Modules over a Scheme

Daniel Murfet

October 5, 2006

In these notes we collect various facts about quasi-coherent sheaves on a scheme. Nearly all of the material is trivial or can be found in [Gro60]. These notes are not intended as an introduction to the subject.

Contents

1 Basic Properties

Definition 1. Let X be a scheme. We denote the category of sheaves of $\mathcal{O}_X$-modules by $\mathcal{O}_X\mathbf{Mod}$ or $\mathfrak{Mod}(X)$. The full subcategories of qausi-coherent and coherent modules are denoted by $\mathfrak{Qco}(X)$ and $\mathfrak{Coh}(X)$ respectively. $\mathfrak{Mod}(X)$ is a grothendieck abelian category, and it follows from (AC,Lemma 39) and (H, II 5.7) that $\mathfrak{Qco}(X)$ is an abelian subcategory of $\mathfrak{Mod}(X)$. If X is noetherian, then $\mathfrak{Coh}(X)$ is also an abelian subcategory.

The fact that $\mathfrak{Qco}(X)$ is an abelian subcategory of $\mathfrak{Mod}(X)$ means that the following operations preserve the quasi-coherent property:

- If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of quasi-coherent sheaves and $\mathscr{H}$ is a quasi-coherent submodule of $\mathscr{G}$, then $\phi^{-1}\mathscr{H}$ is a quasi-coherent submodule of $\mathscr{F}$.

- If $\mathscr{F}$ is a quasi-coherent sheaf and $\mathscr{G}_1, \ldots, \mathscr{G}_n$ are quasi-coherent submodules then the finite union $\sum_i \mathscr{G}_i$ is quasi-coherent (we mean the categorical union in $\mathfrak{Mod}(X)$, but it follows that this submodule is also the union in $\mathfrak{Qco}(X)$). To see this, realise the union as the image of a morphism out of a finite coproduct.

- Finite limits and colimits of quasi-coherent modules are quasi-coherent.

- If $\mathscr{F}$ is a quasi-coherent sheaf and $\mathscr{G}_1, \ldots, \mathscr{G}_n$ are quasi-coherent submodules then the finite intersection $\bigcap_i \mathscr{G}_i$ is quasi-coherent (we mean the categorical intersection in $\mathfrak{Mod}(X)$). This follows from the fact that binary intersections are pullbacks.

If X is a noetherian scheme then all these statements are true with "quasi-coherent" replaced by "coherent". In fact we will see in Proposition 25 that arbitrary colimits of quasi-coherent sheaves are quasi-coherent. In particular, the union of *any* family of quasi-coherent submodules of a quasi-coherent module is quasi-coherent. Here's another clever trick:

Lemma 1. *Let X be a scheme. Let $\{\mathscr{F}_i\}_{i \in I}$ be a nonempty set of sheaves of $\mathcal{O}_X$-modules and suppose that the coproduct $\bigoplus_i \mathscr{F}_i$ is quasi-coherent. Then $\mathscr{F}_i$ is quasi-coherent for every $i \in I$. If X is noetherian the same is true of coherent modules.*

Proof. Let $u_i : \mathscr{F}_i \longrightarrow \bigoplus_i \mathscr{F}_i$ be the injections, and induce projections $p_i : \bigoplus_i \mathscr{F}_i \longrightarrow \mathscr{F}_i$ in the usual way such that $p_i u_i = 1$. Therefore $\mathscr{F}_i$ is the image of the composite $p_i u_i$, and since this is a morphism of quasi-coherent sheaves it follows from (5.7) that $\mathscr{F}_i$ is quasi-coherent. The same is true of coherent sheaves if X is noetherian. $\square$

Lemma 2. *Let X be a scheme and $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of quasi-coherent sheaves of modules on X. Then*

(i) ϕ is a monomorphism $\Leftrightarrow$ ϕ_U is injective for all open affine $U \subseteq X$.

(ii) ϕ is an epimorphism $\Leftrightarrow$ ϕ_U is surjective for all open affine $U \subseteq X$.

(iii) ϕ is an isomorphism $\Leftrightarrow$ ϕ_U is bijective for all open affine $U \subseteq X$.

Proof. Let $U \subseteq X$ be a nonempty affine open subset and let $a : U \longrightarrow Spec\mathcal{O}_X(U)$ be the canonical isomorphism. We have a commutative diagram

$$\begin{array}{ccc} a_*(\mathscr{F}|_U) & \xrightarrow{a_*\phi|_U} & a_*(\mathscr{G}|_U) \\ \Big\Uparrow & & \Big\Uparrow \\ \widetilde{\mathscr{F}(U)} & \xrightarrow[\widetilde{\phi_U}]{} & \widetilde{\mathscr{G}(U)} \end{array}$$

The functor $\widetilde{-} : \mathcal{O}_X(U)\mathbf{Mod}$ is exact and fully faithful, and therefore preserves and reflects monomorphisms, epimorphisms and isomorphisms. Therefore ϕ_U is a monomorphism, epimorphism or isomorphism iff. $\phi|_U : \mathscr{F}|_U \longrightarrow \mathscr{G}|_U$ has that property, which completes the proof. $\square$

Proposition 3. *Let A be a commutative ring and set $X = SpecA$. Then we have an adjoint pair of functors*

$$A\mathbf{Mod} \underset{\Gamma(-)}{\overset{\widetilde{-}}{\rightleftarrows}} \mathfrak{Mod}(X) \qquad \widetilde{-} \dashv \Gamma(-)$$

The functor $\widetilde{-}$ is additive, exact and fully faithful. For an A-module M the unit $\eta : M \longrightarrow \widetilde{M}(X)$ is an isomorphism and for a sheaf of modules $\mathscr{F}$ on X the counit $\varepsilon : \Gamma(\mathscr{F})^{\sim} \longrightarrow \mathscr{F}$ is an isomorphism if and only if $\mathscr{F}$ is quasi-coherent.

Proof. See our solution to (H,Ex.5.3) for the adjunction. Clearly if ε is an isomorphism then $\mathscr{F}$ is quasi-coherent, and the converse follows from (H,5.4). $\square$

Lemma 4. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. If $U \subseteq X$ is affine with canonical isomorphism $f : U \longrightarrow Spec\mathcal{O}_X(U)$ then there is an isomorphism of sheaves of modules on $Spec\mathcal{O}_X(U)$ natural in $\mathscr{F}$*

$$f_*(\mathscr{F}|_U) \cong \mathscr{F}(U)^{\sim}$$

If X is noetherian and $\mathscr{F}$ is coherent then it follows that $\mathscr{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module. In this case $\mathscr{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module for every $x \in X$.

Lemma 5. *Let X be a scheme and suppose have a sequence of quasi-coherent sheaves of modules on X*

$$\mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \tag{1}$$

*this sequence is exact in $\mathfrak{Mod}(X)$ if and only if for every open affine $U \subseteq X$ the following sequence is exact in $\mathcal{O}_X(U)$***Mod**

$$\mathscr{F}'(U) \xrightarrow{\varphi_U} \mathscr{F}(U) \xrightarrow{\psi_U} \mathscr{F}''(U)$$

Proof. Using (MRS,Lemma 38) and Lemma 4 we reduce to showing that for any affine open $U \subseteq X$ the sequence $\mathscr{F}'(U)^{\sim} \longrightarrow \mathscr{F}(U)^{\sim} \longrightarrow \mathscr{F}''(U)^{\sim}$ is exact in $\mathfrak{Mod}(Spec\mathcal{O}_X(U))$ if and only if $\mathscr{F}'(U) \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{F}''(U)$ is exact in $\mathcal{O}_X(U)$**Mod**. But $\tilde{-}$ is exact and fully faithful, so it preserves and reflects exact sequences. □

Lemma 6. *Let X be a scheme and $\{\mathscr{F}_i\}_{i\in I}$ a nonempty family of quasi-coherent sheaves of modules on X. If $U \subseteq X$ is affine then there is a canonical isomorphism of $\mathcal{O}_X(U)$-modules*

$$\bigoplus_i \mathscr{F}_i(U) \longrightarrow \left(\bigoplus_i \mathscr{F}_i\right)(U)$$

Proof. Let P be the presheaf coproduct $P(U) = \bigoplus_i \mathscr{F}_i(U)$. There is a canonical morphism of presheaves of modules $\psi : P \longrightarrow \bigoplus_i \mathscr{F}_i$ into the sheaf coproduct, and we claim that ψ_U is an isomorphism for any open affine $U \subseteq X$. To this end consider the following isomorphisms of sheaves of modules

$$f_*((\bigoplus_i \mathscr{F}_i)|_U) \cong f_*(\bigoplus_i (\mathscr{F}_i|_U)) \cong \bigoplus_i f_*(\mathscr{F}_i|_U) \cong \bigoplus_i \widetilde{\mathscr{F}_i(U)} \cong (\bigoplus_i \mathscr{F}_i(U))^{\sim}$$

Evaluating this isomorphism on global sections gives the desired result. □

Lemma 7. *Let X be a scheme and $\mathscr{F}, \mathscr{G}$ quasi-coherent sheaves of modules on X. For affine open $U \subseteq X$ there is a canonical isomorphism of $\mathcal{O}_X(U)$-modules $\tau : \mathscr{F}(U) \otimes \mathscr{G}(U) \longrightarrow (\mathscr{F} \otimes \mathscr{G})(U)$ natural in both variables with*

$$\tau(f \otimes g) = f \dot{\otimes} g$$

Proof. Let $a : Spec\mathcal{O}_X(U) \longrightarrow U$ be the canonical isomorphism and consider the following isomorphism of sheaves of modules on U

$$\begin{aligned}(\mathscr{F} \otimes \mathscr{G})|_U &\cong \mathscr{F}|_U \otimes \mathscr{G}|_U \\ &\cong a_*\widetilde{\mathscr{F}(U)} \otimes a_*\widetilde{\mathscr{G}(U)} \\ &\cong a_*(\widetilde{\mathscr{F}(U)} \otimes \widetilde{\mathscr{G}(U)}) \\ &\cong a_*(\mathscr{F}(U) \otimes \mathscr{G}(U))^{\sim}\end{aligned}$$

This gives the desired isomorphism τ, which is clearly natural in both variables $\mathscr{F}, \mathscr{G}$. □

Lemma 8. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. Given an affine open subset $U \subseteq X$ a nonempty subset $\{s_i\}_{i\in I} \subseteq \mathscr{F}(U)$ generates the sheaf of modules $\mathscr{F}|_U$ if and only if the s_i generate $\mathscr{F}(U)$ as an $\mathcal{O}_X(U)$-module.*

Proof. We reduce immediately to the case where $X = SpecA$ is affine and $\mathscr{F} = M^{\sim}$. In this case we need only observe that elements $s_i \in M$ generate M as an A-module if and only if their images generate $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$. □

2 Ideals

In this section all rings and algebras are commutative.

Lemma 9. *Let A be a ring and $\mathfrak{a} \subseteq A$ an ideal. Then*

(a) If B is an A-algebra then $\mathfrak{a}B$ is an ideal of B and $\mathscr{J}_{\mathfrak{a}} \cdot \mathcal{O}_{SpecB} = \mathscr{J}_{\mathfrak{a}B}$.

(b) If S is a graded A-algebra then $\mathfrak{a}S$ is a homogenous ideal of S and $\mathscr{J}_{\mathfrak{a}} \cdot \mathcal{O}_{ProjS} = \mathscr{J}_{\mathfrak{a}S}$.

