : \rho^*_{W,V}(\mathscr{M}(V)^{\sim}) \cong (\mathscr{M}(V) \otimes_{\mathcal{O}_X(f^{-1}V)} \mathcal{O}_X(f^{-1}W))^{\sim} \cong \mathscr{M}(W)^{\sim}$$
$$[V, \dot{m/s}] \dot{\otimes} \dot{b/t} \mapsto (b \cdot m|_W)\dot{/}ts|_W$$

Let $X_{V,W}$ be the affine open subset $(\psi_V)^{-1}(f^{-1}W)$ of X_V, denote by $\rho'_{W,V} : X_W \longrightarrow X_{V,W}$ the isomorphism induced by $\rho_{W,V}$, and let $\psi'_{V,W} : X_{V,W} \longrightarrow f^{-1}W$ be the isomorphism induced by ψ_V. Notice that $\psi'_{V,W} = \psi_W \circ (\rho'_{W,V})^{-1}$. We have an isomorphism of sheaves of modules on $f^{-1}W$

$$\begin{aligned}
\mathscr{M}_W &= (\psi_W)_*(\mathscr{M}(W)^{\sim}) \cong (\psi_W)_*\rho^*_{W,V}(\mathscr{M}(V)^{\sim}) \\
&\cong (\psi_W)_*(\rho'_{W,V})^{-1}_*(\mathscr{M}(V)^{\sim}|_{X_{V,W}}) \\
&= (\psi_W(\rho'_{W,V})^{-1})_*(\mathscr{M}(V)^{\sim}|_{X_{V,W}}) \\
&= (\psi'_{V,W})_*(\mathscr{M}(V)^{\sim}|_{X_{V,W}}) = (\psi_V)_*(\mathscr{M}(V)^{\sim})|_{f^{-1}W} \\
&= \mathscr{M}_V|_{f^{-1}W}
\end{aligned}$$

using (MRS,Proposition 107) and (MRS,Proposition 111). So for every affine open inclusion $W \subseteq V$ we have an isomorphism

$$\varphi_{V,W} : \mathscr{M}_V|_{f^{-1}W} \longrightarrow \mathscr{M}_W$$
$$\dot{m/s} \mapsto m|_W \dot{/} s|_W$$

Clearly $\varphi_{U,U} = 1$ and if $Q \subseteq W \subseteq V$ are open affine subsets then $\varphi_{V,Q} = \varphi_{W,Q} \circ \varphi_{V,W}|_{f^{-1}Q}$. This means that for open affine $U, V \subseteq Y$ the isomorphisms $\varphi^{-1}_{U,W}\varphi_{V,W}$ for open affine $W \subseteq U \cap V$ glue together to give an isomorphism of sheaves of modules

$$\varphi_{V,U} : \mathscr{M}_V|_{f^{-1}U\cap f^{-1}V} \longrightarrow \mathscr{M}_U|_{f^{-1}U\cap f^{-1}V}$$
$$\varphi_{U,W} \circ \varphi_{V,U}|_{f^{-1}W} = \varphi_{V,W} \text{ for affine open } W \subseteq U \cap V$$

The notation is unambiguous, since this definition agrees with the earlier one if $U \subseteq V$. By construction these isomorphisms can be glued (GS,Proposition 1) to give a canonical sheaf of modules $\widetilde{\mathscr{M}}$ on X and a canonical isomorphism of sheaves of modules $\mu_V : \widetilde{\mathscr{M}}|_{f^{-1}V} \longrightarrow \mathscr{M}_V$ for every open affine $V \subseteq Y$. These isomorphisms are compatible in the following sense: we have $\mu_V = \varphi_{U,V} \circ \mu_U$ on $f^{-1}U \cap f^{-1}V$ for any open affine $U, V \subseteq Y$. It is clear that $\widetilde{\mathscr{M}}$ is quasi-coherent since the modules $\mathscr{M}(U)^{\sim}$ are. □

Proposition 17. *If $\beta : \mathscr{M} \longrightarrow \mathscr{N}$ is a morphism of quasi-coherent sheaves of $\mathscr{A}(X)$-modules then there is a canonical morphism $\widetilde{\beta} : \widetilde{\mathscr{M}} \longrightarrow \widetilde{\mathscr{N}}$ of sheaves of modules on X and this defines an additive functor $\widetilde{-} : \mathfrak{Qco}(\mathscr{A}(X)) \longrightarrow \mathfrak{Mod}(X)$.*

