# Modules over Projective Schemes

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**Definition 1.** Let $S$ be a graded ring, set $X = ProjS$ and let $M$ a graded $S$-module. We define a sheaf of modules $M^{\sim}$ on $X$ as follows. For each $\mathfrak{p} \in ProjS$ we have the local ring $S_{(\mathfrak{p})}$ and the $S_{(\mathfrak{p})}$-module $M_{(\mathfrak{p})}$ (GRM,Definition 4). Let $\Gamma(U, M^{\sim})$ be the set of all functions $s : U \longrightarrow \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$ with $s(\mathfrak{p}) \in M_{(\mathfrak{p})}$ for each $\mathfrak{p}$, which are *locally fractions*. That is, for every $\mathfrak{p} \in U$ there is an open neighborhood $\mathfrak{p} \in V \subseteq U$ and $m \in M, f \in S$ of the same degree, such that for every $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$. It is easy to check that $M^{\sim}$ is a sheaf of modules with the obvious restriction maps and the action $(r \cdot s)(\mathfrak{p}) = r(\mathfrak{p}) \cdot m(\mathfrak{p})$.

If $\phi : M \longrightarrow N$ is a morphism of graded $S$-modules then for every $\mathfrak{p} \in ProjS$ we have a canonical morphism of $S_{(\mathfrak{p})}$-modules $\phi_{(\mathfrak{p})} : M_{(\mathfrak{p})} \longrightarrow N_{(\mathfrak{p})}$ (GRM,Definition 4) and we define a morphism of sheaves of modules

$$\widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$$
$$\widetilde{\phi}_V(s)(\mathfrak{p}) = \phi_{(\mathfrak{p})}(s(\mathfrak{p}))$$

This defines an additive functor $(-)^{\sim} : S\mathbf{GrMod} \longrightarrow \mathfrak{Mod}(X)$, where $S\mathbf{GrMod}$ is the complete, cocomplete abelian category of graded $S$-modules (GRM,Proposition 21).

**Remark 1.** Let $S$ be a graded ring, set $X = ProjS$ and let $f \in S_+$ be homogenous. The ring morphism $S_{(f)} \longrightarrow \Gamma(D_+(f), \mathcal{O}_X)$ defined by $a/f^n \mapsto a\dot{/}f^n$ induces a morphism of schemes $\varphi : D_+(f) \longrightarrow SpecS_{(f)}$ which is precisely the isomorphism of (H, 2.5$b$).

**Proposition 1.** *Let $S$ be a graded ring, set $X = ProjS$ and let $M$ a graded $S$-module. Then*

*(a) For $\mathfrak{p} \in X$ there is a canonical isomorphism of $S_{(\mathfrak{p})}$-modules $\widetilde{M}_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$ natural in $M$.*

*(b) For homogenous $f \in S_+$ let $\varphi : D_+(f) \longrightarrow SpecS_{(f)}$ be the canonical isomorphism. There is a canonical isomorphism of sheaves of modules $\varphi_*(M^{\sim}|_{D_+(f)}) \cong (M_{(f)})^{\sim}$ natural in $M$.*

*(c) The sheaf $M^{\sim}$ is quasi-coherent. If $S$ is noetherian and $M$ finitely generated, $M^{\sim}$ is coherent.*

*Proof.* $(a)$ Define a map $(M^{\sim})_{\mathfrak{p}} \longrightarrow M_{(\mathfrak{p})}$ by $(U, s) \mapsto s(\mathfrak{p})$. The fact that this is an isomorphism follows as in the proof of (H, 2.5$a$). It is easy to check that this maps the action of $\mathcal{O}_{X,\mathfrak{p}}$ to the action of $S_{(\mathfrak{p})}$ in a way compatible with the ring isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong S_{(\mathfrak{p})}$. Naturality in $M$ is obvious. (TODO) Copy the rest of the written proof. The iso of $(b)$ should be the one arising from adjoitness. □

**Lemma 2.** *Let $S$ be a graded ring and set $X = ProjS$. Then the functor $(-)^{\sim} : S\mathbf{GrMod} \longrightarrow \mathfrak{Mod}(X)$ is exact.*

*Proof.* A sequence in $S\mathbf{GrMod}$ is exact iff. it is exact as a sequence of $S$-modules. As in the affine case, we show the functor $\widetilde{\cdot}$ is exact by using the natural isomorphism $\widetilde{M}_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$ and the fact that homogenous localisation is exact (GRM,Lemma 6). □

Throughout this note the tensor product $M \otimes_S N$ has the canonical grading. If $\phi : M \longrightarrow M'$ is a morphism of graded modules, the induced morphism $M \otimes_S N \longrightarrow M' \otimes_S N$ is also a morphism of graded modules. Similarly for a morphism $N \longrightarrow N'$. Let $\varphi : S \longrightarrow T$ be a morphism of graded rings. If $M$ is a graded $S$-module, then $M \otimes_S T$ becomes a graded $T$-module by considering $T$ as a graded $S$-module and using the canonical grading on the tensor product. If $\phi : M \longrightarrow M'$ is a morphism of graded modules, the induced morphism $M \otimes_S T \longrightarrow M' \otimes_S T$ is also a morphism of graded modules.

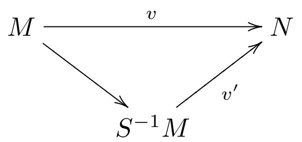
**Definition 2.** Let $\varphi : S \longrightarrow T$ be a morphism of graded rings and let $U$ be the open subset $\{\mathfrak{p} \,|\, \mathfrak{p} \not\supseteq \varphi(S_+)\}$ of $ProjT$. In (H,Ex.2.14) we defined a morphism of schemes $\Phi : U \longrightarrow ProjS$ defined by $\Phi(\mathfrak{p}) = \varphi^{-1}\mathfrak{p}$ on points and on sections by

$$\Phi_V^{\#} : \mathcal{O}_{ProjS}(V) \longrightarrow \mathcal{O}_{ProjT}(\Phi^{-1}V)$$
$$\Phi_V^{\#}(s)(\mathfrak{p}) = \varphi_{(\mathfrak{p})}(s(\varphi^{-1}\mathfrak{p}))$$

where $\varphi_{(\mathfrak{p})} : S_{(\varphi^{-1}\mathfrak{p})} \longrightarrow T_{(\mathfrak{p})}$ is the ring morphism defined by $a/s \mapsto \varphi(a)/\varphi(s)$.

