Relative Affine Schemes

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The fundamental $\text{Spec}(-)$ construction associates an affine scheme to any ring. In this note we study the relative version of this construction, which associates a scheme to any sheaf of algebras. The contents of this note are roughly EGA II §1.2, §1.3.

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1 Affine Morphisms

Definition 1. Let $f : X \rightarrow Y$ be a morphism of schemes. Then we say $f$ is an affine morphism or that $X$ is affine over $Y$, if there is a nonempty open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of $Y$ by open affine subsets $V_\alpha$ such that for every $\alpha$, $f^{-1}V_\alpha$ is also affine. If $X$ is empty (in particular if $Y$ is empty) then $f$ is affine. Any morphism of affine schemes is affine. Any isomorphism is affine, and the affine property is stable under composition with isomorphisms on either end.

Example 1. Any closed immersion $X \rightarrow Y$ is an affine morphism by our solution to (Ex 4.3).

Remark 1. A scheme $X$ affine over $S$ is not necessarily affine (for example $X = S$) and if an affine scheme $X$ is an $S$-scheme, it is not necessarily affine over $S$. However, if $S$ is separated then an $S$-scheme $X$ which is affine is affine over $S$.

Lemma 1. An affine morphism is quasi-compact and separated. Any finite morphism is affine.

Proof. Let $f : X \rightarrow Y$ be affine. Then $f$ is separated since any morphism of affine schemes is separated, and the separatedness condition is local. Since any affine scheme is quasi-compact it is clear that $f$ is quasi-compact. It follows from (Ex. 3.4) that a finite morphism is an affine morphism of a very special type.

The next two result show that being affine is a property local on the base:

Lemma 2. If $f : X \rightarrow Y$ is affine and $V \subseteq Y$ is open then $f^{-1}V \rightarrow V$ is also affine.

Proof. Let $\{Y_\alpha\}_{\alpha \in \Lambda}$ be a nonempty affine open cover of $Y$ such that $f^{-1}Y_\alpha$ is affine for every $\alpha \in \Lambda$. For every point $y \in V$ find $\alpha$ such that $y \in V \cap Y_\alpha$. If $Y_\alpha \cong \text{Spec}B_\alpha$ then there is $g \in B_\alpha$ with $y \in D(g) \subseteq V \cap Y_\alpha$. If $f^{-1}Y_\alpha \cong \text{Spec}A_\alpha$ and $\varphi : B_\alpha \rightarrow A_\alpha$ corresponds to $f^{-1}Y_\alpha \rightarrow Y_\alpha$ then $D(\varphi(g)) = f^{-1}D(g)$ is an affine open subset of $f^{-1}V$.

Lemma 3. If $f : X \rightarrow Y$ is a morphism of schemes and $\{Y_\alpha\}_{\alpha \in \Lambda}$ is a nonempty open cover of $Y$ such that $f^{-1}Y_\alpha \rightarrow Y_\alpha$ is affine for every $\alpha$, then $f$ is affine.

Proof. We can take a cover of $Y$ consisting of affine open sets contained in some $Y_\alpha$ whose inverse image is affine. This shows that $f$ is affine.
**Proposition 4.** A morphism \( f : X \to Y \) is affine if and only if for every open affine subset \( U \subseteq Y \) the open subset \( f^{-1}U \) is also affine.

**Proof.** The condition is clearly sufficient, so suppose that \( f \) is affine. Using Lemma 2 we can reduce to the case where \( Y = \text{Spec} A \) is affine and we only have to show that \( X \) is affine. Cover \( Y \) with open affines \( Y_\alpha = \text{Spec} B_\alpha \) with \( f^{-1}Y_\alpha \) affine. If \( x \in Y_\alpha \) then there is \( g \in A \) with \( x \in D(g) \subseteq Y_\alpha \), and since \( D(g) \) corresponds to \( D(h) \) for some \( h \in B_\alpha \) the open set \( f^{-1}D(g) \) is affine. Therefore we can assume \( Y_\alpha = D(g_\alpha) \) and further that there is only a finite number \( D(g_1), \ldots, D(g_n) \) with the \( g_i \) generating the unit ideal of \( A \). Let \( t_i \) be the image of \( g_i \) under the canonical ring morphism \( A \to \mathcal{O}_X(X) \). Then the \( t_i \) generate the unit ideal of \( \mathcal{O}_X(X) \) and the open sets \( X_{t_i} = f^{-1}D(g_i) \) cover \( X \), so by Ex. 2.17 the scheme \( X \) is affine.

**Proposition 5.** Affine morphisms are stable under pullback. That is, suppose we have a pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{p} & & \downarrow{q} \\
G & \xrightarrow{f} & H
\end{array}
\]

If \( f \) is an affine morphism then so is \( g \).

**Proof.** By Lemma 3 it suffices to find an open neighborhood \( V \) of each point \( y \in Y \) such that \( g^{-1}V \to V \) is affine. Given \( y \in Y \) let \( S \) be an affine open neighborhood of \( q(y) \) and set \( U = f^{-1}S, V = q^{-1}S \). By assumption \( U \) is affine and \( g^{-1}V = p^{-1}U \) so we have a pullback diagram

\[
\begin{array}{ccc}
g^{-1}V & \xrightarrow{g^V} & V \\
\downarrow & & \downarrow \\
U & \to & S
\end{array}
\]

If \( W \subseteq V \) is affine then \( g^V W = U \times_S W \) is a pullback of affine schemes and therefore affine. Therefore \( g^V \) is an affine morphism, as required.

