Matsumura: Commutative Algebra

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These notes closely follow Matsumura's book [Mat80] on commutative algebra. Proofs are the ones given there, sometimes with slightly more detail. Our focus is on the results needed in algebraic geometry, so some topics in the book do not occur here or are not treated in their full depth. In particular material the reader can find in the more elementary [AM69] is often omitted. References on dimension theory are usually to Robert Ash's webnotes since the author prefers this approach to that of [AM69].

Contents

1 General Rings

Throughout these notes all rings are commutative, and unless otherwise specified all modules are left modules. A local ring A is a commutative ring with a single maximal ideal (we do not require A to be noetherian).

Lemma 1 (Nakayama). *Let A be a ring, M a finitely generated A-module and I an ideal of A. Suppose that $IM = M$. Then there exists an element $a \in A$ of the form $a = 1 + x, x \in I$ such that $aM = 0$. If moreover I is contained in the Jacobson radical, then $M = 0$.*

Corollary 2. *Let A be a ring, M an A-module, N and N' submodules of M and I an ideal of A. Suppose that $M = N + IN'$, and that either (a) I is nilpotent or (b) I is contained in the Jacobson radical and N' is finitely generated. Then $M = N$.*

Proof. In case (a) we have $M/N = I(M/N) = I^2(M/N) = \cdots = 0$. In (b) apply Nakayama's Lemma to M/N. $\square$

In particular let $(A, \mathfrak{m}, k)$ be a local ring and M an A-module. Suppose that either $\mathfrak{m}$ is nilpotent or M is finitely generated. Then a subset G of M generates M iff. its image in $M/\mathfrak{m}M = M \otimes_A k$ generates $M \otimes_A k$ as a k-vector space. In fact if N is submodule generated by G, and if the image of G generates $M \otimes_A k$, then $M = N + \mathfrak{m}M$ whence $M = N$ by the Corollary. Since $M \otimes_A k$ is a finitely generated vector space over the field k, it has a finite basis, and if we take an arbitrary preimage of each element this collection generates M. A set of elements which becomes a basis in $M/\mathfrak{m}M$ (and therefore generates M) is called a *minimal basis.* If M is a finitely generated free A-module, then it is clear that

$$rank_A M = rank_k(M/\mathfrak{m}M)$$

In fact, of $rank_A M = n \geq 1$ and $\{x_1, \ldots, x_n\}$ is a basis of M, then $\{x_1 + \mathfrak{m}M, \ldots, x_n + \mathfrak{m}M\}$ is a basis of the k-module $M/\mathfrak{m}M$. Or equivalently, the $x_i \otimes 1$ are a basis of the k-module $M \otimes k$.

Let A be a ring and $\alpha : \mathbb{Z} \longrightarrow A$. The kernel is (n) for some integer $n \geq 0$ which we call the *characteristic* of A. The characteristic of a field is either 0 or a prime number, and if A is local the characteristic $ch(A)$ is either 0 or a power of a prime number ($\mathfrak{m}$ is a primary ideal and the contraction of primary ideals are primary, and 0 and (p^n) are the only primary ideals in $\mathbb{Z}$).

Lemma 3. *Let A be an integral domain with quotient field K, all localisations of A can be viewed as subrings of K and in this sense $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ where the intersection is over all maximal ideals.*

Proof. Given $x \in K$ we put $D = \{a \in A \,|\, ax \in A\}$, we call D the ideal of denominators of x. The element x is in A iff. $D = A$ and $x \in A_{\mathfrak{p}}$ iff. $D \not\subseteq \mathfrak{p}$. Therefore if $x \notin A$, there exists a maximal ideal $\mathfrak{m}$ such that $D \subseteq \mathfrak{p}$ and $x \notin A_{\mathfrak{m}}$ for this $\mathfrak{m}$. $\square$

Lemma 4. *Let A be a ring and $S \subseteq T$ multiplicatively closed subsets. Then*

(a) There is a canonical isomorphism of $S^{-1}A$-algebras $T^{-1}A \cong T^{-1}(S^{-1}A)$ defined by $a/t \mapsto (a/1)/(t/1)$.

(b) If M is an A-module then there is a canonical isomorphism of $S^{-1}A$-modules $T^{-1}M \cong T^{-1}(S^{-1}M)$ defined by $m/t \mapsto (m/1)/(t/1)$.

Proof. (a) Just using the universal property of localisation we can see $T^{-1}A \cong T^{-1}(S^{-1}A)$ as $S^{-1}A$-algebras via the map $a/t \mapsto (a/1)/(t/1)$. (b) is also easily checked. $\square$

Lemma 5. *Let A be an integral domain with quotient field K and B a subring of K containing A. If Q is the quotient field of B then there is a canonical isomorphism of B-algebras $K \cong Q$.*

If $\phi : A \longrightarrow B$ is a ring isomorphism and $S \subseteq A$ is multiplicatively closed (denote also by S the image in B) then there is an isomorphism of rings $S^{-1}A \cong S^{-1}B$ making the following diagram commute

$$\begin{tikzcd} S^{-1}A \arrow[r, Rightarrow] & S^{-1}B \\ A \arrow[u] \arrow[r, Rightarrow, "\phi"'] & B \arrow[u] \end{tikzcd}$$

Lemma 6. *Let A be a ring, $S \subseteq A$ a multiplicatively closed subset and $\mathfrak{p}$ a prime ideal with $\mathfrak{p} \cap S = \emptyset$. Let $B = S^{-1}A$. Then there is a canonical ring isomorphism $B_{\mathfrak{p}B} \cong A_{\mathfrak{p}}$.*

Proof. $A \longrightarrow B \longrightarrow B_{\mathfrak{p}B}$ sends elements of A not in $\mathfrak{p}$ to units, so we have an induced ring morphism $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}B}$ defined by $a/s \mapsto (a/1)/(s/1)$ and it is easy to check this is an isomorphism. $\square$

Let $\psi : A \longrightarrow B$ be a morphism of rings and I an ideal of A. The extended ideal IB consists of sums $\sum \psi(a_i)b_i$ with $a_i \in I, b_i \in B$. Consider the exact sequence of A-modules

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

Tensoring with B gives an exact sequence of B-modules

$$I \otimes_A B \longrightarrow A \otimes_A B \longrightarrow (A/I) \otimes_A B \longrightarrow 0$$

The image of $I \otimes_A B$ in $B \cong A \otimes_A B$ is simply IB. So there is an isomorphism of B-modules $B/IB \cong (A/I) \otimes_A B$ defined by $b + IB \mapsto 1 \otimes b$. In fact, this is an isomorphism of rings as well. Of course, for any two A-algebras E, F twisting gives a ring isomorphism $E \otimes_A F \cong F \otimes_A E$.

Lemma 7. *Let $\phi : A \longrightarrow B$ be a morphism of rings, S a multiplicatively closed subset of A and set $T = \phi(S)$. Then for any B-module M there is a canonical isomorphism of $S^{-1}A$-modules natural in M*

$$\alpha : S^{-1}M \longrightarrow T^{-1}M$$
$$\alpha(m/s) = m/\phi(s)$$

In particular there is a canonical isomorphism of $S^{-1}A$-algebras $S^{-1}B \cong T^{-1}B$.

Proof. One checks easily that α is a well-defined isomorphism of $S^{-1}A$-modules. In the case $M = B$ the $S^{-1}A$-module $S^{-1}B$ becomes a ring in the obvious way, and α preserves this ring structure. □

In particular, let S be a multiplicatively closed subset of a ring A, let I be an ideal of A and let T denote the image of S in A/I. Then there is a canonical isomorphism of rings

$$T^{-1}(A/I) \cong A/I \otimes_A S^{-1}A \cong S^{-1}A/I(S^{-1}A)$$
$$(a + I)/(s + I) \mapsto a/s + I(S^{-1}A)$$

Definition 1. A ring A is *catenary* if for each pair of prime ideals $\mathfrak{q} \subset \mathfrak{p}$ the height of the prime ideal $\mathfrak{p}/\mathfrak{q}$ in $A/\mathfrak{q}$ is finite and is equal to the length of any maximal chain of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$. Clearly the catenary property is stable under isomorphism, and any quotient of a catenary ring is catenary. If $S \subseteq A$ is a multiplicatively closed subset and A is catenary, then so is $S^{-1}A$.

Lemma 8. *Let A be a ring. Then the following are equivalent:*

(i) *A is catenary;*

(ii) *$A_{\mathfrak{p}}$ is catenary for every prime ideal $\mathfrak{p}$;*

(iii) *$A_{\mathfrak{m}}$ is catenary for every maximal ideal $\mathfrak{m}$.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. $(iii) \Rightarrow (i)$ If $\mathfrak{q} \subset \mathfrak{p}$ are primes, find a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$ and pass to the catenary ring $A_{\mathfrak{m}}$ to see that the required property is satisfied for $\mathfrak{q}, \mathfrak{p}$. □

Lemma 9. *Let A be a noetherian ring. Then A is catenary if for every pair of prime ideals $\mathfrak{q} \subset \mathfrak{p}$ we have $ht.(\mathfrak{p}/\mathfrak{q}) = ht.\mathfrak{p} - ht.\mathfrak{q}$.*

Proof. Since A is noetherian, all involved heights are finite. Suppose A satisfies the condition and let $\mathfrak{q} \subset \mathfrak{p}$ be prime ideals. Obviously $ht.(\mathfrak{p}/\mathfrak{q})$ is finite, and there is at least one maximal chain between $\mathfrak{p}$ and $\mathfrak{q}$ with length $ht.(\mathfrak{p}/\mathfrak{q})$. Let

$$\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{p}$$

be a maximal chain of length n. Then by assumption $1 = ht.(\mathfrak{q}_i/\mathfrak{q}_{i-1}) = ht.\mathfrak{q}_i - ht.\mathfrak{q}_{i-1}$ for $1 \leq i \leq n$. Hence $ht.\mathfrak{p} = ht.\mathfrak{q} + n$, so $n = ht.(\mathfrak{p}/\mathfrak{q})$, as required. □

Definition 2. A ring A is *universally catenary* if A is noetherian and every finitely generated A-algebra is catenary. Equivalently, a noetherian ring A is universally catenary if A is catenary and $A[x_1, \dots, x_n]$ is catenary for $n \geq 1$.

Lemma 10. *Let A be a ring and $S \subseteq A$ a multiplicatively closed subset. Then there is a canonical ring isomorphism $S^{-1}(A[x]) \cong (S^{-1}A)[x]$. In particular if $\mathfrak{q}$ is a prime ideal of $A[x]$ and $\mathfrak{p} = \mathfrak{q} \cap A$ then $A[x]_{\mathfrak{q}} \cong A_{\mathfrak{p}}[x]_{\mathfrak{q}A_{\mathfrak{p}}[x]}$.*

Proof. The ring morphism $A \longrightarrow S^{-1}A$ induced $A[x] \longrightarrow (S^{-1}A)[x]$ which sends elements of $S \subseteq A[x]$ to units. So there is an induced ring morphism $\varphi : S^{-1}(A[x]) \longrightarrow (S^{-1}A)[x]$ defined by

$$\varphi\left(\frac{a_0 + a_1 x + \cdots + a_n x^n}{s}\right) = \frac{a_0}{s} + \frac{a_1}{s}x + \cdots + \frac{a_n}{s}x^n$$

This is easily checked to be an isomorphism. In the second claim, there is an isomorphism $A_{\mathfrak{p}}[x] \cong A[x]_{\mathfrak{p}}$, where the second ring denotes $(A - \mathfrak{p})^{-1}(A[x])$, and $\mathfrak{q}A_{\mathfrak{p}}[x]$ denotes the prime ideal of $A_{\mathfrak{p}}[x]$ corresponding to $\mathfrak{q}A[x]_{\mathfrak{p}}$. Using the isomorphism φ it is clear that $\mathfrak{q}A_{\mathfrak{p}}[x] \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Using Lemma 6, there is clearly an isomorphism of rings $A[x]_{\mathfrak{q}} \cong A_{\mathfrak{p}}[x]_{\mathfrak{q}A_{\mathfrak{p}}[x]}$. □

If R is a ring and M an R-module, then let $\mathcal{Z}(M)$ denote the set of zero-divisors in M. That is, all elements $r \in R$ with $rm = 0$ for some nonzero $m \in M$.

Lemma 11. *Let R be a nonzero reduced noetherian ring. Then $\mathcal{Z}(R) = \bigcup_i \mathfrak{p}_i$, with the union being taken over all minimal prime ideals $\mathfrak{p}_i$.*

Proof. Since R is reduced, $\bigcap_i \mathfrak{p}_i = 0$. If $ab = 0$ with $b \neq 0$, then $b \notin \mathfrak{p}_j$ for some j, and therefore $a \in \mathfrak{p}_j \subseteq \bigcup_i \mathfrak{p}_i$. The reverse inclusion follows from the fact that no minimal prime can contain a regular element (since otherwise by Krull's PID Theorem it would have height ≥ 1). □

Lemma 12. *Let R be a nonzero reduced noetherian ring. Assume that every element of R is either a unit or a zero-divisor. Then $dim(R) = 0$.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal primes of R. Then by Lemma 11, $\mathcal{Z}(R) = \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. Let $\mathfrak{p}$ be a prime ideal. Since $\mathfrak{p}$ is proper, $\mathfrak{p} \subseteq \mathcal{Z}(R)$ and therefore $\mathfrak{p} \subseteq \mathfrak{p}_i$ for some i. Since $\mathfrak{p}_i$ is minimal, $\mathfrak{p} = \mathfrak{p}_i$, so the $\mathfrak{p}_i$ are the only primes in R. Since these all have height zero, it is clear that $dim(R) = 0$. □

Lemma 13. *Let R be a reduced ring, $\mathfrak{p}$ a minimal prime ideal of R. Then $R_{\mathfrak{p}}$ is a field.*

Proof. If $\mathfrak{p} = 0$ this is trivial, so assume $\mathfrak{p} \neq 0$. Since $\mathfrak{p}R_{\mathfrak{p}}$ is the only prime ideal in $R_{\mathfrak{p}}$, it is also the nilradical. So if $x \in \mathfrak{p}$ then $tx^n = 0$ for some $t \notin \mathfrak{p}$ and $n > 0$. But this implies that tx is nilpotent, and therefore zero since R is reduced. Therefore $\mathfrak{p}R_{\mathfrak{p}} = 0$ and $R_{\mathfrak{p}}$ is a field. □

Let $A_1, \dots, A_n$ be rings. Let A be the product ring $A = \prod_{i=1}^n A_i$. Ideals of A are in bijection with sequences $I_1, \dots, I_n$ with I_i an ideal of A_i. This sequence corresponds to

$$I_1 \times \cdots \times I_n$$

This bijection identifies the prime ideals of A with sequences $I_1, \dots, I_n$ in which every $I_i = A_i$ except for a single I_j which is a prime ideal of A_j. So the primes look like

$$A_1 \times \cdots \times \mathfrak{p}_i \times \cdots \times A_n$$

for some i and some prime ideal $\mathfrak{p}_i$ of A_i. Given i and a prime ideal $\mathfrak{p}_i$ of A_i, let $\mathfrak{p}$ be the prime ideal $A_1 \times \cdots \times \mathfrak{p}_i \times \cdots A_n$. Then the projection of rings $A \longrightarrow A_i$ gives rise to a ring morphism

$$A_{\mathfrak{p}} \longrightarrow (A_i)_{\mathfrak{p}_i}$$
$$(a_1, \dots, a_i, \dots, a_n)/(b_1, \dots, b_i, \dots, b_n) \mapsto a_i/b_i$$

It is easy to check that this is an isomorphism. An *orthogonal set of idempotents* in a ring A is a set $e_1, \dots, e_r$ with $1 = e_1 + \cdots + e_r$, $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$. If $A = \prod_{i=1}^n A_i$ is a product of rings, then the elements $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ are clearly such a set.

Conversely if $e_1, \dots, e_r$ is an orthogonal set of idempotents in a ring A, then the ideal $e_i A$ becomes a ring with identity e_i. The map

$$\begin{aligned} A &\longrightarrow e_1 A \times \cdots \times e_r A \\ a &\mapsto (e_1 a, \dots, e_r a) \end{aligned}$$

is a ring isomorphism.

Proposition 14. *Any nonzero artinian ring A is a finite direct product of local artinian rings.*

Proof. See [Eis95] Corollary 2.16. This shows that there is a finite list of maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ (allowing repeats) and a ring isomorphism $A \longrightarrow \prod_{i=1}^n A_{\mathfrak{m}_i}$ defined by $a \mapsto (a/1, \dots, a/1)$. ☐

Proposition 15. *Let $\varphi : A \longrightarrow B$ be a surjective morphism of rings, M an A-module and $\mathfrak{p} \in SpecB$. There is a canonical morphism of $B_{\mathfrak{p}}$-modules natural in M*

$$\begin{gathered} \kappa : Hom_A(B, M)_{\mathfrak{p}} \longrightarrow Hom_{A_{\varphi^{-1}\mathfrak{p}}}(B_{\mathfrak{p}}, M_{\varphi^{-1}\mathfrak{p}}) \\ \kappa(u/s)(b/t) = u(b)/\varphi^{-1}(st) \end{gathered}$$

If A is noetherian, this is an isomorphism.

Proof. Let a morphism of A-modules $u : B \longrightarrow M$, $s, t \in B \setminus \mathfrak{p}$ and $b \in B$ be given. Choose $k \in A$ with $\varphi(k) = st$. We claim the fraction $u(b)/k \in M_{\varphi^{-1}\mathfrak{p}}$ doesn't depend on the choice of k. If we have $\varphi(l) = st$ also, then

$$ku(b) = u(kb) = u(\varphi(k)b) = u(\varphi(l)b) = u(lb) = lu(b)$$

so $u(b)/l = u(b)/k$, as claimed. Throughout the proof, given $x \in B$ we write $\varphi^{-1}(x)$ for an arbitrary element in the inverse image of x. One checks the result does not depend on this choice. We can now define a morphism of $B_{\mathfrak{p}}$-modules $\kappa(u/s)(b/t) = u(b)/\varphi^{-1}(st)$ which one checks is well-defined and natural in M.

Now assume that A is noetherian. In showing that κ is an isomorphism, we may as well assume φ is the canonical projection $A \longrightarrow A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. In that case the prime ideal $\mathfrak{p}$ is $\mathfrak{q}/\mathfrak{a}$ for some prime $\mathfrak{q}$ of A containing $\mathfrak{a}$, and if we set $S = A \setminus \mathfrak{q}$ and $T = \varphi(S)$ we have by Lemma 7 an isomorphism

$$\begin{aligned} Hom_A(B, M)_{\mathfrak{p}} &= T^{-1} Hom_A(A/\mathfrak{a}, M) \\ &\cong S^{-1} Hom_A(A/\mathfrak{a}, M) \\ &\cong Hom_{S^{-1}A}(S^{-1}(A/\mathfrak{a}), S^{-1}M) \\ &\cong Hom_{S^{-1}A}(T^{-1}(A/\mathfrak{a}), S^{-1}M) \\ &= Hom_{A_{\varphi^{-1}\mathfrak{p}}}(B_{\mathfrak{p}}, M_{\varphi^{-1}\mathfrak{p}}) \end{aligned}$$

We have use the fact that A is noetherian to see that $A/\mathfrak{a}$ is finitely presented, so we have the second isomorphism in the above sequence. One checks easily that this isomorphism agrees with κ, completing the proof. ☐

Remark 1. The right adjoint $Hom_A(B, -)$ to the restriction of scalars functor exists for any morphism of rings $\varphi : A \longrightarrow B$, but as we have just seen, this functor is not *local* unless the ring morphism is surjective. This explains why the right adjoint $f^!$ to the direct image functor in algebraic geometry essentially only exists for closed immersions.

2 Flatness

Definition 3. Let A be a ring and M an A-module. We say M is flat if the functor $- \otimes_A M : A\mathbf{Mod} \longrightarrow A\mathbf{Mod}$ is exact (equivalently $M \otimes_A -$ is exact). Equivalently M is flat if whenever we have an injective morphism of modules $N \longrightarrow N'$ the morphism $N \otimes_A M \longrightarrow N' \otimes_A M$ is injective. This property is stable under isomorphism.

We say M is *faithfully flat* if a morphism $N \longrightarrow N'$ is injective if and only if $N \otimes_A M \longrightarrow N' \otimes_A M$ is injective. This property is also stable under isomorphism. An A-algebra $A \longrightarrow B$ is *flat* if B is a flat A-module and we say $A \longrightarrow B$ is a *flat morphism.*

Example 1. Nonzero free modules are faithfully flat.