We make a few small observations:

- If $u : A \longrightarrow B$ is a ring morphism, $M$ an $A$-module and $N$ a $B$-module, $v : M \longrightarrow N$ a morphism of $A$-modules (where $N$ becomes an $A$-module via $u$), and if $S \subseteq A$ is a multiplicatively closed set such that $u(s)$ is a unit for all $s \in S$, then $v$ factorises uniquely as




  where $v'$ is the morphism of $A$-modules defined by $v'(m/s) = u(s)^{-1} \cdot v(m)$.

- Let $S$ be a graded ring, $\mathfrak{p}$ a homogenous prime of $S$. Then there is a canonical morphism of rings $\varphi_{\mathfrak{p}} : S_{(\mathfrak{p})} \longrightarrow S_{\mathfrak{p}}$ defined by $a/s \mapsto a/s$. In fact this morphism is injective, since if $a, s \in S$ are homogenous of degree $e \geq 0$ and $s \notin \mathfrak{p}$ and $a/s = 0$ in $S_{\mathfrak{p}}$, then there is $t \notin \mathfrak{p}$ with $ta = 0$. Some homogenous component $t_d$ of $t$ does not belong to $\mathfrak{p}$, and $t_d a = 0$ implies $a/s = 0$ in $S_{(\mathfrak{p})}$. Similarly if $M$ is a graded $S$-module there is an injective morphism $M_{(\mathfrak{p})} \longrightarrow M_{\mathfrak{p}}$ of $S_{(\mathfrak{p})}$-modules. If $A$ is any ring considered as a graded ring and $\mathfrak{p}$ a prime ideal of $A$ (which is of course homogenous) then $\varphi_{\mathfrak{p}} : A_{(\mathfrak{p})} \longrightarrow A_{\mathfrak{p}}$ is an isomorphism and if $M$ is an $A$-module considered as a graded $A$-module then $M_{(\mathfrak{p})} \longrightarrow M_{\mathfrak{p}}$ is an isomorphism.

- Let $M, N$ be graded $S$-modules. The composite of $M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \longrightarrow M_{\mathfrak{p}} \otimes_{S_{\mathfrak{p}}} N_{\mathfrak{p}}$ (induced by the inclusions of $M_{(\mathfrak{p})}, N_{(\mathfrak{p})}, S_{(\mathfrak{p})}$) with the canonical isomorphism $M_{\mathfrak{p}} \otimes_{S_{\mathfrak{p}}} N_{\mathfrak{p}} \longrightarrow (M \otimes_S N)_{\mathfrak{p}}$ gives a morphism of $S_{(\mathfrak{p})}$-modules

$$\lambda_{\mathfrak{p}} : M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \longrightarrow (M \otimes_S N)_{(\mathfrak{p})}$$
$$m/s \otimes n/t \mapsto (m \otimes n)/st$$

**Proposition 3.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra, set $X = ProjS$ and let $M, N$ be graded $S$-modules. We claim that there is an isomorphism of sheaves of modules on $X$ natural in $M, N$*

$$\Lambda : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow (M \otimes_S N)^{\sim}$$
$$m\dot{/}s \,\dot{\otimes}\, n\dot{/}t \mapsto (m \otimes n)\dot{/}st$$

*Proof.* Let $Q$ be the presheaf of $\mathcal{O}_X$-modules on $X$ defined by

$$Q(U) = \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$$

Previous notes show that for $\mathfrak{p} \in X$ there is an isomorphism

$$\alpha_{\mathfrak{p}} : Q_{\mathfrak{p}} \longrightarrow M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})}$$
$$(U, s \otimes t) \mapsto s(\mathfrak{p}) \otimes t(\mathfrak{p})$$

Which maps the action of $\mathcal{O}_{X,\mathfrak{p}}$ to the action of $S_{(\mathfrak{p})}$. It is easy to check that the following defines a morphism of $\mathcal{O}_X$-modules:

$$\Lambda : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow (M \otimes_S N)^{\sim}$$
$$\Lambda_U(s)(\mathfrak{p}) = \lambda_{\mathfrak{p}}\alpha_{\mathfrak{p}}(s(\mathfrak{p}))$$

Since $S$ is generated by $S_1$ as an $S_0$-algebra, the open sets $D_+(f)$ for $f \in S_1$ cover $X$, and it suffices to show that $\Lambda|_{D_+(f)}$ is an isomorphism for all $f \in S_1$. The important step in the proof is showing that the following is an isomorphism of $S_{(f)}$-modules:

$$\lambda_f : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \longrightarrow (M \otimes_S N)_{(f)}$$
$$\lambda_f(m/f^j \otimes n/f^k) = (m \otimes n)/f^{j+k}$$

The fact that this is a well-defined map follows from the same argument used above for $\lambda_{\mathfrak{p}}$. For $m, n \in \mathbb{Z}$ define a $\mathbb{Z}$-bilinear map $M_m \times N_n \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ by mapping $(x, y)$ to $x/f^m \otimes y/f^n$ (if $m < 0$ replace $x/f^m$ by $f^{-m}x/1$). This induces $M_m \otimes_{\mathbb{Z}} N_n \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ and hence a morphism $M \otimes_{\mathbb{Z}} N \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$. This morphism maps elements of the form $(sx) \otimes y - x \otimes (sy)$ to zero, and hence induces

$$\gamma_f : M \otimes_S N \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$$
$$\gamma_f(m \otimes n) = \sum_{d,e \in \mathbb{Z}} m_d/f^d \otimes n_e/f^e$$

This is a morphism of $S$-modules relative to the canonical ring morphism $S \longrightarrow S_{(f)}$ which maps $s \in S_q$ to $s/f^q$. Since this morphism maps powers of $f$ to units, we obtain a morphism of $S$-modules $(M \otimes_S N)_f \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ which maps $(m \otimes n)/f^j$ to $\gamma_f(m \otimes n)$. The restriction to $(M \otimes_S N)_{(f)}$ gives a morphism of $S_{(f)}$-modules

$$\lambda'_f : (M \otimes_S N)_{(f)} \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$$
$$\lambda'_f((m \otimes n)/f^j) = \sum_{d,e \in \mathbb{Z}} m_d/f^d \otimes n_e/f^e$$