**Lemma 6.** Let \( X \) be a scheme and \( \mathcal{L} \) an invertible sheaf with global section \( f \in \Gamma(X, \mathcal{L}) \). Then the inclusion \( X_f \to X \) is affine.

**Proof.** The open set \( X_f \) is defined in (MOS, Lemma 29), and it follows from Remark (MOS, Remark 2) that we can find an affine open cover of \( X \) whose intersections with \( X_f \) are all affine. This shows that the inclusion \( X_f \to X \) is an affine morphism of schemes.

If \( X \) is a scheme over \( S \) with structural morphism \( f : X \to S \) then we denote by \( \mathcal{A}(X) \) the canonical \( \mathcal{O}_S \)-algebra \( f_*\mathcal{O}_X \). When there is no chance of confusion, if \( X \) is an \( S \)-scheme affine over \( S \) then it follows from Lemma 1 and (5.18) that \( \mathcal{A}(X) \) is a quasi-coherent sheaf of \( \mathcal{O}_S \)-algebras. If \( X, Y \) are two schemes over \( S \) with structural morphisms \( f, g \) then a morphism \( h : X \to Y \) of schemes over \( S \) consists of a continuous map \( \psi : X \to Y \) and a morphism of sheaves of rings \( \theta : \mathcal{O}_Y \to h_*\mathcal{O}_X \). The composite \( g_*\mathcal{O}_Y \to g_*h_*\mathcal{O}_X = f_*\mathcal{O}_X \) is a morphism of \( \mathcal{O}_S \)-algebras \( \mathcal{A}(h) : \mathcal{A}(Y) \to \mathcal{A}(X) \). This defines a contravariant functor

\[
\mathcal{A}(-) : \text{Sch}/S \to \text{Alg}(S)
\]

from the category of schemes over \( S \) to the category of sheaves of commutative \( \mathcal{O}_S \)-algebras.
We begin with the case where $S$ is a ringed space $g:Y\to X$, where $X$ is a scheme. Then by (Ex 2.4) we have a bijection $\text{Hom}_S(X,Y) \cong \text{Hom}_{\mathfrak{Alg}(S)}(\mathfrak{a}(Y),\mathfrak{a}(X))$.

Proof. We begin with the case where $Y = \text{Spec} A$ and $S = \text{Spec} B$ are affine for a $B$-algebra $A$. We know from (Ex 2.4) that there is a bijection $\text{Hom}_S(X,Y) \cong \text{Hom}_B(A,\mathcal{O}_X(X))$. Let $f:Y\to S$ be the structural morphism, so that $\mathfrak{a}(Y) = f_*\mathcal{O}_Y$. Then by (5.2) we have $\mathfrak{a}(Y) \cong \tilde{A}$ as sheaves of algebras (see our Modules over a Ringed Space notes for the properties of the functor $\sim: \mathcal{B}\text{Alg} \to \mathfrak{Alg}(S)$). Therefore the adjunction $\sim \dashv \Gamma$ (these are functors between $\mathcal{B}\text{Alg}$ and $\mathfrak{Alg}(S)$) gives us a bijection

$$\text{Hom}_{\mathfrak{Alg}(S)}(\mathfrak{a}(Y),\mathfrak{a}(X)) \cong \text{Hom}_{\mathfrak{Alg}(S)}(\tilde{A},\mathfrak{a}(X)) \cong \text{Hom}_B(A,\mathcal{O}_X(X))$$

It is not difficult to check that this has the desired form, so we have the desired bijection in this special case. One extends easily to the case where $Y \cong \text{Spec} A$ and $S \cong \text{Spec} B$.

In the general case, cover $S$ with open affines $S_\alpha$ with $g^{-1}S_\alpha$ an affine open subset of $Y$ (by Proposition 4 we may as well assume $\{S_\alpha\}_{\alpha \in \Lambda}$ is the set of all affine subsets of $S$). Let $f_\alpha:f^{-1}S_\alpha \to S_\alpha, g_\alpha : g^{-1}S_\alpha \to S_\alpha$ be induced by $f,g$ respectively. Then by the special case already treated, the following maps are bijections

$$\text{Hom}_{S_\alpha}(f^{-1}S_\alpha,g^{-1}S_\alpha) \to \text{Hom}_{\mathfrak{Alg}(S_\alpha)}(\mathfrak{a}(Y)|_{S_\alpha},\mathfrak{a}(X)|_{S_\alpha})$$

(1)

Suppose that $h,h': X \to Y$ are morphisms of schemes over $S$ with $\mathfrak{a}(h) = \mathfrak{a}(h')$. Then $\mathfrak{a}(h)|_{S_\alpha} = \mathfrak{a}(h')|_{S_\alpha}$ and therefore $h_\alpha = h'_\alpha$. Since the $g^{-1}S_\alpha$ cover $Y$ it follows that $h = h'$. This shows that the map $\text{Hom}_S(X,Y) \to \text{Hom}_{\mathfrak{Alg}(S)}(\mathfrak{a}(Y),\mathfrak{a}(X))$ is injective. To see that it is surjective, let $\phi : \mathfrak{a}(Y) \to \mathfrak{a}(X)$ be given, and let $h_\alpha : f^{-1}S_\alpha \to g^{-1}S_\alpha$ correspond to $\phi|_{S_\alpha}$ via (1). Since every affine open subset of $S$ occurs among the $S_\alpha$, we can use injectivity of (1) to see that the $h_\alpha$ can be glued to give $h : X \to Y$ with $\mathfrak{a}(h) = \phi$, which completes the proof. □

Let $\mathfrak{Sch}/A$ denote the full subcategory of $\mathfrak{Sch}/S$ consisting of all schemes affine over $S$. Then there is an induced functor $\mathfrak{a}(\cdot) : \mathfrak{Sch}/A \to \mathfrak{Qco}\mathfrak{Alg}(S)$ into the full subcategory of all quasi-coherent sheaves of commutative $\mathcal{O}_S$-algebras.
Corollary 8. The functor $\mathcal{A}(-) : \text{Sch}/S \to \text{QcoAlg}(S)$ is fully faithful. In particular, if $X,Y$ are $S$-schemes affine over $S$ then an $S$-morphism $h : X \to Y$ is an isomorphism if and only if $\mathcal{A}(h) : \mathcal{A}(Y) \to \mathcal{A}(X)$ is an isomorphism.

