More Noether Normalisation

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January 6, 2005

The following is Theorem 8, Section 4, Chapter V of Volume 1 of Zariski &Samuel.

Theorem 1 (Noether Normalisation). *Let $A = k[s_1, \ldots, s_n]$ be a finitely generated domain over an infinite field k, and let K be the quotient field of A. Let d be the transcendence degree of K/k. There are two cases:*

Case $d = 0$ *: A is integral over k;*

Case $d \geq 1$ *: There exist d linear combinations $y_1, \ldots, y_d$ of the s_i which are algebraically independent over k and such that A is integral over $k[y_1, \ldots, y_d]$.*

Moreover if K is separably generated over k, the y_j may be chosen in such a way that K is a separable extension of $k(y_1, \ldots, y_d)$, so $\{y_1, \ldots, y_d\}$ is a separating transcendence basis of K/k.

Proof. The claim for $d = 0$ is trivial, so assume $d \geq 1$. Notice that if A is integral over $k[y_1, \ldots, y_d]$ then K is algebraic over $k(y_1, \ldots, y_d)$. Since tr.deg.$K/k = d$ it follows that the y_i are algebraically independent and $k[y_1, \ldots, y_d]$ is a polynomial ring. We assume K/k is separably generated (it is easy to go through and remove this hypothesis in the case K/k is not).

Notice that $K = k(s_1, \ldots, s_n)$ so that $1 \leq d \leq n$. If $d = n$ then the s_i are algebraically independent over k and the result is trivial. So we can assume $d < n$ and by relabeling $s_1, \ldots, s_d$ is a separating transcendence basis for K/k (Theorem 30, Ch. II.13). Hence $s_1, \ldots, s_d$ are algebraically independent over k and $s_1, \ldots, s_d, s_n$ is an algebraically *dependent* set over k for any $d+1 \leq j \leq n$. Let $f(x_1, \ldots, x_d, x_{d+1}) \in k[x_1, \ldots, x_{d+1}]$ be the *irreducible relation* $f(s_1, \ldots, s_d, s_n) = 0$ (see p.11 of our EFT notes). Then f is an irreducible polynomial in $k[x_1, \ldots, x_d][x_{d+1}]$, hence also as an element of $k(x_1, \ldots, x_d)[x_{d+1}]$ (see Lemma 1 on p.11 of our EFT notes). But we can identify $k(x_1, \ldots, x_d)$ with $k(s_1, \ldots, s_d)$, proving that $f(s_1, \ldots, s_d, x_{d+1})$ is the minimal polynomial of s_n over the field $k(s_1, \ldots, s_d)$ (up to a unit of this field, anyway). Since s_n is separable algebraic over $k(s_1, \ldots, s_d)$, it follows that

$$\frac{\partial f}{\partial x_{d+1}}(s_1, \ldots, s_d, s_n) \neq 0$$

Let F be the homogenous part of f of highest degree. If the degree of F is e, then for any constants $\lambda_1, \ldots, \lambda_d \in k$ the polynomial $f(x_1 + \lambda_1 x_{d+1}, \ldots, x_d + \lambda_d x_{d+1}, x_{d+1})$ expands to

$$F(\lambda_1, \ldots, \lambda_d, 1)x_{d+1}^e + b_1 x_{d+1}^{e-1} + \cdots + b_e \tag{1}$$

where $b_i \in k[x_1, \ldots, x_d]$ for $1 \leq i \leq e$. Put $q_j = \partial f / \partial x_j(s_1, \ldots, s_d, s_n) \in A$ for $1 \leq j \leq d+1$. We want the λ_i to satisfy the following two equations:

$$F(\lambda_1, \ldots, \lambda_d, 1) \neq 0 \tag{2}$$

$$\sum_{j=1}^{d} \lambda_j q_j + q_{d+1} \neq 0 \tag{3}$$

Let X denote the set of tuples $(\lambda_1, \ldots, \lambda_d, 1) \in k^{d+1}$ which satisfy (**??**). Then X is nonempty since $(0, \ldots, 0, 1) \in X$. Assume that F is zero on every tuple in X. We show that this implies $F = 0$, which is a contradiction.