Lemma 16. *We have the following fundamental properties of flatness:*

- Transitivity: *If $\phi : A \longrightarrow B$ is a flat morphism of rings and N a flat B-module, then N is also flat over A.*
- Change of Base: *If $\phi : A \longrightarrow B$ is a morphism of rings and M is a flat A-module, then $M \otimes_A B$ is a flat B-module.*
- Localisation: *If A is a ring and S a multiplicatively closed subset, then $S^{-1}A$ is flat over A.*

Proof. The second and third claims are done in our Atiyah & Macdonald notes. To prove the first claim, let $M \longrightarrow M'$ be a monomorphism of A-modules and consider the following commutative diagram of abelian groups

$$\begin{tikzcd}
M \otimes_A N \arrow[r] \arrow[d, Rightarrow] & M' \otimes_A N \arrow[d, Rightarrow] \\
M \otimes_A (B \otimes_B N) \arrow[r] \arrow[d, Rightarrow] & M' \otimes_A (B \otimes_B N) \arrow[d, Rightarrow] \\
(M \otimes_A B) \otimes_B N \arrow[r] & (M' \otimes_A B) \otimes_B N
\end{tikzcd}$$

Since B is a flat A-module and N is a flat B-module the bottom row is injective, hence so is the top row. $\square$

Lemma 17. *Let $\phi : A \longrightarrow B$ be a morphism of rings and N a B-module which is flat over A. If S is a multiplicatively closed subset of B, then $S^{-1}N$ is flat over A. In particular any localisation of a flat A-module is flat.*

Proof. If $M \longrightarrow M'$ is a monomorphism of A-modules then we have a commutative diagram

$$\begin{tikzcd}
M \otimes_A S^{-1}N \arrow[r] \arrow[d, Rightarrow] & M' \otimes_A S^{-1}N \arrow[d, Rightarrow] \\
(M \otimes_A N) \otimes_B S^{-1}B \arrow[r] & (M' \otimes_A N) \otimes_B S^{-1}B
\end{tikzcd}$$

The bottom row is clearly injective, and hence so is the top row, which shows that $S^{-1}N$ is flat over A. $\square$

Lemma 18. *Let A be a ring and M, N flat A-modules. Then $M \otimes_A N$ is also flat over A.*

Lemma 19. *Let $\phi : A \longrightarrow B$ be a flat morphism of rings and S a multiplicatively closed subset of A. Then $T = \phi(S)$ is a multiplicatively closed subset of B and $T^{-1}B$ is flat over $S^{-1}A$.*

Proof. This follows from Lemma 7 and stability of flatness under base change. $\square$

Lemma 20. *Let $A \longrightarrow B$ be a morphism of rings. Then the functor $-\otimes_A B : A\mathbf{Mod} \longrightarrow B\mathbf{Mod}$ preserves projectives.*

Proof. The functor $-\otimes_A B$ is left adjoint to the restriction of scalars functor. This latter functor is clearly exact, so since any functor with an exact right adjoint must preserve projectives, $P \otimes_A B$ is a projective B-module for any projective A-module P. $\square$

Lemma 21. *Let $A \longrightarrow B$ be a flat morphism of rings. If I is an injective B-module then it is also an injective A-module.*

Proof. The restriction of scalars functor has an exact left adjoint $-\otimes_A B : A\mathbf{Mod} \longrightarrow B\mathbf{Mod}$, and therefore preserves injectives. $\square$

Lemma 22. *Let $\phi : A \longrightarrow B$ be a flat morphism of rings, and let M, N be A-modules. Then there is an isomorphism of B-modules $Tor_i^A(M,N) \otimes_A B \cong Tor_i^B(M \otimes_A B, N \otimes_A B)$. If A is noetherian and M finitely generated over A, there is an isomorphism of B-modules $Ext^i_A(M,N) \otimes_A B \cong Ext^i_B(M \otimes_A B, N \otimes_A B)$.*

Proof. Let $X : \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$ be a projective resolution of the A-module M. Since B is flat, the sequence

$$X \otimes_A B : \cdots \longrightarrow X_1 \otimes_A B \longrightarrow X_0 \otimes_A B \longrightarrow M \otimes_A B \longrightarrow 0$$

is a projective resolution of $M \otimes_A B$. The chain complex of B-modules $(X \otimes_A B) \otimes_B (B \otimes_A N)$ is isomorphic to $(X \otimes_A N) \otimes_A B$. The exact functor $-\otimes_A B$ commutes with taking homology so there is an isomorphism of B-modules $Tor_i^A(M,N) \otimes_A B \cong Tor_i^B(M \otimes_A B, N \otimes_A B)$, as required.

If A is noetherian and M finitely generated we can assume that the X_i are finite free A-modules. Then $Ext^i_A(M,N)$ is the i-cohomology module of the sequence

$$0 \longrightarrow Hom(X_0, N) \longrightarrow Hom(X_1, N) \longrightarrow Hom(X_2, N) \longrightarrow \cdots$$

Since tensoring with B is exact, $Ext^i_A(M,N) \otimes_A B$ is isomorphic as a B-module to the i-th cohomology of the following sequence

$$0 \longrightarrow Hom(X_0, N) \otimes_A B \longrightarrow Hom(X_1, N) \otimes_A B \longrightarrow \cdots$$

After a bit of work, we see that this cochain complex is isomorphic to $Hom_B(X \otimes_A B, N \otimes_A B)$, and the i-th cohomology of this complex is $Ext^i_B(M \otimes_A B, N \otimes_A B)$, as required. $\square$

In particular for a ring A and prime ideal $\mathfrak{p} \subseteq A$ we have isomorphisms of $A_\mathfrak{p}$-modules for $i \geq 0$

$$Tor_i^{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \cong Tor_i^A(M,N)_\mathfrak{p}$$
$$Ext^i_{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \cong Ext^i_A(M,N)_\mathfrak{p}$$

the latter being valid for A noetherian and M finitely generated.

Lemma 23. *Let A be a ring and M an A-module. Then the following are equivalent*

(i) M is a flat A-module;

(ii) $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for each prime ideal $\mathfrak{p}$;

(iii) $M_\mathfrak{m}$ is a flat $A_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m}$.

Proof. See [AM69] or any book on commutative algebra. $\square$

Proposition 24. *Let $(A, \mathfrak{m}, k)$ be a local ring and M an A-module. Suppose that either $\mathfrak{m}$ is nilpotent or M is finitely generated over A. Then M is free $\Leftrightarrow$ M is projective $\Leftrightarrow$ M is flat.*

Proof. It suffices to show that if M is flat then it is free. We prove that any minimal basis of M is a basis of M. If $M/\mathfrak{m}M = 0$ then $M = 0$ and M is trivially free. Otherwise it suffices to show that if $x_1, \ldots, x_n \in M$ are elements whose images in $M/\mathfrak{m}M = M \otimes_A k$ are linearly independent over k, then they are linearly independent over A. We use induction on n. For $n = 1$ let $ax = 0$. Then there exist $y_1, \ldots, y_r \in M$ and $b_1, \ldots, b_r \in A$ such that $ab_i = 0$ for all i and $x = \sum b_i y_i$. Since $x + \mathfrak{m}M \neq 0$ not all b_i are in $\mathfrak{m}$. Suppose $b_1 \notin \mathfrak{m}$. Then b_1 is a unit in A and $ab_1 = 0$, hence $a = 0$.

Suppose $n > 1$ and $\sum_{i=1}^n a_i x_i = 0$. Then there exist $y_1, \ldots, y_r \in M$ and $b_{ij} \in A (1 \leq j \leq r)$ such that $x_i = \sum_j b_{ij} y_j$ and $\sum_i a_i b_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we have $b_{nj} \notin \mathfrak{m}$ for at least one j. Since $a_1 b_{1j} + \cdots + a_n b_{nj} = 0$ and b_{nj} is a unit, we have

$$a_n = \sum_{i=1}^{n-1} c_i a_i \qquad c_i = -b_{ij}/b_{nj}$$

Then

$$0 = \sum_{i=1}^{n} a_i x_i = a_1(x_1 + c_1 x_n) + \cdots + a_{n-1}(x_{n-1} + c_{n-1} x_n)$$

Since the residues of $x_1 + c_1 x_n, \ldots, x_{n-1} + c_{n-1} x_n$ are linearly independent over k, by the inductive hypothesis we get $a_1 = \cdots = a_{n-1} = 0$ and $a_n = \sum c_i a_i = 0$. $\square$

Corollary 25. *Let A be a ring and M a finitely generated A-module. Then the following are equivalent*

(i) M is a flat A-module;

(ii) $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module for each prime ideal $\mathfrak{p}$;

(iii) $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m}$.

Proof. This is immediate from the previous two results. $\square$

Proposition 26. *Let A be a ring and M a finitely presented A-module. Then M is flat if and only if it is projective.*

Proof. See Stenstrom Chapter 1, Corollary 11.5. $\square$

Corollary 27. *Let A be a noetherian ring, M a finitely generated A-module. Then the following conditions are equivalent*

(i) M is projective;

(ii) M is flat;

(ii) $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module for each prime ideal $\mathfrak{p}$;

(iii) $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m}$.

Proof. Since A is noetherian, M is finitely presented, so $(i) \Leftrightarrow (ii)$ is an immediate consequence of Proposition 26. The rest of the proof follows from Corollary 25. $\square$

Lemma 28. *Let $A \longrightarrow B$ be a flat morphism of rings, and let I, J be ideals of A. Then $(I \cap J)B = IB \cap JB$ and $(I : J)B = (IB : JB)$ if J is finitely generated.*

Proof. Consider the exact sequence of A-modules

$$I \cap J \longrightarrow A \longrightarrow A/I \oplus A/J$$

Tensoring with B we get an exact sequence

$$(I \cap J) \otimes_A B = (I \cap J)B \longrightarrow B \longrightarrow B/IB \oplus B/JB$$

This means $(I \cap J)B = IB \cap JB$. For the second claim, suppose firstly that J is a principal ideal aA and use the exact sequence

$$(I : aA) \xrightarrow{i} A \xrightarrow{f} A/I$$

where i is the injection and $f(x) = ax + I$. Tensoring with B we get the formula $(I : a)B = (IB : a)$. In the general case, if $J = (a_1, \dots, a_n)$ we have $(I : J) = \bigcap_i (I : a_i)$ so that

$$(I : J)B = \bigcap (I : a_i)B = \bigcap (IB : a_i) = (IB : JB)$$

$\square$

Example 2. Let $A = k[x, y]$ be a polynomial ring over a field k and put $B = A/(x) \cong k[y]$. Then B is not flat over A since $y \in A$ is regular but is not regular on B. Let $I = (x + y)$ and $J = (y)$. Then $I \cap J = (xy + y^2)$ and $IB = JB = yB$, $(I \cap J)B = y^2 B \neq IB \cap JB$.

Example 3. Let k be a field, put $A = k[x, y]$ and let K be the quotient field of A. Let B be the subring $k[x, y/x]$ of K (i.e. the k-subalgebra generated by x and $z = y/x$). Then $A \subset B \subset K$. Let $I = xA$, $J = yA$. Then $I \cap J = xyA$ and $(I \cap J)B = x^2 zB$, $IB \cap JB = xzB$ so B is not flat over A. The map $SpecB \longrightarrow SpecA$ corresponding to $A \longrightarrow B$ is the projection to the (x, y)-plane of the surface $F : xz = y$ in (x, y, z)-space. Note F contains the whole z-axis so it does not look “flat” over the (x, y)-plane.

Proposition 29. *Let $\varphi : A \longrightarrow B$ be a morphism of rings, M an A-module and N a B-module. Then for every $\mathfrak{p} \in SpecB$ there is a canonical isomorphism of $B_{\mathfrak{p}}$-modules natural in both variables*

$$\kappa : M_{\mathfrak{p} \cap A} \otimes_{A_{\mathfrak{p} \cap A}} N_{\mathfrak{p}} \longrightarrow (M \otimes_A N)_{\mathfrak{p}}$$
$$\kappa(m/s \otimes n/t) = (m \otimes n)/\varphi(s)t$$

Proof. Fix $\mathfrak{p} \in SpecB$ and $\mathfrak{q} = \mathfrak{p} \cap A$. There is a canonical ring morphism $A_{\mathfrak{q}} \longrightarrow B_{\mathfrak{p}}$ and we make $N_{\mathfrak{p}}$ into an $A_{\mathfrak{q}}$-module using this morphism. One checks that the following map is well-defined and $A_{\mathfrak{q}}$-bilinear

$$\varepsilon : M_{\mathfrak{q}} \times N_{\mathfrak{p}} \longrightarrow (M \otimes_A N)_{\mathfrak{p}}$$
$$\varepsilon(m/s, n/t) = (m \otimes n)/\varphi(s)t$$

We show that in fact this is a tensor product of $A_{\mathfrak{q}}$-modules. Let Z be an abelian group and $\psi : M_{\mathfrak{q}} \times N_{\mathfrak{p}} \longrightarrow Z$ an $A_{\mathfrak{q}}$-bilinear map.

$$\begin{tikzcd} M_{\mathfrak{q}} \times N_{\mathfrak{p}} \arrow[r, "\psi"] \arrow[d, "\varepsilon"'] & Z \\ (M \otimes_A N)_{\mathfrak{p}} \arrow[ur, dotted, "\phi"'] & \end{tikzcd} \qquad (1)$$

We have to define a morphism of abelian groups ϕ unique making this diagram commute. For $s \notin \mathfrak{p}$ we define an A-bilinear morphism $\phi'_s : M \times N \longrightarrow Z$ by $\phi'_s(m, n) = \psi(m/1, n/s)$. This induces a morphism of abelian groups

$$\phi''_s : M \otimes_A N \longrightarrow Z$$
$$\phi''_s(m \otimes b) = \psi(m/1, b/s)$$

We make some observations about these morphisms

- Suppose $w/s = w'/s'$ in $(M \otimes_A N)_{\mathfrak{p}}$, with say $w = \sum_i m_i \otimes n_i, w' = \sum_i m'_i \otimes n'_i$ and $t \notin \mathfrak{p}$ such that $ts'w = tsw'$. That is, $\sum_i m_i \otimes ts'n_i = \sum_i m'_i \otimes tsn'_i$. Applying $\phi''_{tss'}$ to both sides of this equality gives $\phi''_s(w) = \phi''_{s'}(w')$.

- For $w/s, w'/s' \in (M \otimes_A N)_{\mathfrak{p}}$ we have $\phi''_s(w) + \phi''_{s'}(w') = \phi''_{ss'}(s'w + sw')$.

It follows that $\phi(w/s) = \phi''_s(w)$ gives a well-defined morphism of abelian groups $\phi : (M \otimes_A N)_{\mathfrak{p}} \longrightarrow Z$ which is clearly unique making (1) commute. By uniqueness of the tensor product there is an induced isomorphism of abelian groups $\kappa : M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} N_{\mathfrak{p}} \longrightarrow (M \otimes_A N)_{\mathfrak{p}}$ with $\kappa(m/s \otimes n/t) = (m \otimes n)/\varphi(s)t$. One checks that this is a morphism of $B_{\mathfrak{p}}$-modules. The inverse is defined by $(m \otimes n)/t \mapsto m/1 \otimes n/t$. Naturality in both variables is easily checked. $\square$

Corollary 30. *Let $\varphi : A \longrightarrow B$ be a morphism of rings, M an A-module and $\mathfrak{p} \in SpecB$. Then there is a canonical isomorphism of $B_{\mathfrak{p}}$-modules $M_{\mathfrak{p}\cap A} \otimes_{A_{\mathfrak{p}\cap A}} B_{\mathfrak{p}} \longrightarrow (M \otimes_A B)_{\mathfrak{p}}$ natural in M.*

We will not actually use the next result in these notes, so the reader not familiar with homological δ-functors can safely skip it. Alternatively one can provide a proof by following the one given in Matsumura (the proof we give is more elegant, provided you know about δ-functors).

Proposition 31. *Let $\varphi : A \longrightarrow B$ be a morphism of rings, M an A-module and N a B-module. Then for every $\mathfrak{p} \in SpecB$ and $i \geq 0$ there is a canonical isomorphism of $B_{\mathfrak{p}}$-modules natural in M*

$$\kappa_i : \underline{Tor}_i^A(N, M)_{\mathfrak{p}} \longrightarrow \underline{Tor}_i^{A_{\mathfrak{p}\cap A}}(N_{\mathfrak{p}}, M_{\mathfrak{p}\cap A})$$

Proof. Fix $\mathfrak{p} \in SpecB$ and a B-module N and set $\mathfrak{q} = \mathfrak{p} \cap A$. Then N is a B-A-bimodule and $N_{\mathfrak{p}}$ is a $B_{\mathfrak{p}}$-$A_{\mathfrak{q}}$-bimodule so by (TOR,Section 5.1) the abelian group $\underline{Tor}_i^A(N, M)$ acquires a canonical B-module structure, and $\underline{Tor}_i^{A_{\mathfrak{q}}}(N_{\mathfrak{p}}, M_{\mathfrak{q}})$ acquires a canonical $B_{\mathfrak{p}}$-module structure for any A-module M and $i \geq 0$. Using (TOR,Lemma 14) and (DF,Definition 23) we have two homological δ-functors between $A\mathbf{Mod}$ and $B_{\mathfrak{p}}\mathbf{Mod}$

$$\{\underline{Tor}_i^A(N, -)_{\mathfrak{p}}\}_{i\geq 0}, \{\underline{Tor}_i^{A_{\mathfrak{q}}}(N_{\mathfrak{p}}, (-)_{\mathfrak{q}})\}_{i\geq 0}$$

For $i > 0$ these functors all vanish on free A-modules, so by (DF,Theorem 74) both δ-functors are universal. For $i = 0$ we have the canonical natural equivalence of Proposition 29

$$\kappa_0 : \underline{Tor}_0^A(N, -)_{\mathfrak{p}} \cong (N \otimes_A -)_{\mathfrak{p}} \cong N_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} (-)_{\mathfrak{q}} \cong \underline{Tor}_0^{A_{\mathfrak{q}}}(N_{\mathfrak{p}}, (-)_{\mathfrak{q}})$$

By universality this lifts to a canonical isomorphism of homological δ-functors κ. In particular for each $i \geq 0$ we have a canonical natural equivalence $\kappa_i : \underline{Tor}_i^A(N, -)_{\mathfrak{p}} \longrightarrow \underline{Tor}_i^{A_{\mathfrak{q}}}(N_{\mathfrak{p}}, (-)_{\mathfrak{q}})$, as required. $\square$

We know from Lemma 23 that flatness is a local property. We are now ready to show that *relative* flatness (i.e. flatness with respect to a morphism of rings) is also local. This is particularly important in algebraic geometry. The reader who skipped Proposition 31 will also have to skip the implication $(iii) \Rightarrow (i)$ in the next result, but this will not affect their ability to read the rest of these notes.

Corollary 32. *Let $A \longrightarrow B$ be a morphism of rings and N a B-module. Then the following conditions are equivalent*

(i) N is flat over A.

(ii) $N_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}\cap A}$ for all prime ideals $\mathfrak{p}$ of B.

(iii) $N_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}\cap A}$ for all maximal ideals $\mathfrak{m}$ of B.

Proof. $(i) \Rightarrow (ii)$ If N is flat over A then $N_{\mathfrak{p}\cap A}$ is flat over $A_{\mathfrak{p}\cap A}$ for any prime $\mathfrak{p}$ of B. By an argument similar to the one given in Lemma 19 we see that $N_{\mathfrak{p}}$ is isomorphic as a $B_{\mathfrak{p}\cap A}$-module to a localisation of $N_{\mathfrak{p}\cap A}$. Applying Lemma 17 to the ring morphism $A_{\mathfrak{p}\cap A} \longrightarrow B_{\mathfrak{p}\cap A}$ we see that $N_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}\cap A}$, as required. $(ii) \Rightarrow (iii)$ is trivial. $(iii) \Rightarrow (i)$ For every A-module M and maximal ideal $\mathfrak{m}$ of B we have by Proposition 31

$$\underline{Tor}_1^A(N, M)_{\mathfrak{m}} \cong \underline{Tor}_1^{A_{\mathfrak{m}\cap A}}(N_{\mathfrak{m}}, M_{\mathfrak{m}\cap A}) = 0$$

since by assumption $N_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}\cap A}$. Therefore $\underline{Tor}_1^A(N, M) = 0$ for every A-module M, which implies that N is flat over A and completes the proof. $\square$

Lemma 33. *Let $A \longrightarrow B$ be a morphism of rings. Then the following conditions are equivalent*

(i) B is flat over A;

(ii) $B_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}\cap A}$ for all prime ideals $\mathfrak{p}$ of B;

(iii) $B_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}\cap A}$ for all maximal ideals $\mathfrak{m}$ of B.