2 The Spec Construction

In this section $X$ is a scheme and $\mathcal{B}$ is a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. Then for an affine open subset $U \subseteq X$ the scheme $\text{Spec} \mathcal{B}(U)$ is canonically a scheme over $U$, via the morphism $\pi_U : \text{Spec} \mathcal{B}(U) \to \text{Spec} \mathcal{O}_X(U) \cong U$. It is clear that $\pi_U$ is affine, and given another affine open set $V$ we denote $\pi_U^{-1}(U \cap V)$ by $X_{U,V}$. If $U \subseteq V$ are affine open subsets, then let $\rho_{U,V} : \text{Spec} \mathcal{B}(U) \to \text{Spec} \mathcal{B}(V)$ be induced by the restriction $\mathcal{B}(V) \to \mathcal{B}(U)$. Our aim in this section is to glue the schemes $\text{Spec} \mathcal{B}(U)$ to define a “canonical” scheme affine over $X$ associated to $\mathcal{B}$. We begin with an important technical Lemma (which will also be used in our construction of $\text{Proj}$).

Lemma 9. Let $X$ be a scheme and $U \subseteq V$ affine open subsets. If $\mathcal{F}$ is a quasi-coherent sheaf of modules on $X$ then the following morphism of $\mathcal{O}_X(U)$-modules is an isomorphism

$$\mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \to \mathcal{F}(U)$$

$$a \otimes b \mapsto b \cdot a|_U$$

Proof. We reduce immediately to the case where $X = V$, and then to the case where $X = \text{Spec} A$ is actually an affine scheme. So there is a ring $B$ and an open immersion $i : Y = \text{Spec} B \to X$ whose open image in $X$ is $U$. Factor $U \to X$ as an isomorphism $j : U \cong Y$ followed by $i$. Let $\mathcal{F}$ be a quasi-coherent sheaf of modules on $X$ and induce a morphism of $\mathcal{O}_X(U)$-modules $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U) \to \mathcal{F}(U)$ with $a \otimes b \mapsto b \cdot a|_U$. We have to show this is an isomorphism. Using various isomorphisms defined in our Section 2.5 notes we have an isomorphism of abelian groups

$$\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U) \cong \mathcal{F}(X) \otimes_{A} B \cong (\mathcal{F}(X) \otimes_{A} B)(Y)$$

$$\cong i^* \mathcal{F}(Y) \cong i^* \mathcal{F}(Y)$$

$$\cong (j_* \mathcal{F}|_U)(Y) \cong \mathcal{F}(U)$$

Using the explicit calculations of these isomorphisms, one can check that it sends $a \otimes b$ to $b \cdot a|_U$, which proves our claim.

Proposition 10. Let $X$ be a scheme and $U \subseteq V$ affine open subsets. If $\mathcal{B}$ is a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras then the following diagram is a pullback

$$\begin{array}{ccc}
\text{Spec} \mathcal{B}(U) & \xrightarrow{\rho_{U,V}} & \text{Spec} \mathcal{B}(V) \\
\pi_U \downarrow & & \downarrow \pi_V \\
U & \to & V
\end{array}$$

In particular $\rho_{U,V}$ is an open immersion inducing an isomorphism of $\text{Spec} \mathcal{B}(U)$ with $\pi_V^{-1} U$.

Proof. It follows from Lemma 9 that the following diagram is a pushout of rings

$$\begin{array}{ccc}
\mathcal{O}_X(V) & \to & \mathcal{O}_X(U) \\
\downarrow & & \downarrow \\
\mathcal{B}(V) & \to & \mathcal{B}(U)
\end{array}$$

That is, the morphism $\mathcal{B}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \to \mathcal{B}(U)$ given by $a \otimes b \mapsto b \cdot a|_U$ is an isomorphism of $\mathcal{O}_X(V)$-algebras (and $\mathcal{O}_X(U)$-algebras). Applying $\text{Spec}$ gives the desired pullback. Since open
immersions are stable under pullback, we see that $\rho_{U,V}$ is an open immersion. Since $\pi^{-1}_V U$ is another candidate for the pullback $U \times_V \text{Spec} \mathcal{B}(V)$ it follows that the open image of $\rho_{U,V}$ is $\pi^{-1}_V U$.