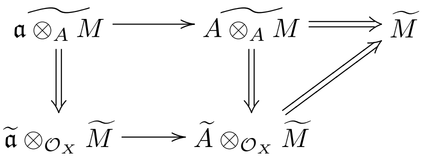
Proof. Both statements are easily checked, using (SIAS,Lemma 1), (SIPS,Lemma 1) and (MRS,Lemma 46). □

Lemma 10. *Let X be an affine scheme and $Spec\mathcal{O}_X(X) \cong X$ the canonical isomorphism. If $\mathscr{K}$ is a quasi-coherent sheaf of ideals on X then $\mathscr{K}$ corresponds to the ideal sheaf $\mathscr{J}_{\mathscr{K}(X)}$ on $Spec\mathcal{O}_X(X)$.*

Proof. Let $f : X \longrightarrow Spec\mathcal{O}_X(X)$ be the canonical isomorphism. Then $\mathscr{K}$ corresponds to the image of $f_*\mathscr{K} \longrightarrow f_*\mathcal{O}_X \cong \mathcal{O}_{Spec\mathcal{O}_X(X)}$, and since $f_*\mathscr{K}$ is quasi-coherent it is equivalent as a subobject of $\mathcal{O}_{Spec\mathcal{O}_X(X)}$ to $\mathscr{K}(X)^{\sim}$, which is equivalent by definition to $\mathscr{J}_{\mathscr{K}(X)}$. □

Proposition 11. *Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal and set $X = SpecA$. If M is an A-module then $\widetilde{\mathfrak{a}}\widetilde{M} = \widetilde{\mathfrak{a}M}$. In particular we have $\widetilde{\mathfrak{a}}\widetilde{\mathfrak{b}} = \widetilde{\mathfrak{a}\mathfrak{b}}$ and $\widetilde{\mathfrak{a}}^n = \widetilde{\mathfrak{a}^n}$ for ideals $\mathfrak{a}, \mathfrak{b}$ and $n \geq 1$.*

Proof. The functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ is exact, so we can identify $\widetilde{\mathfrak{a}}$ and $\widetilde{\mathfrak{a}M}$ with submodules of $\mathcal{O}_X, \widetilde{M}$ respectively. The fact that $\widetilde{\mathfrak{a}}\widetilde{M} = \widetilde{\mathfrak{a}M}$ follows from commutativity of the following diagram



□

Corollary 12. *Let X be a scheme and $\mathscr{J}$ a quasi-coherent sheaf of ideals on X. If $\mathscr{F}$ is a quasi-coherent sheaf of modules on X, then so is $\mathscr{J}\mathscr{F}$. In particular any product of quasi-coherent sheaves of ideals is quasi-coherent. If X is noetherian the same statements are true for coherent sheaves.*

Proof. Let $U \subseteq X$ be an affine open subset with canonical isomorphism $f : U \longrightarrow Spec\mathcal{O}_X(U)$. Then using (MRS,Proposition 52), Proposition 11 and Lemma 10 we have

$$\begin{aligned} f_*(\mathscr{J}\mathscr{F})|_U &= f_*(\mathscr{J}|_U\mathscr{F}|_U) \\ &= (\mathscr{J}|_U \cdot \mathcal{O}_{Spec\mathcal{O}_X(U)})f_*(\mathscr{F}|_U) \\ &= \widetilde{\mathscr{J}(U)}f_*(\mathscr{F}|_U) \\ &\cong \widetilde{\mathscr{J}(U)}\widetilde{\mathscr{F}(U)} \\ &\cong (\mathscr{J}(U)\mathscr{F}(U))^{\sim} \end{aligned}$$

which shows that $\mathscr{J}\mathscr{F}$ is quasi-coherent. If X is noetherian and $\mathscr{J}, \mathscr{F}$ are coherent then $\mathscr{J}(U), \mathscr{F}(U)$ are finitely generated, so it is not hard to see that $\mathscr{J}(U)\mathscr{F}(U)$ is finitely generated, and therefore $\mathscr{J}\mathscr{F}$ is coherent. □

Proposition 13. *Let X be a scheme and $\mathscr{J}$ a quasi-coherent sheaf of ideals on X. If $\mathscr{F}$ is a quasi-coherent sheaf of modules on X, then for any affine open $U \subseteq X$ we have $(\mathscr{J}\mathscr{F})(U) = \mathscr{J}(U)\mathscr{F}(U)$ as submodules of $\mathscr{F}(U)$. In particular we have $\mathscr{J}^n(U) = \mathscr{J}(U)^n$ for any $n \geq 1$.*

Proof. The first claim follows from evaluating the isomorphism in the proof of Corollary 12 on global sections. To prove $\mathscr{J}^n(U) = \mathscr{J}(U)^n$ for all $n \geq 1$ is a simple induction. □

Proposition 14. *Let X be a scheme and $\mathscr{F},\mathscr{G}$ submodules of a sheaf of $\mathcal{O}_X$-modules $\mathscr{H}$. If $x \in X$ then we have $(\mathscr{F}:\mathscr{G})_x \subseteq (\mathscr{F}_x : \mathscr{G}_x)$ as ideals of $\mathcal{O}_{X,x}$. If X is noetherian, $\mathscr{F},\mathscr{H}$ quasi-coherent and $\mathscr{G}$ coherent this is an equality.*

Proof. The inclusion $(\mathscr{F}:\mathscr{G})_x \subseteq (\mathscr{F}_x : \mathscr{G}_x)$ is easily checked. Now suppose that X is noetherian, $\mathscr{F},\mathscr{H}$ quasi-coherent and $\mathscr{G}$ coherent. For the reverse inclusion, we can reduce to the following situation: $X = SpecA$ for a commutative noetherian ring A, M, N are A-submodules of an A-module S with N finitely generated, $\mathfrak{p} \in SpecA$ and we have to show that the ring isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}$ identifies the ideals $(M^{\sim} : N^{\sim})_{\mathfrak{p}}$ and $(M_{\mathfrak{p}} : N_{\mathfrak{p}})$. It is a standard result of commutative algebra that $(M_{\mathfrak{p}} : N_{\mathfrak{p}}) = (M : N)_{\mathfrak{p}}$. Given $a \in (M : N)$ and $s \notin \mathfrak{p}$ it is easy to see that the section $\dot{a/s} \in \Gamma(D(s), \mathcal{O}_X)$ has the property that $\dot{a/s} N^{\sim}(V) \subseteq M^{\sim}(V)$ for any open $V \subseteq D(s)$, and therefore $\dot{a/s} \in \Gamma(D(s), (M^{\sim} : N^{\sim}))$ which completes the proof. ☐

Corollary 15. *Let A be a commutative noetherian ring and M, N A-submodules of an A-module S. If N is finitely generated then we have $(M^{\sim} : N^{\sim}) = (M : N)^{\sim}$ as sheaves of ideals on $X = SpecA$.*

Proof. It suffices to show that $(M^{\sim} : N^{\sim})_{\mathfrak{p}} = (M : N)^{\sim}{}_{\mathfrak{p}}$ as ideals of $\mathcal{O}_{X,\mathfrak{p}}$ for every point $\mathfrak{p} \in X$. But the proof of Proposition 14 shows that the isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}$ identifies both these ideals with $(M : N)_{\mathfrak{p}}$, so they must be equal. ☐

Corollary 16. *Let X be a noetherian scheme and $\mathscr{F},\mathscr{G}$ quasi-coherent submodules of a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{H}$. If $\mathscr{G}$ is coherent then $(\mathscr{F}:\mathscr{G})$ is a coherent sheaf of ideals on X.*

Proof. If $U \subseteq X$ is an affine open subset and $f : U \longrightarrow Spec\mathcal{O}_X(U)$ the canonical isomorphism, then we have using (MRS,Lemma 53) and Corollary 15 an isomorphism of sheaves of modules

$$\begin{aligned} f_*(\mathscr{F}:\mathscr{G})|_U &= f_*(\mathscr{F}|_U : \mathscr{G}|_U) \\ &\cong (\mathscr{F}|_U : \mathscr{G}|_U) \cdot \mathcal{O}_Y \\ &= (f_*\mathscr{F}|_U : f_*\mathscr{G}|_U) \\ &= (\mathscr{F}(U)^{\sim} : \mathscr{G}(U)^{\sim}) \\ &= (\mathscr{F}(U) : \mathscr{G}(U))^{\sim} \end{aligned}$$

This shows that $(\mathscr{F}:\mathscr{G})$ is coherent, since $\mathcal{O}_X(U)$ is noetherian and therefore the ideal $(\mathscr{F}(U) : \mathscr{G}(U))$ is finitely generated. ☐

3 Special Functors

Let $f : X \longrightarrow Y$ be a closed immersion of schemes. The direct image functor f_* has a right adjoint $f^! : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ (MRS,Proposition 97) and in this section we study the properties of this functor.

Proposition 17. *Let $f : X \longrightarrow Y$ be a closed immersion of schemes, $V \subseteq Y$ an open subset and $g : f^{-1}V \longrightarrow V$ be the induced morphism. Then for a sheaf of modules $\mathscr{F}$ on Y there is a canonical isomorphism of sheaves of modules on $f^{-1}V$ natural in $\mathscr{F}$*

$$\theta : (f^!\mathscr{F})|_{f^{-1}V} \longrightarrow g^!(\mathscr{F}|_V)$$

That is, there is a canonical natural equivalence $\theta : (-|_{f^{-1}V})f^! \longrightarrow g^!(-|_V)$.

Proof. Set $U = Y \setminus f(X)$ and $U' = V \setminus g(f^{-1}V)$. Then $U' = U \cap V$ and for open $W \subseteq f^{-1}V$ we have

$$\Gamma(W, f^!\mathscr{F}) = \{s \in \Gamma(f(W) \cup U, \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{F})) \,|\, s|_U = 0\}$$
$$\Gamma(W, g^!\mathscr{F}|_V) = \{s \in \Gamma(f(W) \cup U', \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathscr{F})) \,|\, s|_{U'} = 0\}$$

Since $f(W) \cup U' \subseteq f(W) \cup U$ we can define the map $\theta_W : \Gamma(W, f^! \mathscr{F}) \longrightarrow \Gamma(W, g^! \mathscr{F}|_V)$ by $\theta_W(s) = s|_{f(W)\cup U'}$. This is an isomorphism, since in the restriction we only remove part of the open set where s is zero. This defines the isomorphism θ, which is clearly natural in $\mathscr{F}$. □

Proposition 18. *Suppose there is a commutative diagram of ringed spaces*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle k}\Downarrow & & \Downarrow{\scriptstyle h} \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

with f, g closed embeddings. Then for any sheaf of modules $\mathscr{F}$ on Y there is a canonical isomorphism of sheaves of modules natural in $\mathscr{F}$

$$\mu : k_* f^! \mathscr{F} \longrightarrow g^! h_* \mathscr{F}$$
$$\mu_Q(s)_T = s_{h^{-1}T} \circ \omega_{(hf)^{-1}T}$$

where $\omega : (k^{-1})_* \mathcal{O}_{X'} \longrightarrow \mathcal{O}_X$ is the canonical isomorphism. That is, there is a canonical natural equivalence $\mu : k_* f^! \longrightarrow g^! h_*$.