2.1 Faithful Flatness

Theorem 34. *Let A be a ring and M an A-module. The following conditions are equivalent:*

(i) M is faithfully flat over A;

(ii) M is flat over A, and for any nonzero A-module N we have $N \otimes_A M \neq 0$;

(iii) M is flat over A, and for any maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}M \neq M$.

Proof. $(i) \Rightarrow (ii)$ Let N be an A-module and $\varphi : N \longrightarrow 0$ the zero map. Then if M is faithfully flat and $N \otimes_A M = 0$ we have $\varphi \otimes_A M = 0$ which means that φ is injective and therefore $N = 0$. $(ii) \Rightarrow (iii)$ Since $A/\mathfrak{m} \neq 0$ we have $(A/\mathfrak{m}) \otimes_A M = M/\mathfrak{m}M \neq 0$ by hypothesis. $(iii) \Rightarrow (ii)$ Let N be a nonzero A-module and pick $0 \neq x \in N$. Let $\varphi : A \longrightarrow N$ be $1 \mapsto x$. If $I = Ker\varphi$ then there is an injective morphism of modules $A/I \longrightarrow N$. Let $\mathfrak{m}$ be a maximal ideal containing I. Then $M \supset \mathfrak{m}M \supseteq IM$ so $(A/I) \otimes_A M = M/IM \neq 0$. Since M is flat the morphism $(A/I) \otimes_A M \longrightarrow N \otimes_A M$ is injective so $N \otimes_A M \neq 0$. $(ii) \Rightarrow (i)$ Let $\psi : N \longrightarrow N'$ be a morphism of modules with kernel $K \longrightarrow N$. If $N \otimes_A M \longrightarrow N' \otimes_A M$ is injective then $K \otimes_A M = 0$, which is only possible if $K = 0$. ☐

Corollary 35. *Let A and B be local rings, and $\psi : A \longrightarrow B$ a local morphism of rings. Let M be a nonzero finitely generated B-module. Then*

$$M \text{ is flat over } A \Longleftrightarrow M \text{ is faithfully flat over } A$$

In particular, B is flat over A if and only if it is faithfully flat over A.

Proof. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of A and B, respectively. Then $\mathfrak{m}M \subseteq \mathfrak{n}M$ since ψ is local, and $\mathfrak{n}M \neq M$ by Nakayama, so the assertion follows from the Theorem. ☐

Lemma 36. *We have the following fundamental properties of flatness:*

- Transitivity: *If $\phi : A \longrightarrow B$ is a faithfully flat morphism of rings and N a faithfully flat B-module, then N is a faithfully flat A-module.*
- Change of Base: *If $\phi : A \longrightarrow B$ is a morphism of rings and M is a faithfully flat A-module, then $M \otimes_A B$ is a faithfully flat B-module.*
- Descent: *If $\phi : A \longrightarrow B$ is a ring morphism and M is a faithfully flat B-module which is also faithfully flat over A, then B is faithfully flat over A.*

Proof. The diagram in the proof of transitivity for flatness makes it clear that faithful flatness is also transitive. Similarly the flatness under base change proof in our Atiyah & Macdonald notes shows that faithful flatness is also stable under base change. The descent property is also easily checked. ☐

Proposition 37. *Let $\psi : A \longrightarrow B$ be a faithfully flat morphism of rings. Then*

(i) For any A-module N, the map $N \longrightarrow N \otimes_A B$ defined by $x \mapsto x \otimes 1$ is injective. In particular ψ is injective and A can be viewed as a subring of B.

(ii) For any ideal I of A we have $IB \cap A = I$.

(iii) The map $\Psi : Spec(B) \longrightarrow Spec(A)$ is surjective.

(iv) If B is noetherian then so is A.

Proof. (i) Let $0 \neq x \in N$. Then $0 \neq Ax \subseteq N$ and since B is flat we see that $Ax \otimes_A B$ is isomorphic to the submodule $(x \otimes 1)B$ of $N \otimes_A B$. It follows from Theorem 34 that $x \otimes 1 \neq 0$.

(ii) By change of base, $B \otimes_A (A/I) = B/IB$ is faithfully flat over A/I. Now the assertion follows from (i). For (iii) let $\mathfrak{p} \in Spec(A)$. The ring $B_\mathfrak{p} = B \otimes_A A_\mathfrak{p}$ is faithfully flat over $A_\mathfrak{p}$ so by (ii) $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$. Take a maximal ideal $\mathfrak{m}$ of $B_\mathfrak{p}$ containing $\mathfrak{p}B_\mathfrak{p}$. Then $\mathfrak{m} \cap A_\mathfrak{p} \supseteq \mathfrak{p}A_\mathfrak{p}$, therefore $\mathfrak{m} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ since $\mathfrak{p}A_\mathfrak{p}$ is maximal. Putting $\mathfrak{q} = \mathfrak{m} \cap B$, we get

$$\mathfrak{q} \cap A = (\mathfrak{m} \cap B) \cap A = \mathfrak{m} \cap A = (\mathfrak{m} \cap A_\mathfrak{p}) \cap A = \mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$$

as required. (iv) Follows immediately from (ii). $\square$

Theorem 38. *Let $\psi : A \longrightarrow B$ be a morphism of rings. The following conditions are equivalent.*

(1) ψ is faithfully flat;

(2) ψ is flat, and $\Psi : Spec(B) \longrightarrow Spec(A)$ is surjective;

(3) ψ is flat, and for any maximal ideal $\mathfrak{m}$ of A there is a maximal ideal $\mathfrak{n}$ of B lying over $\mathfrak{m}$.

Proof. (1) $\Rightarrow$ (2) was proved above. (2) $\Rightarrow$ (3) By assumption there exists $\mathfrak{q} \in Spec(B)$ with $\mathfrak{q} \cap A = \mathfrak{m}$. If $\mathfrak{n}$ is any maximal ideal of B containing $\mathfrak{q}$ then $\mathfrak{n} \cap A = \mathfrak{m}$ as $\mathfrak{m}$ is maximal. (3) $\Rightarrow$ (1) The existence of $\mathfrak{n}$ implies $\mathfrak{m}B \neq B$, so B is faithfully flat over A by Theorem 34. $\square$

Definition 4. In algebraic geometry we say a morphism of schemes $f : X \longrightarrow Y$ is *flat* if the local morphisms $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ are flat for all $x \in X$. We say the morphism is *faithfully flat* if it is flat and surjective.

Lemma 39. *Let A be a ring and B a faithfully flat A-algebra. Let M be an A-module. Then*

(i) M is flat (resp. faithfully flat) over $A \Leftrightarrow M \otimes_A B$ is so over B,

(ii) If A is local and M finitely generated over A we have M is A-free $\Leftrightarrow M \otimes_A B$ is B-free.

Proof. (i) Let $N \longrightarrow N'$ be a morphism of A-modules. Both claims follow from commutativity of the following diagram

$$\begin{tikzcd}
(N \otimes_A M) \otimes_A B \arrow[r] \arrow[d, Leftrightarrow] & (N' \otimes_A M) \otimes_A B \arrow[d, Leftrightarrow] \\
N \otimes_A (M \otimes_A B) \arrow[r] \arrow[d, Leftrightarrow] & N' \otimes_A (M \otimes_A B) \arrow[d, Leftrightarrow] \\
N \otimes_A (B \otimes_B (M \otimes_A B)) \arrow[r] \arrow[d, Leftrightarrow] & N' \otimes_A (B \otimes_B (M \otimes_A B)) \arrow[d, Leftrightarrow] \\
(N \otimes_A B) \otimes_B (M \otimes_A B) \arrow[r] & (N' \otimes_A B) \otimes_B (M \otimes_A B)
\end{tikzcd}$$

(ii) The functor $- \otimes_A B$ preserves coproducts, so the implication $(\Rightarrow)$ is trivial. $(\Leftarrow)$ follows from (i) because, under the hypothesis, freeness of M is equivalent to flatness as we saw in Proposition 24. $\square$

2.2 Going-up and Going-down

Definition 5. Let $\phi : A \longrightarrow B$ be a morphism of rings. We say that the *going-up theorem* holds for ϕ if the following condition is satisfied:

(GU) For any $\mathfrak{p}, \mathfrak{p}' \in Spec(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$ and for any prime $\mathfrak{q} \in Spec(B)$ lying over $\mathfrak{p}$, there exists $\mathfrak{q}' \in Spec(B)$ lying over $\mathfrak{p}'$ such that $\mathfrak{q} \subset \mathfrak{q}'$.

Similarly we say that the *going-down theorem* holds for ϕ if the following condition is satisfied:

(GD) For any $\mathfrak{p}, \mathfrak{p}' \in Spec(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$ and for any prime $\mathfrak{q}' \in Spec(B)$ lying over $\mathfrak{p}'$, there exists $\mathfrak{q} \in Spec(B)$ lying over $\mathfrak{p}$ such that $\mathfrak{q} \subset \mathfrak{q}'$.

Lemma 40. *The condition (GD) is equivalent to the following condition (GD'): For any $\mathfrak{p} \in Spec(A)$ and any minimal prime overideal $\mathfrak{q}$ of $\mathfrak{p}B$ we have $\mathfrak{q} \cap A = \mathfrak{p}$.*

Proof. (GD) $\Rightarrow$ (GD') Clearly $\mathfrak{q} \cap A \supseteq \mathfrak{p}$. If this inclusion is proper then by (GD) there exists a prime $\mathfrak{q}_1$ of B with $\mathfrak{q}_1 \subset \mathfrak{q}$ and $\mathfrak{q}_1 \cap A = \mathfrak{p}$, contradicting minimality of $\mathfrak{q}$. (GD') $\Rightarrow$ (GD) Suppose primes $\mathfrak{p} \subset \mathfrak{p}'$ of A are given and $\mathfrak{q}' \cap A = \mathfrak{p}'$. We can shrink $\mathfrak{q}'$ to a prime $\mathfrak{q}$ minimal among all prime ideals containing $\mathfrak{p}B$, and by assumption $\mathfrak{q} \cap A = \mathfrak{p}$, which completes the proof. □

Let $\mathfrak{a}$ be any proper radical ideal in a noetherian ring B. Then $\mathfrak{a}$ is the intersection of all its minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and the closed irreducible sets $V(\mathfrak{p}_1) \subseteq V(\mathfrak{a})$ are the irreducible components of the closed set $V(\mathfrak{a})$ in the noetherian space $Spec(B)$.

Let $\phi : A \longrightarrow B$ a morphism of rings, put $X = Spec(A), Y = Spec(B)$ and let $\Psi : Y \longrightarrow X$ the corresponding morphism of affine schemes, and suppose B is noetherian. Then (GD') can be formulated geometrically as follows: let $\mathfrak{p} \in X$, put $X' = V(\mathfrak{p}) \subseteq X$ and let Y' be an arbitrary irreducible component of $\Psi^{-1}(X')$ (which we assume is nonempty). Then Ψ maps Y' generically onto X' in the sense that the generic point of Y' is mapped to the generic point $\mathfrak{p}$ of X'.

Theorem 41. *Let $\phi : A \longrightarrow B$ be a flat morphism of rings. Then the going-down theorem holds for ϕ.*

Proof. Let $\mathfrak{p}' \subset \mathfrak{p}$ be prime ideals of A and let $\mathfrak{q}$ be a prime ideal of B lying over $\mathfrak{p}$. Then $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ by Lemma 33, hence faithfully flat since $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ is local. Therefore $Spec(B_{\mathfrak{q}}) \longrightarrow Spec(A_{\mathfrak{p}})$ is surjective. Let $\mathfrak{q}''$ be a prime ideal of $B_{\mathfrak{q}}$ lying over $\mathfrak{p}'A_{\mathfrak{p}}$. Then $\mathfrak{q}' = \mathfrak{q}'' \cap B$ is a prime ideal of B lying over $\mathfrak{p}'$ and contained in $\mathfrak{q}$. □

Theorem 42. *Let B be a ring and A a subring over which B is integral. Then*

(i) The canonical map $Spec(B) \longrightarrow Spec(A)$ is surjective.

(ii) If two prime ideals $\mathfrak{q} \subseteq \mathfrak{q}'$ lie over the same prime ideal of A then they are equal.

(iii) The going-up theorem holds for $A \subseteq B$.

(iv) If A is a local ring and $\mathfrak{m}$ its maximal ideal, then the prime ideals of B lying over $\mathfrak{m}$ are precisely the maximal ideals of B.

(v) If A and B are integral domains and A is integrally closed, then the going-down theorem holds for $A \subseteq B$.

Proof. See [AM69] or [Mat80] Theorem 5. □

2.3 Constructible Sets

Definition 6. A topological space X is *noetherian* if the descending chain condition holds for the closed sets in X. The spectrum $Spec(A)$ of a noetherian ring A is noetherian. If a space is covered by a finite number of noetherian subspaces then it is noetherian. Any subspace of a noetherian space is noetherian. A noetherian space is quasi-compact. In a noetherian space X any nonempty closed set Z is uniquely decomposed into a finite number of irreducible closed sets $Z = Z_1 \cup \cdots \cup Z_n$ such that $Z_i \nsubseteq Z_j$ for $i \neq j$. The Z_i are called the *irreducible components* of Z.

Lemma 43 (Noetherian Induction). *Let X be a noetherian topological space, and $\mathscr{P}$ a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y (in particular $\mathscr{P}$ holds for the empty set). Then $\mathscr{P}$ holds for X.*

Proof. Suppose that $\mathscr{P}$ does not hold for X, and let $\mathscr{Z}$ be the set of all proper closed subsets of X which do not satisfy $\mathscr{P}$. Then since X is noetherian $\mathscr{Z}$ has a minimal element Y. Since Y is minimal, every proper closed subset of Y must satisfy $\mathscr{P}$, and therefore Y satisfies $\mathscr{P}$, contradicting the fact that $Y \in \mathscr{Z}$. □

Lemma 44. *Let X be a noetherian topological space, and $\mathscr{P}$ a property of general subsets of X. Assume that for any subset Y of X, if $\mathscr{P}$ holds for every proper subset Y' of Y with $\overline{Y'} \subset \overline{Y}$, then $\mathscr{P}$ holds for Y (in particular $\mathscr{P}$ holds for the empty set). Then $\mathscr{P}$ holds for X.*

Proof. Suppose that $\mathscr{P}$ does not hold for X, and let $\mathscr{Z}$ be the set of all closures $\overline{Q}$ of proper subsets Q of X with $\overline{Q} \subset X$ and $\mathscr{P}$ not holding for Q. Let $\overline{Q}$ be a minimal element of $\mathscr{Z}$. If Q' is any proper subset of Q with $\overline{Q'} \subset \overline{Q}$ then Q' must satisfy $\mathscr{P}$, otherwise $\overline{Q'}$ would contradict minimality of $\overline{Q}$ in $\mathscr{Z}$. But by assumption this implies that Q satisfies $\mathscr{P}$, which is a contradiction. □

Definition 7. Let X be a topological space and Z a subset of X. We say Z is *locally closed* in X if it satisfies the following equivalent properties

(i) Every point $z \in Z$ has an open neighborhood U in X such that $U \cap Z$ is closed in U.

(ii) Z is the intersection of an open set in X and a closed set in X.

(iii) Z is an open subset of its closure.

Definition 8. Let X be a noetherian space. We say a subset Z of X is a *constructible* set in X if it is a finite union of locally closed sets in X, so $Z = \bigcup_{i=1}^m (U_i \cap F_i)$ with U_i open and F_i closed. The set $\mathscr{F}$ of all constructible subsets of X is the smallest collection of subsets of X containing all the open sets which is closed with respect to the formation of finite intersections and complements. It follows that all open and closed sets are constructible, and $\mathscr{F}$ is also closed under finite unions.

We say that a subset Z is *pro-constructible* (resp. *ind-constructible*) if it is the intersection (resp. union) of an arbitrary collection of constructible sets in X.

Proposition 45. *Let X be a noetherian space and Z a subset of X. Then Z is constructible if and only if the following condition is satisfied.*

> $(*)$ *For each irreducible closed set X_0 in X, either $X_0 \cap Z$ is not dense in X_0, or $X_0 \cap Z$ contains a nonempty open set of X_0.*

Proof. Assume that Z is constructible and $Z \cap X_0$ nonempty. Then we can write $X_0 \cap Z = \bigcup_{i=1}^m U_i \cap F_i$ for U_i open in X, F_i closed and irreducible in X (by taking irreducible components) and $U_i \cap F_i$ nonempty for all i. Then $\overline{U_i \cap F_i} = F_i$ since F_i is irreducible, therefore $\overline{X_0 \cap Z} = \bigcup_i F_i$. If $X_0 \cap Z$ is dense in X_0, we have $X_0 = \bigcup_i F_i$ so that some F_i, say F_1, is equal to X_0. Then $U_1 \cap X_0 = U_1 \cap F_1$ is a nonempty open subset of X_0 contained in $X_0 \cap Z$.

Next we prove the converse. We say that a subset T of X has the property $\mathscr{P}$ if whenever a subset Z of T satisfies $(*)$ it is constructible. We need to show that X has the property $\mathscr{P}$, for which we use the form of noetherian induction given in Lemma 44. Suppose that Y is a subset of X with $\mathscr{P}$ holding for every proper subset Y' of Y with $\overline{Y'} \subset \overline{Y}$. We need to show that $\mathscr{P}$ holds for Y. Let Z be a nonempty subset of Y satisfying $(*)$, and let $\overline{Z} = F_1 \cup \ldots \cup F_r$ be the decomposition of $\overline{Z}$ into irreducible components. Since $Z = Z \cap F_1 \cup \cdots \cup Z \cap F_r$ we have

$$F_1 = F_1 \cap \overline{Z} = F_1 \cap \left(\overline{Z \cap F_1} \cup \ldots \cup \overline{Z \cap F_r} \right) = (F_1 \cap \overline{Z \cap F_1}) \cup \cdots \cup (F_1 \cap \overline{Z \cap F_r})$$

Since F_1 is irreducible and not contained in any other F_i we must have $F_1 = \overline{Z \cap F_1}$, so $F_1 \cap Z$ is dense in F_1, whence by $(*)$ there exists a proper closed subset F' of F_1 such that $F_1 \setminus F' \subseteq Z$. Then, putting $F^* = F' \cup F_2 \cup \cdots \cup F_r$ we have $Z = (F_1 \setminus F') \cup (Z \cap F^*)$. The set $F_1 \setminus F'$ is locally closed in X, so to complete the proof it suffices to show that $Z \cap F^*$ is constructible in X. Since $\overline{Z \cap F^*} \subseteq F^* \subset \overline{Z} \subseteq \overline{Y}$, by the inductive hypothesis $\mathscr{P}$ holds for $Z \cap F^*$, so it suffices to show that $Z \cap F^*$ satisfies $(*)$. If X_0 is irreducible and $\overline{Z \cap F^* \cap X_0} = X_0$, the closed set F^* must contain X_0 and so $Z \cap F^* \cap X_0 = Z \cap X_0$, which contains a nonempty open subset of X_0 since Z satisfies $(*)$, and clearly $Z \cap X_0$ is dense in X_0. $\square$

Lemma 46. *Let $\phi : A \longrightarrow B$ be a morphism of rings and $f : Spec(B) \longrightarrow Spec(A)$ the corresponding morphism of schemes. Then f dominant if and only if $Ker\phi \subseteq nil(A)$. In particular if A is reduced, the f dominant if and only if ϕ is injective.*

Proof. Let $X = Spec(A)$ and $Y = Spec(B)$. The closure $\overline{f(Y)}$ is the closed set $V(I)$ defined by the ideal $I = \bigcap_{\mathfrak{p} \in Y} \phi^{-1}\mathfrak{p} = \phi^{-1} \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$, which is $\phi^{-1}(nil(B))$. Clearly $Ker\phi \subseteq I$. Suppose that $f(Y)$ is dense in X. Then $V(I) = X$, whence $I = nil(A)$ and so $Ker\phi \subseteq nil(A)$. Conversely, suppose $Ker\phi \subseteq nil(A)$. Then it is clear that $I = \phi^{-1}(nil(B)) = nil(A)$, which means that $\overline{f(Y)} = V(I) = X$. $\square$

3 Associated Primes

This material can be found in [AM69] Chapter 11, webnotes of Robert Ash or in [Mat80] itself. There is not much relevant to add here, apart from a few small comments.