For open affine $U \subseteq X$ we denote by $\overline{\mathcal{B}(U)}$ the sheaf of algebras on $U$ obtained by taking the direct image along $\text{Spec} \mathcal{O}_X(U) \cong U$ of the sheaf of algebras on $\text{Spec} \mathcal{O}_X(U)$ corresponding to the $\mathcal{O}_X(U)$-algebra $\mathcal{B}(U)$. There is an isomorphism of sheaves of algebras on $U$

$$\delta_U : \mathcal{A}(\text{Spec} \mathcal{B}(U)) \cong \overline{\mathcal{B}(U)}$$

defined as the direct image of the isomorphism of sheaves on $\text{Spec} \mathcal{O}_X(U)$ obtained from the algebra analogue of (5.2d) (see our Modules over a Ringed Space notes). The sheaves of algebras are compatible with restriction in the following sense

**Lemma 11.** Let $X$ be a scheme, $W \subseteq U$ affine open subsets and $\mathcal{B}$ a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. There is a canonical isomorphism of sheaves of algebras on $W$

$$\epsilon_{W,U} : \overline{\mathcal{B}(U)}|_W \rightarrow \overline{\mathcal{B}(W)}$$

$$a/s \mapsto a|_W/s|_W$$

**Proof.** We have the following commutative diagram

$$\begin{array}{ccc}
\text{Spec} \mathcal{O}_X(W) & \xrightarrow{j} & \text{Spec} \mathcal{O}_X(U) \\
\downarrow{k} & & \downarrow{i} \\
W & \xrightarrow{\iota} & U
\end{array}$$

Since $j$ is an open immersion it induces an isomorphism $\text{Spec} \mathcal{O}_X(W) \cong \text{Im} j$. Let $g : \text{Im} j \rightarrow \text{Spec} \mathcal{O}_X(W)$ be the inverse of this isomorphism, and let $s : \text{Im} j \rightarrow W$ be the morphism induced by $l$. Let $\epsilon_{W,U}$ be the following isomorphism of sheaves of algebras (using some results from our Modules over Ringed space notes)

$$\epsilon_{W,U} : \overline{\mathcal{B}(U)}|_W = (g_* (\overline{\mathcal{B}(U)}|_{\text{Im} j})) = k_* (s_* (\overline{\mathcal{B}(U)}|_{\text{Im} j})) \cong k_* (\mathcal{B}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(W)) \cong k_* \mathcal{B}(W) = \overline{\mathcal{B}(W)}$$

Suppose we are given an open set $T \subseteq W$, $a \in \mathcal{B}(U)$ and $s \in \mathcal{O}_X(U)$ with $l^{-1} T \subseteq D(s)$. Then using the explicit forms of the above isomorphisms, one checks that the section $a/s \in \overline{\mathcal{B}(U)}(T)$ maps to the section $a|_W/s|_W$ of $\overline{\mathcal{B}(W)}(T)$.

Since $\mathcal{B}$ is quasi-coherent, there is an isomorphism $\mu_U : \overline{\mathcal{B}(U)} \cong \mathcal{B}|_U$ of sheaves of algebras on $U$. Together with $\delta_U$ this gives an isomorphism of sheaves of algebras on $U$

$$\beta_U = \mu_U \delta_U : \mathcal{A}(\text{Spec} \mathcal{B}(U)) \cong \overline{\mathcal{B}(U)} \cong \mathcal{B}|_U$$

In particular for affine $W \subseteq U$ and $b \in \mathcal{B}(U)$ we have the following action of $\beta_U$:

$$(\beta_U)_W : \mathcal{O}_{\text{Spec} \mathcal{B}(U)}(X_{U,W}) \rightarrow \mathcal{B}(W)$$

$$b/1 \mapsto b|_W$$

(2)

The schemes $X_{U,V}$ and $X_{V,U}$ are both affine over $U \cap V$ by Lemma 2. And on $U \cap V$ we have an isomorphism of sheaves of algebras

$$\mathcal{A}(X_{U,V}) = \mathcal{A}(\text{Spec} \mathcal{B}(U))|_{U \cap V} \cong \mathcal{B}|_{U \cap V} \cong \mathcal{A}(\text{Spec} \mathcal{B}(V))|_{U \cap V} = \mathcal{A}(X_{V,U})$$
Therefore by Corollary 8 there is an isomorphism $\theta_{U,V} : X_{U,V} \longrightarrow X_{V,U}$ of schemes over $U \cap V$ with $\mathcal{A}(\theta_{U,V}) = (\beta_U)^{-1}|_{U \cap V}(\beta_U)|_{U \cap V}$. If $U \subseteq V$ then $X_{U,V} = Spec(B(U))$ and we claim that the following diagram commutes

$$
\begin{align*}
Spec(B(U)) & \xrightarrow{\theta_{U,V}} X_{V,U} \\
\rho_{U,V} & \downarrow \quad \downarrow \\
Spec(B(V)) & 
\end{align*}
$$

The two legs agree on global sections of $Spec(B(V))$ by (2), and they are therefore equal. It is clear that $\theta_{U,V} = \theta_{V,U}^{-1}$ and for affine opens $U, V, W \subseteq X$ we have $\theta_{U,V}(X_{U,V} \cap X_{U,W}) = X_{V,U} \cap X_{V,W}$ since $\theta_{U,V}$ is a morphism of schemes over $U \cap V$. So to glue the $Spec(B(U))$ it only remains to check that $\theta_{U,W} = \theta_{V,W} \circ \theta_{U,V}$ on $X_{U,V} \cap X_{U.W}$. But these are all morphisms of schemes affine over $U \cap V \cap W$, so by using the injectivity of $\mathcal{A}(-)$ the verification is straightforward.

Thus our family of schemes and patches satisfies the conditions of the Glueing Lemma (Ex. 2.12), and we have a scheme $Spec(\mathcal{A})$ together with open immersions $\psi_U : Spec(B(U)) \longrightarrow Spec(\mathcal{A})$ for each affine open subset $U \subseteq X$. These morphisms have the following properties:

(a) The open sets $Im\psi_U$ cover $Spec(\mathcal{A})$.