It is easily checked that $\lambda'_f$ is inverse to $\lambda_f$, completing the proof that $\lambda_f$ is an isomorphism. Let $\varphi : D_+(f) \longrightarrow Spec(S_{(f)})$ be the canonical isomorphism of schemes. Combining earlier notes, there is an isomorphism of $\mathcal{O}_{Spec(S_{(f)})}$-modules

$$\begin{aligned}\kappa : \varphi_*\left((\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})|_{D_+(f)}\right) &\cong \varphi_*(\widetilde{M}|_{D_+(f)} \otimes_{\mathcal{O}_X|_{D_+(f)}} \widetilde{N}|_{D_+(f)}) \\ &\cong \varphi_*(\widetilde{M}|_{D_+(f)}) \otimes_{\mathcal{O}_{SpecS_{(f)}}} \varphi_*(\widetilde{N}|_{D_+(f)}) \\ &\cong \widetilde{M_{(f)}} \otimes_{\mathcal{O}_{SpecS_{(f)}}} \widetilde{N_{(f)}} \\ &\cong (M_{(f)} \otimes_{S_{(f)}} N_{(f)})^{\sim}\end{aligned}$$

If $V$ is an open subset of $Spec(S_{(f)})$, $\mathfrak{p} \in \varphi^{-1}V$ and $s \in (\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})(\varphi^{-1}V)$ with $s(\mathfrak{p}) = (U, \sum_i a_i \otimes b_i)$ with $U \subseteq \varphi^{-1}V$ and $a_i \in \widetilde{M}(U), b_i \in \widetilde{N}(U)$ such that $a_i(\mathfrak{p}) = m_i/s_i \in M_{(\mathfrak{p})}$ and $b_i(\mathfrak{p}) = n_i/t_i \in N_{(\mathfrak{p})}$, then

$$\kappa_V(s)(\varphi(\mathfrak{p})) = \sum_i (m_i/f^{j_i} \otimes n_i/f^{k_i})/(s_i t_i/f^{j_i+k_i})$$

where $m_i, s_i$ are of degree $j_i$ and $n_i, t_i$ of degree $k_i$ for all $i$. To show that $\Lambda|_{D_+(f)}$ is an isomorphism, it suffices to show that $\varphi_*\Lambda|_{D_+(f)}$ is an isomorphism. For this it suffices to check commutativity of the following diagram:

$$\begin{array}{ccc} \varphi_*(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) & \xrightarrow{\varphi_*\Lambda|_{D_+(f)}} & \varphi_*((M \otimes_S N)^{\sim}|_{D_+(f)}) \\ \kappa \Big\Downarrow & & \Big\Downarrow \omega \\ (M_{(f)} \otimes_{S_{(f)}} N_{(f)})^{\sim} & \underset{\widetilde{\lambda_f}}{\Longrightarrow} & \left((M \otimes_S N)_{(f)}\right)^{\sim} \end{array}$$

But given $V, \mathfrak{p}$ and $s$ as above, we have

$$\begin{aligned}
\widetilde{\lambda_f}_V(\kappa_V(s))(f(\mathfrak{p})) &= (\lambda_f)_{\varphi(\mathfrak{p})}\left(\sum_i (m_i/f^{j_i} \otimes n_i/f^{k_i})/(s_i t_i/f^{j_i+k_i})\right) \\
&= \sum_i \lambda_f \left(m_i/f^{j_i} \otimes n_i/f^{k_i}\right)/(s_i t_i/f^{j_i+k_i}) \\
&= \sum_i \left((m_i \otimes n_i)/f^{j_i+k_i}\right)/(s_i t_i/f^{j_i+k_i}) \\
&= \sum_i \gamma_{\mathfrak{p}}^{-1}(m_i \otimes n_i/s_i t_i) \\
&= \omega_V(\Lambda_{\varphi^{-1}V}(s))(f(\mathfrak{p}))
\end{aligned}$$

as required. Hence $\Lambda : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow (M \otimes_S N)^{\sim}$ is an isomorphism of $\mathcal{O}_X$-modules. It is straightforward to check that $\Lambda$ is natural in $N$ and $M$. $\square$

**Corollary 4.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra. If $M, N$ are graded $S$-modules and $\mathfrak{p}$ homogenous prime of $S$ not containing $S_+$ then there is an isomorphism of $S_{(\mathfrak{p})}$-modules $\lambda_{\mathfrak{p}} : M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \longrightarrow (M \otimes_S N)_{(\mathfrak{p})}$ natural in $M, N$.*

*Proof.* We know that under the stated hypotheses $\Lambda$ is an isomorphism. Hence the induced morphisms on the stalks are isomorphisms, implying that $\lambda_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in X$. Naturality of this isomorphism with respect to morphisms of graded modules is easy to check. $\square$

**Proposition 5.** *Let $\varphi : S \longrightarrow T$ be a morphism of graded rings, where $S$ is generated by $S_1$ as an $S_0$-algebra. If $\mathfrak{p}$ is a homogenous prime ideal of $T$ not containing $\varphi(S_+)$ then for every graded $S$-module $M$ there is a canonical isomorphism of $T_{(\mathfrak{p})}$-modules*

$$\begin{gathered}
\kappa : M_{(\varphi^{-1}\mathfrak{p})} \otimes_{S_{(\varphi^{-1}\mathfrak{p})}} T_{(\mathfrak{p})} \longrightarrow (M \otimes_S T)_{(\mathfrak{p})} \\
m/s \otimes r/t \mapsto (m \otimes r)/\varphi(s)t
\end{gathered}$$

*Proof.* Note that $T_{(\mathfrak{p})}$ becomes a $S_{(\varphi^{-1}\mathfrak{p})}$-module via the ring morphism $\varphi_{(\mathfrak{p})} : S_{(\varphi^{-1}\mathfrak{p})} \longrightarrow T_{(\mathfrak{p})}$. Define a map

$$\begin{gathered}
\varepsilon : M_{(\varphi^{-1}\mathfrak{p})} \times T_{(\mathfrak{p})} \longrightarrow (M \otimes_S T)_{(\mathfrak{p})} \\
(m/s, r/t) \mapsto (m \otimes r)/\varphi(s)t
\end{gathered}$$