Lemma 47. *Let A be a ring and M an A-module. Let $\mathfrak{a}$ be an ideal in A that is maximal among all annihilators of nonzero elements of M. Then $\mathfrak{a}$ is prime.*

Proof. Say $\mathfrak{a} = Ann(x)$. Given $ab \in \mathfrak{a}$ we must show that $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$. Assume $a \notin \mathfrak{a}$. Then $ax \neq 0$. We note that $Ann(ax) \supseteq \mathfrak{a}$. By hypothesis it cannot properly be larger. Hence $Ann(ax) = \mathfrak{a}$. Now b annihilates ax; hence $b \in \mathfrak{a}$. $\square$

Lemma 48. *Let A be a noetherian ring and M an A-module. If $0 \neq a \in M$ then $Ann(a)$ is contained in an associated prime of M.*

Proposition 49. *Let A be a noetherian ring and M a nonzero finitely generated A-module. A maximal ideal $\mathfrak{m}$ is an associated prime of M if and only if no element of $\mathfrak{m}$ is regular on M.*

Proof. One implication is obvious. If $x \in \mathfrak{m}$ is not regular on M, say $x \in Ann(b)$ for some nonzero b, then x is contained in an associated prime of M. Thus $\mathfrak{m}$ is contained in the finite union of the associated primes of M, and since $\mathfrak{m}$ is maximal it must be one of them. $\square$

Proposition 50. *Let A be a nonzero noetherian ring, I an ideal, and M a nonzero finitely generated A-module. If there exist elements $x, y \in I$ with x regular on A and y regular on M, then there exists an element of I regular on both A and M.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the associated primes of A and $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ the associated primes of M. By assumption I is not contained in any of these primes. But if no element of I is regular on both A and M, then I is contained in the union $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \cup \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_m$, and therefore contained in one of these primes, which is a contradiction. $\square$

4 Dimension

This is covered in [AM69], so we restrict ourselves here to mentioning some of the major points. Recall that an ideal $\mathfrak{q} \subseteq R$ in a ring is *primary* if it is proper and if whenever $xy \in \mathfrak{q}$ we have either $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some $n > 0$. Then the radical of $\mathfrak{q}$ is a prime ideal $\mathfrak{p}$, and we say $\mathfrak{q}$ is a $\mathfrak{p}$-*primary* ideal. If $\mathfrak{a}$ is an ideal and $\mathfrak{b} \supseteq \mathfrak{a}$ is $\mathfrak{p}$-primary, then in the ring $R/\mathfrak{a}$ the ideal $\mathfrak{b}/\mathfrak{a}$ is $\mathfrak{p}/\mathfrak{a}$-primary. A *minimal primary decomposition* of an ideal $\mathfrak{b}$ is an expression

$$\mathfrak{b} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

where $\cap_{j \neq i} \mathfrak{q}_j \not\subseteq \mathfrak{q}_i$ for all i, and the primes $\mathfrak{p}_i = r(\mathfrak{q}_i)$ are all distinct. If $\mathfrak{a}$ is an ideal contained in $\mathfrak{b}$, then

$$\mathfrak{b}/\mathfrak{a} = \mathfrak{q}_1/\mathfrak{a} \cap \cdots \cap \mathfrak{q}_n/\mathfrak{a}$$

is a minimal primary decomposition of $\mathfrak{b}/\mathfrak{a}$ in $A/\mathfrak{a}$.

Let A be a nonzero ring. Recall that *dimension* of an A-module M is the Krull dimension of the ring $A/Ann(M)$ and is defined for all modules M (-1 if $M = 0$). The *rank* is defined for free A-modules, and is the common size of any basis (0 if $M = 0$). Throughout these notes $dim(M)$ will denote the dimension, *not* the rank.

Definition 9. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. An *ideal of definition* is an $\mathfrak{m}$-primary ideal. Recall that the dimension of A is the size of the smallest collection of elements of A which generates an $\mathfrak{m}$-primary ideal. Recall that $rank_k(\mathfrak{m}/\mathfrak{m}^2)$ is equal to the size of the smallest set of generators for $\mathfrak{m}$ as an ideal, so always $d \leq rank_k(\mathfrak{m}/\mathfrak{m}^2)$.

A *system of parameters* is a set of d elements generating an $\mathfrak{m}$-primary ideal. If $d = rank_k(\mathfrak{m}/\mathfrak{m}^2)$, or equivalently there is a system of parameters generating $\mathfrak{m}$, we say that A is a *regular local ring* and we call such a system of parameters a *regular system of parameters*.

Proposition 51. *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension $d \geq 1$ and let $x_1, \ldots, x_d$ be a system of parameters of A. Then*

$$dim(A/(x_1, \ldots, x_i)) = d - i = dim(A) - i$$

for each $1 \leq i \leq d$.

Proof. Put $\overline{A} = A/(x_1, \ldots, x_i)$. If $i = d$ then the zero ideal in $\overline{A}$ is an ideal of definition, so clearly $dim(\overline{A}) = 0$. If $1 \leq i < d$ then $dim(\overline{A}) \leq d - i$ since $\overline{x}_{i+1}, \ldots, \overline{x}_d$ generate an ideal of definition of $\overline{A}$. Let $dim(\overline{A}) = p$. If $p = 0$ then $(x_1, \ldots, x_i)$ must be an ideal of definition, contradicting $i < d$. So $p \geq 1$, and if $\overline{y}_1, \ldots, \overline{y}_p$ is a system of parameters of $\overline{A}$, then $x_1, \ldots, x_i, y_1, \ldots, y_p$ generate an ideal of definition of A, so that $p + i \geq d$. That is, $p \geq d - i$. □

Definition 10. Let A be a nonzero ring and I a proper ideal. The *height* of I, denoted $ht.I$, is the minimum of the heights of the prime ideals containing I

$$ht.I = inf\{ht.\mathfrak{p} \,|\, \mathfrak{p} \supseteq I\}$$

This is a number in $\{0, 1, 2, \ldots, \infty\}$. Equivalently we can take the infimum over the heights of primes minimal over I. Clearly $ht.0 = 0$ and if $I \subseteq J$ are proper ideals then it is clear that $ht.I \leq ht.J$. If I is a prime ideal then $ht.I$ is the usual height of a prime ideal. If A is a noetherian ring then $ht.I < \infty$ for every proper ideal I, since $A_\mathfrak{p}$ is a local noetherian ring and $ht.\mathfrak{p} = dim(A_\mathfrak{p})$.

Lemma 52. *Let A be a nonzero ring and I a proper ideal. Then we have*

$$ht.I = inf\{ht.IA_\mathfrak{m} \,|\, \mathfrak{m} \text{ is a maximal ideal and } I \subseteq \mathfrak{m}\}$$

Lemma 53. *Let A be a noetherian ring and suppose we have an exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in which M', M, M'' are nonzero and finitely generated. Then $dimM = max\{dimM', dimM''\}$.

Proof. We know that $Supp(M) = Supp(M') \cup Supp(M'')$ and for all three modules the dimension is the supremum of the coheights of prime ideals in the support. So the result is straightforward to check. □

4.1 Homomorphism and Dimension

Let $\phi : A \longrightarrow B$ be a morphism of rings. If $\mathfrak{p} \in Spec(A)$ then put $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Let $B_{\mathfrak{p}}$ denote the ring $T^{-1}B$ where $T = \phi(A - \mathfrak{p})$. There is an isomorphism of A-algebras $B_{\mathfrak{p}} \cong B \otimes_A A_{\mathfrak{p}}$. There is a commutative diagram of rings

$$\begin{tikzcd}
B \arrow[r] \arrow[drr] & B_{\mathfrak{p}} \arrow[r] & B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \arrow[d, Rightarrow] \\
 & & B \otimes_A \kappa(\mathfrak{p})
\end{tikzcd}$$

The vertical isomorphism is defined by $b/\phi(s) + \mathfrak{p}B_{\mathfrak{p}} \mapsto b \otimes (1/s + \mathfrak{p}A_{\mathfrak{p}})$. We call $Spec(B \otimes_A \kappa(\mathfrak{p}))$ the *fibre* over $\mathfrak{p}$ of the map $\Phi : Spec(B) \longrightarrow Spec(A)$. Since primes of $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ clearly correspond to primes $\mathfrak{q}$ of B with $\mathfrak{q} \cap A = \mathfrak{p}$, it is easy to see that the ring morphism $B \longrightarrow B \otimes_A \kappa(\mathfrak{p})$ gives rise to a continuous map $Spec(B \otimes_A \kappa(\mathfrak{p})) \longrightarrow SpecB$ which gives a homeomorphism between $\Phi^{-1}(\mathfrak{p})$ and $Spec(B \otimes_A \kappa(\mathfrak{p}))$. See [AM69] Chapter 3.

Lemma 54. *Let $\mathfrak{q}$ be a prime ideal of B with $\mathfrak{q} \cap A = \mathfrak{p}$ and let $\mathfrak{P}$ be the corresponding prime ideal of $B \otimes_A \kappa(\mathfrak{p})$. Then there is an isomorphism of A-algebras*

$$B_{\mathfrak{q}} \otimes_A \kappa(\mathfrak{p}) \cong (B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{P}}$$
$$b/t \otimes (a/s + \mathfrak{p}A_{\mathfrak{p}}) \mapsto (b \otimes (a/1 + \mathfrak{p}A_{\mathfrak{p}}))/(t \otimes (s/1 + \mathfrak{p}A_{\mathfrak{p}}))$$

Proof. It is not difficult to see that there is an isomorphism of rings $B_{\mathfrak{q}} \cong (B_{\mathfrak{p}})_{\mathfrak{q}B_{\mathfrak{p}}}$ defined by $b/t \mapsto (b/1)/(t/1)$. Consider the prime ideal $\mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$. We know that there is a ring isomorphism

$$(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{P}} \cong (B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})_{\mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}} \cong (B_{\mathfrak{p}})_{\mathfrak{q}B_{\mathfrak{p}}}/\mathfrak{p}(B_{\mathfrak{p}})_{qB_{\mathfrak{p}}} \cong B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$$

by the comments following Lemma 7. It is not hard to check there is a ring isomorphism $B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} \cong B_{\mathfrak{q}} \otimes_A \kappa(\mathfrak{p})$ defined by $b/t + \mathfrak{p}B_{\mathfrak{q}} \mapsto b/t \otimes 1$ (the inverse of $b/t \otimes (a/s + \mathfrak{p}A_{\mathfrak{p}})$ is $b\phi(a)/t\phi(s) + \mathfrak{p}B_{\mathfrak{q}}$). So by definition of $\mathfrak{P}$ there is an isomorphism of rings $(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{P}} \cong B_{\mathfrak{q}} \otimes_A \kappa(\mathfrak{p})$, and this is clearly an isomorphism of A-algebras. $\square$

In particular if $\phi : A \longrightarrow B$ is a ring morphism, $\mathfrak{p} \in Spec(A)$ and $\mathfrak{q} \in Spec(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$, then there is an isomorphism of rings $(B/\mathfrak{p}B)_{\mathfrak{q}/\mathfrak{p}B} \cong B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$, so we have $ht.(\mathfrak{q}/\mathfrak{p}B) = dim(B_{\mathfrak{q}} \otimes_A \kappa(\mathfrak{p}))$.

Theorem 55. *Let $\phi : A \longrightarrow B$ be a morphism of noetherian rings. Let $\mathfrak{q} \in Spec(B)$ and put $\mathfrak{p} = \mathfrak{q} \cap A$. Then*

(1) $ht.\mathfrak{q} \leq ht.\mathfrak{p} + ht.(\mathfrak{q}/\mathfrak{p}B)$. In other words $dim(B_{\mathfrak{q}}) \leq dim(A_{\mathfrak{p}}) + dim(B_{\mathfrak{q}} \otimes_A \kappa(\mathfrak{p}))$.

(2) We have equality in (1) if the going-down theorem holds for ϕ (in particular if ϕ is flat).

(3) If $\Phi : Spec(B) \longrightarrow Spec(A)$ is surjective and if the going-down theorem holds, then we have $dim(B) \geq dim(A)$ and $ht.I = ht.(IB)$ for any proper ideal I of A.

Proof. (1) Replacing A by $A_{\mathfrak{p}}$ and B by $B_{\mathfrak{q}}$ we may suppose that $(A, \mathfrak{p})$ and $(B, \mathfrak{q})$ are noetherian local rings such that $\mathfrak{q} \cap A = \mathfrak{p}$, and we must show that $dim(B) \leq dim(A) + dim(B/\mathfrak{p}B)$. Let I be a $\mathfrak{p}$-primary ideal of A. Then $\mathfrak{p}^n \subseteq I$ for some $n > 0$, so $\mathfrak{p}^n B \subseteq IB \subseteq \mathfrak{p}B$. Thus the ideals $\mathfrak{p}B$ and IB have the same radical, and so by definition $dim(B/\mathfrak{p}B) = dim(B/IB)$. If $dimA = 0$ then we can take $I = 0$ and the result is trivial. So assume $dimA = r \geq 1$ and let $I = (a_1, \ldots, a_r)$ for a system of parameters $a_1, \ldots, a_r$. If $dim(B/IB) = 0$ then IB is an $\mathfrak{q}$-primary ideal of B and so $dim(B) \leq r$, as required. Otherwise if $dim(B/IB) = s \geq 1$ let $b_1 + IB, \ldots, b_s + IB$ be a system of parameters of B/IB. Then $b_1, \ldots, b_s, a_1, \ldots, a_r$ generate an ideal of definition of B. Hence $dim(B) \leq r + s$.

(2) We use the same notation as above. If $ht.(\mathfrak{q}/\mathfrak{p}B) = s \geq 0$ then there exists a prime chain of length s, $\mathfrak{q} = \mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_s$ such that $\mathfrak{q}_s \supseteq \mathfrak{p}B$. As $\mathfrak{p} = \mathfrak{q} \cap A \supseteq \mathfrak{q}_i \cap A \supseteq \mathfrak{p}$ all the $\mathfrak{q}_i$

lie over $\mathfrak{p}$. If $ht.\mathfrak{p} = r \geq 0$ then there exists a prime chain $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$ in A, and by going-down there exists a prime chain $\mathfrak{q}_s = \mathfrak{t}_0 \supset \mathfrak{t}_1 \supset \cdots \supset \mathfrak{t}_r$ of B such that $\mathfrak{t}_i \cap A = \mathfrak{p}_i$. Then

$$\mathfrak{q} = \mathfrak{q}_0 \supset \cdots \supset \mathfrak{q}_s \supset \mathfrak{t}_1 \supset \cdots \supset \mathfrak{t}_r$$

is a prime chain of length $r+s$, therefore $ht.\mathfrak{q} \geq r+s$.

(3) (i) follows from (2) since $dim(A) = sup\{ht.\mathfrak{p} \,|\, \mathfrak{p} \in Spec(A)\}$. (ii) Since Φ is surjective IB is a proper ideal. Let $\mathfrak{q}$ be a minimal prime over IB such that $ht.\mathfrak{q} = ht.(IB)$ and put $\mathfrak{p} = \mathfrak{q} \cap A$. Then $ht.(\mathfrak{q}/\mathfrak{p}B) = 0$, so by (2) we find that $ht.(IB) = ht.\mathfrak{q} = ht.\mathfrak{p} \geq ht.I$. For the reverse inclusion, let $\mathfrak{p}$ be a minimal prime ideal over I such that $ht.\mathfrak{p} = ht.I$ and take a prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$. Replacing $\mathfrak{q}$ if necessary, we may assume that $\mathfrak{q}$ is a minimal prime ideal over $\mathfrak{p}B$. Then $ht.I = ht.\mathfrak{p} = ht.\mathfrak{q} \geq ht.(IB)$. ☐

Theorem 56. *Let A be a nonzero subring of B, and suppose that B is integral over A. Then*

(1) $dim(A) = dim(B)$.

(2) Let $\mathfrak{q} \in Spec(B)$ and set $\mathfrak{p} = \mathfrak{q} \cap A$. Then we have $coht.\mathfrak{p} = coht.\mathfrak{q}$ and $ht.\mathfrak{q} \leq ht.\mathfrak{p}$.

(3) If the going-down theorem holds between A and B, then for any ideal J of B with $J \cap A \neq A$ we have $ht.J = ht.(J \cap A)$.

Proof. (1) By Theorem 42 the going-up theorem holds for $A \subseteq B$ and $Spec(B) \longrightarrow Spec(A)$ is surjective, so we can lift any chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ in A to a chain of prime ideals $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_n$ in B. On the other hand, if $\mathfrak{q} \subseteq \mathfrak{q}'$ are prime ideals of B and $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$, then $\mathfrak{q} = \mathfrak{q}'$, so any chain of prime ideals in B restricts to a chain of the same length in A. Hence $dim(A) = dim(B)$.

(2) Since $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$ it is clear from (1) that $coht.\mathfrak{p} = coht.\mathfrak{q}$. If $\mathfrak{q} = \mathfrak{q}_0 \supset \cdots \supset \mathfrak{q}_n$ is a chain of prime ideals in B then intersecting with A gives a chain of length n descending from $\mathfrak{p}$. Hence $ht.\mathfrak{q} \leq ht.\mathfrak{p}$.

(3) Given the going-down theorem, it is clear that $ht.\mathfrak{q} = ht.\mathfrak{p}$ in (2). Let J be a proper ideal of B with $J \cap A \neq A$ and let $\mathfrak{q}$ be such that $ht.J = ht.\mathfrak{q}$. Then $ht.(J \cap A) \leq ht.(\mathfrak{q} \cap A) = ht.\mathfrak{q} = ht.J$. On the other hand, B/J is integral over $B/J \cap A$, so every prime ideal $\mathfrak{p}$ of A containing $J \cap A$ can be lifted to a prime ideal $\mathfrak{q}$ of B containing J. In particular we can lift a prime ideal $\mathfrak{p}$ with $ht.(J \cap A) = ht.\mathfrak{p}$, to see that $ht.J \leq ht.\mathfrak{q} = ht.\mathfrak{p} = ht.(J \cap A)$, as required. ☐

5 Depth

Definition 11. Let A be a ring, M an A-module and $a_1, \ldots, a_r$ a sequence of elements of A. We say $a_1, \ldots, a_r$ is an *M-regular sequence* (or simply *M-sequence*) if the following conditions are satisfied:

(1) For each $2 \leq i \leq r$, a_i is regular on $M/(a_1, \ldots, a_{i-1})M$ and a_1 is regular on M.

(2) $M \neq (a_1, \ldots, a_n)M$.

If $a_1, \ldots, a_r$ is an M-regular sequence then so is $a_1, \ldots, a_i$ for any $i \leq r$. When all a_i belong to an ideal I we say $a_1, \ldots, a_r$ is an *M-regular sequence in I*. If, moreover, there is no $b \in I$ such that $a_1, \ldots, a_r, b$ is M-regular, then $a_1, \ldots, a_r$ is said to be a maximal M-regular sequence in I. Notice that the notion of M-regular depends on the order of the elements in the sequence. If M, N are isomorphic A-modules then a sequence is regular on M iff. it is regular on N.

Lemma 57. *A sequence $a_1, \ldots, a_r$ with $r \geq 2$ is M-regular if and only if a_1 is regular on M and $a_2, \ldots, a_r$ is an M/a_1M-regular sequence. If the sequence $a_1, \ldots, a_r$ is a maximal M-regular sequence in I then $a_2, \ldots, a_r$ is a maximal M/a_1M-regular sequence in I.*

Proof. The key point is that for ideals $\mathfrak{a} \subseteq \mathfrak{b}$ there is a canonical isomorphism of A-modules $M/\mathfrak{b}M \cong N/\mathfrak{b}N$ where $N = M/\mathfrak{a}M$. If $a_1, \dots, a_r$ is M-regular then a_1 is regular on M, a_2 is regular on $N = M/a_1M$ and for $3 \leq i \leq r$, a_i is regular on

$$M/(a_1, \dots, a_{i-1})M \cong N/(a_2, \dots, a_{i-1})N$$

Hence $a_2, \dots, a_r$ is an N-regular sequence. The converse follows from the same argument. $\square$

More generally if $a_1, \dots, a_r$ is an M-regular sequence and we set $N = M/(a_1, \dots, a_r)M$, and if $b_1, \dots, b_s$ is an N-regular sequence, then $a_1, \dots, a_r, b_1, \dots, b_s$ is an M-regular sequence.