(b) For affine open subsets $U, V \subseteq X$ we have $\psi_U(X_{U,V}) = Im\psi_U \cap Im\psi_V$ and $\psi_V|_{X_{V,U}} \theta_{U,V} = \psi_U|_{X_{U,V}}$.

In particular for affine open subsets $U \subseteq V$ we have a commutative diagram

$$
\begin{align*}
Spec(\mathcal{A}(V)) & \xrightarrow{\psi_V} Spec(\mathcal{A}) \\
\rho_{U,V} & \downarrow \quad \downarrow \\
Spec(B(U)) & 
\end{align*}
$$

The open sets $Im\psi_U$ are a nonempty open cover of $Spec(\mathcal{A})$, and it is a consequence of (b) above and the fact that $\theta_{U,V}$ is a morphism of schemes over $U \cap V$ that the morphisms $Im\psi_U \cong Spec(B(U)) \longrightarrow U \longrightarrow X$ can be glued (that is, for open affines $U, V$ the corresponding morphisms agree on $Im\psi_U \cap Im\psi_V$). Therefore there is a unique morphism of schemes $\pi : Spec(\mathcal{A}) \longrightarrow X$ with the property that for every affine open subset $U \subseteq X$ the following diagram commutes

$$
\begin{align*}
Spec(\mathcal{A}(U)) & \xrightarrow{\psi_U} Spec(\mathcal{A}) \\
\pi_U & \downarrow \quad \downarrow \pi \\
U & \longrightarrow X 
\end{align*}
$$

In fact it is easy to see that $\pi^{-1}U = Im\psi_U$, the above diagram is also a pullback. Moreover $\pi$ is affine by Lemma 3 since all the morphisms $\pi_U$ are affine.

Our next task is to show that $\mathcal{A}(Spec(\mathcal{A})) \cong \mathcal{A}$. For open affine $U \subseteq X$, let $j_U : Im\psi_U \longrightarrow U$ be the morphism induced by $\pi$ and $\psi_U : Spec(\mathcal{A}(U)) \cong Im\psi_U$ the isomorphism induced by $\psi_U$, so that $j_U \psi_U = \pi_U$. Then there is an isomorphism of sheaves of algebras on $U$

$$
\omega_U : \mathcal{A}(Spec(\mathcal{A}))|_U = (j_U)_* (O_{Spec(\mathcal{A})}|_{Im\psi_U}) \\
\cong (j_U)_* ((\psi_U)_* O_{Spec(B(U))}) \\
= (\pi_U)_* O_{Spec(\mathcal{A}(U))} \\
= \mathcal{A}(Spec(\mathcal{A}(U)))
$$
For open affines \( W \subseteq U \) let \( \pi_U|_{X_{U,W}} : X_{U,W} \to W \) be induced from \( \pi_U \). Then there is an isomorphism of sheaves of algebras on \( W \)

\[
\zeta_{W,U} : \mathcal{A}(\text{Spec} \mathcal{B}(U))|_W = (\pi_U|_{X_{U,W}})_*(\mathcal{O}_{\text{Spec} \mathcal{B}(U)}|_{X_{U,W}}) \\
\cong (\pi_U|_{X_{U,W}})_*(\theta_{W,U})_*\mathcal{O}_{\text{Spec} \mathcal{B}(W)}) \\
= \mathcal{A}(\text{Spec} \mathcal{B}(W))
\]

Despite the complicated notation, if one draws a picture it is straightforward to check that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{A}(\text{Spec} \mathcal{B}(U))|_W & \xrightarrow{\omega_{W|U}} & \mathcal{A}(\text{Spec} \mathcal{B}(U))|_W \\
\downarrow{\zeta_{W,U}} & & \downarrow{\zeta_{W,U}} \\
\mathcal{A}(\text{Spec} \mathcal{B}(W)) & \xrightarrow{\omega_W} & \mathcal{A}(\text{Spec} \mathcal{B}(W))
\end{array}
\] (5)

We claim that the isomorphism \( \zeta_{W,U} \) is compatible with the isomorphism \( \varepsilon_{W,U} \) defined earlier.

**Lemma 12.** Let \( X \) be a scheme, \( W \subseteq U \) affine open subsets and \( \mathcal{B} \) a quasi-coherent sheaf of commutative \( \mathcal{O}_X \)-algebras. Then the following diagram of sheaves of algebras on \( W \) commutes

\[
\begin{array}{ccc}
\mathcal{A}(\text{Spec} \mathcal{B}(U))|_W & \xrightarrow{\delta_{U|W}} & \mathcal{B}(U)|_W \\
\downarrow{\zeta_{W,U}} & & \downarrow{\varepsilon_{W,U}} \\
\mathcal{A}(\text{Spec} \mathcal{B}(W)) & \xrightarrow{\delta_W} & \mathcal{B}(W) \\
& & \xrightarrow{\mu_W} \mathcal{B}|_W
\end{array}
\]

**Proof.** First we check that the square on the left commutes by beginning at \( \mathcal{B}(U)|_W \) and showing at the two morphisms to \( \mathcal{A}(\text{Spec} \mathcal{B}(W)) \) agree. For this, we need only check they agree on sections of the form \( a/s \), and this is straightforward. We can use the same trick to check commutativity of the triangle on the right. \( \square \)

For an affine open subset \( U \subseteq X \) let \( \phi_U : \mathcal{A}(\text{Spec} \mathcal{B}(|U|) \to \mathcal{B}|_U \) be the isomorphism of sheaves of algebras on \( U \) given by the composite \( \phi_U = \beta_U \omega_U \). Lemma 12 and (5) show that \( \phi_U|_W = \phi_W \) for open affines \( W \subseteq U \), so together the \( \phi_U \) give an isomorphism of sheaves of algebras \( \phi : \mathcal{A}(\text{Spec} \mathcal{B}) \to \mathcal{B} \) with \( \phi|_U = \phi_U \).