Proof. Set $U = Y \setminus f(X)$ and $V = Y' \setminus g(X')$. For an open set $Q \subseteq X'$ we have using (MRS,Proposition 86) an isomorphism of $\mathcal{O}_{X'}(Q)$-modules

$$\begin{aligned} \mu_Q : \Gamma(Q, k_* f^! \mathscr{F}) &= \Gamma(f(k^{-1}Q) \cup U, \mathscr{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathscr{F})) \\ &\cong \Gamma(h^{-1}(g(Q) \cup V), \mathscr{H}om_{\mathcal{O}_Y}(f_*(k^{-1})_* \mathcal{O}_{X'}, \mathscr{F})) \\ &= \Gamma(g(Q) \cup V, h_* \mathscr{H}om_{\mathcal{O}_Y}((h^{-1})_* g_* \mathcal{O}_{X'}, \mathscr{F})) \\ &\cong \Gamma(g(Q) \cup V, \mathscr{H}om_{\mathcal{O}_{Y'}}(g_* \mathcal{O}_{X'}, h_* \mathscr{F})) \\ &= \Gamma(Q, g^! h_* \mathscr{F}) \end{aligned}$$

One checks that together these morphisms define an isomorphism of sheaves of modules μ natural in $\mathscr{F}$, as required. □

Corollary 19. *Let $f : X \longrightarrow Y$ be a closed immersion of noetherian schemes and $\mathscr{G}$ a sheaf of modules on Y. Then for $x \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules natural in $\mathscr{G}$*

$$\lambda : f^!(\mathscr{G})_x \longrightarrow Hom_{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x}, \mathscr{G}_{f(x)})$$
$$\lambda(V, s)(T, r) = (Q, s_Q(r|_{T\cap V}))$$

where $Q = f(T \cap V) \cup U$.

Proof. Since f is a finite morphism of noetherian schemes, the sheaf $f_* \mathcal{O}_X$ is coherent (H, II Ex.5.5) and therefore locally finitely presented by Lemma 34. So the existence of the claimed isomorphism is a special case of (MRS,Proposition 96). □

Proposition 20. *Let $\phi : A \longrightarrow B$ be a surjective morphism of noetherian rings and $f : X \longrightarrow Y$ the corresponding closed immersion of affine schemes. For any A-module M there is a canonical isomorphism of sheaves of modules natural in M*

$$\zeta : Hom_A(B, M)^{\sim} \longrightarrow f^!(\widetilde{M})$$
$$\zeta_V(\dot{u/s}) \mapsto \dot{1/s} \cdot (\widetilde{u}\rho)|_{f(V)\cup U}$$

where $\rho : f_* \mathcal{O}_X \longrightarrow \widetilde{B}$ is the canonical isomorphism of sheaves of modules on Y.

Proof. We define the additive functor $Hom_A(B,-) : A\mathbf{Mod} \longrightarrow B\mathbf{Mod}$ in the usual way, and it is easy to check this is right adjoint to the restriction of scalars functor. We set $X = SpecB, Y = SpecA$ in what follows. There is a canonical morphism of B-modules

$$Hom_A(B,M) \longrightarrow Hom_{\mathcal{O}_Y}(\widetilde{B},\widetilde{M}) \cong Hom_{\mathcal{O}_Y}(f_*\mathcal{O}_X,\widetilde{M}) = \Gamma(X, f^!(\widetilde{M}))$$

and therefore an induced morphism of sheaves of modules $\zeta : Hom_A(B,M)^\sim \longrightarrow f^!(M^\sim)$. Given $\mathfrak{p} \in SpecB$ there is an isomorphism of $\mathcal{O}_{X,\mathfrak{p}}$-modules

$$\begin{aligned}(Hom_A(B,M)^\sim)_\mathfrak{p} &\cong Hom_A(B,M)_\mathfrak{p}\\ &\cong Hom_{A_{\phi^{-1}\mathfrak{p}}}(B_\mathfrak{p}, M_{\phi^{-1}\mathfrak{p}})\\ &\cong Hom_{\mathcal{O}_{Y,f(\mathfrak{p})}}(\mathcal{O}_{X,\mathfrak{p}}, \widetilde{M}_{f(\mathfrak{p})})\\ &\cong f^!(\widetilde{M})_\mathfrak{p}\end{aligned}$$

where we use Corollary 19 and (MAT,Proposition 15). One checks this agrees with $\zeta_\mathfrak{p}$, which is therefore an isomorphism. This shows that ζ is an isomorphism of sheaves of modules, which is clearly natural in M. □

Corollary 21. *Let $f : X \longrightarrow Y$ be a closed immersion of noetherian schemes. If $\mathscr{F}$ is a quasi-coherent sheaf of modules on Y then $f^!\mathscr{F}$ is also quasi-coherent. If $\mathscr{F}$ is coherent then so is $f^!\mathscr{F}$.*

Proof. Let $\mathscr{F}$ be a quasi-coherent sheaf of modules on Y, and let $x \in X$ be given. Let V be an affine open neighborhood of $f(x)$ and set $U = f^{-1}V$, which is also affine. Let $g : U \longrightarrow V, F : Spec\mathcal{O}_X(U) \longrightarrow Spec\mathcal{O}_Y(V)$ be the induced closed immersions and $k : U \longrightarrow Spec\mathcal{O}_X(U), h : V \longrightarrow Spec\mathcal{O}_Y(V)$ the canonical isomorphisms. Then using Proposition 17, Proposition 18 and Proposition 20 we have an isomorphism of sheaves of modules

$$\begin{aligned}k_*(f^!\mathscr{F}|_U) &\cong k_*(g^!\mathscr{F}|_V)\\ &\cong F^!h_*(\mathscr{F}|_V)\\ &\cong F^!(\mathscr{F}(V)^\sim)\\ &\cong Hom_{\mathcal{O}_Y(V)}(\mathcal{O}_X(U),\mathscr{F}(V))^\sim\end{aligned}$$

which shows that $f^!\mathscr{F}$ is quasi-coherent. If $\mathscr{F}$ is coherent then $f^!\mathscr{F}$ is coherent, since if A is a commutative noetherian ring and M, N finitely generated A-modules, the A-module $Hom_A(M,N)$ is also finitely generated. □

Remark 1. Combining Corollary 21 with (H, II 5.18) and (H, II Ex.5.5) we see that if $f : X \longrightarrow Y$ is a closed immersion of noetherian schemes, the three adjoints $f^*, f_*, f^!$ all preserve quasi-coherent and coherent sheaves.

4 Open Subsets

Lemma 22. *If X is a scheme and $U \subseteq X$ open, restriction preserves quasi-coherent and coherent modules. There are induced additive functors $\mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(U)$ and $\mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(U)$.*

Proof. This follows from the proof of (H,5.4). Note that X need not be noetherian in the coherent case. For any scheme $\mathfrak{Qco}(X)$ is an abelian subcategory of $\mathfrak{Mod}(X)$, so it is clear that $(-)|_U : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(U)$ is exact. If X is noetherian then $(-)|_U : \mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(U)$ is also exact. □

Lemma 23. *If X is a scheme and $U \subseteq X$ open with concentrated inclusion $i : U \longrightarrow X$, then the functor $(-)|_U : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(U)$ has a right adjoint given by direct image i_*. So we have an adjoint pair of functors*

$$\mathfrak{Qco}(U) \underset{-|_U}{\overset{i_*}{\rightleftarrows}} \mathfrak{Qco}(X)$$

Proof. It follows from (CON, Proposition 18) that i_* restricts to give a functor $\mathfrak{Qco}(U) \longrightarrow \mathfrak{Qco}(X)$, and it is easy to check that $-|_U$ is left adjoint to i_*. In particular this applies when X is concentrated and $U \subseteq X$ quasi-compact. □

Lemma 24. *Let X be a scheme and $\{U_i\}_{i \in I}$ a nonempty open cover. Then a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is quasi-coherent (resp. coherent) if and only if $\mathscr{F}|_{U_i}$ is a quasi-coherent (resp. coherent) sheaf of modules on U_i for all $i \in I$.*

Proof. Once again, this is very easily checked. □

Proposition 25. *Let X a scheme. Then finite limits and arbitrary colimits of quasi-coherent modules are quasi-coherent. In other words the abelian category $\mathfrak{Qco}(X)$ is cocomplete and the inclusion $\mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ preserves colimits.*

Proof. We have already checked $\mathfrak{Qco}(X)$ is closed under finite limits and finite colimits. Using Lemma 22 and (MRS, Proposition 37) we reduce easily to the case where $X = SpecA$ is affine. In this case $\mathfrak{Qco}(X)$ is equivalent to A**Mod**, so is cocomplete, and the inclusion $\mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ preserves all colimits since the functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ does. □

Corollary 26. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. Then $\mathscr{F}$ is quasi-coherent if and only if every point $x \in X$ has an open neighborhood U such that $\mathscr{F}|_U$ is the cokernel of a morphism of free modules on U. If X is noetherian, then $\mathscr{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank.*

Proof. Suppose that $\mathscr{F}$ is quasi-coherent. Given $x \in X$ find an affine open neighborhood U of x. Then in the usual way we can write $\mathscr{F}(U)$ as the cokernel of a morphism $\mathcal{O}_X(U)^I \longrightarrow \mathcal{O}_X(U)^J$ for possibly infinite index sets I, J. This leads to an exact sequence

$$\mathcal{O}_X|_U^I \longrightarrow \mathcal{O}_X|_U^J \longrightarrow \mathscr{F}|_U$$

as required. If X is noetherian and $\mathscr{F}$ coherent, then $\mathcal{O}_X(U)$ is noetherian and $\mathscr{F}(U)$ finitely generated so we can take I, J to be finite sets. The reverse implications follow immediately from Proposition 25 and (H,5.7). □

5 Locally Free Sheaves

Proposition 27. *Let X be a nonempty scheme. A sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is locally free of finite rank n if and only if every point $x \in X$ has an open affine neighborhood U together with an isomorphism $f : U \longrightarrow SpecA$ such that*

$$f_*(\mathscr{F}|_U) \cong \widetilde{A^n}$$

as $\mathcal{O}_{SpecA}$-modules, where A^n is the free A-module of rank n.

Proof. The result is trivial in the case $n = 0$, since $A^0 = 0$. So suppose $\mathscr{F}$ is locally free of finite positive rank n and let $x \in X$ be given. By assumption there is an open neighborhood V of x such that $\mathscr{F}|_V$ is free of rank n. Let U be an affine open subset of V containing x and $g : U \longrightarrow SpecA$ an isomorphism. The inclusion $\mathcal{O}_X|_U \longrightarrow \mathcal{O}_X|_V$ is right adjoint to the restriction functor. Hence restriction and $\widetilde{\ }$ both preserve coproducts, and we have

$$\begin{aligned} g_*(\mathscr{F}|_U) &= g_*\left((\mathscr{F}|_V)|_U\right) \\ &= g_*\left((\bigoplus_{i=1}^n \mathcal{O}_X|_V)|_U\right) \\ &= g_*\left(\bigoplus_{i=1}^n \mathcal{O}_X|_U\right) \\ &\cong \bigoplus_{i=1}^n \mathcal{O}_{SpecA} \cong \widetilde{A^n} \end{aligned}$$

The converse is clear, since if the condition is satisfied then $\mathscr{F}$ is free of rank n on an open cover of X. In fact the above proof works just as well if $\mathscr{F}$ is locally free of infinite rank, we just end up with $g_*(\mathscr{F}|_U) \cong (\bigoplus_{i\in I} A)^{\sim}$ for an infinite set I. □

Corollary 28. *Let X be a scheme. If $\mathscr{F}$ is locally free it is quasi-coherent, and if $\mathscr{F}$ is locally finitely free, then $\mathscr{F}$ is coherent.*