Lemma 58. *If $a_1, \dots, a_r$ is an A-regular sequence and M is a flat A-module, then $a_1, \dots, a_r$ is also M-regular provided $(a_1, \dots, a_r)M \neq M$.*

Proof. Left multiplication by a_1 defines a monomorphism $A \longrightarrow A$ since a_1 is A-regular. Tensoring with M and using the fact that M is flat we see that left multiplication by a_1 also gives a monomorphism $M \longrightarrow M$, as required. Similarly tensoring with the monomorphism $a_2 : A/a_1 \longrightarrow A/a_1$ we get a monomorphism $M/a_1M \longrightarrow M/a_1M$, and so on. $\square$

Lemma 59. *Let A be a ring and M an A-module. Given an integer $n \geq 1$, a sequence $a_1, \dots, a_r$ is M-regular if and only if it is M^n-regular.*

Proof. Suppose the sequence $a_1, \dots, a_r$ is M-regular. We prove it is M^n-regular by induction on r. The case $r = 1$ is trivial, so assume $r > 1$. By the inductive hypothesis the sequence $a_1, \dots, a_{r-1}$ is M^n-regular. Let $L = (a_1, \dots, a_{r-1})M$. Then $(a_1, \dots, a_{r-1})M^n = L^n$ and there is an isomorphism of A-modules $M^n/L^n \cong (M/L)^n$. So we need only show that a_r is regular on $(M/L)^n$. Since by assumption it is regular on M/L, this is not hard to check. Clearly $(a_1, \dots, a_r)M^n \neq M^n$, so the sequence $a_1, \dots, a_r$ is M^n-regular, as required. The converse is similarly checked. $\square$

Lemma 60. *Let A be a nonzero ring, M an A-module and $a_1, \dots, a_r \in A$. If $a_1, \dots, a_r \in A_\mathfrak{m}$ is $M_\mathfrak{m}$-regular for every maximal ideal $\mathfrak{m}$ of A then the sequence $a_1, \dots, a_r$ is M-regular.*

Proof. This follows from the fact that given an A-module M an element $a \in A$ is regular on M if and only if its image in $A_\mathfrak{m}$ is regular on $M_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A. $\square$

Lemma 61. *Suppose that $a_1, \dots, a_r$ is M-regular and $a_1\xi_1 + \cdots + a_r\xi_r = 0$ for $\xi_i \in M$. Then $\xi_i \in (a_1, \dots, a_r)M$ for all i.*

Proof. By induction on r. For $r = 1$, $a_1\xi_1 = 0$ implies that $\xi_1 = 0$. Let $r > 1$. Since a_r is regular on $M/(a_1, \dots, a_{r-1})M$ we have $\xi_r = \sum_{i=1}^{r-1} a_i\eta_i$, so $\sum_{i=1}^{r-1} a_i(\xi_i + a_r\eta_i) = 0$. By the inductive hypothesis for $i < r$ we have $\xi_i + a_r\eta_i \in (a_1, \dots, a_{r-1})M$ so that $\xi_i \in (a_1, \dots, a_r)M$. $\square$

Theorem 62. *Let A be a ring and M an A-module, and let $a_1, \dots, a_r$ be an M-regular sequence. Then for every sequence $n_1, \dots, n_r$ of integers > 0 the sequence $a_1^{n_1}, \dots, a_r^{n_r}$ is M-regular.*

Proof. Suppose we can prove the following statement

> $(*)$ Given an integer $n > 0$, an A-module M and any M-regular sequence $a_1, \dots, a_r$ the sequence $a_1^n, a_2, \dots, a_r$ is also M-regular.

We prove the rest of the Theorem by induction on r. For $r = 1$ this follows immediately from $(*)$. Let $r > 1$ and suppose $a_1, \dots, a_r$ is M-regular. Then by $(*)$ $a_1^{n_1}, a_2, \dots, a_r$ is M-regular. Hence $a_2, \dots, a_r$ is $M/a_1^{n_1}M$-regular. By the inductive hypothesis $a_2^{n_2}, \dots, a_r^{n_r}$ is $M/a_1^{n_1}M$-regular and therefore $a_1^{n_1}, \dots, a_r^{n_r}$ is M-regular by Lemma 57.

So it only remains to prove $(*)$, which we do by induction on n. The case $n = 1$ is trivial, so let $n > 1$ be given, along with an A-module M and an M-regular sequence $a_1, \dots, a_r$. By the inductive hypothesis $a_1^{n-1}, a_2, \dots, a_r$ is M-regular and clearly a_1^n is regular on M. Since $(a_1^n, a_2, \dots, a_r) \subseteq (a_1^{n-1}, a_2, \dots, a_r)$ it is clear that $M \neq (a_1^n, a_2, \dots, a_r)M$. Let $i > 1$ and assume that $a_1^n, a_2, \dots, a_{i-1}$ is an M-regular sequence. We need to show that a_i is regular on $M/(a_1^n, a_2, \dots, a_{i-1})M$. Suppose that $a_i\omega = a_1^n\xi_1 + a_2\xi_2 + \cdots + a_{i-1}\xi_{i-1}$. Then $\omega = a_1^{n-1}\eta_1 + a_2\eta_2 + \cdots + a_{i-1}\eta_{i-1}$ by the inductive hypothesis. So

$$a_1^{n-1}(a_1\xi_1 - a_i\eta_1) + a_2(\xi_2 - a_i\eta_2) + \cdots + a_{i-1}(\xi_{i-1} - a_i\eta_{i-1}) = 0$$

Hence $a_1\xi_1 - a_i\eta_1 \in (a_1^{n-1}, a_2, \ldots, a_{i-1})M$ by Lemma 61. It follows that $a_i\eta_1 \in (a_1, a_2, \ldots, a_{i-1})M$, hence $\eta_1 \in (a_1, \ldots, a_{i-1})M$ and so $\omega \in (a_1^n, a_2, \ldots, a_{i-1})M$, as required. This proves $(*)$ and therefore completes the proof. □

Let A be a ring. There is an isomorphism of $A[x_1, \ldots, x_n]$-modules $M \otimes_A A[x_1, \ldots, x_n] \cong M[x_1, \ldots, x_n]$ where the latter module consists of polynomials in $x_1, \ldots, x_n$ with coefficients in M (see our Polynomial Ring notes). For any $f(x_1, \ldots, x_n) \in M[x_1, \ldots, x_n]$ and tuple $(a_1, \ldots, a_n) \in A^n$ we can define an element of M

$$f(a_1, \ldots, a_n) = \sum_\alpha a_1^{\alpha_1} \cdots a_n^{\alpha_n} \cdot f(\alpha)$$

For an element $r \in R$ and $h \in M[x_1, \ldots, x_n]$

$$(f + h)(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) + h(a_1, \ldots, a_n)$$
$$(r \cdot f)(a_1, \ldots, a_n) = r \cdot f(a_1, \ldots, a_n)$$

For an ideal I in R the R-submodule $IM[x_1, \ldots, x_n]$ consists of all polynomials whose coefficients are in the R-submodule $IM \subseteq M$.

Let us review the definition of the associated graded ring and modules. Let A be a ring and I an ideal of A. Then the abelian group

$$gr^I(A) = A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

becomes a graded ring in a fairly obvious way. For an A-module M we have the graded $gr^I(A)$-module

$$gr^I(M) = M/IM \oplus IM/I^2M \oplus I^2M/I^3M \oplus \cdots$$

If A is noetherian and M is a finitely generated A-module, then $gr^I(A)$ is a noetherian ring and if $gr^I(M)$ is a finitely generated $gr^I(A)$-module.

Given elements $a_1, \ldots, a_n \in A$ and $I = (a_1, \ldots, a_n)$, we define a morphism of abelian groups $\psi : M[x_1, \ldots, x_n] \longrightarrow gr^I(M)$ as follows: if f is homogenous of degree $m \geq 0$, define $\psi(f)$ to be the image of $f(a_1, \ldots, a_n)$ in $I^mM/I^{m+1}M$. This defines a morphism of groups $M[x_1, \ldots, x_n]_m \longrightarrow I^mM/I^{m+1}M$ and together these define the morphism of groups ψ. Since ψ maps $IM[x_1, \ldots, x_n]$ to zero it induces a morphism of abelian groups $\phi : (M/IM)[x_1, \ldots, x_n] \longrightarrow gr^I(M)$, and

Proposition 63. *Let A be a ring and M an A-module. Let $a_1, \ldots, a_n \in A$ and set $I = (a_1, \ldots, a_n)$. Then the following conditions are equivalent*

(a) For every $m > 0$ and for every homogenous polynomial $f(x_1, \ldots, x_n) \in M[x_1, \ldots, x_n]$ of degree m such that $f(a_1, \ldots, a_n) \in I^{m+1}M$, we have $f \in IM[x_1, \ldots, x_n]$.

(b) If $f(x_1, \ldots, x_n) \in M[x_1, \ldots, x_n]$ is homogenous and $f(a_1, \ldots, a_n) = 0$ then the coefficients of f are in IM.

(c) The morphism of abelian groups $\phi : (M/IM)[x_1, \ldots, x_n] \longrightarrow gr^I(M)$ defined by mapping a homogenous polynomial $f(x_1, \ldots, x_n)$ of degree m to $f(a_1, \ldots, a_n) \in I^mM/I^{m+1}M$ is an isomorphism.

Proof. It is easy to see that $(a) \Leftrightarrow (c)$ and $(a) \Rightarrow (b)$. To show $(b) \Rightarrow (a)$ let $f \in M[x_1, \ldots, x_n]$ be a homogenous polynomial of degree $m > 0$ and suppose $f(a_1, \ldots, a_n) \in I^{m+1}M$. Any element of $I^{m+1}M$ can be written as a sum of terms of the form $a_1^{\alpha_1} \cdots a_n^{\alpha_n} \cdot m$ with $\sum_i \alpha_i = m+1$. By shifting one of the a_i across we can write $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$ for a homogenous polynomial $g \in M[x_1, \ldots, x_n]$ of degree m all of whose coefficients belong to IM. Hence $(f-g)(a_1, \ldots, a_n) = 0$ so by (b) the coefficients of $f - g$ belong to IM, and this implies implies that the coefficients of f also belong to IM, as required. □

Definition 12. Let A be a ring and M an A-module. A sequence $a_1, \ldots, a_n \in A$ is *M-quasiregular* if it satisfies the equivalent conditions of the Proposition. Obviously this concept does not depend on the order of the elements. But $a_1, \ldots, a_i$ for $i < n$ need not be M-quasiregular.

Recall that for an A-module M, a submodule $N \subseteq M$ and $x \in A$ the notation $(N : x)$ means $\{m \in M \mid xm \in N\}$. This is a submodule of M. If A is a ring, I an ideal and M an A-module, recall that M is *separated in the I-adic topology* when $\bigcap_n I^n M = 0$.

Theorem 64. *Let A be a ring, M a nonzero A-module, $a_1, \ldots, a_n \in A$ and $I = (a_1, \ldots, a_n)$. Then*

- *(i) If $a_1, \ldots, a_n$ is M-quasiregular and $x \in A$ is such that $(IM : x) = IM$, then $(I^m M : x) = I^m M$ for all $m > 0$.*
- *(ii) If $a_1, \ldots, a_n$ is M-regular then it is M-quasiregular.*
- *(iii) If $M, M/a_1M, M/(a_1, a_2)M, \ldots, M/(a_1, \ldots, a_{n-1})M$ are separated in the I-adic topology, then the converse of (ii) is also true.*

Proof. (i) By induction on m, with the case $m = 1$ true by assumption. Suppose $m > 1$ and $x\xi \in I^m M$. By the inductive hypothesis $\xi \in I^{m-1}M$. Hence there exists a homogenous polynomial $f \in M[x_1, \ldots, x_n]$ of degree $m - 1$ such that $\xi = f(a_1, \ldots, a_n)$. Since $x\xi = xf(a_1, \ldots, a_n) \in I^m M$ the coefficients of f are in $(IM : x) = IM$. Therefore $\xi = f(a_1, \ldots, a_n) \in I^m M$.

(ii) By induction on n. For $n = 1$ this is easy to check. Let $n > 1$ and suppose $a_1, \ldots, a_n$ is M-regular. Then by the induction hypothesis $a_1, \ldots, a_{n-1}$ is M-quasiregular. Let $f \in M[x_1, \ldots, x_n]$ be homogenous of degree $m > 0$ such that $f(a_1, \ldots, a_n) = 0$. We prove that $f \in IM[x_1, \ldots, x_n]$ by induction on m (the case $m = 0$ being trivial). Write

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_{n-1}) + x_n h(x_1, \ldots, x_n)$$

Then g and h are homogenous of degrees m and $m - 1$ respectively. By (i) we have

$$h(a_1, \ldots, a_n) \in ((a_1, \ldots, a_{n-1})^m M : a_n) = (a_1, \ldots, a_{n-1})^m M \subseteq I^m M$$

Since by assumption $a_1, \ldots, a_n$ is regular on M, so a_n is regular on $M/(a_1, \ldots, a_{n-1})M$ and hence $((a_1, \ldots, a_{n-1})M : a_n) = (a_1, \ldots, a_{n-1})M$. So by the induction hypothesis on m we have $h \in IM[x_1, \ldots, x_n]$ (by the argument of Proposition 63). Since $h(a_1, \ldots, a_n) \in (a_1, \ldots, a_{n-1})^m M$ there exists $H \in M[x_1, \ldots, x_{n-1}]$ which is homogenous of degree m such that $h(a_1, \ldots, a_n) = H(a_1, \ldots, a_{n-1})$. Putting

$$G(x_1, \ldots, x_{n-1}) = g(x_1, \ldots, x_{n-1}) + a_n H(x_1, \ldots, x_{n-1})$$

we have $G(a_1, \ldots, a_{n-1}) = 0$, so by the inductive hypothesis on n we have $G \in IM[x_1, \ldots, x_n]$, hence $g \in IM[x_1, \ldots, x_n]$ and so $f \in IM[x_1, \ldots, x_n]$.

(iii) By induction on $n \geq 1$. Assume that $a_1, \ldots, a_n$ is M-quasiregular and the modules $M, M/a_1M, \ldots, M/(a_1, \ldots, a_{n-1})M$ are all separated in the I-adic topology. If $a_1\xi = 0$ then $\xi \in IM$, hence $\xi = \sum a_i \eta_i$ and $\sum a_1 a_i \eta_i = 0$, hence $\eta_i \in IM$ and so $\xi \in I^2 M$. In this way we see that $\xi \in \bigcap_t I^t M = 0$. Thus a_1 is regular on M, and this also takes care of the case $n = 1$ since $M \neq IM$ by the separation condition. So assume $n > 1$. By Lemma 57 it suffices to show that $a_2, \ldots, a_n$ is an N-regular sequence, where $N = M/a_1M$. Since there is an isomorphism of A-modules for $2 \leq i \leq n - 1$

$$M/(a_1, \ldots, a_i)M \cong N/(a_2, \ldots, a_i)N$$

The modules $N, N/a_2N, \ldots, N/(a_2, \ldots, a_{n-1})N$ are separated in the I-adic topology. So by the inductive hypothesis it suffices to show that the sequence $a_2, \ldots, a_n$ is N-quasiregular.

It suffices to show that if $f(x_2, \ldots, x_n) \in M[x_2, \ldots, x_n]$ is homogenous of degree $m \geq 1$ with $f(a_2, \ldots, a_n) \in a_1M$ then the coefficients of f belong to IM. Put $f(a_2, \ldots, a_n) = a_1\omega$. We claim that $\omega \in I^{m-1}M$. Let $0 \le i \le m-1$ be the largest integer with $\omega \in I^i M$. Then $\omega = g(a_1, \dots, a_n)$ for some homogenous polynomial of degree i, and

$$f(a_2, \dots, a_n) = a_1 g(a_1, \dots, a_n) \tag{2}$$

If $i < m-1$ then $g \in IM[x_1, \dots, x_n]$ and so $\omega \in I^{i+1}M$, which is a contradiction. Hence $i = m-1$ and so $\omega \in I^{m-1}M$. Again using (2) we see that $f(x_2, \dots, x_n) - x_1 g(x_1, \dots, x_n) \in IM[x_1, \dots, x_n]$. Since f does not involve x_1 we have $f \in IM[x_1, \dots, x_n]$, as required. □

The theorem shows that, under the assumptions of *(iii)* any permutation of an M-regular sequence is M-regular.

Corollary 65. *Let A be a noetherian ring, M a finitely generated A-module and let $a_1, \dots, a_n$ be contained in the Jacobson radical of A. Then $a_1, \dots, a_n$ is M-regular if and only if it is M-quasiregular. In particular if $a_1, \dots, a_n$ is M-regular so is any permutation of the sequence.*

Proof. From [AM69] we know that for any ideal I contained in the Jacobson radical, the I-adic topology on any finitely generated A-module is separated. □

If A is a ring and M an A-module, then any M-regular sequence $a_1, \dots, a_n \in A$ gives rise to a strictly increasing chain of submodules $a_1M, (a_1, a_2)M, \dots, (a_1, \dots, a_n)M$. Hence the chain of ideals $(a_1), (a_1, a_2), \dots, (a_1, \dots, a_n)$ must also be strictly increasing.

Lemma 66. *Let A be a noetherian ring and M an A-module. Any M-regular sequence $a_1, \dots, a_n$ in an ideal I can be extended to a maximal M-regular sequence in I.*

Proof. If $a_1, \dots, a_n$ is not maximal in I, we can find $a_{n+1} \in I$ such that $a_1, \dots, a_n, a_{n+1}$ is an M-regular sequence. Either this process terminates at a maximal M-regular sequence in I, or it produces a strictly ascending chain of ideals

$$(a_1) \subset (a_1, a_2) \subset (a_1, a_2, a_3) \subset \cdots$$

Since A is noetherian, we can exclude this latter possibility. □

Theorem 67. *Let A be a noetherian ring, M a finitely generated A-module and I an ideal of A with $IM \neq M$. Let $n > 0$ be an integer. Then the following are equivalent:*

(1) $Ext^i_A(N, M) = 0$ for $i < n$ and every finitely generated A-module N with $Supp(N) \subseteq V(I)$.

(2) $Ext^i_A(A/I, M) = 0$ for $i < n$.

(3) There exists a finitely generated A-module N with $Supp(N) = V(I)$ and $Ext^i_A(N, M) = 0$ for $i < n$.

(4) There exists an M-regular sequence $a_1, \dots, a_n$ of length n in I.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial. With I fixed we show that (3) $\Rightarrow$ (4) for every finitely generated module M with $IM \neq M$ by induction on n. We have $0 = Ext^0_A(N, M) \cong Hom_A(N, M)$. Since M is finitely generated and nonzero, the set of associated primes of M is finite and nonempty. If no elements of I are M-regular, then I is contained in the union of these associated primes, and hence $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in Ass(M)$ (see [AM69] for details). By definition there is a monomorphism of A-modules $\phi : A/\mathfrak{p} \longrightarrow M$. There is an isomorphism of A-modules

$$(A/\mathfrak{p})_{\mathfrak{p}} \cong A/\mathfrak{p} \otimes_A A_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = k$$

It is not hard to check this is an isomorphism of $A_{\mathfrak{p}}$-modules as well. Since $\phi_{\mathfrak{p}}$ is a monomorphism and $k \neq 0$ it follows that $Hom_{A_{\mathfrak{p}}}(k, M_{\mathfrak{p}}) \neq 0$. Since $\mathfrak{p} \in V(I) = Supp(N)$ we have $N_{\mathfrak{p}} \neq 0$ and so the k-module $N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}} \neq 0$ is nonzero and therefore free, so $Hom_k(N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}}, k) \neq 0$. Since $k \cong (A/\mathfrak{p})_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$-modules it follows that $Hom_A(N, A/\mathfrak{p})_{\mathfrak{p}} \cong Hom_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}, (A/\mathfrak{p})_{\mathfrak{p}}) \neq 0$. Since $A/\mathfrak{p}$ is isomorphic to a submodule of M it follows that $Hom_A(N, M) \neq 0$, which is a contradiction, therefore there exists an M-regular element $a_1 \in I$, which takes care of the case $n = 1$. If $n > 1$ then put $M_1 = M/a_1M$. From the exact sequence

$$0 \longrightarrow M \xrightarrow{a_1} M \longrightarrow M_1 \longrightarrow 0 \tag{3}$$

we get the long exact sequence

$$\cdots \longrightarrow Ext^i_A(N, M) \longrightarrow Ext^i_A(N, M_1) \longrightarrow Ext^{i+1}_A(N, M) \longrightarrow \cdots$$

which shows that $Ext^i_A(N, M_1) = 0$ for $0 \le i < n-1$. By the inductive hypothesis on n there exists an M_1-regular sequence $a_2, \dots, a_n$ in I. The sequence $a_1, \dots, a_n$ is then an M-regular sequence in I.