In summary:

**Definition 2.** Let \( X \) be a scheme and \( \mathcal{B} \) a commutative quasi-coherent sheaf of \( \mathcal{O}_X \)-algebras. Then we can canonically associate to \( \mathcal{B} \) a scheme \( \pi : \text{Spec} \mathcal{B} \to X \) affine over \( X \). For every open affine subset \( U \subseteq X \) there is an open immersion \( \psi_U : \text{Spec} \mathcal{B}(U) \to \text{Spec} \mathcal{B} \) with the property that the diagram (4) is a pullback and the diagram (3) commutes for any open affines \( U \subseteq V \). There is also a canonical isomorphism of sheaves of algebras \( \mathcal{A}(\text{Spec} \mathcal{B}) \cong \mathcal{B} \).
Corollary 13. Let $S$ be a scheme. The functor $\mathcal{A}(-) : \mathbf{Sch}/_A S \rightarrow \mathbf{Qco}(S)$ is an equivalence. In particular schemes $X, Y$ affine over $S$ are $S$-isomorphic if and only if $\mathcal{A}(X) \cong \mathcal{A}(Y)$.

Proof. We know from Corollary 8 that the functor is fully faithful, and the above construction together with the fact that $\mathcal{A}(\text{Spec}(B)) \cong B$ shows that it is representative. Therefore it is an equivalence.

If $A$ is a commutative ring, then the composite $\Gamma(-)\mathcal{A}(-)$ gives an equivalence $\mathbf{Sch}/_A \text{Spec}A \rightarrow A\mathbf{Alg}$. Any quasi-coherent sheaf of commutative algebras on $\text{Spec}A$ is isomorphic to $B$ for some commutative $A$-algebra $B$. The morphism $\text{Spec}B \rightarrow \text{Spec}A$ is affine and $\mathcal{A}(\text{Spec}B) \cong \tilde{B}$, so it follows that $\text{Spec}(\tilde{B}) \cong \text{Spec}B$ as schemes over $A$. In particular any scheme $X$ affine over $\text{Spec}A$ is $A$-isomorphic to $\text{Spec}B$ for some commutative $A$-algebra $B$. Therefore

Lemma 14. Let $S$ be an affine scheme. Then an $S$-scheme $X$ is affine over $S$ if and only if $X$ is an affine scheme.

3 The Sheaf Associated to a Module

Let $X$ be a scheme and $\mathcal{B}$ a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Let $\mathbf{Mod}(\mathcal{B})$ denote the category of all sheaves of $\mathcal{B}$-modules and $\mathbf{Qco}(\mathcal{B})$ the full subcategory of quasi-coherent sheaves of $\mathcal{B}$-modules (SOA, Definition 1). Note that these are precisely the sheaves of $\mathcal{B}$-modules that are quasi-coherent as sheaves of $\mathcal{O}_X$-modules, so in this section there is no harm in simply calling these sheaves “quasi-coherent” (SOA, Proposition 19). In this section we define for every finite morphism $f : X \rightarrow Y$ a functor

$$\sim : \mathbf{Qco}(\mathcal{A}(X)) \rightarrow \mathbf{Mod}(X)$$

which is the relative version of the functor $A\mathbf{Mod} \rightarrow \mathbf{Mod}(\text{Spec}A)$ for a ring $A$. Note that $\mathbf{Qco}(\mathcal{A}(X))$ is an abelian category (SOA, Corollary 20).

Lemma 15. Let $X$ be a scheme and $U \subseteq V$ affine open subsets. If $\mathcal{B}$ is a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras, and $\mathcal{M}$ a quasi-coherent sheaf of $\mathcal{B}$-modules, then the following morphism of $\mathcal{B}(U)$-modules is an isomorphism

$$\mathcal{M}(V) \otimes_{\mathcal{B}(V)} \mathcal{B}(U) \rightarrow \mathcal{M}(U)$$

$$a \otimes b \mapsto b \cdot a|_U$$

(6) (7)

Proof. Such a morphism of $\mathcal{B}(U)$-modules certainly exists. We know from Lemma 9 that there are isomorphisms of $\mathcal{O}_X(U)$-modules

$$\mathcal{B}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \cong \mathcal{B}(U)$$

$$\mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \cong \mathcal{M}(U)$$

So at least we have an isomorphism of abelian groups

$$\mathcal{M}(V) \otimes_{\mathcal{B}(V)} \mathcal{B}(U) \cong \mathcal{M}(V) \otimes_{\mathcal{B}(V)} (\mathcal{B}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U))$$

$$\cong (\mathcal{M}(V) \otimes_{\mathcal{B}(V)} \mathcal{B}(V)) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U)$$

$$\cong \mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U)$$

$$\cong \mathcal{M}(U)$$

It is easily checked that this map agrees with (6), which is therefore an isomorphism.