Proof. For every affine open $V \subseteq Y$, $\beta_V : \mathscr{M}(V) \longrightarrow \mathscr{N}(V)$ is a morphism of $\Gamma(V, \mathscr{A}(X)) = \mathcal{O}_X(f^{-1}V)$-modules, and therefore gives a morphism $(\beta_V)^{\sim} : \mathscr{M}(V)^{\sim} \longrightarrow \mathscr{N}(V)^{\sim}$ of sheaves of modules on X_V. Let $b_V : \mathscr{M}_V \longrightarrow \mathscr{N}_V$ be the morphism $(\psi_V)_*(\beta_V)^\sim$. One checks easily that for open affine $U, V \subseteq Y$ and $T = f^{-1}(U \cap V)$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{M}_V|_T & \xrightarrow{b_V|_T} & \mathscr{N}_V|_T \\ \Downarrow & & \Downarrow \\ \mathscr{M}_U|_T & \xrightarrow[b_U|_T]{} & \mathscr{N}_U|_T \end{array}$$

Therefore there is a unique morphism of sheaves of modules $\beta^\sim$ with the property that for every affine open $V \subseteq X$ the following diagram commutes (GS,Proposition 6)

$$\begin{array}{ccc} \widetilde{\mathscr{M}}|_{f^{-1}V} & \xrightarrow{\widetilde{\beta}|_{f^{-1}V}} & \widetilde{\mathscr{N}}|_{f^{-1}V} \\ {\scriptstyle \mu_V}\Downarrow & & \Downarrow{\scriptstyle \mu_V} \\ \mathscr{M}_V & \xrightarrow[b_V]{} & \mathscr{N}_V \end{array}$$

Using this unique property it is easy to check that $\widetilde{-}$ defines an additive functor. □

Proposition 18. *The additive functor* $\widetilde{-} : \mathfrak{Qco}(\mathscr{A}(X)) \longrightarrow \mathfrak{Mod}(X)$ *is exact.*

Proof. Since $\mathfrak{Qco}(\mathscr{A}(X))$ is an abelian subcategory of $\mathfrak{Mod}(\mathscr{A}(X))$ (SOA,Corollary 20), a sequence is exact in the former category if and only if it is exact in the latter, which is if and only if it is exact as a sequence of sheaves of abelian groups. So suppose we have an exact sequence of quasi-coherent $\mathscr{A}(X)$-modules

$$\mathscr{M}' \xrightarrow{\varphi} \mathscr{M} \xrightarrow{\psi} \mathscr{M}''$$

This is exact in $\mathfrak{Mod}(X)$, so using (MOS,Lemma 5) we have for every open affine $V \subseteq Y$ an exact sequence of $\mathcal{O}_X(f^{-1}V)$-modules

$$\mathscr{M}'(V) \xrightarrow{\varphi_V} \mathscr{M}(V) \xrightarrow{\psi_V} \mathscr{M}''(V)$$

Since the functor $\widetilde{-} : \mathcal{O}_X(f^{-1}V)\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X_V)$ is exact, we have an exact sequence of sheaves of modules on X_V

$$\mathscr{M}'(V)^\sim \xrightarrow{\widetilde{\varphi_V}} \mathscr{M}(V)^\sim \xrightarrow{\widetilde{\psi_V}} \mathscr{M}''(V)^\sim$$

Applying $(\psi_V)_*$ and using the natural isomorphism $(\psi_V)_*\mathscr{M}(V)^\sim \cong \mathscr{M}^\sim|_{f^{-1}V}$ we see that the following sequence of sheaves of modules on $f^{-1}V$ is exact

$$\mathscr{M}'^\sim|_{f^{-1}V} \xrightarrow{\widetilde{\varphi}|_{f^{-1}U}} \mathscr{M}^\sim|_{f^{-1}V} \xrightarrow{\widetilde{\psi}|_{f^{-1}V}} \mathscr{M}''^\sim|_{f^{-1}V}$$

It now follows from (MRS,Lemma 38) that the functor $\widetilde{-}$ is exact. □

Definition 3. Let $f : X \longrightarrow Y$ be a finite morphism of noetherian schemes and $\mathscr{F}$ a quasi-coherent sheaf of modules on Y. Then $\mathscr{A}(X)$ coherent (H, II Ex.5.5) and therefore the sheaf of $\mathcal{O}_Y$-modules $\mathscr{H}om_{\mathcal{O}_Y}(\mathscr{A}(X), \mathscr{F})$ is quasi-coherent (MOS,Corollary 44). This sheaf becomes a quasi-coherent sheaf of $\mathscr{A}(X)$-modules with the action $(a \cdot \phi)_W(t) = \phi_W(a|_{f^{-1}W}t)$. Therefore we have a quasi-coherent sheaf of modules on X

$$f!(\mathscr{F}) = \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{A}(X), \mathscr{F})^\sim$$

This defines an additive functor $f!(-) : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Qco}(X)$.

Remark 2. For any closed immersion $f : X \longrightarrow Y$ of schemes there is a right adjoint $f^! : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ to the direct image functor f_* (MRS,Proposition 97). In the special case where X, Y are noetherian we have just defined another functor $f! : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Qco}(X)$ and as the notation suggests, these two functors are naturally equivalent.