(4) $\Rightarrow$ (1) By induction on n with I fixed. For $n = 1$ we have $a_1 \in I$ regular on M and so (3) gives an exact sequence of R-modules

$$0 \longrightarrow Hom_A(N, M) \xrightarrow{a_1} Hom_A(N, M)$$

Where a_1 denotes left multiplication by a_1. Since $Supp(N) = V(Ann(N)) \subseteq V(I)$ it follows that $I \subseteq r(Ann(N))$, and so $a_1^r N = 0$ for some $r > 0$. It follows that a_1^r annihilates $Hom_A(N, M)$ as well, but since the action of a_1^r on $Hom_A(N, M)$ gives an injective map it follows that $Hom_A(N, M) = 0$. Now assume $n > 1$ and put $M_1 = M/a_1M$. Then $a_2, \dots, a_n$ is an M_1-regular sequence, so by the inductive hypothesis $Ext^i_A(N, M_1) = 0$ for $i < n-1$. So the long exact sequence corresponding to (3) gives an exact sequence for $0 \le i < n$

$$0 \longrightarrow Ext^i_A(N, M) \xrightarrow{a_1} Ext^i_A(N, M)$$

Here a_1 denotes left multiplication by a_1, which is equal to $Ext^i_A(\alpha, M) = Ext^i(N, \beta)$ where α, β are the endomorphisms given by left multiplication by a_1 on N, M respectively. Assume that a_1^r annihilates N with $r > 0$. Then α^r is the zero map, so $Ext^i_A(\alpha, M)^r = 0$ and so a_1^r also annihilates $Ext^i_A(N, M)$. Since the a_1 is regular on this module, it follows that $Ext^i_A(N, M) = 0$ for $i < n$, as required. □

Corollary 68. *Let A be a noetherian ring, M a finitely generated A-module, and I an ideal of A with $IM \neq M$. If $a_1, \dots, a_n$ a maximal M-regular sequence in I, then $Ext^i_A(A/I, M) = 0$ for $i < n$ and $Ext^n_A(A/I, M) \neq 0$.*

Proof. We already know that $Ext^i_A(A/I, M) = 0$ for $i < n$, so with I fixed we prove by induction on n that $Ext^n_A(A/I, M) \neq 0$ for any finitely generated module M with $IM \neq M$ admitting a maximal M-regular sequence of length n. For $n = 1$ we have $a_1 \in I$ regular on M and an exact sequence (3) where $M_1 = M/a_1M$. Part of the corresponding long exact sequence is

$$Ext^0_A(A/I, M) \longrightarrow Ext^0_A(A/I, M_1) \longrightarrow Ext^1_A(A/I, M)$$

We know from the Theorem 67 that $Ext^0_A(A/I, M) = 0$, so it suffices to show that we have $Hom_A(A/I, M_1) \neq 0$. But if $Hom_A(A/I, M_1) = 0$ then it follows from the proof of (3) $\Rightarrow$ (4) above that there would be $b \in I$ regular on M_1, so a_1, b is an M-regular sequence. This is a contradiction since the sequence a_1 was maximal, so we conclude that $Ext^1_A(A/I, M) \neq 0$.

Now assume $n > 1$ and let $a_1, \dots, a_n$ be a maximal M-regular sequence in I. Then $a_2, \dots, a_n$ is a maximal M_1-regular sequence in I, so by the inductive hypothesis $Ext^{n-1}_A(A/I, M_1) \neq 0$. So from the long exact sequence for (3) we conclude that $Ext^n_A(A/I, M) \neq 0$ also. □

It follows that under the conditions of the Corollary every maximal M-regular sequence in I has a common length, and you can find this length by looking at the sequence of abelian groups

$$Hom_A(A/I, M), Ext^1_A(A/I, M), Ext^2_A(A/I, M), \dots, Ext^n_A(A/I, M), \dots$$

If there are M-regular sequences in I, then this sequence will start off with $n-1$ zero groups, where $n \ge 1$ is the common length of every maximal M-regular sequence. The nth group will be nonzero, and we can't necessarily say anything about the rest of the sequence. Notice that since any M-regular sequence can be extended to a maximal one, any M-regular sequence has length $\leq n$. There are no M-regular sequences in I if and only if the first term of this sequence is nonzero.

Definition 13. Let A be a noetherian ring, M a finitely generated A-module, and I an ideal of A. If $IM \neq M$ then we define the I-*depth* of M to be

$$depth_I(M) = inf\{i \,|\, Ext^i_A(A/I, M) \neq 0\}$$

So $depth_I(M) = 0$ if and only if there are no M-regular sequences in I, and otherwise it is the common length of all maximal M-regular sequences in I, or equivalently the supremum of the lengths of M-regular sequences in I. We define $depth_I(M) = \infty$ if $IM = M$. In particular $depth_I(0) = \infty$. Isomorphic modules have the same I-depth. When $(A, \mathfrak{m})$ is a local ring we write $depth(M)$ or $depth_A M$ for $depth_{\mathfrak{m}}(M)$ and call it simply the *depth* of M. Thus $depth(M) = \infty$ iff. $M = 0$ and $depth(M) = 0$ iff. $\mathfrak{m} \in Ass(M)$.

Lemma 69. *Let $\phi : A \longrightarrow B$ be a surjective local morphism of local noetherian rings, and let M be a finitely generated B-module. Then $depth_A(M) = depth_B(M)$.*

Proof. It is clear that $depth_A(M) = \infty$ iff. $depth_B(M) = \infty$, so we may as well assume both depths are finite. Given a sequence of elements $a_1, \ldots, a_n \in \mathfrak{m}_A$ it is clear that they are an M-regular sequence iff. the images $\phi(a_1), \ldots, \phi(a_n) \in \mathfrak{m}_B$ are an M-regular sequence. Given an M-regular sequence $b_1, \ldots, b_n$ in $\mathfrak{m}_B$ you can choose inverse images $a_1, \ldots, a_n \in \mathfrak{m}_A$ and these form an M-regular sequence. This makes it clear that $depth_A(M) = depth_B(M)$. $\square$

Lemma 70. *Let A be a noetherian ring and M a finitely generated A-module. Then for any ideal I and integer $n \geq 1$ we have $depth_I(M) = depth_I(M^n)$.*

Proof. We have $IM^n = (IM)^n$ so $depth_I(M) = \infty$ if and only if $depth_I(M^n) = \infty$. In the finite case the result follows immediately from Lemma 59. $\square$

Lemma 71. *Let A be a noetherian ring, M a finitely generated A-module and $\mathfrak{p}$ a prime ideal. Then $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ if and only if $\mathfrak{p} \in Ass_A(M)$.*

Proof. We have $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ iff. $\mathfrak{p}A_\mathfrak{p} \in Ass_{A_\mathfrak{p}}(M_\mathfrak{p})$ which by [Ash] Chapter 1, Lemma 1.4.2 is iff. $\mathfrak{p} \in Ass_A(M)$. So the associated primes are precisely those with $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$. $\square$

Lemma 72. *Let A be a noetherian ring, and M a finitely generated A-module. For any prime $\mathfrak{p}$ we have $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq depth_\mathfrak{p}(M)$.*

Proof. If $depth_\mathfrak{p}(M) = \infty$ then $\mathfrak{p}M = M$, and this implies that $(\mathfrak{p}A_\mathfrak{p})M_\mathfrak{p} = M_\mathfrak{p}$ so $depth_{A_\mathfrak{p}}(M) = \infty$. If $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ then $\mathfrak{p}A_\mathfrak{p} \in Ass(M_\mathfrak{p})$ which can only occur if $\mathfrak{p} \in Ass(M)$, and this implies that $Hom_A(A/\mathfrak{p}, M) \neq 0$, so $depth_\mathfrak{p}(M) = 0$ (since we have already excluded the possibility of $\mathfrak{p}M = M$). So we can reduce to the case where $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) = n$ with $0 < n < \infty$ and $\mathfrak{p}M \neq M$. We have seen earlier in notes that there is an isomorphism of groups for $i \geq 0$

$$Ext^i_{A_\mathfrak{p}}((A/\mathfrak{p})_\mathfrak{p}, M_\mathfrak{p}) \cong Ext^i_A(A/\mathfrak{p}, M)_\mathfrak{p}$$

As an $A_\mathfrak{p}$-module $(A/\mathfrak{p})_\mathfrak{p} \cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ and by assumption $Ext^n_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) \neq 0$, so it follows that $Ext^n_A(A/\mathfrak{p}, M) \neq 0$ and hence $depth_\mathfrak{p}(M) \leq n$. $\square$

Definition 14. Let A be a noetherian ring and M a finitely generated A-module. Then we define the *grade* of M, denoted $grade(M)$, to be $depth_I(A)$ where I is the ideal $Ann(M)$. So $grade(M) = \infty$ if and only if $M = 0$. Isomorphic modules have the same grade.

If I is an ideal of A then we call $grade(A/I) = depth_I(A)$ the *grade of* I and denote it by $G(I)$. So the grade of A is ∞ and the grade of any proper ideal I is the common length of the maximal A-regular sequences in I (zero if none exist).

Lemma 73. *Let A be a noetherian ring and M a nonzero finitely generated A-module. Then*

$$grade(M) = inf\{i \,|\, Ext^i(M, A) \neq 0\}$$

Proof. Put $I = Ann(M)$. Since M and A/I are both finitely generated A-modules whose supports are equal to $V(I)$ it follows from Theorem 67 that for any $n > 0$ we have $Ext^i(A/I, A) = 0$ for all $i < n$ if and only if $Ext^i(M, A) = 0$ for all $i < n$. In particular $Ext^0(M, A) \neq 0$ if and only if $Ext^0(A/I, A) \neq 0$. By definition

$$grade(M) = depth_I(A) = inf\{i \,|\, Ext^i(A/I, M) \neq 0\}$$

so the claim is straightforward to check. □

The following result is a generalisation of Krull's Principal Ideal Theorem.

Lemma 74. *Let A be a noetherian ring and $a_1, \ldots, a_r$ an A-regular sequence. Then every minimal prime over $(a_1, \ldots, a_r)$ has height r, and in particular $ht.(a_1, \ldots, a_r) = r$.*

Proof. By assumption $I = (a_1, \ldots, a_r)$ is a proper ideal. If $r = 1$ then this is precisely Krull's PID Theorem. For $r > 1$ we proceed by induction. If $a_1, \ldots, a_r$ is an A-regular sequence then set $J = (a_1, \ldots, a_{r-1})$. Clearly $a_r + J$ is a regular element of R/J which is not a unit, so every minimal prime over $(a_r + J)$ in R/J has height 1. But these are precisely the primes in R minimal over I. So if $\mathfrak{p}$ is any prime ideal minimal over I there is a prime $J \subseteq \mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q}$ minimal over J. By the inductive hypothesis $ht.\mathfrak{q} = r - 1$ so $ht.\mathfrak{p} \geq r$. We know the height is $\leq r$ by another result of Krull. □

For any nonzero ring A the sequence $x_1, \ldots, x_n$ in $A[x_1, \ldots, x_n]$ is clearly a maximal A-regular sequence. So in some sense regular sequences in a ring generalise the notion of independent variables.

Lemma 75. *Let A be a noetherian ring, M a nonzero finitely generated A-module and I a proper ideal. Then $grade(M) \leq proj.dim.M$ and $G(I) \leq ht.I$.*

Proof. For a nonzero module M the projective dimension is the largest $i \geq 0$ for which there exists a module N with $Ext^i(M, N) \neq 0$. So clearly $grade(M) \leq proj.dim.M$. The second claim is trivial if $G(I) = 0$ and otherwise $G(I)$ is the length r of a maximal A-regular sequence $a_1, \ldots, a_r$ in I. But then $r = ht.(a_1, \ldots, a_r) \leq ht.I$, so the proof is complete. □

Proposition 76. *Let A be a noetherian ring, M, N finitely generated A-modules with M nonzero, and suppose that $grade(M) = k$ and $proj.dim.N = \ell < k$. Then*

$$Ext^i_A(M, N) = 0 \qquad (0 \leq i < k - \ell)$$

Proof. Induction on ℓ. If $\ell = -1$ then this is trivial. If $\ell = 0$ then is a direct summand of a free module. Since our assertion holds for A by definition, it holds for N also. If $\ell > 0$ take an exact sequence $0 \longrightarrow N' \longrightarrow L \longrightarrow N \longrightarrow 0$ with L free. Then $proj.dim.N' = \ell - 1$ and our assertion is proved by induction. □

Lemma 77 (Ischebeck). *Let $(A, \mathfrak{m})$ be a noetherian local ring and let M, N be nonzero finitely generated A-modules. Suppose that $depth(M) = k, dim(N) = r$. Then*

$$Ext^i_A(N, M) = 0 \qquad (0 \leq i < k - r)$$

Proof. By induction on integers r with $r < k$ (we assume $k > 0$). If $r = 0$ then $Supp(N) = \{\mathfrak{m}\}$ so the assertion follows from Theorem 67. Let $r > 0$. First we prove the result in the case where $N = A/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$. We can pick $x \in \mathfrak{m} \setminus \mathfrak{p}$ and then the following sequence is exact

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N' \longrightarrow 0$$

where $N' = A/(\mathfrak{p} + Ax)$ has dimension $< r$. Then using the induction hypothesis we get exact sequences of A-modules

$$0 = Ext^i(N', M) \longrightarrow Ext^i_A(N, M) \xrightarrow{x} Ext^i_A(N, M) \longrightarrow Ext^{i+1}_A(N', M) = 0$$

for $0 \leq i < k - r$, and so $Ext^i_A(N, M) = 0$ by Nakayama, since the module $Ext^i_A(N, M)$ is finitely generated (see our Ext notes). This proves the result for modules of the form $N = A/\mathfrak{p}$.

For general N we use know from [Ash] Chapter 1, Theorem 1.5.10 that there is a chain of modules $0 = N_0 \subset \cdots \subset N_s = N$ such that for $1 \leq j \leq s$ we have an isomorphism of A-modules $N_j/N_{j-1} \cong A/\mathfrak{p}_j$ where the $\mathfrak{p}_j$ are prime ideals of A. Lemma 53 shows that $dimN_1 \leq dimN_2 \leq \cdots \leq dimN = r$, so since $N_1 \cong A/\mathfrak{p}_1$ we have already shown the result holds for N_1. Consider the exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow A/\mathfrak{p}_2 \longrightarrow 0$$

By Lemma 53 we know that $r \geq dimA/\mathfrak{p}_2$, so the result holds for $A/\mathfrak{p}_2$ and the following piece of the long exact Ext sequence shows that the result is true for N_2 as well

$$Ext^i_A(A/\mathfrak{p}_2, M) \longrightarrow Ext^i_A(N_2, M) \longrightarrow Ext^i_A(N_1, M)$$

Proceeding in this way proves the result for all N_j and hence for N, completing the proof. □

Theorem 78. *Let $(A, \mathfrak{m})$ be a noetherian local ring and let M be a nonzero finitely generated A-module. Then $depth(M) \leq dim(A/\mathfrak{p})$ for every $\mathfrak{p} \in Ass(M)$.*

Proof. If $\mathfrak{p} \in Ass(M)$ then $Hom_A(A/\mathfrak{p}, m) \neq 0$, hence $depth(M) \leq dim(A/\mathfrak{p})$ by Lemma 77. □

Lemma 79. *Let A be a ring and let E, F be finitely generated A-modules. Then $Supp(E \otimes_A F) = Supp(E) \cap Supp(F)$.*

Proof. See [AM69] Chapter 3, Exercise 19. □

The Dimension Theorem for modules (see [AM69] Chapter 11) shows that for a nonzero finitely generated module M over a noetherian local ring A, the dimension of M is zero iff. M is of finite length, and otherwise $dim(M)$ is the smallest $r \geq 1$ for which there exists elements $a_1, \ldots, a_r \in \mathfrak{m}$ with $M/(a_1, \ldots, a_n)M$ of finite length.

Proposition 80. *Let A be a noetherian local ring and M a finitely generated A-module. Let $a_1, \ldots, a_r$ be an M-regular sequence. Then*

$$dimM/(a_1, \ldots, a_r)M = dimM - r$$

In particular if $a_1, \ldots, a_r$ is an A-regular sequence, then the dimension of the ring $A/(a_1, \ldots, a_r)$ is $dimA - r$.

Proof. Let $N = M/(a_1, \ldots, a_r)M$. Then N is a nonzero finitely generated A-module, so if $k = dim(N)$ then $0 \leq k < \infty$. If $k = 0$ then it is clear from the preceding comments that $dimM/(a_1, \ldots, a_r)M \geq dimM - r$. If $k \geq 1$ and $b_1, \ldots, b_k \in \mathfrak{m}$ are elements such that the module

$$N/(b_1, \ldots, b_k)N \cong M/(a_1, \ldots, a_r, b_1, \ldots, b_k)M$$

is of finite length, then since the a_i all belong to $\mathfrak{m}$ we conclude that $dim(M) \leq r + k$. Hence we at least have the inequality $dimM/(a_1, \ldots, a_n)M \geq dim(M) - r$. On the other hand, suppose $f \in \mathfrak{m}$ is an M-regular element. We have $Supp(M/fM) = Supp(M) \cap Supp(A/fA) = Supp(M) \cap V(f)$ by Lemma 79, and f is not in any minimal element of $Supp(M)$ since these coincide with the minimal elements of $Ass(M)$, and f is regular on M. Since

$$dimM = sup\{coht.\mathfrak{p} \,|\, \mathfrak{p} \in Supp(M)\}$$

it follows easily that $dim(M/fM) < dimM$. Proceeding by induction on r we see that

$$dimM/(a_1, \ldots, a_r)M \leq dimM - r$$

as required. □

Corollary 81. *Let $(A, \mathfrak{m})$ be a noetherian local ring and M a nonzero finitely generated A-module. Then $depth M \leq dim M$.*

Proof. This is trivial if $depth M = 0$. Otherwise let $a_1, \ldots, a_r$ be a maximal M-regular sequence in $\mathfrak{m}$, so $depth M = r$. Then we know from Proposition 80 that $r = dim M - dim M/(a_1, \ldots, a_r)M$, so of course $r \leq dim M$. $\square$

Lemma 82. *Let A be a noetherian ring, M a finitely generated A-module and I an ideal. Let $a_1, \ldots, a_r$ be an M-regular sequence in I and assume $IM \neq M$. Then*

$$depth_I(M/(a_1, \ldots, a_r)M) = depth_I(M) - r$$

Proof. Let $N = M/(a_1, \ldots, a_r)M$. It is clear that $IM = M$ iff. $IN = N$ so both depths are finite. If $depth_I(N) = 0$ then the sequence $a_1, \ldots, a_r$ must be a maximal M-regular sequence in I, so $depth_I(M) = r$ and we are done. Otherwise let $b_1, \ldots, b_s$ be a maximal N-regular sequence in I. Then $a_1, \ldots, a_r, b_1, \ldots, b_s$ is a maximal M-regular sequence in I, so $depth_I(M) = r + s = r + depth_I(N)$, as required. $\square$

Lemma 83. *Let A be a noetherian local ring, $a_1, \ldots, a_r$ an A-regular sequence. If $I = (a_1, \ldots, a_r)$ then*

$$depth_{A/I}(A/I) = depth_A(A) - r$$

Proof. A sequence $b_1, \ldots, b_s \in \mathfrak{m}$ is A/I-regular iff. $b_1 + I, \ldots, b_s + I \in \mathfrak{m}/I$ is A/I-regular, so it is clear that $depth_{A/I}(A/I) = depth_A(A/I)$. By Lemma 82, $depth_A(A/I) = depth_A(A) - r$, as required. $\square$

Proposition 84. *Let A be a noetherian ring, M a finitely generated A-module and I a proper ideal. Then*

$$depth_I(M) = inf\{depth M_{\mathfrak{p}} \,|\, \mathfrak{p} \in V(I)\}$$

Proof. Let n denote the value of the right hand side. If $n = 0$ then $depth M_{\mathfrak{p}} = 0$ for some $\mathfrak{p} \supseteq I$ and then $I \subseteq \mathfrak{p} \in Ass(M)$, since $\mathfrak{p}A_{\mathfrak{p}} \in Ass(M_{\mathfrak{p}})$ implies $\mathfrak{p} \in Ass(M)$. Thus $depth_I(M) = 0$, since there can be no M-regular sequences in $\mathfrak{p}$. If $0 < n < \infty$ then I is not contained in any associated prime of M, and so it is not contained in their union, which is the set of elements not regular on M. Hence there exists $a \in I$ regular on M. Moreover $IM \neq M$ since otherwise we would have $(\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}} = M_{\mathfrak{p}}$ and hence $depth M_{\mathfrak{p}} = \infty$ for any $\mathfrak{p} \supseteq I$, which would contradict the fact that $n < \infty$. Put $M' = M/aM$. Then for any $\mathfrak{p} \supseteq I$ with $M_{\mathfrak{p}} \neq 0$ the element $a/1 \in A_{\mathfrak{p}}$ is an $M_{\mathfrak{p}}$-regular sequence in $\mathfrak{p}A_{\mathfrak{p}}$, so

$$depth M'_{\mathfrak{p}} = depth M_{\mathfrak{p}}/aM_{\mathfrak{p}} = depth M_{\mathfrak{p}} - 1$$

and $depth_I(M') = depth_I(M) - 1$ by the Lemma 82. Therefore our assertion is proved by induction on n.