Lemma 29. *Let X be a scheme and $\mathscr{L}$ an invertible sheaf. If $U \subseteq X$ is open and $f \in \mathscr{L}(U)$ then $X_f = \{x \in U \,|\, germ_x f \notin \mathfrak{m}_x \mathscr{L}_x\}$ is an open set.*

Proof. If $X_f = \emptyset$ then this is trivial. If $x \in X_f$ then let $x \in V \subseteq X$ be an affine neighborhood with isomorphisms $f : V \longrightarrow SpecB$ and $\alpha : \mathcal{O}_X|_V \longrightarrow \mathscr{L}|_V$. Let $f' \in \mathcal{O}_{SpecB}(f(U \cap V))$ correspond to $f|_{U\cap V} \in \mathscr{L}(U \cap V)$. It would suffice to show that $f(X_f \cap V)$ is open in $SpecB$. But for $y \in SpecB$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_{SpecB,y} & \xrightarrow{(f_*\alpha)_y} & (f_*\mathscr{L}|_V)_y \\ \Downarrow & & \Downarrow \\ \mathcal{O}_{X,f^{-1}y} & \xrightarrow[\alpha_{f^{-1}y}]{} & \mathscr{L}_{f^{-1}y} \end{array}$$

It is not difficult to check that if R is a ring and M a free R-module of rank 1, $\alpha : R \longrightarrow M$ corresponding to a basis element, then $\alpha(\mathfrak{a}) = \mathfrak{a}M$ for any ideal $\mathfrak{a}$. Hence

$$\begin{aligned} f(X_f \cap V) &= \{y \in f(U \cap V) \,|\, germ_{f^{-1}y} f \notin \mathfrak{m}_{f^{-1}y} \mathscr{L}_{f^{-1}y}\} \\ &= \{y \in f(U \cap V) \,|\, germ_{f^{-1}y} f^{\#}_{f(U\cap V)}(f') \notin \mathfrak{m}_{f^{-1}y}\} \\ &= \{y \in f(U \cap V) \,|\, germ_y f' \notin \mathfrak{m}_y\} \end{aligned}$$

But from earlier notes we know that this final set is open. Hence X_f is open, since it is a union of open sets. □

Remark 2. In the previous Lemma if $f \in \mathscr{L}(U)$ and $V \subseteq U$ is affine with $\mathscr{L}|_V$ free, and if $g : V \longrightarrow SpecB$ is an isomorphism, then $g(X_f \cap V) = D(f')$ where $f' \in B$ corresponds to $f|_V \in \mathscr{L}(V) \cong \mathcal{O}_X(V)$.

Lemma 30. *Let X be a locally ringed space and $\mathscr{L}$ an invertible sheaf with section $s \in \mathscr{L}(U)$. Then for $x \in X$ we have $x \in X_s$ if and only if $germ_x s$ is a $\mathcal{O}_{X,x}$-basis for $\mathscr{L}_x$.*

Proof. By assumption if θ is a basis for $\mathscr{L}_x$ then $germ_x s_i = \lambda \cdot \theta$ with $\lambda \notin \mathfrak{m}_x$. Therefore λ is a unit and $germ_x s_i$ is a basis. The converse is just as easy. □

Lemma 31. *Let X be a locally ringed space, $\mathscr{L}$ an invertible sheaf and $f \in \Gamma(X, \mathscr{L})$ a global section. Then there is a canonical isomorphism $\mathscr{L}|_{X_f} \cong \mathcal{O}_X|_{X_f}$ of sheaves of modules.*

Proof. Let $\varphi : \mathcal{O}_X \longrightarrow \mathscr{L}$ be the morphism corresponding to the global section f. By Lemma 30 this induces an isomorphism on the stalks $\mathcal{O}_{X,x} \longrightarrow \mathscr{L}_x$ for every $x \in X_f$, and therefore the restriction $\varphi|_{X_f}$ must be an isomorphism. □

Lemma 32. *Let X be a scheme and $\mathscr{L}$ an invertible sheaf. If $s_1, \ldots, s_n \in \Gamma(X, \mathscr{L})$ then these global sections generate $\mathscr{L}$ if and only if the open sets X_{s_i} cover X.*

Lemma 33. *Let A be a local ring and M, M' A-modules with M finitely generated. If there is an isomorphism of A-modules $M \otimes_A M' \cong A$ then there is an isomorphism of A-modules $M \cong A$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A. If $M \otimes_A M' \cong A$ then tensoring with $k = A/\mathfrak{m}$ we obtain an isomorphism of k-vector spaces

$$M/\mathfrak{m}M \otimes_k M'/\mathfrak{m}M' \cong k$$

It follows that $M/\mathfrak{m}M$ and $M'/\mathfrak{m}M'$ both have dimension 1 as vector spaces. Therefore by Nakayama's Lemma M is generated by some element $z \in M$. In fact M is free on this element since if $a \in A$ annihilates z it must annihilate $M \otimes_A M' \cong A$, and therefore $a = 0$. This shows that $M \cong A$ as A-modules. □

Lemma 34. *Let X be a noetherian scheme and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. Then $\mathscr{F}$ is coherent if and only if it is locally finitely presented.*

Proof. This follows immediately from Corollary 26, and is the scheme version of the fact that over a noetherian ring the conditions of "finitely generated" and "finitely presented" are equivalent. □

Proposition 35. *Let X be a nonempty noetherian scheme and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Then*

(i) If $\mathscr{F}_x$ is a free $\mathcal{O}_{X,x}$-module of finite rank n then there is an open neighborhood U of x such that $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module of rank n.

(ii) $\mathscr{F}$ is locally free of finite rank n if and only if $\mathscr{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank n for every $x \in X$. In particular $\mathscr{F}$ is invertible if and only if $\mathscr{F}_x \cong \mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-modules for every $x \in X$.

(iii) $\mathscr{F}$ is invertible if and only if there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$ as $\mathcal{O}_X$-modules.

Proof. (i) and (ii) follow immediately from (MRS,Corollary 90) and Lemma 34, while (iii) follows from (MRS,Lemma 83), Lemma 33 and (ii). In the reverse implication of (iii) we do not require that $\mathscr{G}$ be coherent: that is, if $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$ for *any* sheaf of modules $\mathscr{G}$ then $\mathscr{F}$ is invertible. □

Proposition 36. *Let A be a noetherian ring and $X = SpecA$. If M is a finitely generated A-module then $M^\sim$ is a locally free sheaf if and only if M is a projective module.*

Proof. It follows from Proposition 35 that $M^\sim$ is a locally free sheaf if and only if $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module for every prime ideal $\mathfrak{p} \in SpecA$. So the result follows immediately from (MAT,Corollary 27). □

Remark 3. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a locally finitely free sheaf of modules on X. It is not in general true that $\mathscr{F}$ is a projective object in the abelian category $\mathfrak{Mod}(X)$. Take for example $\mathscr{F} = \mathcal{O}_X$. If $\phi : \mathscr{G} \longrightarrow \mathscr{H}$ is an epimorphism then for $\mathscr{F}$ to be projective we would require the morphism $\phi_X : \mathscr{G}(X) \longrightarrow \mathscr{H}(X)$ to be surjective, which is not generally the case. The problem is that projectivity in $\mathfrak{Mod}(X)$ is a "global" condition. If we replace the usual Hom with $\mathscr{H}om$ then things are better.

Lemma 37. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a locally finitely free sheaf of modules on X. Then the functor $\mathscr{H}om(\mathscr{F}, -) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ is exact.*

Proof. We already know the functor $\mathscr{H}om(\mathscr{F}, -)$ is left exact (MRS,Lemma 72), so it suffices to show that if $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is an epimorphism of sheaves of modules then so is $\mathscr{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}')$. But $\mathscr{F}$ is locally finitely presented so by (MRS,Proposition 89) we have a commutative diagram for every $x \in X$

$$\begin{array}{ccc} \mathscr{H}om(\mathscr{F}, \mathscr{G})_x & \longrightarrow & \mathscr{H}om(\mathscr{F}, \mathscr{G}')_x \\ \Downarrow & & \Downarrow \\ Hom_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) & \longrightarrow & Hom_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}'_x) \end{array}$$

The bottom row of this diagram is surjective since $\mathscr{F}_x$ is a projective $\mathcal{O}_{X,x}$-module, and we infer that the top row is also surjective. This shows that $\mathscr{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}')$ is an epimorphism and completes the proof. □

6 Exponential Tensor Product

Lemma 38. *Let X be a scheme, $\mathscr{L}$ an invertible sheaf on X and $s \in \Gamma(X, \mathscr{L})$. Let $X_s = \{x \in X \,|\, germ_x s \notin \mathfrak{m}_x \mathscr{L}_x\}$ be the canonical open set associated with s. Then for $n > 0$ we have $X_s = X_{s^n}$ for the global section $s^n \in \Gamma(X, \mathscr{L}^{\otimes n})$.*

Proof. Given $x \in X$ let U be an affine open neighborhood with $\mathscr{L}|_U$ free, and let $f \in \mathcal{O}_X(U)$ correspond to $s|_U$. Then $X_s \cap U = D_f$ and it is not hard to see that $X_{s^n} \cap U = D_{f^n} = D_f$, as required. □

Lemma 39. *Let $X = SpecA$ be an affine scheme and M an A-module. For $d > 0$ there is a canonical isomorphism of sheaves of modules $(M^{\otimes d})^{\sim} \cong \widetilde{M}^{\otimes d}$ natural in M. Under this isomorphism we have*

$$(m_1 \otimes \cdots \otimes m_d)\dot{/}s \mapsto m_1\dot{/}s \,\dot{\otimes}\, m\dot{/}1 \cdots \dot{\otimes}\, m_d\dot{/}1$$

Proof. The case $n = 2$ is handled in (H,5.2) and the rest is a simple induction. Naturality is not hard to check. □

Lemma 40. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. If $U \subseteq X$ is an affine open subset and $d > 0$ then there is a canonical isomorphism of $\mathcal{O}_X(U)$-modules $\tau : \mathscr{F}(U)^{\otimes d} \longrightarrow \mathscr{F}^{\otimes d}(U)$ natural in $\mathscr{F}$ defined by*

$$\tau(f_1 \otimes \cdots \otimes f_d) = f_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, f_d$$

Proof. For $d = 1$ this is trivial, so assume $d > 1$. If U is affine, let $a : Spec\mathcal{O}_X(U) \longrightarrow U$ be the canonical isomorphism. We have an isomorphism of sheaves of modules on U

$$\begin{aligned} \mathscr{F}^{\otimes d}|_U &\cong (\mathscr{F}|_U)^{\otimes d} \\ &\cong (a_* \widetilde{\mathscr{F}(U)})^{\otimes d} \\ &\cong a_*(\widetilde{\mathscr{F}(U)}^{\otimes d}) \\ &\cong a_*(\mathscr{F}(U)^{\otimes d})^{\sim} \end{aligned}$$

This gives rise to the desired isomorphism τ which is clearly natural in $\mathscr{F}$. □

7 Sheaf Hom

Lemma 41. *Let $X = SpecA$ be an affine scheme and M an A-module. Then M is finitely presented if and only if $\widetilde{M}$ is a globally finitely presented sheaf of modules.*

Proposition 42. *Let $X = SpecA$ be an affine scheme and let M, N be A-modules. Then there is a canonical morphism of sheaves of modules natural in M, N*

$$\begin{aligned} \lambda : Hom_A(M, N)^{\sim} &\longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \\ \phi\dot{/}s &\mapsto 1\dot{/}s \cdot \widetilde{\phi}|_U \end{aligned}$$

If M is finitely presented then this is an isomorphism.

