More Noether Normalisation

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The following is Theorem 8, Section 4, Chapter V of Volume 1 of Zariski & Samuel.

**Theorem 1 (Noether Normalisation).** Let $A = k[s_1, \ldots, s_n]$ be a finitely generated domain over an infinite field $k$, and let $K$ be the quotient field of $A$. Let $d$ be the transcendence degree of $K/k$. There are two cases:

**Case** $d = 0$ : $A$ is integral over $k$;

**Case** $d \geq 1$ : There exist $d$ linear combinations $y_1, \ldots, y_d$ of the $s_i$ which are algebraically independent over $k$ and such that $A$ is integral over $k[y_1, \ldots, y_d]$.

Moreover if $K$ is separably generated over $k$, the $y_j$ may be chosen in such a way that $K$ is a separable extension of $k(y_1, \ldots, y_d)$, so $\{y_1, \ldots, y_d\}$ is a separating transcendence basis of $K/k$.

**Proof.** The claim for $d = 0$ is trivial, so assume $d \geq 1$. Notice that if $A$ is integral over $k[y_1, \ldots, y_d]$ then $K$ is algebraic over $k(y_1, \ldots, y_d)$. Since $\text{tr.deg.} K/k = d$ it follows that the $y_i$ are algebraically independent and $k[y_1, \ldots, y_d]$ is a polynomial ring. We assume $K/k$ is separably generated (it is easy to go through and remove this hypothesis in the case $K/k$ is not).

Notice that $K = k(s_1, \ldots, s_n)$ so that $1 \leq d \leq n$. If $d = n$ then the $s_i$ are algebraically independent over $k$ and the result is trivial. So we can assume $d < n$ and by relabeling $s_1, \ldots, s_d$ is a separating transcendence basis for $K/k$ (Theorem 30, Ch. II.13). Hence $s_1, \ldots, s_d$ are algebraically independent over $k$ and $s_1, \ldots, s_d, s_n$ is an algebraically dependent set over $k$ for any $d + 1 \leq j \leq n$.

Let $f(x_1, \ldots, x_d, x_{d+1}) \in k[x_1, \ldots, x_{d+1}]$ be the irreducible relation $f(s_1, \ldots, s_d, s_n) = 0$ (see p.11 of our EFT notes). Then $f$ is an irreducible polynomial in $k[x_1, \ldots, x_d][x_{d+1}]$, hence also an element of $k(x_1, \ldots, x_d)[x_{d+1}]$ (see Lemma 1 on p.11 of our EFT notes). But we can identify $k(x_1, \ldots, x_d)$ with $k(s_1, \ldots, s_d)$, proving that $f(s_1, \ldots, s_d, x_{d+1})$ is the minimal polynomial of $s_n$ over the field $k(s_1, \ldots, s_d)$ (up to a unit of this field, anyway). Since $s_n$ is separable algebraic over $k(s_1, \ldots, s_d)$, it follows that

$$\frac{\partial f}{\partial x_{d+1}}(s_1, \ldots, s_d, s_n) \neq 0$$

Let $F$ be the homogenous part of $f$ of highest degree. If the degree of $F$ is $e$, then for any constants $\lambda_1, \ldots, \lambda_d \in k$ the polynomial $f(x_1 + \lambda_1 x_{d+1}, \ldots, x_d + \lambda_d x_{d+1}, x_{d+1})$ expands to

$$F(\lambda_1, \ldots, \lambda_d, 1)x_{d+1} + b_1 x_{d+1}^{e-1} + \cdots + b_e$$

(1)

where $b_i \in k[x_1, \ldots, x_d]$ for $1 \leq i \leq e$. Put $q_j = \partial f/\partial x_j(s_1, \ldots, s_d, s_n) \in A$ for $1 \leq j \leq d + 1$. We want the $\lambda_i$ to satisfy the following two equations:

$$F(\lambda_1, \ldots, \lambda_d, 1) \neq 0$$

(2)

$$\sum_{j=1}^{d} \lambda_j q_j + q_{d+1} \neq 0$$

(3)