One checks that this map is well-defined and $S_{(\varphi^{-1}\mathfrak{p})}$-bilinear, so it induces a morphism of $T_{(\mathfrak{p})}$-modules

$$\begin{gathered}
\kappa : M_{(\varphi^{-1}\mathfrak{p})} \otimes_{S_{(\varphi^{-1}\mathfrak{p})}} T_{(\mathfrak{p})} \longrightarrow (M \otimes_S T)_{(\mathfrak{p})} \\
m/s \otimes r/t \mapsto (m \otimes r)/\varphi(s)t
\end{gathered}$$

The tricky part is defining the inverse to $\kappa$. Let $f \in S_1$ be such that $f \notin \varphi^{-1}\mathfrak{p}$ (this is possible since the $D_+(f)$ for $f \in S_1$ cover $Proj S$). For $i \in \mathbb{Z}$ and $j \geq 0$ define a $\mathbb{Z}$-bilinear map $M_i \times T_j \longrightarrow M_{(\varphi^{-1}\mathfrak{p})} \otimes_{S_{(\varphi^{-1}\mathfrak{p})}} T_{(\mathfrak{p})}$ by mapping $(m, r)$ to $m/f^i \otimes r/\varphi(f)^j$. Inducing morphisms out of the tensor products over $\mathbb{Z}$ gives a morphism of abelian groups $M \otimes_{\mathbb{Z}} T \longrightarrow M_{(\varphi^{-1}\mathfrak{p})} \otimes_{S_{(\varphi^{-1}\mathfrak{p})}} T_{(\mathfrak{p})}$, which maps elements of the form $(m \cdot s) \otimes t - m \otimes (s \cdot t)$ to zero, and thus induces a morphism of $T$-modules $M \otimes_S T \longrightarrow M_{(\varphi^{-1}\mathfrak{p})} \otimes_{S_{(\varphi^{-1}\mathfrak{p})}} T_{(\mathfrak{p})}$. The $T$-module structure on the second group is given by the canonical ring morphism $T \longrightarrow T_{(\mathfrak{p})}$, which maps $t \in T_q$ to $t/\varphi(f)^q$.

The morphism $T \longrightarrow T_{(\mathfrak{p})}$ maps homogenous elements of $T$ not in $\mathfrak{p}$ to units. So we end up with a morphism of abelian groups

$$\begin{gathered}
\kappa' : (M \otimes_S T)_{(\mathfrak{p})} \longrightarrow M_{(\varphi^{-1}\mathfrak{p})} \otimes_{S_{(\varphi^{-1}\mathfrak{p})}} T_{(\mathfrak{p})} \\
\kappa'((m \otimes r)/q) = m/f^i \otimes r\varphi(f)^i/q
\end{gathered}$$

where $m \in M_i, r \in T_j$ and $q \in T_{i+j}$. It is not difficult to check that $\kappa, \kappa'$ are mutually inverse, completing the proof that $\kappa$ is an isomorphism of $T_{(\mathfrak{p})}$-modules. $\square$

**Proposition 6.** *Let $\varphi : S \longrightarrow T$ be a morphism of graded rings where $S$ is generated by $S_1$ as an $S_0$-algebra, and let $f : U \longrightarrow ProjS$ be the induced morphism of schemes. If $M$ is a graded $S$-module there is an isomorphism of sheaves of modules on $U$ natural in $M$*

$$\eta : f^*(\widetilde{M}) \longrightarrow (M \otimes_S T)^{\sim}|_U$$
$$[T, m\dot{/}s] \dot{\otimes}\, b\dot{/}t \mapsto (m \otimes b)\dot{/}\varphi(s)t$$

*Proof.* Let $Z$ be the presheaf of $\mathcal{O}_U$ modules defined for $W \subseteq U$ by

$$Z(W) = (f^{-1}\widetilde{M})(W) \otimes_{f^{-1}\mathcal{O}_X(W)} \mathcal{O}_Y(W)$$

For $\mathfrak{p} \in U$ there is an isomorphism of groups

$$\tau_{\mathfrak{p}} : Z_{\mathfrak{p}} \cong \widetilde{M}_{\varphi^{-1}\mathfrak{p}} \otimes_{\mathcal{O}_{X,\varphi^{-1}\mathfrak{p}}} \mathcal{O}_{Y,\mathfrak{p}} \cong M_{(\varphi^{-1}\mathfrak{p})} \otimes_{S_{(\varphi^{-1}\mathfrak{p})}} T_{(\mathfrak{p})} \cong (M \otimes_S T)_{(\mathfrak{p})}$$

If $W \subseteq U$ and $a \in (f^{-1}\widetilde{M})(W), b \in \mathcal{O}_Y(W)$ with $a(\mathfrak{p}) = (Q, (W, t)), W \supseteq f(Q), t \in \widetilde{M}(W)$ then $\tau_{\mathfrak{p}}(V, a \otimes b) = (m \otimes r)/\varphi(s)t$ where $t(\varphi^{-1}\mathfrak{p}) = m/s$ and $b(\mathfrak{p}) = r/t$. We define

$$\eta : f^*(\widetilde{M}) \longrightarrow (M \otimes_S T)^{\sim}|_U$$
$$\eta_W(s)(\mathfrak{p}) = \tau_{\mathfrak{p}}(s(\mathfrak{p}))$$

One checks that $\eta$ is a well-defined morphism of $\mathcal{O}_U$-modules, which is an isomorphism since $\tau_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in U$. Naturality in $M$ is easily checked. $\square$

**Proposition 7.** *Let $\varphi : S \longrightarrow T$ be a morphism of graded rings and $f : U \longrightarrow ProjS$ the induced morphism of schemes. If $N$ is a graded $T$-module then there is an isomorphism of sheaves of modules on $ProjS$ natural in $N$*

$$\eta : ({}_SN)^{\sim} \longrightarrow f_*(\widetilde{N}|_U)$$
$$n\dot{/}s \mapsto n\dot{/}\varphi(s)$$