Proof. There is a canonical morphism of A-modules $Hom_A(M, N) \longrightarrow Hom_{\mathcal{O}_X}(M^{\sim}, N^{\sim})$ given by the functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$. Using the adjunction between this functor and the global sections functor $\Gamma(-) : \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$ we obtain a canonical morphism of sheaves of modules $\lambda : Hom_A(M, N)^{\sim} \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(M^{\sim}, N^{\sim})$ with the required action. Naturality in both variables is easily checked. Now suppose that M is finitely presented, so that for every $\mathfrak{p} \in X$

we have a canonical isomorphism of $A_\mathfrak{p}$-modules $Hom_A(M,N)_\mathfrak{p} \cong Hom_{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$. One checks easily that the following diagram commutes

$$\begin{array}{ccc} Hom_A(M,N)^{\sim}{}_\mathfrak{p} & \xrightarrow{\lambda_\mathfrak{p}} & \mathscr{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})_\mathfrak{p} \\ \Downarrow & & \Downarrow \\ Hom_{A_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p}) & \Longrightarrow & Hom_{\mathcal{O}_{X,\mathfrak{p}}}(\widetilde{M}_\mathfrak{p},\widetilde{N}_\mathfrak{p}) \end{array}$$

where the right hand side is the isomorphism of (MRS,Proposition 89). Therefore $\lambda_\mathfrak{p}$ is an isomorphism for every $\mathfrak{p} \in X$, which shows that λ is an isomorphism of sheaves of modules. □

Corollary 43. *Let X be a scheme, $\mathscr{F},\mathscr{G}$ quasi-coherent sheaves of modules with $\mathscr{F}$ locally finitely presented. Then $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is quasi-coherent.*

Proof. The question is local so we may assume that $X = Spec(A)$ is affine and that $\mathscr{F} = M^{\sim}, \mathscr{G} = N^{\sim}$ for modules M, N with M finitely presented. The claim now follows from Proposition 42. □

Corollary 44. *Let X be a noetherian scheme, $\mathscr{F},\mathscr{G}$ quasi-coherent sheaves of modules. Then*

(i) If $\mathscr{F}$ is coherent then $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is quasi-coherent.

(ii) If $\mathscr{F},\mathscr{G}$ are both coherent then $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is coherent.

Proof. Recall that on a noetherian scheme the locally finitely presented sheaves of modules are the same as the coherent sheaves, so (i) is Corollary 43. (ii) follows from the fact that if M, N are two finitely generated modules over a noetherian ring A, the module $Hom_A(M,N)$ is also finitely generated. □

8 Extension of Coherent Sheaves

Proposition 45. *Let $X = SpecA$ be a noetherian affine scheme and $\mathscr{F}$ a quasi-coherent sheaf. Let $\{\alpha_i : \mathscr{G}_i \longrightarrow \mathscr{F}\}_{i\in I}$ the family of all coherent submodules of $\mathscr{F}$. This is a direct family of subobjects and $\mathscr{F} = \sum_i \mathscr{G}_i = \varinjlim \mathscr{G}_i$. In particular $\mathscr{F}(U) = \bigcup_i \mathscr{G}_i(U)$ for any open set U.*

Proof. When we write $\sum_i \mathscr{G}_i$ we mean the categorical union, which in the case of a noetherian space X and a direct family of submodules is actually the pointwise union (MRS,Lemma 5).

For each $i \in I$ the A-module $\mathscr{G}_i(X)$ is a finitely generated submodule of $\mathscr{F}(X)$ with $\mathscr{G}_i(X)^{\sim} \cong \mathscr{G}_i$. Moreover the following diagram commutes

$$\begin{array}{ccc} \mathscr{G}_i & \longrightarrow & \mathscr{F} \\ \Downarrow & & \Downarrow \\ \widetilde{\mathscr{G}_i(X)} & \longrightarrow & \widetilde{\mathscr{F}(X)} \end{array}$$

It is not hard to see that every finitely generated submodule of $\mathscr{F}(X)$ is of the form $\mathscr{G}(X)$ for some coherent submodule $\mathscr{G}$ of $\mathscr{F}$. So $\{\mathscr{G}_i(X)\}_{i\in I}$ is the set of finitely generated submodules of $\mathscr{F}(X)$ (note that all these submodules are distinct). Since $\mathscr{F}(X)$ is the direct limit of its finitely generated submodules and the functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ preserves colimits, it follows that $\mathscr{F} = \varinjlim \mathscr{G}_i$ and therefore $\mathscr{F} = \sum_i \mathscr{G}_i$. □

It is clear that we can also apply the Proposition to any scheme isomorphic to $SpecA$ for a noetherian ring A.

Lemma 46. *Let X be a locally noetherian scheme and $\mathscr{F}$ a coherent sheaf of modules on X. Any quasi-coherent submodule of $\mathscr{F}$ is coherent.*

Proof. The question is local, and on an affine noetherian scheme this follows from the fact that over a noetherian ring, any submodule of a finitely generated module is finitely generated. ☐

Proposition 47. *Let X be a noetherian scheme and $\mathscr{F}$ a coherent sheaf. If $\mathscr{F} = \varinjlim \mathscr{G}_i$ for a direct family of quasi-coherent submodules $\mathscr{G}_i$ then $\mathscr{F} = \mathscr{G}_\ell$ for some index ℓ.*

Proof. First we prove it in the case where $X = SpecA$ is an affine noetherian scheme. The functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ is fully faithful, hence reflects colimits. The A-modules $\mathscr{G}_i(X)$ form a direct family of submodules of $\mathscr{F}(X)$, and since $\mathscr{F}(X)^\sim$ is the direct limit of the $\mathscr{G}_i(X)^\sim$ it follows that $\mathscr{F}(X)$ is the direct limit of the $\mathscr{G}_i(X)$ and therefore $\mathscr{F}(X) = \sum_i \mathscr{G}_i(X)$. So we have reduced to proving that if a finitely generated A-module M over a noetherian ring A is the sum $\sum_i N_i$ of a nonempty direct family of submodules, then $M = N_k$ for some k. But this is obvious, so we have completed the proof in the affine case. It is therefore also true in the case of $X \cong SpecA$ with A noetherian.

Now let X be an arbitrary noetherian scheme. Since X is quasi-compact we can cover it with a finite collection of affine open sets $U_1, \ldots, U_n$. If $\mathscr{F} = \varinjlim \mathscr{G}_i$ then by the comment preceeding (MRS,Lemma 5) we see that $\mathscr{F} = \sum_i \mathscr{G}_i$. That is, $\mathscr{F}(U) = \bigcup_i \mathscr{G}_i(U)$ for all open U. It is immediate that $\mathscr{F}|_{U_k} = \sum_i \mathscr{G}_i|_{U_k}$ and therefore $\mathscr{F}|_{U_k} = \varinjlim_i \mathscr{G}_i|_{U_k}$ for $1 \le k \le n$. Since $\mathscr{F}|_{U_k}$ is coherent we have $\mathscr{F}|_{U_k} = \mathscr{G}_{i_k}|_{U_k}$ for some index i_k. Let ℓ be such that $i_k \le \ell$ for all k. Then it is clear that $\mathscr{F} = \mathscr{G}_\ell$, as required. ☐

Lemma 48. *Let $X = SpecA$ be a noetherian affine scheme, U an open subset and $\mathscr{F}$ a coherent $\mathcal{O}_X|_U$-module. Then there is a coherent $\mathcal{O}_X$-module $\mathscr{F}'$ with $\mathscr{F}'|_U = \mathscr{F}$.*

Proof. If $i : U \longrightarrow X$ is the inclusion then $i_*\mathscr{F}$ is quasi-coherent by (H, II 5.8). Therefore by Proposition 45, $i_*\mathscr{F} = \sum_i \mathscr{G}_i$ is the union of all its coherent submodules. It is clear that

$$\mathscr{F} = (i_*\mathscr{F})|_U = \sum_i \mathscr{G}_i|_U = \varinjlim \mathscr{G}_i|_U$$

Therefore by Proposition 47 we see that $\mathscr{F} = \mathscr{G}_i|_U$ for some index i, which completes the proof. ☐

Lemma 49. *Let $X = SpecA$ be a noetherian affine scheme, U an open subset and $\mathscr{F}$ a coherent $\mathcal{O}_X|_U$-module. Suppose that we are given a quasi-coherent sheaf $\mathscr{G}$ on X such that $\mathscr{F}$ is a submodule of $\mathscr{G}|_U$. Then there is a coherent submodule $\mathscr{F}'$ of $\mathscr{G}$ with $\mathscr{F}'|_U = \mathscr{F}$.*

Proof. Let $i : U \longrightarrow X$ be the inclusion and let $\rho : \mathscr{G} \longrightarrow i_*(\mathscr{G}|_U)$ be the morphism of modules defined by restriction $\rho_V : \mathscr{G}(V) \longrightarrow \mathscr{G}(U \cap V)$. By assumption $\mathscr{F}$ is a submodule of $\mathscr{G}|_U$, so it follows that $i_*\mathscr{F}$ is a submodule of $i_*(\mathscr{G}|_U)$. Let $\mathscr{H}$ be the submodule of $\mathscr{G}$ given by taking the inverse image (MRS,Lemma 7) of $i_*\mathscr{F}$ along ρ, which fits into the following pullback diagram

$$\begin{array}{ccc} \mathscr{H} & \longrightarrow & i_*\mathscr{F} \\ \downarrow & & \downarrow \\ \mathscr{G} & \xrightarrow[\rho]{} & i_*(\mathscr{G}|_U) \end{array}$$

Since the functor $|_U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)$ is exact it preserves finite limits, so applying $|_U$ to the above pullback diagram and using the fact that $\rho|_U = 1$ we see that $\mathscr{H}|_U = \mathscr{F}$. The subcategory $\mathfrak{Qco}(X)$ of quasi-coherent modules is an abelian subcategory of $\mathfrak{Mod}(X)$ and $i_*\mathscr{F}, \mathscr{G}, i_*(\mathscr{G}|_U)$ are all quasi-coherent, so it follows that $\mathscr{H}$ is also quasi-coherent. If $\{\mathscr{H}_i\}_{i \in I}$ is the collection of coherent submodules of $\mathscr{H}$ then $\mathscr{H} = \sum_i \mathscr{H}_i$ and by the method of Lemma 48 we see that $\mathscr{F} = \mathscr{H}_i|_U$ for some index i, which completes the proof. ☐

Of course it is immediate that we can also apply the Lemma in the case where X is isomorphic to $SpecA$ for some noetherian ring A. In the next Proposition we will use the following simple pasting procedure:

Lemma 50. *Let $(X, \mathcal{O}_X)$ be a ringed space and suppose that $X = U \cup V$ for two open sets U, V. Suppose that $\mathscr{G}$ is a sheaf of modules on X and $\mathscr{F}, \mathscr{F}'$ are submodules of $\mathscr{G}|_U, \mathscr{G}|_V$ on U, V respectively. If $\mathscr{F}|_{U\cap V} = \mathscr{F}'|_{U\cap V}$ then there is a unique submodule $\mathscr{F}''$ of $\mathscr{G}$ with $\mathscr{F}''|_U = \mathscr{F}$ and $\mathscr{F}''|_V = \mathscr{F}'$. Moreover if X is a scheme and both $\mathscr{F}, \mathscr{F}'$ are quasi-coherent (resp. coherent), then so is $\mathscr{F}''$.*