If $n = \infty$ then $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \supseteq I$. If $IM \neq M$ then $Supp(M/IM)$ is nonempty, since $Ass(M/IM) \subseteq Supp(M/IM)$ and $Ass(M/IM) = \emptyset$ iff. $M/IM = 0$. If $\mathfrak{p} \in Supp(M/IM) = Supp(M) \cap V(I)$ then $(M/IM)_{\mathfrak{p}} \neq 0$ and so $M_{\mathfrak{p}}/IM_{\mathfrak{p}} \neq 0$, which is a contradiction. Hence $IM = M$ and therefore $depth_I(M) = \infty$.

$\square$

5.1 Cohen-Macaulay Rings

Definition 15. Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finitely generated A-module. We know that $depth M \leq dim M$ provided M is nonzero. We say that M is *Cohen-Macaulay* if $M = 0$ or if $depth M = dim M$. If the noetherian local ring A is Cohen-Macaulay as an A-module then we call A a *Cohen-Macaulay* ring. So a noetherian local ring is Cohen-Macaulay if its dimension is equal to the common length of the maximal A-regular sequences in $\mathfrak{m}$. The Cohen-Macaulay property is stable under isomorphisms of modules and rings.

Example 4. Let A be a noetherian local ring. If $dim(A) = 0$ then A is Cohen-Macaulay, since $\mathfrak{m}$ is an associated prime of A and therefore no element of $\mathfrak{m}$ is regular. If $dim(A) = d \geq 1$ then A is Cohen-Macaulay if and only if there is an A-regular sequence in $\mathfrak{m}$ of length d.

Recall that for a module M over a noetherian ring A, the elements of $Ass(M)$ which are not minimal are called the *embedded* primes of M. Since a noetherian ring has descending chain condition on prime ideals, every associated prime of M contains a minimal associated prime.

Theorem 85. *Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finitely generated A-module. Then*

(i) If M is a Cohen-Macaulay module and $\mathfrak{p} \in Ass(M)$, then we have $depthM = dim(A/\mathfrak{p})$. Consequently M has no embedded primes.

(ii) If $a_1, \ldots, a_r$ is an M-regular sequence in $\mathfrak{m}$ and $M' = M/(a_1, \ldots, a_r)M$ then M is Cohen-Macaulay $\Leftrightarrow$ M' is Cohen-Macaulay.

(iii) If M is Cohen-Macaulay, then for every $\mathfrak{p} \in Spec(A)$ the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is Cohen-Macaulay and if $M_\mathfrak{p} \neq 0$ we have $depth_\mathfrak{p}(M) = depth_{A_\mathfrak{p}}(M_\mathfrak{p})$.

Proof. (i) Since $Ass(M) \neq \emptyset$, M is nonzero and so $depthM = dimM$. Since $\mathfrak{p} \in Supp(M)$ we have $\mathfrak{p} \supseteq Ann(M)$ and therefore $dimM \geq dim(A/\mathfrak{p})$ and $dim(A/\mathfrak{p}) \geq depthM$ by Theorem 78. If $\mathfrak{p} \in Ass(M)$ were an embedded prime, there would be a minimal prime $\mathfrak{q} \in Ass(M)$ with $\mathfrak{q} \subset \mathfrak{p}$. But since $coht.\mathfrak{p} = coht.\mathfrak{q}$ are both finite this is impossible.

(ii) By Nakayama we have $M = 0$ iff. $M' = 0$. Suppose $M \neq 0$. Then $dimM' = dimM - r$ by Proposition 80 and $depthM' = depthM - r$ by Lemma 82.

(iii) We may assume that $M_\mathfrak{p} \neq 0$. Hence $\mathfrak{p} \supseteq Ann(M)$. We know that $dimM_\mathfrak{p} \geq depth_{A_\mathfrak{p}} M_\mathfrak{p} \geq depth_\mathfrak{p}(M)$. So we will prove $depth_\mathfrak{p}(M) = dimM_\mathfrak{p}$ by induction on $depth_\mathfrak{p}(M)$. If $depth_\mathfrak{p}(M) = 0$ then no element of $\mathfrak{p}$ is regular on M, so by the usual argument $\mathfrak{p}$ is contained in some $\mathfrak{p}' \in Ass(M)$. But $Ann(M) \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$ and the associated primes of M are the minimal primes over the ideal $Ann(M)$ by (i). Hence $\mathfrak{p} = \mathfrak{p}'$, and so $\mathfrak{p}$ is a minimal element of $Supp(M)$. The dimension of $M_\mathfrak{p}$ is the length of the longest chain in $Supp(M_\mathfrak{p})$. If $\mathfrak{p}_0 A_\mathfrak{p} \subset \cdots \subset \mathfrak{p}_s A_\mathfrak{p} = \mathfrak{p} A_\mathfrak{p}$ is a chain of length $s = dimM_\mathfrak{p}$ then $\mathfrak{p}_0 A_\mathfrak{p}$ is minimal and therefore $\mathfrak{p}_0 \in Ass(M)$. It follows that $\mathfrak{p}_0 = \mathfrak{p}$ and so $s = 0$, as required.

Now suppose $depth_\mathfrak{p}(M) > 0$, take an M-regular element $a \in \mathfrak{p}$ and set $M_1 = M/aM$. The element $a/1 \in A_\mathfrak{p}$ is then $M_\mathfrak{p}$-regular. Therefore we have

$$dim(M_1)_\mathfrak{p} = dimM_\mathfrak{p}/aM_\mathfrak{p} = dimM_\mathfrak{p} - 1$$

and $depth_\mathfrak{p}(M_1) = depth_\mathfrak{p}(M) - 1$. Since M_1 is Cohen-Macaulay by (ii), by the inductive hypothesis we have $dim(M_1)_\mathfrak{p} = depth_\mathfrak{p}(M_1)$, which completes the proof. ☐

Corollary 86. *Let $(A, \mathfrak{m})$ be a noetherian local ring and $a_1, \ldots, a_r$ an A-regular sequence in $\mathfrak{m}$. Let A' be the ring $A/(a_1, \ldots, a_r)$. Then A is a Cohen-Macaulay ring if and only if A' is a Cohen-Macaulay ring.*

Proof. Let $I = (a_1, \ldots, a_r)$. It suffices to show that A' is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay module over A. The dimension of A' as an A-module, the Krull dimension of A' and the dimension of A' as a module over itself are all equal. So it suffices to observe that a sequence $b_1, \ldots, b_s \in \mathfrak{m}$ is an A'-regular sequence iff. $b_1 + I, \ldots, b_s + I \in \mathfrak{m}/I$ is an A'-regular sequence, so $depth_A A' = depth_{A'} A'$. ☐

Corollary 87. *Let A be a Cohen-Macaulay local ring and $\mathfrak{p}$ a prime ideal. Then $A_\mathfrak{p}$ is a Cohen-Macaulay local ring and $ht.\mathfrak{p} = dimA_\mathfrak{p} = depth_\mathfrak{p}(A)$.*

Proof. This all follows immediately from Theorem 85. In the statement, by $dimA_\mathfrak{p}$ we mean the Krull dimension of the ring. ☐

Lemma 88. *Let A be a noetherian ring, I a proper ideal and $a \in I$ a regular element. Then $ht.I/(a) = ht.I - 1$.*

Proof. The minimal primes over the ideal $I/(a)$ of $A/(a)$ correspond to the minimal primes over I, and we know from [Ash] Chapter 5, Corollary 5.4.8 that for any prime $\mathfrak{p}$ containing a, $ht.\mathfrak{p}/(a) = ht.\mathfrak{p} - 1$, so the proof is straightforward. □

Lemma 89. *Let A be a Cohen-Macaulay local ring and I a proper ideal with $ht.I = r \geq 1$. Then we can choose $a_1, \dots, a_r \in I$ in such a way that $ht.(a_1, \dots, a_i) = i$ for $1 \leq i \leq r$.*

Proof. We claim that there exists a regular element $a \in I$. Otherwise, if every element of I was a zero divisor on A, then I would be contained in the union of the finite number of primes in $Ass(A)$, and hence contained in some $\mathfrak{p} \in Ass(M)$. By Theorem 85 (i) these primes are all minimal, so $I \subseteq \mathfrak{p}$ implies $ht.I = 0$, a contradiction.

Now we proceed by induction on r. For $r = 1$ let $a \in I$ be regular. It follows from Krull's PID Theorem that $ht.(a) = 1$. Now assume $r > 1$ and let $a \in I$ be regular. Then by Corollary 86 the ring $A' = A/(a)$ is Cohen-Macaulay, and by Lemma 88, $ht.I/(a) = r - 1$, so by the inductive hypothesis there are $a_1, \dots, a_{r-1} \in I$ with $ht.(a, a_1, \dots, a_i)/(a) = i$ for $1 \leq i \leq r - 1$. Hence

$$ht.(a, a_1, \dots, a_i) = i + 1$$

for $1 \leq i \leq r - 1$, as required. □

Theorem 90. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. Then*

(i) For every proper ideal I of A we have

$$ht.I = depth_I(A) = G(I)$$
$$ht.I + dim(A/I) = dimA$$

(ii) A is catenary.

(iii) For every sequence $a_1, \dots, a_r \in \mathfrak{m}$ the following conditions are equivalent

(1) The sequence $a_1, \dots, a_r$ is A-regular.

(2) $ht.(a_1, \dots, a_r) = i$ for $1 \leq i \leq r$.

(3) $ht.(a_1, \dots, a_r) = r$.

(4) There is a system of parameters of A containing $\{a_1, \dots, a_r\}$.

Proof. (iii) (1) ⇒ (2) is immediate by Lemma 74. (2) ⇒ (3) is trivial. (3) ⇒ (4) If $dimA = r \geq 1$ then $(a_1, \dots, a_r)$ must be $\mathfrak{m}$-primary, so this is trivial. If $dimA > r$ then $\mathfrak{m}$ is not minimal over $(a_1, \dots, a_r)$, so we can take $a_{r+1} \in \mathfrak{m}$ which is not in any minimal prime ideal of $(a_1, \dots, a_r)$. Then by construction $ht.(a_1, \dots, a_{r+1}) \geq r + 1$, and therefore $ht.(a_1, \dots, a_{r+1}) = r + 1$ by Krull's Theorem. Continuing in this way we produce the desired system of parameters. Note that these implications are true for any noetherian local ring. (4) ⇒ (1) It suffices to show that every system of parameters $x_1, \dots, x_n$ of a Cohen-Macaulay ring A is an A-regular sequence, which we do by induction on n. Let $I = (x_1, \dots, x_n)$ and put $A' = A/(x_1)$. If $n = 1$ and (x_1) is $\mathfrak{m}$-primary then it suffices to show that x_1 is regular. If not, then $x_1 \in \mathfrak{p}$ for some $\mathfrak{p} \in Ass(A)$, which implies that $\mathfrak{m} = \mathfrak{p}$ is a minimal prime over 0 (since by Theorem 85 every prime of $Ass(M)$ is minimal), contradicting the fact that $dimA = 1$. Now assume $n > 1$. Since A is Cohen-Macaulay the dimensions $dim(A/\mathfrak{p})$ for $\mathfrak{p} \in Ass(A)$ all agree, and hence they are all equal to $n = dimA$. For any $\mathfrak{p} \in Ass(A)$ the ideal $\mathfrak{p} + I$ is $\mathfrak{m}$-primary since

$$r(I + \mathfrak{p}) = r(r(I) + r(\mathfrak{p})) = r(\mathfrak{m} + \mathfrak{p}) = \mathfrak{m}$$

Thus $\mathfrak{p} + I/\mathfrak{p}$ is an $\mathfrak{m}/\mathfrak{p}$-primary ideal in the ring $A/\mathfrak{p}$, which has dimension n, so $\mathfrak{p} + I/\mathfrak{p}$ cannot be generated by fewer than n elements. This shows that $x_1 \notin \mathfrak{p}$ for any $\mathfrak{p} \in Ass(A)$, and therefore x_1 is A-regular. Put $A' = A/(x_1)$. By Corollary 86, A' is a Cohen-Macaulay ring, and it has dimension $n-1$ by Proposition 80. The images of $x_2, \dots, x_n$ in A' form a system of parameters for A'. Thus the residues $x_2 + (x_1), \dots, x_n + (x_1)$ form an A'-regular sequence (A' as an A'-module) by the inductive hypothesis, and therefore $x_2, \dots, x_n$ is an A'-regular sequence (A' as an A-module). Hence $x_1, \dots, x_n$ is an A-regular sequence, and we are done.

(i) Let I be a proper ideal of A. If $ht.I = 0$ then there is a prime $\mathfrak{p}$ minimal over I with $ht.\mathfrak{p} = 0$. Since $\mathfrak{p}$ is minimal over 0, we have $\mathfrak{p} \in Ass(A)$ and every element of I annihilates some nonzero element of A. Therefore no A-regular sequence can exist in I, and $G(I) = 0$. Now assume $ht.I = r$ with $r \geq 1$. Using Lemma 89 we produce $a_1, \dots, a_r \in I$ with $ht.(a_1, \dots, a_i) = i$ for $1 \leq i \leq r$. Then the sequence $a_1, \dots, a_r$ is A-regular by (iii). Hence $r \leq G(I)$. Conversely, if $b_1, \dots, b_s$ is an A-regular sequence in I then $ht.(b_1, \dots, b_s) = s \leq ht.I$ by Lemma 74. Hence $ht.I = G(I)$.

We first prove the second formula for prime ideals $\mathfrak{p}$. Put $dimA = depthA = n$ and $ht.\mathfrak{p} = r$. If $r = 0$ then $dim(A/\mathfrak{p}) = depthA = n$ by Theorem 85 (i). If $r \geq 1$ then since $A_\mathfrak{p}$ is a Cohen-Macaulay local ring and $ht.\mathfrak{p} = dimA_\mathfrak{p} = depth_\mathfrak{p}(A)$ we can find an A-regular sequence $a_1, \dots, a_r$ in $\mathfrak{p}$. Then $A/(a_1, \dots, a_r)$ is a Cohen-Macaulay ring of dimension $n - r$, and $\mathfrak{p}$ is a minimal prime of $(a_1, \dots, a_r)$. Therefore $dim(A/\mathfrak{p}) = n - r$ by Theorem 85 (i), so the result is proved for prime ideals. Now let I be an arbitrary proper ideal with $ht.I = r$. We have

$$\begin{aligned} dim(A/I) &= sup\{dim(A/\mathfrak{p}) \,|\, \mathfrak{p} \in V(I)\} \\ &= sup\{dimA - ht.\mathfrak{p} \,|\, \mathfrak{p} \in V(I)\} \end{aligned}$$

There exists a prime ideal $\mathfrak{p}$ minimal over I with $ht.\mathfrak{p} = r$, so it is clear that $dim(A/I) = dimA - r$, as required.

(ii) If $\mathfrak{q} \subset \mathfrak{p}$ are prime ideals of A, then since $A_\mathfrak{p}$ is Cohen-Macaulay we have $dimA_\mathfrak{p} = ht.\mathfrak{q}A_\mathfrak{p} + dimA_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}$, i.e. $ht.\mathfrak{p} - ht.\mathfrak{q} = ht.(\mathfrak{p}/\mathfrak{q})$. Therefore A is catenary. □

Definition 16. We say a noetherian ring A is *Cohen-Macaulay* if $A_\mathfrak{p}$ is a Cohen-Macaulay local ring for every prime ideal of A. A local noetherian ring is Cohen-Macaulay in this new sense iff. it is Cohen-Macaulay in the original sense. The Cohen-Macaulay property is stable under ring isomorphism.

Lemma 91. *Let $A \subseteq B$ be nonzero noetherian rings with B integral over A and suppose that B is a flat A-module. If A is Cohen-Macaulay then so is B.*

Proof. Let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q} \cap A$. By Lemma 33, $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ and so using Lemma 58 it follows that $depth_{B_\mathfrak{q}}(B_\mathfrak{q}) \geq depth_{A_\mathfrak{p}}(A_\mathfrak{p}) = dim(A_\mathfrak{p})$. By Theorem 56 we have $dim(B_\mathfrak{q}) \leq dim(A_\mathfrak{p})$, and hence $depth_{B_\mathfrak{q}}(B_\mathfrak{q}) \geq dim(B_\mathfrak{q})$, which shows that $B_\mathfrak{q}$ is Cohen-Macaulay. □

Definition 17. Let A be a noetherian ring and I a proper ideal, and let $Ass_A(A/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be the associated primes of I. We say that I is *unmixed* if $ht.\mathfrak{p}_i = ht.I$ for all i. In that case all the $\mathfrak{p}_i$ are minimal, and A/I has no embedded primes. We say that the *unmixedness theorem holds in* A if the following is true: for $r \geq 0$ if I is a proper ideal of height r generated by r elements, then I is unmixed. Note that such an ideal is unmixed if and only if A/I has no embedded primes, and for $r = 0$ the condition means that A has no embedded primes.