*Proof.* For $\mathfrak{p} \in U$ define a morphism of abelian groups $z_{\mathfrak{p}} : ({}_SN)_{(\varphi^{-1}\mathfrak{p})} \longrightarrow N_{(\mathfrak{p})}$ by $n/s \mapsto n/\varphi(s)$. One checks that $z_{\mathfrak{p}}$ is compatible with the ring morphism $\varphi_{(\mathfrak{p})} : S_{(\varphi^{-1}\mathfrak{p})} \longrightarrow T_{(\mathfrak{p})}$. For $V \subseteq X$ and $s \in ({}_SN)^{\sim}(V)$ and $\mathfrak{q} \in f^{-1}V$ we define

$$\eta_V(s)(\mathfrak{q}) = z_{\mathfrak{q}}(s(\varphi^{-1}\mathfrak{q}))$$

Since $\eta_V(s)$ is clearly regular, this is well-defined. One checks that $\eta$ is a morphism of $\mathcal{O}_X$-modules as in the affine case. Since the open sets $D_+(f)$ for $f \in S_+$ form a basis for $X$, to show that $\eta$ is an isomorphism it suffices to show that $\eta_{D_+(f)}$ is an isomorphism for all $f \in S_+$. But using (5.11$b$) and the fact that $f^{-1}D_+(f) = D_+(\varphi(f))$ for $f \in S_+$ we obtain a commutative diagram

$$\begin{array}{ccc} ({}_SN)^{\sim}(D_+(f)) & \xrightarrow{\eta_{D_+(f)}} & (f_*\widetilde{N}|_U)(D_+(f)) \\ \Downarrow & & \Downarrow \\ ({}_SN)_{(f)} & \Longrightarrow & N_{(\varphi(f))} \end{array}$$

Where the bottom map is the isomorphism of abelian groups $n/f^m \mapsto n/\varphi(f)^m$. This completes the proof that $\eta : ({}_SN)^{\sim} \longrightarrow f_*(\widetilde{N}|_U)$ is an isomorphism, which one easily checks is natural in $N$. In particular, considering $T$ as a graded $T$-module, $f_*(\mathcal{O}_Y|_U) \cong \widetilde{T}$ as $\mathcal{O}_X$-modules. $\square$

**Definition 3.** Let $A$ be a ring and $S$ a graded $A$-algebra with structural morphism $\varphi : A \longrightarrow S$. If we grade $A$-canonically then any $A$-module $M$ is a graded $A$-module, and therefore $M \otimes_A S$ is canonically a graded $S$-module. If $\phi : M \longrightarrow M'$ is a morphism of $A$-modules then $\phi \otimes_A S : M \otimes_A S \longrightarrow M' \otimes_A S$ preserves grade, so we have an addtiive functor

$$- \otimes_A S : A\mathbf{Mod} \longrightarrow S\mathbf{GrMod}$$

It is not difficult to check that this morphism is left adjoint to the functor $(-)_0 : S\mathbf{GrMod} \longrightarrow A\mathbf{Mod}$ which picks out the degree 0 subgroup, which is an $A$-module. The unit $M \longrightarrow (M \otimes_A S)_0$ is the map $m \mapsto m \otimes 1$.

**Proposition 8.** *Let $A$ be a ring, $S$ a graded $A$-algebra and $\mathfrak{p}$ a homogenous prime ideal of $S$. Then for every $A$-module $M$ there is a canonical isomorphism of $S_{(\mathfrak{p})}$-modules*

$$\kappa : M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} S_{(\mathfrak{p})} \longrightarrow (M \otimes_A S)_{(\mathfrak{p})}$$
$$m/s \otimes r/t \mapsto (m \otimes r)/\varphi(s)t$$

*where $\mathfrak{q} = \mathfrak{p} \cap A$.*

*Proof.* Let $\varphi : A \longrightarrow S$ be the structural morphism and note that $S_{(\mathfrak{p})}$ becomes an $A_{\mathfrak{q}}$-module via the canonical ring morphism $A_{\mathfrak{q}} \longrightarrow S_{(\mathfrak{p})}$. Define a map

$$\varepsilon : M_{\mathfrak{q}} \times S_{(\mathfrak{p})} \longrightarrow (M \otimes_A S)_{(\mathfrak{p})}$$
$$(m/s, r/t) \mapsto (m \otimes r)/\varphi(s)t$$

One checks that this map is well-defined and $A_{\mathfrak{q}}$-bilinear, so it induces a morphism of $S_{(\mathfrak{p})}$-modules

$$\kappa : M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} S_{(\mathfrak{p})} \longrightarrow (M \otimes_A S)_{(\mathfrak{p})}$$
$$m/s \otimes r/t \mapsto (m \otimes r)/\varphi(s)t$$

To show that $\kappa$ is an isomorphism we construct its inverse. Let $t \notin \mathfrak{p}$ be homogenous of degree $d \geq 0$ and define a bilinear map $M \times S_d \longrightarrow M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} S_{(\mathfrak{p})}$ by $(m, s) \mapsto m/1 \otimes s/t$. This induces a morphism of abelian groups

$$\phi'_t : M \otimes_{\mathbb{Z}} S_d \longrightarrow M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} S_{(\mathfrak{p})}$$
$$m \otimes s \mapsto m/1 \otimes s/t$$

If $\alpha : M \otimes_{\mathbb{Z}} S \longrightarrow M \otimes_A S$ is the canonical morphism of abelian groups, then for $d \geq 0$ the graded subgroup $(M \otimes_A S)_d$ is $\alpha((M \otimes_{\mathbb{Z}} S)_d)$ where $(M \otimes_{\mathbb{Z}} S)_d$ is the image of the monomorphism of abelian groups $M \otimes_{\mathbb{Z}} S_d \longrightarrow M \otimes_{\mathbb{Z}} S$ (GRM,Section 6). Therefore there is an isomorphism of abelian groups $(M \otimes_A S)_d \cong M \otimes_{\mathbb{Z}} S_d/K$ where $K$ is the subgroup generated by elements of the form $(a \cdot x) \otimes y - x \otimes (a \cdot y)$ for $a \in A, x \in M, y \in S_d$. But $\phi'_t$ takes the value zero on such elements, so we have a morphism of abelian groups

$$\phi_t : (M \otimes_A S)_d \longrightarrow M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} S_{(\mathfrak{p})}$$
$$m \otimes s \mapsto m/1 \otimes s/t$$

We define

$$\kappa' : (M \otimes_A S)_{(\mathfrak{p})} \longrightarrow M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} S_{(\mathfrak{p})}$$
$$z/t \mapsto \phi_t(z)$$