Proof. Define $\mathscr{F}''(W) = \{s \in \mathscr{G}(W) \,|\, s|_{W\cap U} \in \mathscr{F}(W \cap U) \text{ and } s|_{W\cap V} \in \mathscr{F}'(W \cap V)\}$. It is not hard to check this is a sheaf of $\mathcal{O}_X$-modules with the required property. □

Proposition 51. *Let X be a noetherian scheme, U an open subset and $\mathscr{F}$ a coherent sheaf on U. Suppose that we are given a quasi-coherent sheaf $\mathscr{G}$ on X such that $\mathscr{F}$ is a submodule of $\mathscr{G}|_U$. Then there is a coherent submodule $\mathscr{F}'$ of $\mathscr{G}$ with $\mathscr{F}'|_U = \mathscr{F}$.*

Proof. Let $V_1, \ldots, V_n$ be a cover of X by open affine subsets. We can apply Lemma 49 to the scheme V_1 with open subset $V_1 \cap U$ and sheaves $\mathscr{F}|_{V_1\cap U}, \mathscr{G}|_{V_1}$ to find a coherent submodule $\mathscr{F}'$ of $\mathscr{G}|_{V_1}$ such that $\mathscr{F}'|_{V_1\cap U} = \mathscr{F}|_{V_1\cap U}$. Since the coherent sheaves $\mathscr{F}, \mathscr{F}'$ agree on $U \cap V_1$ we can paste them to get a coherent submodule $\mathscr{F}_1'$ of $\mathscr{G}|_{U_1}$ where $U_1 = U \cup V_1$, such that $\mathscr{F}_1'|_U = \mathscr{F}$ and $\mathscr{F}_1'|_{V_1} = \mathscr{F}'$. Applying this argument again with U replaced by U_1 and V_1 replaced by V_2 to extend to a coherent sheaf on $U_2 = V_2 \cup U_1$, and eventually this produces the desired coherent sheaf $\mathscr{F}_n'$ on all of X. □

Corollary 52. *Let X be a noetherian scheme, U an open subset and $\mathscr{F}$ a coherent sheaf on U. Then there is a coherent sheaf $\mathscr{F}'$ on X with $\mathscr{F}'|_U = \mathscr{F}$.*

Proof. Take $\mathscr{G} = i_*\mathscr{F}$ in the Proposition. □

Lemma 53. *Let $(X, \mathcal{O}_X)$ be a ringed space and U an open subset. If $\mathscr{F}$ is a sheaf of modules on X which can be written as the union $\mathscr{F} = \sum_i \mathscr{G}_i$ of a nonempty collection of submodules, then $\mathscr{F}|_U = \sum_i \mathscr{G}_i|_U$.*

Proof. Realise the union as the image of a coproduct $\bigoplus_i \mathscr{G}_i \longrightarrow \mathscr{F}$ and use the fact that functor $-|_U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)$ is exact and preserves all colimits. □

Lemma 54. *Let X be a noetherian scheme and let $\mathscr{F}$ be a quasi-coherent sheaf which can be written as the finite union $\mathscr{F} = \mathscr{G}_1 \cup \cdots \cup \mathscr{G}_n$ of coherent submodules. Then $\mathscr{F}$ is coherent.*

Proof. Using Lemma 53 we reduce easily to the affine case, which follows easily from the fact that a module which is the sum of a finite number of finitely generated submodules is finitely generated. □

Corollary 55. *Let X be a noetherian scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then the coherent submodules $\{\mathscr{G}_i\}_{i\in I}$ of $\mathscr{F}$ form a direct family of submodules with $\mathscr{F} = \sum_i \mathscr{G}_i = \varinjlim \mathscr{G}_i$.*

Proof. Since X is quasi-compact we can cover it with a finite number of affine open sets $U_1, \ldots, U_n$. Suppose $\mathscr{G}_i, \mathscr{G}_j$ are coherent submodules of $\mathscr{F}$. For $1 \le k \le n$ it follows from Proposition 45 that there is a coherent submodule $\mathscr{H}_k$ of $\mathscr{F}|_{U_k}$ with $\mathscr{G}_i|_{U_k} \le \mathscr{H}_k$ and $\mathscr{G}_j|_{U_k} \le \mathscr{H}_k$. By Proposition 51 there is an index i_k such that $\mathscr{H}_k = \mathscr{G}_{i_k}|_{U_k}$. We already know the union $\mathscr{G}' = \mathscr{G}_{i_1} \cup \cdots \cup \mathscr{G}_{i_n}$ is a quasi-coherent submodule of $\mathscr{F}$, and since this is a finite union, $\mathscr{G}'$ is coherent and $\mathscr{G}_i \le \mathscr{G}'$, $\mathscr{G}_j \le \mathscr{G}'$ which completes the proof that the family $\{\mathscr{G}_i\}_{i\in I}$ is directed.

To complete the proof, it suffices to show that $\mathscr{F}|_{U_k} = \sum_i (\mathscr{G}_i|_{U_k})$ for $1 \le k \le n$. So by Proposition 45 it suffices to show that every coherent submodule of $\mathscr{F}|_{U_k}$ is of the form $\mathscr{G}_i|_{U_k}$ for some index i. Since this follows immediately from Proposition 51, the proof is complete. □

8.1 Extension in general

Definition 2. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules on X. We say that $\mathscr{F}$ is *of finite type* if for every $x \in X$ there is an open neighborhood U of x such that $\mathscr{F}|_U$ is generated as a $\mathcal{O}_X|_U$-module by a *finite* set of sections over U. Equivalently, $\mathscr{F}|_U$ is a quotient of $(\mathcal{O}_X|_U)^p$ for some finite $p > 0$. This property is stable under isomorphism. If $\mathscr{F}$ is of finite type then $\mathscr{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module for every $x \in X$.

Remark 4. It is clear that any quotient of a sheaf of modules of finite type is also of finite type. Also any finite direct sum of sheaves of modules of finite type is of finite type. Consequently, any finite sum of submodules of finite type is also of finite type. Any locally finitely presented sheaf of modules is of finite type. If X is a noetherian scheme and $\mathscr{F}$ a quasi-coherent sheaf then $\mathscr{F}$ is coherent if and only if it is of finite type.

Lemma 56. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a locally free sheaf of modules on X. Then $\mathscr{F}$ is locally finitely free if and only if it is of finite type.*

Proof. Suppose that $\mathscr{F}$ is locally free of finite type. Then for each $x \in X$ the $\mathcal{O}_{X,x}$-module $\mathscr{F}_x$ is free and finitely generated, and therefore of finite rank (provided $\mathcal{O}_{X,x}$ is nonzero). It follows that $\mathscr{F}$ is locally finitely free. The converse is trivial. □

Lemma 57. *Let $(X, \mathcal{O}_X)$ be a ringed space whose underlying space is quasi-compact, and let $u : \mathscr{F} \longrightarrow \mathscr{G}$ be an epimorphism of sheaves of modules on X with $\mathscr{G}$ of finite type. Suppose that $\mathscr{F} = \varinjlim_{\lambda \in \Lambda} \mathscr{F}_\lambda$ is a direct limit of submodules. Then $\mathscr{F}_\mu \longrightarrow \mathscr{G}$ is an epimorphism for some $\mu \in \Lambda$.*

Proof. For each point $x \in X$ let $U(x)$ be an open neighborhood of x and $s_{x,i_1}, \ldots, s_{x,i_{N_x}}$ a finite collection of sections of $\mathscr{G}$ over $U(x)$ which generate $\mathscr{G}|_{U(x)}$ as a sheaf of modules. Since u is an epimorphism and the collection is finite, we can find an open set $x \in V(x) \subseteq U(x)$ and sections $t_{x,i_1}, \ldots, t_{x,i_{N_x}}$ of $\mathscr{F}$ over $V(x)$ such that $s_{x,i_j}|_{V(x)} = u(t_{x,i_j})$ for every $1 \le j \le N_x$. Moreover by shrinking $V(x)$ if necessary we can assume by (MRS,Lemma 4) that there is a single index $\lambda(x)$ such that each t_{x,i_j} belongs to $\Gamma(V(x), \mathscr{F}_{\lambda(x)})$. Since X is quasi-compact we can choose a finite number of points $x_1, \ldots, x_n$ such that the $V(x_i)$ cover X, and also a single index $\mu \in \Lambda$ such that every $\lambda(x_i) \le \mu$. Then it is clear that $\mathscr{F}_\mu \longrightarrow \mathscr{G}$ is an epimorphism, as required. □

Example 1. As a special case, if $(X, \mathcal{O}_X)$ is a ringed space with quasi-compact underlying topological space, and $\mathscr{F}$ a sheaf of modules of finite type which is a direct limit $\mathscr{F} = \varinjlim_\lambda \mathscr{F}_\lambda$ of submodules, then $\mathscr{F} = \mathscr{F}_\lambda$ for some λ. When we say "is a direct limit of submodules" we really mean "is a direct limit of a direct family of submodules", so that the index λ is unique with this property.

Let $(X, \mathcal{O}_X)$ be a ringed space, $U \subseteq X$ an open subset and $\mathscr{F}$ a sheaf of modules on X. Let $i : U \longrightarrow X$ be the canonical injection, and $\mathscr{G}$ a submodule of $\mathscr{F}|_U$. Then $i_*\mathscr{G}$ is a submodule of $i_*(\mathscr{F}|_U)$. Let $\overline{\mathscr{G}}$ denote the submodule $\rho^{-1} i_* \mathscr{G}$ of $\mathscr{F}$ where ρ is the canonical morphism

$$\rho : \mathscr{F} \longrightarrow i_*(\mathscr{F}|_U)$$

It is clear from the definitions that for an open subset $V \subseteq X$

$$\Gamma(V, \overline{\mathscr{G}}) = \{ s \in \Gamma(V, \mathscr{F}) \,|\, s|_{V \cap U} \in \Gamma(V \cap U, \mathscr{G}) \}$$

Therefore $\overline{\mathscr{G}}|_U = \mathscr{G}$ as submodules of $\mathscr{F}|_U$, and $\overline{\mathscr{G}}$ is the largest submodule of $\mathscr{G}$ with this property (that is, it contains any other such submodule). We call $\overline{\mathscr{G}}$ the *canonical extension* of the submodule $\mathscr{G}$ of $\mathscr{F}|_U$ to a submodule of $\mathscr{F}$.

Proposition 58. *Let X be a scheme, U an open subset of X such that the injection $i : U \longrightarrow X$ is quasi-compact. Then*

(i) For any quasi-coherent sheaf of modules $\mathscr{G}$ on U, $j_*\mathscr{G}$ is quasi-coherent.

(ii) *For any quasi-coherent sheaf of modules $\mathscr{F}$ on X and quasi-coherent submodule $\mathscr{G}$ of $\mathscr{F}|_U$, the canonical extension $\overline{\mathscr{G}}$ is quasi-coherent.*

Proof. (i) The morphism i is quasi-compact by assumption, and quasi-separated since it is an open immersion. The functor j_* therefore preserves quasi-coherence by (CON,Proposition 18).

(ii) By construction $\overline{\mathscr{G}}$ is the pullback along a morphism ρ of quasi-coherent sheaves, of a quasi-coherent submodule $i_*(\mathscr{G})$. Since the quasi-coherent sheaves form an abelian subcategory of $\mathfrak{Mod}(X)$, it is therefore clear that $\overline{\mathscr{G}}$ is quasi-coherent. □

Example 2. If X is a quasi-separated scheme whose underlying space is noetherian, the inclusion of every open subset is a quasi-compact morphism, so the result applies in this case. In particular it is true for a noetherian scheme X.