Throughout the remainder of this section, $f : X \rightarrow Y$ is a finite morphism and $\mathcal{A}(X) = f_*\mathcal{O}_X$ the corresponding commutative quasi-coherent sheaf of $\mathcal{O}_Y$-algebras.
Proposition 16. Let $\mathcal{M}$ a quasi-coherent sheaf of $\mathcal{A}(X)$-modules. There is a canonical quasi-coherent sheaf of modules $\mathcal{M}$ on $X$ with the property that for every affine open subset $V \subseteq Y$ there is an isomorphism

$$\mu_V : \mathcal{M}|_{f^{-1}V} \longrightarrow (\psi_V)_*(\mathcal{M}(V)^\sim)$$

where $\psi_V : Spec\mathcal{O}_X(f^{-1}V) \longrightarrow f^{-1}V$ is the canonical isomorphism.

Proof. Let $\mathcal{U}$ be the set of all affine subsets of $Y$, so that $\mathcal{U}' = \{f^{-1}V\}_{V \in \mathcal{U}}$ is an indexed affine open cover of $X$. For each $V \in \mathcal{U}$ we have the $\Gamma(V, \mathcal{A}(X)) = \mathcal{O}_X(f^{-1}V)$-module $\mathcal{M}(V)$ and therefore a sheaf of modules $\mathcal{M}(V)$ on $X = Spec\mathcal{O}_X(f^{-1}V)$. Taking the direct image along $\psi_V$ we have a sheaf of modules $\mathcal{M}_V = (\psi_V)_*(\mathcal{M}(V)^\sim)$ on $f^{-1}V$. We want to glue the sheaves $\mathcal{M}_V$.

Let $W \subseteq V$ be affine open subsets of $Y$ and $\rho_{W,V} : X_W \longrightarrow X_V$ the canonical open immersion. Using Lemma 15 we have an isomorphism of sheaves of modules on $X_W$

$$\alpha_{W,V} : \rho_{W,V}^*(\mathcal{M}(V)^\sim) \cong (\mathcal{M}(V) \otimes_{\mathcal{O}_X(f^{-1}V)} \mathcal{O}_X(f^{-1}W))^\sim \cong \mathcal{M}(W)^\sim$$

Let $X_{V,W}$ be the affine open subset $(\psi_V)^{-1}(f^{-1}W)$ of $X_V$, denote by $\rho_{W,V} : X_W \longrightarrow X_{V,W}$ the isomorphism induced by $\rho_{W,V}$, and let $\psi_{V,W} : X_{V,W} \longrightarrow f^{-1}W$ be the isomorphism induced by $\psi_V$. Notice that $\psi_{V,W} = \psi_W \circ (\rho_{W,V})^{-1}$. We have an isomorphism of sheaves of modules on $f^{-1}W$

$$\mathcal{M}_W = (\psi_W)_*(\mathcal{M}(W)^\sim) \cong (\psi_W)_*\rho_{W,V}^*(\mathcal{M}(V)^\sim)$$

$$\cong (\psi_W)_*(\rho_{W,V}^*(\mathcal{M}(V)^\sim)|_{X_{V,W}})$$

$$= (\psi_W)_*(\rho_{W,V}^*(\mathcal{M}(V)^\sim)|_{X_{V,W}}|_{f^{-1}W})$$

$$= (\mathcal{M}_V|_{f^{-1}W})$$

using (MRS, Proposition 107) and (MRS, Proposition 111). So for every affine open inclusion $W \subseteq V$ we have an isomorphism

$$\varphi_{V,W} : \mathcal{M}_W|_{f^{-1}W} \longrightarrow \mathcal{M}_W$$

$$m/s \mapsto m|_W/s|_W$$

Clearly $\varphi_{U,U} = 1$ and if $Q \subseteq W \subseteq V$ are open affine subsets then $\varphi_{V,Q} = \varphi_{W,Q} \circ \varphi_{V,W}|_{f^{-1}Q}$. This means that for open affine $U, V \subseteq Y$ the isomorphisms $\varphi_{V,W}^{-1} \varphi_{U,W}$ for open affine $W \subseteq U \cap V$ glue together to give an isomorphism of sheaves of modules

$$\varphi_{V,U} : \mathcal{M}_V|_{f^{-1}U \cap f^{-1}V} \longrightarrow \mathcal{M}_U|_{f^{-1}U \cap f^{-1}V}$$

$$\varphi_{U,W} \circ \varphi_{V,U}|_{f^{-1}W} = \varphi_{V,W}$$ for affine open $W \subseteq U \cap V$

The notation is unambiguous, since this definition agrees with the earlier one if $U \subseteq V$. By construction these isomorphisms can be glued (GS, Proposition 1) to give a canonical sheaf of modules $\mathcal{M}$ on $X$ and a canonical isomorphism of sheaves of modules $\mu_V : \mathcal{M}|_{f^{-1}V} \longrightarrow \mathcal{M}_V$ for every open affine $V \subseteq Y$. These isomorphisms are compatible in the following sense: we have $\mu_V = \varphi_{U,V} \circ \mu_U$ on $f^{-1}U \cap f^{-1}V$ for any open affine $U, V \subseteq Y$. It is clear that $\mathcal{M}$ is quasi-coherent since the modules $(\mathcal{M}(U)^\sim)$ are.

Proposition 17. If $\beta : \mathcal{M} \longrightarrow \mathcal{N}$ is a morphism of quasi-coherent sheaves of $\mathcal{A}(X)$-modules then there is a canonical morphism $\tilde{\beta} : \mathcal{M} \longrightarrow \mathcal{N}$ of sheaves of modules on $X$ and this defines an additive functor $\mathcal{O}(\mathcal{Q}(\mathcal{A}(X))) \longrightarrow Mod(\mathcal{X})$.