Let $X$ denote the set of tuples $(\lambda_1, \ldots, \lambda_d, 1) \in k^{d+1}$ which satisfy (??). Then $X$ is nonempty since $(0, \ldots, 0, 1) \in X$. Assume that $F$ is zero on every tuple in $X$. We show that this implies $F = 0$, which is a contradiction.
For \((\lambda_1, \ldots, \lambda_d, 1) \in X\) let \(f\) be the polynomial \(F(\lambda_1, \ldots, \lambda_d, x_{d+1})\). Then we can find infinitely many \(\alpha \in k\) with \((\lambda_1, \ldots, \lambda_d, \alpha) \in X\) and hence \(f(\alpha) = 0\). It follows that
\[
F(\lambda_1, \ldots, \lambda_d, x_{d+1}) = 0
\]
Now let \(f\) be the polynomial \(F(\lambda_1, \ldots, \lambda_{d-1}, x_d, x_{d+1})\). We show that this polynomial is also zero. If not, write \(f\) as a polynomial in \(x_{d+1}\) with coefficients in \(k[x_d]\), and let \(b(x_d)\) be the nonzero leading coefficient. There are infinitely many \(\alpha \in k\) with \((\lambda_1, \ldots, \lambda_{d-1}, \alpha, 1) \in X\), and applying the above argument to these tuples we see that \(F(\lambda_1, \ldots, \alpha, x_{d+1}) = 0\) and hence \(f(\alpha, x_{d+1}) = 0\). It follows that \(b(\alpha) = 0\) for infinitely many \(\alpha\), whence \(b = 0\) and so \(f = 0\) as required. So for any tuple \((\lambda_1, \ldots, \lambda_d, 1) \in X\) we have
\[
F(\lambda_1, \ldots, \lambda_d, x_{d+1}) = 0
\]
In the next step we put \(F\) the above argument to these tuples we see that \(F(\lambda_1, \ldots, \lambda_d, x_{d+1}) = 0\). Proceeding recursively we find eventually that \(F = 0\), giving the desired contradiction. We conclude that there is some tuple \((\lambda_1, \ldots, \lambda_d, 1)\) satisfying equations (??), (??).

Consider the morphism of \(k\)-algebras
\[
\varphi : k[x_1, \ldots, x_{d+1}] \to k[x_1, \ldots, x_{d+1}]
\]
\[
x_1 \mapsto x_1 + \lambda_1 x_{d+1}
\]
\[
\vdots
\]
\[
x_d \mapsto x_d + \lambda_dx_d
\]
\[
x_{d+1} \mapsto x_{d+1}
\]
If we write \(t_i = s_i - \lambda_is_n\) for \(1 \leq i \leq d\) then
\[
0 = f(s_1, \ldots, s_d, s_n) = \varphi(f)(t_1, \ldots, t_d, s_n)
\]
So (??) yields an equation of integral dependence for \(s_n\) over \(k[t_1, \ldots, t_d]\). Hence \(A\) is integral over \(B = k[t_1, \ldots, t_d, s_{d+1}, \ldots, s_{n-1}]\) and so \(K\) is an algebraic extension of the quotient field \(L = k(t_1, \ldots, t_d, s_{d+1}, \ldots, s_{n-1})\).

By Corollary 1 to Definition 3 of Ch.II.5 (p.4 our EFT notes) to show that \(s_n\) is separable over \(k(t_1, \ldots, t_d)\) it suffices to show that
\[
\frac{\partial \varphi(f)}{\partial x_{d+1}}(t_1, \ldots, t_d, s_n) \neq 0
\]
Using the chain rule for polynomials (p.9 EFT notes) we have
\[
\frac{\partial \varphi(f)}{\partial x_{d+1}} = \sum_{j=1}^{d} \lambda_j \varphi \left( \frac{\partial f}{\partial x_j} \right) + \varphi \left( \frac{\partial f}{\partial x_{d+1}} \right)
\]
But for \(1 \leq j \leq d + 1\),
\[
\varphi \left( \frac{\partial f}{\partial x_j} \right)(t_1, \ldots, t_d, s_n) = \frac{\partial f}{\partial x_j}(s_1, \ldots, s_d, s_n)
\]
So \(\partial \varphi(f)/\partial x_{d+1}(t_1, \ldots, t_d, s_n) \neq 0\) by construction of the \(\lambda_i\). Hence \(s_n\) is separable over \(k(t_1, \ldots, t_d)\). Let \(Y\) be the subfield of \(K\) consisting of all elements separable over \(k(t_1, \ldots, t_d)\). Then \(Y\) contains \(s_1, \ldots, s_d\) and hence \(k(s_1, \ldots, s_d)\). But every element of \(K\) is separable over \(k(s_1, \ldots, s_d)\) and hence over \(Y\). It follows that \(Y = K\) and \(t_1, \ldots, t_d\) is a separating transcendence basis for \(K/k\) (hence also for \(L/k\)). It follows that \(L/k\) is separably generated.
Since $s_n$ is separable over $L$ (Lemma 2 of Ch.II.5) it follows from Theorem 10 of Ch.II.5 that $K/L$ is a separable algebraic extension. Clearly $B$ is a finitely generated domain over $k$ and $\text{tr.deg.} L/k = d$. By transitivity of integral and separable algebraic extensions it suffices to prove the theorem for $B$. We continue to apply the above argument until we end up with $B = k[t'_1, \ldots, t'_d]$ for linear combinations $t'_i$ of the $s_i$. 
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