Corollary 59. *Let X be a scheme, U a quasi-compact open subset of X such that the injection $j : U \longrightarrow X$ is quasi-compact. Suppose that every quasi-coherent sheaf of modules on X is a direct limit of quasi-coherent submodules of finite type. Then if $\mathscr{F}$ is a quasi-coherent sheaf of modules on X and $\mathscr{G}$ a quasi-coherent submodule of $\mathscr{F}|_U$ of finite type, then there exists a quasi-coherent submodule $\mathscr{G}'$ of $\mathscr{F}$ of finite type such that $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. The canonical extension $\overline{\mathscr{G}}$ is quasi-coherent and $\overline{\mathscr{G}}|_U = \mathscr{G}$. By assumption we have $\overline{\mathscr{G}} = \varinjlim_\lambda \mathscr{H}_\lambda$ for some quasi-coherent submodules $\mathscr{H}_\lambda$ of finite type. Then

$$\mathscr{G} = \overline{\mathscr{G}}|_U = \varinjlim_\lambda \mathscr{H}_\lambda|_U$$

and it follows from Example 1 that $\mathscr{G} = \mathscr{H}_\lambda|_U$ for some λ. □

Remark 5. If X is an affine scheme then every quasi-coherent sheaf of modules is certainly a direct limit of quasi-coherent submodules of finite type. Therefore for any open subset $U \subseteq X$ with quasi-compact inclusion, the conclusion of Corollary 59 holds.

Remark 6. Let X be a scheme, and suppose that for every affine open subset $U \subseteq X$ the inclusion $U \longrightarrow X$ is quasi-compact. Suppose the conclusion of Corollary 59 holds for these open sets. That is, assume that for every open affine U, quasi-coherent sheaf $\mathscr{F}$ on X and quasi-coherent submodule $\mathscr{G}$ of $\mathscr{F}|_U$ of finite type, there exists a quasi-coherent submodule $\mathscr{G}'$ of $\mathscr{F}$ of finite type such that $\mathscr{G}'|_U = \mathscr{G}$. Then we claim every quasi-coherent sheaf $\mathscr{F}$ on X is the direct limit of quasi-coherent submodules of finite type.

To see this, let $U \subseteq X$ be affine. Then $\mathscr{F}|_U$ is clearly the direct limit of quasi-coherent submodules of finite type (since any module is a direct limit of its finitely generated submodules). By assumption each of these submodules extends to a quasi-coherent submodule $\mathscr{G}_{\lambda,U}$ of $\mathscr{F}$ of finite type. Take all finite sums of such submodules, as U ranges over all affine open subsets of X and λ over the indices on each U. Each of these sums is quasi-coherent of finite type, and $\mathscr{F}$ is the direct limit over all such sums.

Corollary 60. *With the assumptions of Corollary 59, for every quasi-coherent sheaf of modules $\mathscr{G}$ on U of finite type there exists a quasi-coherent sheaf of modules $\mathscr{G}'$ on X of finite type with $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. We simply take $\mathscr{F} = j_*\mathscr{G}$ in Corollary 59. □

Lemma 61. *Let X be a scheme, L a small limit ordinal, $\{V_\lambda\}_{\lambda \in L}$ a cover of X by open affines, U an open subset of X, and for each $\lambda \in L$ we set $W_\lambda = \bigcup_{\mu<\lambda} V_\mu$. Suppose that*

(a) *For each $\lambda \in L$ the open set $V_\lambda \cap W_\lambda$ is quasi-compact.*

(b) *The inclusion $U \longrightarrow X$ is quasi-compact.*

Then for each quasi-coherent sheaf of modules $\mathscr{F}$ on X and quasi-coherent submodule $\mathscr{G}$ of $\mathscr{F}|_U$ of finite type, there exists a quasi-coherent submodule $\mathscr{G}'$ of $\mathscr{F}$ of finite type such that $\mathscr{G}'|_U = \mathscr{G}$.

Proof. For each $\lambda \in L$ we set $U_\lambda = U \cup W_\lambda$. To be clear, a *small* limit ordinal is a limit ordinal in the sense of (BST,Definition 2) (where we work in ZFC, so an ordinal can be an arbitrary conglomerate) which is small (i.e. bijective to a set). The idea is that we start with the sheaf $\mathscr{F}$ and by transfinite recursion extend it along the following chain of open subsets

$$U = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_\lambda \subseteq U_{\lambda+1} \subseteq \cdots$$

and then glue to obtain a sheaf on all of X. To be precise, we define for every $\lambda \in L$ a quasi-coherent submodule $\mathscr{G}'_\lambda$ of $\mathscr{F}|_{U_\lambda}$ of finite type, with the property that $\mathscr{G}'_\lambda|_{U_\mu} = \mathscr{G}'_\mu$ for any $\mu < \lambda$, with $\mathscr{G}'_0 = \mathscr{G}$. Then there is a unique submodule $\mathscr{G}'$ of $\mathscr{F}$ with $\mathscr{G}'|_{U_\lambda} = \mathscr{G}'_\lambda$ for every $\lambda \in L$, and it is clear that this is the desired sheaf.

If $\lambda = 0$ then we define $\mathscr{G}'_\lambda = \mathscr{G}$. Suppose that $\lambda \in L$ is a limit ordinal and that we have defined $\mathscr{G}'_\mu$ with the correct property for all $\mu < \lambda$. Then $U_\lambda = \bigcup_{\mu<\lambda} U_\mu$ so there is a unique submodule $\mathscr{G}'_\mu$ of $\mathscr{F}|_{U_\mu}$ with $\mathscr{G}'_\mu|_{U_\lambda} = \mathscr{G}'_\lambda$ for every $\lambda < \mu$. This sheaf is clearly quasi-coherent of finite type.

If λ is a successor ordinal, say $\lambda = \mu + 1$, then $U_\lambda = U_\mu \cup V_\mu$. By (b) the inclusion $U \longrightarrow X$ is quasi-compact, so $U \cap V_\mu$ is a quasi-compact open subset of the affine scheme V_μ. By (a) the open set $W_\mu \cap V_\mu$ is quasi-compact, and therefore so is the union

$$U_\mu \cap V_\mu = (U \cap V_\mu) \cup (W_\mu \cap V_\mu)$$

An affine scheme is quasi-separated, so the inclusion $U_\mu \cap V_\mu \longrightarrow V_\mu$ is quasi-compact. By Corollary 59 there exists a quasi-coherent submodule $\mathscr{G}''_\mu$ of $\mathscr{F}|_{V_\mu}$ of finite type, such that

$$\mathscr{G}''_\mu|_{U_\mu \cap V_\mu} = \mathscr{G}'_\mu|_{U_\mu \cap V_\mu}$$

Let $\mathscr{G}'_\lambda$ be the unique submodule of $\mathscr{F}|_{U_\lambda}$ with $\mathscr{G}'_\lambda|_{U_\mu} = \mathscr{G}'_\mu$ and $\mathscr{G}'_\lambda|_{V_\mu} = \mathscr{G}''_\mu$. This sheaf is clearly quasi-coherent of finite type. By transfinite recursion, we have defined for every $\lambda \in L$ a quasi-coherent submodule $\mathscr{G}'_\lambda$ with the required properties, so the proof is complete. □

Remark 7. There is a small technical detail in the proof of Lemma 61 that we need to remark upon. In any construction by transfinite recursion, the recursive steps cannot involve any arbitrary choices. But in our construction from μ to $\mu + 1$, we made such an arbitrary choice, because extensions are not necessarily canonical. So before we begin the transfinite recursion, we need to fix for every $\mu \in L$, quasi-compact open subset $T \subseteq V_\mu$ and quasi-coherent submodule $\mathscr{H}$ of $\mathscr{F}|_T$ of finite type, a specific extension of $\mathscr{H}$ to V_μ.

Theorem 62. *Let X be a concentrated scheme and U an open subset whose inclusion $U \longrightarrow X$ is quasi-compact. For any quasi-coherent sheaf of modules $\mathscr{F}$ on X and quasi-coherent submodule $\mathscr{G}$ of $\mathscr{F}|_U$ of finite type, there is a quasi-coherent submodule $\mathscr{G}'$ of $\mathscr{F}$ of finite type with $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. Let $V_0, \ldots, V_n$ be an affine open cover of X. To make this into a cover indexed by a small limit ordinal, set $L = \omega$ and $V_i = \emptyset$ for $i > n$. Since X is quasi-separated any intersection $V_\lambda \cap V_\mu$ is quasi-compact (CON,Proposition 12), and since only finitely many V_μ are nonempty, it follows that $V_\lambda \cap W_\lambda$ is quasi-compact for every $\lambda \in L$. Therefore the conditions of Lemma 61 are satisfied and we reach the desired conclusion. □

Corollary 63. *Let X be a concentrated scheme and U an open subset whose inclusion $U \longrightarrow X$ is quasi-compact. If $\mathscr{G}$ is a quasi-coherent sheaf of modules on U of finite type, there is a quasi-coherent sheaf of modules $\mathscr{G}'$ on X of finite type with $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. Let $i : U \longrightarrow X$ be the inclusion and set $\mathscr{F} = i_*\mathscr{G}$ in Theorem 62. □

Corollary 64. *Let X be a concentrated scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. Then $\mathscr{F}$ is the direct limit of its quasi-coherent submodules of finite type.*

Proof. This follows from Corollary 63 and Remark 6. □

8.2 Categories of quasi-coherent sheaves

Lemma 65. *Let X be a scheme. There is a nonempty set $\mathcal{I}$ of sheaves of modules on X of finite type which is* representative. *That is, every sheaf of modules of finite type is isomorphic to some element of $\mathcal{I}$.*

Proof. The proof is the same as (MRS,Proposition 11), with the obvious modifications. ☐

Proposition 66. *Let X be a concentrated scheme. Then $\mathfrak{Qco}(X)$ is a grothendieck abelian category. In particular it is complete and has enough injectives.*

Proof. We already know from Proposition 25 that $\mathfrak{Qco}(X)$ is a cocomplete abelian category. Since $\mathfrak{Mod}(X)$ is grothendieck the category $\mathfrak{Qco}(X)$ clearly satisfies the condition Ab5, so it only remains to show that $\mathfrak{Qco}(X)$ has a generating family.

By Lemma 65 we can find a *set* of quasi-coherent sheaves of modules of finite type $\{\mathscr{F}_D\}_{D\in\mathcal{G}}$ such that every quasi-coherent sheaf of finite type is isomorphic to some $\mathscr{F}_D$. We claim that $\{\mathscr{F}_D\}_{D\in\mathcal{G}}$ is a generating family for $\mathfrak{Qco}(X)$. If $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ is a nonzero morphism of quasi-coherent sheaves, then since $\mathscr{F}$ is the direct limit of its quasi-coherent submodules of finite type by Corollary 64 we can find some $D \in \mathcal{G}$ and a morphism $\mu : \mathscr{F}_D \longrightarrow \mathscr{F}$ with $\varphi\mu \neq 0$, which completes the proof. ☐

Remark 8. If X is noetherian then we can combine (MRS,Proposition 11) and Lemma 34 to find a set of coherent sheaves $\{\mathscr{F}_D\}_{D\in\mathcal{G}}$ with the property that every coherent sheaf is isomorphic to some $\mathscr{F}_D$. Using Corollary 55 one checks that this is a generating family for $\mathfrak{Qco}(X)$, slightly refining Proposition 66 in this case. Actually if X is concentrated and admits an ample family of invertible sheaves then the tensor powers of the ample family generate $\mathfrak{Qco}(X)$ (AMF,Lemma 8).