Proof. For every affine open $V \subseteq Y$, $\beta_V : \mathcal{M}(V) \longrightarrow \mathcal{N}(V)$ is a morphism of $\Gamma(V, \mathcal{A}(X)) = \mathcal{O}_X(f^{-1}V)$-modules, and therefore gives a morphism $(\beta_V)^\sim : \mathcal{M}(V)^\sim \longrightarrow \mathcal{N}(V)^\sim$ of sheaves of
modules on $X_V$. Let $b_V : \mathcal{M}_V \rightarrow \mathcal{N}_V$ be the morphism $(\psi_V)_*(\beta_V)^\sim$. One checks easily that for open affine $U, V \subseteq Y$ and $T = f^{-1}(U \cap V)$ the following diagram commutes

$$
\begin{array}{c}
\mathcal{M}_V|_T \xrightarrow{b_V|_T} \mathcal{N}_V|_T \\
\downarrow \quad \downarrow \\
\mathcal{M}_U|_T \xrightarrow{b_U|_T} \mathcal{N}_U|_T
\end{array}
$$

Therefore there is a unique morphism of sheaves of modules $\beta^\sim$ with the property that for every affine open $V \subseteq X$ the following diagram commutes (GS, Proposition 6)

$$
\begin{array}{c}
\mathcal{M}|_{f^{-1}V} \xrightarrow{\beta|_{f^{-1}V}} \mathcal{N}|_{f^{-1}V} \\
\mu_V \downarrow \quad \downarrow \mu_V \\
\mathcal{M}|_V \xrightarrow{b_V} \mathcal{N}|_V
\end{array}
$$

Using this unique property it is easy to check that $\sim$ defines an additive functor.

**Proposition 18.** The additive functor $\sim : \mathcal{Qco}(\mathscr{A}(X)) \rightarrow \mathcal{Mod}(X)$ is exact.

**Proof.** Since $\mathcal{Qco}(\mathscr{A}(X))$ is an abelian subcategory of $\mathcal{Mod}(\mathscr{A}(X))$ (SOA, Corollary 20), a sequence is exact in the former category if and only if it is exact in the latter, which is if and only if it is exact as a sequence of sheaves of abelian groups. So suppose we have an exact sequence of quasi-coherent $\mathscr{A}(X)$-modules

$$
\begin{array}{c}
\mathcal{M}' \xrightarrow{\varphi} \mathcal{M} \xrightarrow{\psi} \mathcal{M}''
\end{array}
$$

This is exact in $\mathcal{Mod}(X)$, so using (MOS, Lemma 5) we have for every open affine $V \subseteq Y$ an exact sequence of $\mathcal{O}_X(f^{-1}V)$-modules

$$
\begin{array}{c}
\mathcal{M}'(V) \xrightarrow{\varphi_V} \mathcal{M}(V) \xrightarrow{\psi_V} \mathcal{M}''(V)
\end{array}
$$

Since the functor $\sim : \mathcal{O}_X(f^{-1}V)\mathcal{Mod} \rightarrow \mathcal{Mod}(X_V)$ is exact, we have an exact sequence of sheaves of modules on $X_V$

$$
\begin{array}{c}
\mathcal{M}'(V) \xrightarrow{\sim\varphi_V} \mathcal{M}(V) \xrightarrow{\sim\psi_V} \mathcal{M}''(V)
\end{array}
$$

Applying $(\psi_V)_*$ and using the natural isomorphism $(\psi_V)_*\mathcal{M}(V) \cong \mathcal{M}''|_{f^{-1}V}$ we see that the following sequence of sheaves of modules on $f^{-1}V$ is exact

$$
\begin{array}{c}
\mathcal{M}'|_{f^{-1}V} \xrightarrow{\sim\varphi|_{f^{-1}V}} \mathcal{M}|_{f^{-1}V} \xrightarrow{\sim\psi|_{f^{-1}V}} \mathcal{M}''|_{f^{-1}V}
\end{array}
$$

It now follows from (MRS, Lemma 38) that the functor $\sim$ is exact.

**Definition 3.** Let $f : X \rightarrow Y$ be a finite morphism of noetherian schemes and $\mathcal{F}$ a quasi-coherent sheaf of modules on $Y$. Then $\mathscr{A}(X)$ coherent (H, II Ex.5.5) and therefore the sheaf of $\mathcal{O}_Y$-modules $\mathcal{Hom}_{\mathcal{O}_Y}(\mathscr{A}(X), \mathcal{F})$ is quasi-coherent (MOS, Corollary 44). This sheaf becomes a quasi-coherent sheaf of $\mathscr{A}(X)$-modules with the action $(a \cdot \phi)_W(t) = \phi_W(a|_{f^{-1}W}t)$. Therefore we have a quasi-coherent sheaf of modules on $X$

$$
(f!)(\mathcal{F}) = \mathcal{Hom}_{\mathcal{O}_X}(\mathscr{A}(X), \mathcal{F})^\sim
$$

This defines an additive functor $f!(-) : \mathcal{Qco}(Y) \rightarrow \mathcal{Qco}(X)$.

**Remark 2.** For any closed immersion $f : X \rightarrow Y$ of schemes there is a right adjoint $f^! : \mathcal{Mod}(Y) \rightarrow \mathcal{Mod}(X)$ to the direct image functor $f_*$. (MRS, Proposition 97). In the special case where $X, Y$ are noetherian we have just defined another functor $f! : \mathcal{Qco}(Y) \rightarrow \mathcal{Qco}(X)$ and as the notation suggests, these two functors are naturally equivalent.