To see this is well-defined, suppose that $z/t = z'/t'$ where $t \in S_d, t' \in S_e$ and $z \in (M \otimes_A S)_d, z' \in (M \otimes_A S)_e$. Suppose $z = \sum_i m_i \otimes s_i$ and $z' = \sum_i m'_i \otimes s'_i$ and let $q \notin \mathfrak{p}$ be homogenous with $qt' \cdot z = qt \cdot z'$ in $M \otimes_A S$. Therefore

$$\sum_i m_i \otimes s_i qt' = \sum_i m'_i \otimes s'_i qt$$

Applying $\phi_{qtt'}$ to both sides gives $\phi_t(z) = \phi_{t'}(z')$, which shows that $\kappa'$ is a well-defined morphism of sets. It is a morphism of abelian groups since

$$\phi_t(z) + \phi_{t'}(z') = \phi_{tt'}(t'z + tz')$$

Since $\kappa$ is clearly inverse to $\kappa'$ this completes the proof. □

**Corollary 9.** *Let $A$ be a ring and $S$ a graded $A$-algebra with structural morphism $f : ProjS \longrightarrow SpecA$. For any $A$-module $M$ there is an isomorphism of sheaves of modules on $ProjS$ natural in $M$*

$$\zeta : f^*(\widetilde{M}) \longrightarrow (M \otimes_A S)^{\sim}$$
$$[Q, m\dot{/}s] \dot{\otimes} b\dot{/}t \mapsto (m \otimes b)\dot{/}\varphi(s)t$$

*Proof.* Set $Y = ProjS, X = SpecA$ and for a homogenous prime ideal $\mathfrak{p}$ write $\mathfrak{q} = \mathfrak{p} \cap A$ and using Proposition 8 define $\zeta_{\mathfrak{p}}$ to be the following isomorphism of $\mathcal{O}_{Y,\mathfrak{p}}$-modules

$$\zeta_{\mathfrak{p}} : Z_{\mathfrak{p}} \cong \widetilde{M}_{\mathfrak{q}} \otimes_{\mathcal{O}_{X,\mathfrak{q}}} \mathcal{O}_{Y,\mathfrak{p}} \cong M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} S_{(\mathfrak{p})} \cong (M \otimes_A S)_{(\mathfrak{p})}$$

where $Z$ is the presheaf of modules $Z(U) = f^{-1}\widetilde{M}(U) \otimes \mathcal{O}_Y(U)$ on $ProjS$ which sheafifies to give $f^*(\widetilde{M})$, and $\varphi : A \longrightarrow S$ is the structural morphism. Then for open $U \subseteq ProjS$ we define

$$\zeta_U(s)(\mathfrak{p}) = \zeta_{\mathfrak{p}}(s(\mathfrak{p}))$$

To see that this is a well-defined section of $(M \otimes_A S)^{\sim}$, note that every point $\mathfrak{p} \in U$ has an open neighborhood $\mathfrak{p} \in V \subseteq U$ such that $s|_V$ is of the form

$$s|_V = \sum_{i=1}^{n} [Q, m_i\dot{/}s_i] \dot{\otimes} b_i\dot{/}t_i$$

where $m_i \in M, s_i \in A$ and $b_i, t_i \in S$ are homogenous of the same degree for each $i$ and $Q$ is an open subset of $X$ containing $f(V)$. Using this fact it is not difficult to see that $\zeta_U(s) \in (M \otimes_A S)^{\sim}(U)$. One checks that $\zeta$ is a well-defined morphism of sheaves of modules, which is an isomorphism since $\zeta_{\mathfrak{p}}$ is for every $\mathfrak{p} \in ProjS$. Naturality in $M$ is easily checked. □

**Proposition 10.** *Let $S$ be a finitely generated graded $A$-algebra with structural morphism $f : ProjS \longrightarrow SpecA$. For any graded $S$-module $M$ there is an isomorphism of sheaves of modules on $SpecA$ natural in $M$*

$$\zeta : \Gamma(ProjS, \widetilde{M})^{\sim} \longrightarrow f_*(\widetilde{M})$$
$$n\dot{/}s \mapsto (1\dot{/}\varphi(s)) \cdot n|_{f^{-1}U}$$

*where $\varphi : A \longrightarrow S$ is the structural morphism.*

*Proof.* Set $Y = ProjS$ and $X = SpecA$. The canonical ring morphism $A \longrightarrow \Gamma(Y)$ makes $\Gamma(Y, \widetilde{M})$ into an $A$-module. It follows from (TPC,Proposition 3), (TPC,Proposition 1) and (H,5.8) that the functor $f_* : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ preserve quasi-coherentness. Therefore we have an isomorphism of sheaves of modules on $X$ (MOS,Proposition 3)

$$\Gamma(Y, \widetilde{M})^{\sim} = \Gamma(X, f_*(\widetilde{M}))^{\sim} \cong f_*(\widetilde{M})$$

This is clearly natural in $M$ and has the desired effect on the section $n\dot{/}s$. □

**Definition 4.** Let $S$ be a graded ring and set $X = ProjS$. For any $n \in \mathbb{Z}$ we define the sheaf of modules $\mathcal{O}_X(n) = S(n)^{\sim}$ (GRM,Definition 3) which we will denote by $\mathcal{O}(n)$ if there is no chance of confusion. We call $\mathcal{O}_X(1)$ the *twisting sheaf*. For any sheaf of modules $\mathscr{F}$ we denote $\mathscr{F} \otimes \mathcal{O}_X(n)$ by $\mathscr{F}(n)$, and tensoring with $\mathcal{O}_X(n)$ defines the additive functor $-(n) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$.