Remark 9. This does *not* necessarily mean that arbitrary limits of quasi-coherent modules in $\mathfrak{Mod}(X)$ are quasi-coherent, since we don't know that the inclusion $\mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ preserves all limits.

Definition 3. Let X be a concentrated scheme. The inclusion $i : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ is colimit preserving and therefore has a right adjoint $Q : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Qco}(X)$ which we call the *coherator* (AC,Theorem 49). The coherator Q preserves all limits and sends injectives of $\mathfrak{Mod}(X)$ to injectives in $\mathfrak{Qco}(X)$ (AC,Theorem 26). The unit $1 \longrightarrow Qi$ is a natural equivalence (AC,Proposition 21).

Lemma 67. *Let X be a noetherian scheme. For any quasi-coherent sheaf $\mathscr{F}$ there is a quasi-coherent sheaf $\mathscr{G}$ which is injective in $\mathfrak{Mod}(X)$ together with a monomorphism $\mathscr{F} \longrightarrow \mathscr{G}$.*

Proof. See Hartshorne's Residues and Duality (I, 4.8) and (II, 7.18). ☐

Proposition 68. *If X is a noetherian scheme then the inclusion $\mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ preserves injectives.*

Proof. Let $\mathscr{F}$ be a quasi-coherent sheaf which is injective in $\mathfrak{Qco}(X)$. By Lemma 67 we can find a quasi-coherent sheaf $\mathscr{G}$ which is injective in $\mathfrak{Mod}(X)$ together with a monomorphism $\mathscr{F} \longrightarrow \mathscr{G}$. Since $\mathscr{F}$ is injective in $\mathfrak{Qco}(X)$ this monomorphism splits, and it follows that $\mathscr{F}$ is injective in $\mathfrak{Mod}(X)$, as required. ☐

Corollary 69. *Let $f : X \longrightarrow Y$ be an open immersion of noetherian schemes. Then the functor $f^* : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Qco}(X)$ preserves injectives.*

Proof. We already know that $f^* : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ preserves injectives (MRS,Corollary 28), so this follows from Proposition 68. ☐

8.3 Finiteness Conditions

In our notes on Abelian Categories (AC) we introduced several finiteness conditions (AC,Definition 61) on objects of an abelian category. For example we defined the *finitely generated* and *finitely presented* objects. In a category of modules these agree with the usual definitions (AC,Lemma 94) (AC,Proposition 97) and in this section we examine the corresponding objects of $\mathfrak{Qco}(X)$. The reader should be careful to distinguish between the following three conditions that we have now introduced for sheaves of modules:

(i) A *globally finitely presented* sheaf is one that can be written as the cokernel of a morphism of two finite coproducts of the structure sheaf.

(ii) A *locally finitely presented* sheaf is one that can be *locally* written as such a cokernel.

(iii) A *finitely presented* sheaf is one satisfying the categorical property of (AC,Definition 61). This makes sense in both $\mathfrak{Qco}(X)$ and $\mathfrak{Mod}(X)$ so we will specify the ambient category.

Lemma 70. *Let X be a concentrated scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then $\mathscr{F}$ is finitely generated in $\mathfrak{Qco}(X)$ if and only if it is coherent.*

Proof. Recall that $\mathscr{F}$ is coherent if and only if it is of finite type. Suppose that $\mathscr{F}$ is of finite type. It follows from Example 1 that $\mathscr{F}$ is a finitely generated object of $\mathfrak{Qco}(X)$. In fact an arbitrary sheaf of modules of finite type on a quasi-compact ringed space is a finitely generated object of $\mathfrak{Mod}(X)$ by this argument. Conversely, suppose that $\mathscr{F} \in \mathfrak{Qco}(X)$ is finitely generated. From Corollary 64 we deduce that $\mathscr{F}$ is of finite type. □

Remark 10. Let X be a noetherian scheme. From Lemma 46 and (AC,Lemma 92) we deduce that every coherent sheaf $\mathscr{F}$ is a noetherian object of $\mathfrak{Qco}(X)$, which is therefore a locally noetherian grothendieck abelian category (AC,Definition 63). In particular for any quasi-coherent sheaf $\mathscr{F}$ the following conditions are equivalent:

(i) $\mathscr{F}$ is of finite type.

(ii) $\mathscr{F}$ is coherent.

(iii) $\mathscr{F}$ is finitely generated in $\mathfrak{Qco}(X)$.

(iv) $\mathscr{F}$ is noetherian in $\mathfrak{Qco}(X)$.

(v) $\mathscr{F}$ is compact in $\mathfrak{Qco}(X)$ (AC,Corollary 113).

Proposition 71. *Let X be a concentrated scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules. The following are equivalent:*

(a) The functor $Hom(\mathscr{F}, -) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ preserves direct limits. In other words, $\mathscr{F}$ is a finitely presented object of $\mathfrak{Qco}(X)$.

(b) The functor $\mathscr{H}om(\mathscr{F}, -) : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ preserves direct limits.

Proof. $(b) \Rightarrow (a)$ The underlying topology of X is quasi-noetherian so the global sections functor $\Gamma(X, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$ preserves direct limits (COS,Proposition 23). Composing with $\mathscr{H}om(\mathscr{F}, -)$ we deduce (a). For $(a) \Rightarrow (b)$ let $U \subseteq X$ be a quasi-compact open subset with inclusion $i : U \longrightarrow X$. This is a concentrated scheme so the functor

$$i_* : \mathfrak{Qco}(U) \longrightarrow \mathfrak{Qco}(X)$$

exists and preserves direct limits (CON,Proposition 18) (HDIS,Proposition 37). From this fact and the adjunction isomorphism

$$Hom_{\mathfrak{Qco}(U)}(\mathscr{F}|_U, -) \longrightarrow Hom_{\mathfrak{Qco}(X)}(\mathscr{F}, i_*(-))$$

we deduce that $Hom(\mathscr{F}|_U, -) : \mathfrak{Qco}(U) \longrightarrow \mathbf{Ab}$ also preserves direct limits. That is, any quasi-compact restriction of $\mathscr{F}$ is finitely presented. Now let $\{\mathscr{G}_\lambda, \varphi_{\mu\lambda}\}_{\lambda\in\Lambda}$ be a direct system in $\mathfrak{Qco}(X)$ with direct limit $\mathscr{G} = \varinjlim_\lambda \mathscr{G}_\lambda$. To show that the morphisms $\mathscr{H}om(\mathscr{F}, \mathscr{G}_\lambda) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})$ are a direct limit in $\mathfrak{Mod}(X)$ it suffices to check after applying $\Gamma(U, -)$ for every quasi-compact open $U \subseteq X$, since such subsets form a basis of X. But the morphisms

$$Hom(\mathscr{F}|_U, \mathscr{G}_\lambda|_U) \longrightarrow Hom(\mathscr{F}|_U, \mathscr{G}|_U)$$

are known to be a direct limit in $\mathbf{Ab}$ because $\mathscr{F}|_U$ is finitely presented. This establishes (b) and completes the proof. □

Corollary 72. *Let X be a concentrated scheme, U a quasi-compact open subset and $\mathscr{F}$ a finitely presented object of $\mathfrak{Qco}(X)$. Then $\mathscr{F}|_U$ is finitely presented in $\mathfrak{Qco}(U)$.*

The most interesting consequence of this proposition is that being a finitely presented object is local, which is slightly surprising given the very global form of the definition.

Lemma 73. *Let X be a concentrated scheme. An object $\mathscr{F} \in \mathfrak{Qco}(X)$ is finitely presented if and only if every $x \in X$ has a quasi-compact open neighborhood U such that $\mathscr{F}|_U$ is finitely presented in $\mathfrak{Qco}(U)$.*

Proof. Every quasi-compact open subset of X is concentrated and we observed in the proof of Proposition 71 that restricting to a quasi-compact open subset of a concentrated scheme preserves the property of being finitely presented, so it is easy to see that X has a basis of quasi-compact open subsets U_λ for which $\mathscr{F}|_{U_\lambda} \in \mathfrak{Qco}(U_\lambda)$ is a finitely presented object. For each λ the functor $Hom(\mathscr{F}|_{U_\lambda}, (-)|_{U_\lambda}) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ preserves direct limits so using the argument of Proposition 71 part $(a) \Rightarrow (b)$ one checks that $\mathscr{H}om(\mathscr{F}, -) : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ preserves direct limits. This implies that $\mathscr{F}$ is finitely presented in $\mathfrak{Qco}(X)$, as required. □

Lemma 74. *Let $X = Spec(A)$ be an affine scheme. A quasi-coherent sheaf $\mathscr{F}$ is finitely presented in $\mathfrak{Qco}(X)$ if and only if $\Gamma(X, \mathscr{F})$ is a finitely presented A-module.*

Proof. This is an immediate consequence of (AC,Proposition 97). □

Proposition 75. *Let X be a concentrated scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then $\mathscr{F}$ is finitely presented in $\mathfrak{Qco}(X)$ if and only if it is locally finitely presented.*

Proof. If $\mathscr{F}$ is locally finitely presented then every point $x \in X$ has an affine open neighborhood U for which $\Gamma(U, \mathscr{F})$ is a finitely presented $\Gamma(U, \mathcal{O}_X)$-module. From Lemma 74 and Lemma 73 we deduce that $\mathscr{F}$ is a finitely presented object of $\mathfrak{Qco}(X)$. Conversely suppose that $\mathscr{F}$ is finitely presented in $\mathfrak{Qco}(X)$. Given $x \in X$ let $x \in U$ be an affine open neighborhood and observe that $\mathscr{F}|_U$ is finitely presented in $\mathfrak{Qco}(U)$. We infer from Lemma 74 that $\mathscr{F}|_U$ is actually globally finitely presented on U. Hence $\mathscr{F}$ is locally finitely presented and the proof is complete. □

Here is an amusing consequence in the world of commutative rings.

Corollary 76. *Let A be a commutative ring, M an A-module and $f_1, \ldots, f_n \in A$ elements generating the unit ideal. Suppose that each M_{f_i} is a finitely presented A_{f_i}-module. Then M is a finitely presented A-module.*

Next we turn to an analysis of the compact objects of $\mathfrak{Qco}(X)$. For the proof of the next proposition and its corollary we simply copy the argument given in Proposition 71.

Proposition 77. *Let X be a concentrated scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules. The following are equivalent:*

(a) The functor $Hom(\mathscr{F}, -) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ preserves coproducts. In other words, $\mathscr{F}$ is a compact object of $\mathfrak{Qco}(X)$.

(b) The functor $\mathscr{H}om(\mathscr{F}, -) : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ preserves coproducts.

Corollary 78. *Let X be a concentrated scheme, U a quasi-compact open subset and $\mathscr{F}$ a compact object of $\mathfrak{Qco}(X)$. Then $\mathscr{F}|_U$ is compact in $\mathfrak{Qco}(U)$.*

Lemma 79. *Let X be a concentrated scheme. An object $\mathscr{F} \in \mathfrak{Qco}(X)$ is compact if and only if every $x \in X$ has a quasi-compact open neighborhood U such that $\mathscr{F}|_U$ is compact in $\mathfrak{Qco}(U)$.*

Proposition 80. *Let X be a concentrated scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then $\mathscr{F}$ is compact in $\mathfrak{Qco}(X)$ if and only if every point $x \in X$ has an open affine neighborhood $x \in U$ such that $\Gamma(U, \mathscr{F})$ is a compact $\Gamma(U, \mathcal{O}_X)$-module.*

References

[Gro60] A. Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4, 228. MR MR0217083 (36 #177a)