For $(\lambda_1, \ldots, \lambda_d, 1) \in X$ let f be the polynomial $F(\lambda_1, \ldots, \lambda_d, x_{d+1})$. Then we can find infinitely many $\alpha \in k$ with $(\lambda_1, \ldots, \lambda_d, \alpha) \in X$ and hence $f(\alpha) = 0$. It follows that

$$F(\lambda_1, \ldots, \lambda_d, x_{d+1}) = 0$$

Now let f be the polynomial $F(\lambda_1, \ldots, \lambda_{d-1}, x_d, x_{d+1})$. We show that this polynomial is also zero. If not, write f as a polynomial in x_{d+1} with coefficients in $k[x_d]$, and let $b(x_d)$ be the nonzero leading coefficient. There are infinitely many $\alpha \in k$ with $(\lambda_1, \ldots, \lambda_{d-1}, \alpha, 1) \in X$, and applying the above argument to these tuples we see that $F(\lambda_1, \ldots, \alpha, x_{d+1}) = 0$ and hence $f(\alpha, x_{d+1}) = 0$. It follows that $b(\alpha) = 0$ for infinitely many α, whence $b = 0$ and so $f = 0$ as required. So for any tuple $(\lambda_1, \ldots, \lambda_d, 1) \in X$ we have

$$F(\lambda_1, \ldots, \lambda_{d-1}, \lambda_d, x_{d+1}) = 0$$
$$F(\lambda_1, \ldots, \lambda_{d-1}, x_d, x_{d+1}) = 0$$

In the next step we put $f = F(\lambda_1, \ldots, \lambda_{d-2}, x_{d-1}, x_d, x_{d+1})$, write f as a polynomial in x_d, x_{d+1} with coefficients in $k[x_{d-1}]$, and use the fact that there are infinitely many elements $\alpha \in k$ with $(\lambda_1, \ldots, \lambda_{d-2}, \alpha, \lambda_d, 1) \in X$. Proceeding recursively we find eventually that $F = 0$, giving the desired contradiction. We conclude that there is some tuple $(\lambda_1, \ldots, \lambda_d, 1)$ satisfying equations (**??**), (**??**).

Consider the morphism of k-algebras

$$\begin{aligned} \varphi : k[x_1, \ldots, x_{d+1}] &\longrightarrow k[x_1, \ldots, x_{d+1}] \\ x_1 &\mapsto x_1 + \lambda_1 x_{d+1} \\ &\vdots \\ x_d &\mapsto x_d + \lambda_d x_d \\ x_{d+1} &\mapsto x_{d+1} \end{aligned}$$

If we write $t_i = s_i - \lambda_i s_n$ for $1 \leq i \leq d$ then

$$0 = f(s_1, \ldots, s_d, s_n) = \varphi(f)(t_1, \ldots, t_d, s_n)$$

So (**??**) yields an equation of integral dependence for s_n over $k[t_1, \ldots, t_d]$. Hence A is integral over $B = k[t_1, \ldots, t_d, s_{d+1}, \ldots, s_{n-1}]$ and so K is an algebraic extension of the quotient field $L = k(t_1, \ldots, t_d, s_{d+1}, \ldots, s_{n-1})$.

By Corollary 1 to Definition 3 of Ch.II.5 (p.4 our EFT notes) to show that s_n is separable over $k(t_1, \ldots, t_d)$ it suffices to show that

$$\frac{\partial \varphi(f)}{\partial x_{d+1}}(t_1, \ldots, t_d, s_n) \neq 0$$

Using the chain rule for polynomials (p.9 EFT notes) we have

$$\frac{\partial \varphi(f)}{\partial x_{d+1}} = \sum_{j=1}^{d} \lambda_j \varphi\left(\frac{\partial f}{\partial x_j}\right) + \varphi\left(\frac{\partial f}{\partial x_{d+1}}\right)$$

But for $1 \leq j \leq d+1$,

$$\varphi\left(\frac{\partial f}{\partial x_j}\right)(t_1, \ldots, t_d, s_n) = \frac{\partial f}{\partial x_j}(s_1, \ldots, s_d, s_n)$$

So $\partial \varphi(f)/\partial x_{d+1}(t_1, \ldots, t_d, s_n) \neq 0$ by construction of the λ_i. Hence s_n is separable over $k(t_1, \ldots, t_d)$. Let Y be the subfield of K consisting of all elements separable over $k(t_1, \ldots, t_d)$. Then Y contains $s_1, \ldots, s_d$ and hence $k(s_1, \ldots, s_d)$. But every element of K is separable over $k(s_1, \ldots, s_d)$ and hence over Y. It follows that $Y = K$ and $t_1, \ldots, t_d$ is a separating transcendence basis for K/k (hence also for L/k). It follows that L/k is separably generated.

Since s_n is separable over L (Lemma 2 of Ch.II.5) it follows from Theorem 10 of Ch.II.5 that K/L is a separable algebraic extension. Clearly B is a finitely generated domain over k and tr.deg.$L/k = d$. By transitivity of integral and separable algebraic extensions it suffices to prove the theorem for B. We continue to apply the above argument until we end up with $B = k[t'_1, \ldots, t'_d]$ for linear combinations t'_i of the s_i. □