**Proposition 11.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra and set $X = ProjS$. For $m, n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of modules*

$$\tau^{m,n} : \mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \longrightarrow \mathcal{O}_X(m+n)$$
$$a\dot{/}s \dot{\otimes} b\dot{/}t \mapsto ab\dot{/}st$$

**Lemma 12.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra and set $X = ProjS$. For $n, e, d \in \mathbb{Z}$ the following diagram commutes*

$$\begin{array}{ccc}
\big(\mathcal{O}_X(n) \otimes \mathcal{O}_X(e)\big) \otimes \mathcal{O}_X(d) & \Longrightarrow & \mathcal{O}_X(n) \otimes \big(\mathcal{O}_X(e) \otimes \mathcal{O}_X(d)\big) \\
{\scriptstyle \tau^{n,e}\otimes 1}\Big\Downarrow & & \Big\Downarrow{\scriptstyle 1\otimes\tau^{e,d}} \\
\mathcal{O}_X(n+e) \otimes \mathcal{O}_X(d) & & \mathcal{O}_X(n) \otimes \mathcal{O}_X(e+d) \\
& {\scriptstyle \tau^{n+e,d}}\searrow \quad \swarrow{\scriptstyle \tau^{n,e+d}} & \\
& \mathcal{O}_X(n+e+d) &
\end{array}$$

**Proposition 13.** *Let $\varphi : S \longrightarrow T$ be a morphism of graded rings where $S$ is generated by $S_1$ as an $S_0$-algebra. If $\Phi : U \longrightarrow ProjS$ is the induced morphism of schemes then there is a canonical isomorphism of sheaves of modules on $U$*

$$\beta : \Phi^*\mathcal{O}(1) \longrightarrow \mathcal{O}(1)|_U$$
$$\beta_Q([W, a\dot{/}s] \dot{\otimes} b\dot{/}t) = \varphi(a)b\dot{/}\varphi(s)t$$

*Where $Q \subseteq U, W \supseteq \Phi(Q)$ are open with $Q$ nonempty, $a \in S_{d+1}, s \in S_d$ for some $d \geq 0$ and $b, t \in T_e$ for some $e \geq 0$ be such that $Q \subseteq D_+(t)$ and $W \subseteq D_+(s)$.*

*Proof.* This follows immediately from the definition of the isomorphism $\beta$ and our notes in previous Sections. □

# 1 Sheaf Hom

**Lemma 14.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra, and set $X = ProjS$. For $n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of modules*

$$\lambda : \mathcal{O}(-n) \longrightarrow \mathcal{O}(n)^\vee$$
$$\lambda_U(a\dot{/}s)_V(b\dot{/}t) = ab\dot{/}st$$

*Proof.* There is a canonical isomorphism of sheaves of modules $\tau^{n,-n} : \mathcal{O}(n) \otimes \mathcal{O}(-n) \cong \mathcal{O}_X$. Tensoring both sides with $\mathcal{O}(n)^\vee$ and using (MRS,Lemma 83) we obtain an isomorphism. There is another morphism of sheaves of modules $\lambda' : \mathcal{O}(-n) \longrightarrow \mathcal{O}(n)^\vee$ corresponding under the bijection of (MRS,Proposition 76) to $\tau^{-n,n} : \mathcal{O}(-n) \otimes \mathcal{O}(n) \longrightarrow \mathcal{O}_X$. It is straightforward to check that in fact $\lambda' = \lambda$, which yields the action of $\lambda$ on the special sections in the statement. □

**Proposition 15.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra, and set $X = ProjS$. For sheaves of modules $\mathscr{F}, \mathscr{G}$ on $X$ and $n \in \mathbb{Z}$ there are canonical isomorphisms of sheaves of modules natural in $\mathscr{F}, \mathscr{G}$*

$$\mathscr{H}om(\mathscr{F}, \mathscr{G}(n)) \cong \mathscr{H}om(\mathscr{F}(-n), \mathscr{G}) \cong \mathscr{H}om(\mathscr{F}, \mathscr{G})(n)$$

*Proof.* Using Lemma 14, (MRS,Proposition 75) and (MRS,Proposition 77) we have a canonical isomorphism of sheaves of modules

$$\begin{aligned}
\mathscr{H}om(\mathscr{F}, \mathscr{G}(n)) &\cong \mathscr{H}om(\mathscr{F}, \mathcal{O}(n) \otimes \mathscr{G}) \\
&\cong \mathscr{H}om(\mathscr{F}, \mathcal{O}(-n)^\vee \otimes \mathscr{G}) \\
&\cong \mathscr{H}om(\mathscr{F}, \mathscr{H}om(\mathcal{O}(-n), \mathscr{G})) \\
&\cong \mathscr{H}om(\mathscr{F} \otimes \mathcal{O}(-n), \mathscr{G}) = \mathscr{H}om(\mathscr{F}(-n), \mathscr{G})
\end{aligned}$$

Continuing we have

$$\begin{aligned}\mathscr{H}om(\mathscr{F}(-n),\mathscr{G}) &\cong \mathscr{H}om(\mathcal{O}(-n)\otimes\mathscr{F},\mathscr{G})\\ &\cong \mathscr{H}om(\mathcal{O}(-n),\mathscr{H}om(\mathscr{F},\mathscr{G}))\\ &\cong \mathcal{O}(-n)^\vee\otimes\mathscr{H}om(\mathscr{F},\mathscr{G})\\ &\cong \mathcal{O}(n)\otimes\mathscr{H}om(\mathscr{F},\mathscr{G})\cong\mathscr{H}om(\mathscr{F},\mathscr{G})(n)\end{aligned}$$

as required. □

**Corollary 16.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra, and set $X = ProjS$. For sheaves of modules $\mathscr{F},\mathscr{G}$ on $X$ and $n\in\mathbb{Z}$ there is a canonical isomorphism of $\Gamma(X,\mathcal{O}_X)$-modules natural in $\mathscr{F},\mathscr{G}$*

$$Hom(\mathscr{F},\mathscr{G}(n))\cong Hom(\mathscr{F}(-n),\mathscr{G})$$

**Corollary 17.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra, and set $X = ProjS$. For $m,n\in\mathbb{Z}$ there is a canonical isomorphism of sheaves of modules*

$$\gamma:\mathcal{O}(m-n)\longrightarrow\mathscr{H}om(\mathcal{O}(n),\mathcal{O}(m))$$
$$\gamma_U(\dot{a/s})_V(\dot{b/t}) = \dot{ab/st}$$

*Proof.* For $m,n\in\mathbb{Z}$ we have an isomorphism of sheaves of modules (MRS,Proposition 75)

$$\begin{aligned}\mathscr{H}om(\mathcal{O}(n),\mathcal{O}(m)) &\cong \mathcal{O}(n)^\vee\otimes\mathcal{O}(m)\\ &\cong\mathcal{O}(-n)\otimes\mathcal{O}(m)\\ &\cong\mathcal{O}(m-n)\end{aligned}$$

Call the inverse of this morphism $\gamma_1$. There is another morphism of sheaves of modules $\gamma_2 : \mathcal{O}(m-n)\longrightarrow\mathscr{H}om(\mathcal{O}(n),\mathcal{O}(m))$ corresponding under the bijection of (MRS,Proposition 76) to the morphism $\tau^{m-n,n}:\mathcal{O}(m-n)\otimes\mathcal{O}(n)\longrightarrow\mathcal{O}(m)$. We claim that $\gamma_1=\gamma_2$. To show this, we reduce to a section $\dot{a/s}\in\Gamma(U,\mathcal{O}(m-n))$ with $U$ contained in $D_+(f)$ for some $f\in S_1$. Then using the fact that $\dot{1/f^n}\otimes\dot{af^n/s}\in\Gamma(U,\mathcal{O}(-n)\otimes\mathcal{O}(m))$ maps to $\dot{a/s}$ under $\tau^{-n,m}$, one checks that $\gamma_1$ and $\gamma_2$ agree on $\dot{a/s}$. This isomorphism $\gamma$ now has the required properties. □

**Remark 2.** With the notation of Proposition 17, let $s\in S_{m-n}$ be given. Then multiplication by $s$ defines a morphism of graded $S$-modules $S(n)\longrightarrow S(m)$, which induces a morphism of sheaves of moduels $\mathcal{O}(n)\longrightarrow\mathcal{O}(m)$. This is precisely the morphism $\gamma_X(\dot{s/1})$.

# 2 Quasi-Structures

See (GRM,Definition 10) for the definition of the equivalence relation of "quasi-isomorphism" on graded modules over a graded ring, which we use in the next result.

**Proposition 18.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra. If $M,N$ are quasi-isomorphic graded $S$-modules then there is a canonical isomorphism of sheaves of modules $\widetilde{M}\cong\widetilde{N}$. In particular if $M\sim 0$ then $M^\sim = 0$.*

*Proof.* Let $M$ be a graded $S$-module, $\mathfrak{p}\in ProjS$ and $d\geq 0$. Let $f\in S_1$ be such that $f\notin\mathfrak{p}$ and define a morphism of $S_{(\mathfrak{p})}$-modules

$$\gamma_f : M\{d\}_{(\mathfrak{p})}\longrightarrow M_{(\mathfrak{p})}$$
$$m/s\mapsto m/sf^d$$

This is an isomorphism with inverse $m/s\mapsto f^dm/s$. If $N$ is a graded $S$-module with $M\sim N$, so there is an isomorphism of graded $S$-modules $\phi: M\{d\}\longrightarrow N\{d\}$ for some $d\geq 0$, then the following composite is an isomorphism of $S_{(\mathfrak{p})}$-modules

$$M_{(\mathfrak{p})}\cong M\{d\}_{(\mathfrak{p})}\cong N\{d\}_{(\mathfrak{p})}\cong N_{(\mathfrak{p})}$$
$$m/s\mapsto f^dm/s\mapsto\phi(f^dm)/s\mapsto\phi(f^dm)/sf^d$$

One checks easily that this does not depend on the homogenous $f \in S_1 \setminus \mathfrak{p}$ chosen, so we have a canonical isomorphism of $S_{(\mathfrak{p})}$-modules $\tau_{\mathfrak{p}} : M_{(\mathfrak{p})} \longrightarrow N_{(\mathfrak{p})}$ for every homogenous prime ideal $\mathfrak{p} \in ProjS$. In the usual way one checks that this gives rise to an isomorphism of sheaves of modules

$$\tau : \widetilde{M} \longrightarrow \widetilde{N}$$
$$\tau_U(s)(\mathfrak{p}) = \tau_{\mathfrak{p}}(s(\mathfrak{p}))$$

which completes the proof. □

**Corollary 19.** *Let $S$ be a graded ring generated by $S_1$ as an $S_0$-algebra. If $\phi : M \longrightarrow N$ is a morphism of graded $S$-modules then*

*(i) $\phi$ is a quasi-isomorphism $\Longrightarrow \widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$ is an isomorphism.*

*(ii) $\phi$ is a quasi-monomorphism $\Longrightarrow \widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$ is a monomorphism.*

*(iii) $\phi$ is a quasi-epimorphism $\Longrightarrow \widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$ is an epimorphism.*

*Proof.* The claims $(i), (ii), (iii)$ follow from (GRM,Lemma 26) and exactness of the functor $\widetilde{-}$. □

**Proposition 20.** *Let $S$ be a graded ring finitely generated by $S_1$ as an $S_0$-algebra, and let $M$ be a quasi-finitely generated graded $S$-module. Then*

*(i) $M \sim 0$ if and only if $M^{\sim} = 0$.*

*(ii) If $S$ is noetherian then $M^{\sim}$ is a coherent sheaf of modules on $ProjS$.*

*Proof.* $(i)$ We know from Proposition 18 that if $M \sim 0$ then $M^{\sim} = 0$. For the converse we can reduce to the case where $M$ is finitely generated (GRM,Lemma 24). If $\widetilde{M} = 0$ then $M_{(\mathfrak{p})} = 0$ for every $\mathfrak{p} \in ProjS$, and therefore $M \sim 0$ by (GRM,Proposition 30).

$(ii)$ Using Proposition 18 and (GRM,Lemma 24) we can reduce to the case where $M$ is finitely generated, which is (H,5.11$c$). □

**Corollary 21.** *Let $S$ be a noetherian graded ring finitely generated by $S_1$ as an $S_0$-algebra and $\phi : M \longrightarrow N$ a morphism of quasi-finitely generated graded $S$-modules. Then*

*(i) $\phi$ is a quasi-monomorphism $\Leftrightarrow \widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$ is a monomorphism.*

*(ii) $\phi$ is a quasi-epimorphism $\Leftrightarrow \widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$ is an epimorphism.*

*(iii) $\phi$ is a quasi-isomorphism $\Leftrightarrow \widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$ is an isomorphism.*

*Proof.* Using (GRM,Lemma 25) and Proposition 20 we can prove the reverse implications in Corollary 19 using exactness of $\widetilde{-}$. □