Lemma 92. *Let A be a noetherian ring. If the unmixedness theorem holds in $A_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$, then the unmixedness theorem holds in A.*

Proof. Let I be a proper ideal of height r generated by r elements with $r \geq 0$, and let $I = \mathfrak{q}_1 \cap \cdots \mathfrak{q}_n$ be a minimal primary decomposition with $\mathfrak{q}_i$ being $\mathfrak{p}_i$-primary for $1 \leq i \leq n$. Assume that one of these associated primes, say $\mathfrak{p}_1$, is an embedded prime of I, and let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}_1$. Arrange the $\mathfrak{q}_i$ so that the primes $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are contained in $\mathfrak{m}$ whereas $\mathfrak{p}_{s+1}, \cdots, \mathfrak{p}_n$ are not. Then by [AM69] Proposition 4.9 the following is a minimal primary decomposition of the ideal $IA_\mathfrak{m}$

$$IA_\mathfrak{m} = \mathfrak{q}_1 A_\mathfrak{m} \cap \cdots \cap \mathfrak{q}_s A_\mathfrak{m}$$

So $\{\mathfrak{p}_1 A_\mathfrak{m}, \dots, \mathfrak{p}_s A_\mathfrak{m}\}$ are the associated primes of $IA_\mathfrak{m}$. Since $\mathfrak{p}_1$ is embedded, there is some $1 \leq i \leq s$ with $\mathfrak{p}_i \subset \mathfrak{p}_1$, and therefore $\mathfrak{p}_i A_\mathfrak{m} \subset \mathfrak{p}_1 A_\mathfrak{m}$. But this is a contradiction, since $IA_\mathfrak{m}$ has height r, is generated by r elements, and the unmixedness theorem holds in $A_\mathfrak{m}$. So the unmixedness theorem must hold in A. □

Lemma 93. *Let A be a noetherian ring and assume that the unmixedness theorem holds in A. If $a \in A$ is regular then the unmixedness theorem holds in $A/(a)$.*

Proof. Let I be a proper ideal of A containing a, and supppose the ideal $I/(a)$ has height r and is generated by r elements in $A/(a)$. By Lemma 88 the ideal I has height $r+1$ and is clearly generated by $r+1$ elements in A. Therefore I is unmixed. If $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ are the associated primes of I then the associated primes of $I/(a)$ are $\{\mathfrak{p}_1/(a), \dots, \mathfrak{p}_n/(a)\}$. Since $ht.\mathfrak{p}/(a) = ht.\mathfrak{p} - 1 = ht.I - 1 = ht.I/(a)$ it follows that $I/(a)$ is unmixed, as required. □

Lemma 94. *Let A be a noetherian ring and assume that the unmixedness theorem holds in A. Then if I is a proper ideal with $ht.I = r \geq 1$ we can choose $a_1, \dots, a_r \in I$ such that $ht.(a_1, \dots, a_i) = i$ for $1 \leq i \leq r$.*

Proof. The proof is the same as Lemma 89 except we use the fact that 0 has no embedded primes to show I contains a regular element, and we use Lemma 93. □

Theorem 95. *Let A be a noetherian ring. Then A is Cohen-Macaulay if and only if the unmixedness theorem holds in A.*

Proof. Suppose the unmixedness theorem holds in A and let $\mathfrak{p}$ be a prime ideal of height $r \geq 0$. We know that $r = dimA_{\mathfrak{p}} \geq depth(A_{\mathfrak{p}}) \geq depth_{\mathfrak{p}}A$ by Lemma 72. If $r = 0$ then no regular element can exist in $\mathfrak{p}$, so $depth_{\mathfrak{p}}A = 0$ and consequently $dimA_{\mathfrak{p}} = 0 = depth(A_{\mathfrak{p}})$ so $A_{\mathfrak{p}}$ is Cohen-Macaulay. If $r \geq 1$ then by Lemma 94 we can find $a_1, \dots, a_r \in \mathfrak{p}$ such that $ht.(a_1, \dots, a_i) = i$ for $1 \leq i \leq r$. The ideal $(a_1, \dots, a_i)$ is unmixed by assumption, so a_{i+1} lies in no associated primes of $A/(a_1, \dots, a_i)$. Thus $a_1, \dots, a_r$ is an A-regular sequence in $\mathfrak{p}$, so $depth_{\mathfrak{p}}A \geq r$ and consequently $dimA_{\mathfrak{p}} = r = depth(A_{\mathfrak{p}})$, so again $A_{\mathfrak{p}}$ is Cohen-Macaulay. Hence A is a Cohen-Macaulay ring.

Conversely, suppose A is Cohen-Macaulay. It suffices to show that the unmixedness theorem holds in $A_{\mathfrak{m}}$ for all maximal $\mathfrak{m}$, so we can reduce to the case where A is a Cohen-Macaulay local ring. We know from Theorem 85 that 0 is unmixed. Let $(a_1, \dots, a_r)$ be an ideal of height $r > 0$. Then $a_1, \dots, a_r$ is an A-regular sequence by Theorem 90, hence $A/(a_1, \dots, a_r)$ is Cohen-Macaulay and so $(a_1, \dots, a_r)$ is unmixed. □

Corollary 96. *A noetherian ring A is Cohen-Macaualy if and only if $A_{\mathfrak{m}}$ is a Cohen-Macaulay local ring for every maximal ideal $\mathfrak{m}$.*

Proof. This follows immediately from Theorem 95 and Lemma 92. □

Corollary 97. *Let A be a Cohen-Macaulay ring. If $a_1, \dots, a_r \in A$ are such that $ht.(a_1, \dots, a_i) = i$ for $1 \leq i \leq r$ then $a_1, \dots, a_r$ is an A-regular sequence.*

Theorem 98. *Let A be a Cohen-Macaulay ring. Then the polynomial ring $A[x_1, \dots, x_n]$ is also Cohen-Macaulay. Hence any Cohen-Macaulay ring is universally catenary.*

Proof. It is enough to consider the case $n = 1$. Let $\mathfrak{q}$ be a prime ideal of $B = A[x]$ and put $\mathfrak{p} = \mathfrak{q} \cap A$. We have to show that $B_{\mathfrak{q}}$ is Cohen-Macaulay. It follows from Lemma 10 that $B_{\mathfrak{q}}$ is isomorphic to $A_{\mathfrak{p}}[x]_{\mathfrak{q}A_{\mathfrak{p}}[x]}$ where $\mathfrak{q}A_{\mathfrak{p}}[x]$ is a prime ideal of $A_{\mathfrak{p}}[x]$ contracting to $\mathfrak{p}A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is Cohen-Macaulay we can reduce to showing $B_{\mathfrak{q}}$ is Cohen-Macaulay in the case where A is a Cohen-Macaulay local ring and $\mathfrak{p} = \mathfrak{q} \cap A$ is the maximal ideal. Then $B/\mathfrak{p}B \cong k[x]$ where k is a field. Therefore we have either $\mathfrak{q} = \mathfrak{p}B$ or $\mathfrak{q} = \mathfrak{p}B + fB$ where $f \in B = A[x]$ is a monic polynomial of positive degree. By Theorem 55 we have (Krull dimensions)

$$dim(B_{\mathfrak{q}}) = dim(A) + ht.(\mathfrak{q}/\mathfrak{p}B)$$

If $\mathfrak{q} = \mathfrak{p}B$ then this implies that $dim(B_{\mathfrak{q}}) = dim(A)$. So to show $B_{\mathfrak{q}}$ is Cohen-Macaulay it suffices to show that $depth_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}) \geq dimA$. If $dimA = 0$ this is trivial, so assume $dimA = r \geq 1$ and let $a_1, \dots, a_r$ be an A-regular sequence. As B is flat over A, so is $B_{\mathfrak{q}}$, and therefore $a_1, \dots, a_r$ is also a $B_{\mathfrak{q}}$-regular sequence by Lemma 58. It is then not difficult to check that the images of the a_i in $B_{\mathfrak{q}}$ form a $B_{\mathfrak{q}}$-regular sequence, so $depth_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}) \geq r$, as required.

If $\mathfrak{q} = \mathfrak{p}B + fB$ then $dim(B_{\mathfrak{q}}) = dim(A) + 1$ (since every nonzero prime in $k[x]$ has height 1), and so it suffices to show that $depth_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}) \geq dim(A) + 1$. If $dimA = 0$ then since f is monic it is clearly regular in B and therefore also in $B_{\mathfrak{q}}$, which shows that $depth_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}) \geq 1$. If $dimA = r \geq 1$ let $a_1, \ldots, a_r$ be an A-regular sequence. Since f is monic it follows that f is regular on $B/(a_1, \ldots, a_r)B$. Therefore $a_1, \ldots, a_r, f$ is a B-regular sequence. Applying Lemma 58 we see that this sequence is also $B_{\mathfrak{q}}$-regular, and therefore the images in $B_{\mathfrak{q}}$ form a $B_{\mathfrak{q}}$-regular sequence. This shows that $depth_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}) \geq r + 1$, as required.

It follows from Lemma 8 and Theorem 90 that any Cohen-Macaulay ring is catenary. Therefore if A is Cohen-Macaulay, $A[x_1, \ldots, x_n]$ is catenary for $n \geq 1$, and so any Cohen-Macaulay ring is universally catenary. □

Corollary 99. *If k is a field then $k[x_1, \ldots, x_n]$ is Cohen-Macaulay and therefore universally catenary for $n \geq 1$.*

6 Normal and Regular Rings

6.1 Classical Theory

Definition 18. We say that an integral domain A is *normal* if it is integrally closed in its quotient field. The property of being normal is stable under ring isomorphism. If an integral domain A is normal, then so is $S^{-1}A$ for any multiplicatively closed subset S of A not containing zero.

Proposition 100. *Let A be an integral domain. Then the following are equivalent:*

(i) A is normal;

(ii) $A_{\mathfrak{p}}$ is normal for every prime ideal $\mathfrak{p}$;

(iii) $A_{\mathfrak{m}}$ is normal for every maximal ideal $\mathfrak{m}$.

Proof. See [AM69] Proposition 5.13. □

Definition 19. Let A be an integral domain with quotient field K. An element $u \in K$ is *almost integral* over A if there exists a nonzero element $a \in A$ such that $au^n \in A$ for all $n > 0$.

Lemma 101. *If $u \in K$ is integral over A then it is almost integral over A. The elements of K almost integral over A form a subring of K containing the integral closure of A. If A is noetherian then $u \in K$ is integral if and only if it is almost integral.*

Proof. It is clear that any element of A is almost integral over A. Let $u = b/t \in K$ with $b, t \in A$ nonzero be integral over A, and let

$$u^n + a_1 u^{n-1} + \cdots + a_1 u + a_0 = 0$$

be an equation of integral dependence. We claim that $t^n u^m \in A$ for any $m > 0$. If $m \leq n$ this is trivial, and if $m > n$ then we can write u^m as an A-linear combination of strictly smaller powers of u, so $t^n u^m \in A$ in this case as well. It is easy to check that the almost integral elements form a subring of K.

Now assume that A is noetherian, and let u be almost integral over A. If a is nonzero and $au^n \in A$ for $n \geq 1$ then $A[u]$ is a submodule of the finitely generated A-module $a^{-1}A$, whence $A[u]$ itself is finitely generated over A and so u is integral over A. □

Definition 20. We say that an integral domain A is *completely normal* if every element $u \in K$ which is almost integral over A belongs to A. Clearly a completely normal domain is normal, and for a noetherian ring domain normality and complete normality coincide. The property of being completely normal is stable under ring isomorphism.

Example 5. Any field is completely normal, and if k is a field then the domain $k[x_1, \ldots, x_n]$ is completely normal, since it is noetherian and normal.

Definition 21. We say that a ring A is *normal* if $A_\mathfrak{p}$ is a normal domain for every prime ideal $\mathfrak{p}$. An integral domain is normal in this new sense iff. it is normal in the original sense. The property of being normal is stable under ring isomorphism.

Lemma 102. *Let A be a ring and suppose that $A_\mathfrak{p}$ is a domain for every prime ideal $\mathfrak{p}$. Then A is reduced. In particular a normal ring is reduced.*

Proof. Let $a \in A$ be nilpotent. For any prime ideal $\mathfrak{p}$ we have $a/1 = 0$ in $A_\mathfrak{p}$ so $ta = 0$ for some $t \notin \mathfrak{p}$. Hence $Ann(a)$ cannot be a proper ideal, and so $a = 0$. ☐

Lemma 103. *Let $A_1, \dots, A_n$ be normal domains. Then $A_1 \times \cdots \times A_n$ is a normal ring.*

Proof. Let $A = A_1 \times \cdots \times A_n$. A prime ideal $\mathfrak{p}$ of A is $A_1 \times \cdots \times \mathfrak{p}_i \times \cdots A_n$ for some $1 \le i \le n$ and prime ideal $\mathfrak{p}_i$ of A_i. Moreover $A_\mathfrak{p} \cong (A_i)_{\mathfrak{p}_i}$, which by assumption is a normal domain. Hence A is a normal ring. ☐

Proposition 104. *Let A be a completely normal domain. Then a polynomial ring $A[x_1, \dots, x_n]$ is also completely normal. In particular $k[x_1, \dots, x_n]$ is completely normal for any field k.*

Proof. It is enough to treat the case $n = 1$. Let K denote the quotient field of A. Then the canonical injective ring morphism $A[x] \longrightarrow K[x]$ induces an isomorphism between the quotient field of $A[x]$ and $K(x)$, the quotient field of $K[x]$, so we consider all our rings as subrings of $K(x)$. Let $0 \neq u \in K(x)$ be almost integral over $A[x]$. Since $A[x] \subseteq K[x]$ and $K[x]$ is completely normal, the element u must belong to $K[x]$. Write

$$u = \alpha_r x^r + \alpha_{r+1} x^{r+1} + \cdots + \alpha_d x^d$$

for some $r \ge 0$ and $\alpha_r \neq 0$. Let $f(x) = b_s x^s + b_{s+1} x^{s+1} + \cdots + b_t x^t \in A[x]$ with $b_s \neq 0$ be such that $fu^n \in A[x]$ for all $n > 0$. Then $b_s \alpha_r^n \in A$ for all n so that $\alpha_r \in A$. Then $u - \alpha_r x^r = \alpha_{r+1} x^{r+1} + \cdots$ is almost integral over $A[x]$, so we get $\alpha_{r+1} \in A$ as before, and so on. Therefore $u \in A[x]$. ☐

Proposition 105. *Let A be a normal ring. Then $A[x_1, \dots, x_n]$ is normal.*

Proof. It suffices to consider the case $n = 1$. Let $\mathfrak{q}$ be a prime ideal of $A[x]$ and let $\mathfrak{p} = \mathfrak{q} \cap A$. Then $A[x]_\mathfrak{q}$ is a localisation of $A_\mathfrak{p}[x]$ at a prime ideal, and $A_\mathfrak{p}$ is a normal domain. So we reduce to the case where A is a normal domain with quotient field K. As before we identify the quotient field of $A[x]$ with $K(x)$, the quotient field of $K[x]$. We have to prove that $A[x]$ is integrally closed in $K(x)$. Let $u = p(x)/q(x)$ with $p, q \in A[x]$ be a nonzero element of $K(x)$ which is integral over $A[x]$. Let

$$u^d + f_1(x) u^{d-1} + \cdots + f_d(x) = 0 \qquad f_i \in A[x]$$

be an equation of integral dependence. In order to prove that $u \in A[x]$, consider the subring A_0 of A generated by 1 and the coefficients of p, q and all the f_i. Identify A_0, $A_0[x]$ and the quotient field of $A_0[x]$ with subrings of $K(x)$. Then u is integral over $A_0[x]$. The proof of Proposition 104 shows that u belongs to $K[x]$, and moreover

$$u = \alpha_r x^r + \cdots + \alpha_d x^d$$

where each coefficient $\alpha_i \in K$ is almost integral over A_0. As A_0 is noetherian, α_i is integral over A_0 and therefore integral over A. Therefore $\alpha_i \in A$, which is what we wanted. ☐

Let A be a ring and I an ideal with $\bigcap_{n=1}^\infty I^n = 0$. Then for each nonzero $a \in A$ there is an integer $n \ge 0$ such that $a \in I^n$ and $a \notin I^{n+1}$. We then write $n = ord(a)$ (or $ord_I(a)$) and call it the *order* of a with respect to I. We have $ord(a+b) \ge min\{ord(a), ord(b)\}$ and $ord(ab) \ge ord(a) + ord(b)$. Put $A' = gr^I(A) = \bigoplus_{n \ge 0} I^n/I^{n+1}$. For an element a of A with $ord(a) = n$, we call the sequence in A' with a single a in I^n/I^{n+1} the *leading form* of a and denote it by a^*. Clearly $a^* \neq 0$. We define $0^* = 0$. The map $a \mapsto a^*$ is in general not additive or multiplicative, but for nonzero a, b if $a^* b^* \neq 0$ (i.e. if $ord(ab) = ord(a) + ord(b)$) then we have $(ab)^* = a^* b^*$ and if $ord(a) = ord(b)$ and $a^* + b^* \neq 0$ then we have $(a+b)^* = a^* + b^*$.

Theorem 106 (Krull). *Let A be a nonzero ring, I an ideal and $gr^I(A)$ the associated graded ring. Then*

(1) If $\bigcap_{n=1}^{\infty} I^n = 0$ and $gr^I(A)$ is a domain, so is A.

(2) Suppose that A is noetherian and that I is contained in the Jacobson radical of A. Then if $gr^I(A)$ is a normal domain, so is A.

Proof. We denote the ring $gr^I(A)$ by A' for convenience. (1) Let a, b be nonzero elements of A. Then $a^* \neq 0$ and $b^* \neq 0$, hence $a^*b^* \neq 0$ and therefore $ab \neq 0$.

(2) Since I is contained in the Jacobson radical it is immediate that $\bigcap_{n=1}^{\infty} I^n = 0$ (see [AM69] Corollary 10.19) and so by (1) the ring A is a domain. Let K be the quotient field of A and suppose we are given nonzero $a, b \in A$ with a/b integral over A. We have to prove that $a \in bA$. The A-module A/bA is separated in the I-adic topology by Corollary 10.19 of A & M. In other words

$$bA = \bigcap_{n=1}^{\infty} (bA + I^n)$$

Therefore it suffices to prove the following for every $n \geq 1$:

($*$) For nonzero $a, b \in A$ with a/b integral over A, if $a \in bA + I^{n-1}$ then $a \in bA + I^n$.

Suppose that $a \in bA + I^{n-1}$ for some $n \geq 1$. Then $a = br + a'$ with $r \in A$ and $a' \in I^{n-1}$, and $a'/b = a/b - r$ is integral over A. If $a' = 0$ then $a = br$ and we are done. Otherwise we can reduce to proving ($*$) in the case where $a \in I^{n-1}$.

So we are given an integer $n \geq 1$, nonzero a, b with $a \in I^{n-1}$ and a/b integral over A, and we have to show that $a \in bA + I^n$. Since a/b is almost integral over A there exists nonzero $c \in A$ such that $ca^m \in b^m A$ for all $m > 0$. Since A' is a domain the map $a \mapsto a^*$ is multiplicative, hence we have $c^*(a^*)^m \in (b^*)^m A'$ for all m, and since A' is noetherian (see Proposition 10.22 of A & M) and normal we have $a^* \in b^* A'$. Therefore we can find $d \in A$ with $a^* = b^* d^*$. If $a \in I^n$ then we would be done, so suppose $a \notin I^n$ and therefore $ord(a) = n - 1$. Since $a^* = b^* d^*$ the residue of $a - bd$ in I^{n-1}/I^n is zero, and therefore $a - bd \in I^n$. Hence $a \in bA + I^n$, as required. $\square$

Definition 22. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. Recall that the ring A is said to be regular if $\mathfrak{m}$ can be generated by d elements, or equivalently if $rank_k \mathfrak{m}/\mathfrak{m}^2 = d$. Regularity is stable under ring isomorphism.

Recall that if k is a field, a *graded k-algebra* is a k-algebra R which is also a graded ring in such a way that the graded pieces R_d are k-submodules for every $d \geq 0$. A morphism of graded k-algebras is a morphism of graded rings which is also a morphism of k-modules.

Theorem 107. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. Then A is regular if and only if the graded ring $gr^{\mathfrak{m}}(A) = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic as a graded k-algebra to the polynomial ring $k[x_1, \ldots, x_d]$.*

Proof. The first summand in $gr^{\mathfrak{m}}(A)$ is the field $k = A/\mathfrak{m}$, so this ring becomes a graded k-algebra in a canonical way. For $d = 0$ we interpret the statement as saying A is regular iff. $gr^{\mathfrak{m}}(A)$ is isomorphic as a graded k-algebra to k itself. See the section in [AM69] on regular local rings for the proof. $\square$

Theorem 108. *Let A be a regular local ring of dimension d. Then*

(1) A is a normal domain.

(2) A is a Cohen-Macaulay local ring.

If $d \geq 1$ and $\{a_1, \ldots, a_d\}$ is a regular system of parameters, then

(3) $a_1, \ldots, a_d$ is an A-regular sequence.

(4) $\mathfrak{p}_i = (a_1, \dots, a_i)$ *is a prime ideal of height* i *for each* $1 \leq i \leq d$ *and* $A/\mathfrak{p}_i$ *is a regular local ring of dimension* $d - i$.

(5) Conversely, if I *is a proper ideal of* A *such that* A/I *is regular and has dimension* $d - i$ *for some* $1 \leq i \leq d$*, then there exists a regular system of parameters* $\{y_1, \dots, y_d\}$ *such that* $I = (y_1, \dots, y_i)$*. In particular* I *is prime.*

Proof. (1) Follows from Theorems 106 and 107.

(2) If $d = 0$ this is trivial, and if $d \geq 1$ this follows from (3) below.

(3) From the proof of [AM69] Theorem 11.22 we know that there is an isomorphism of graded k-algebras $\varphi : k[x_1, \dots, x_d] \longrightarrow gr^{\mathfrak{m}}(A)$ defined by $x_i \mapsto a_i \in \mathfrak{m}/\mathfrak{m}^2$. If $f(x_1, \dots, x_d)$ is homogenous of degree $m \geq 0$ then $\varphi(f)$ is the element $\sum_\alpha a_1^{\alpha_1} \cdots a_n^{\alpha_n} f(\alpha)$ of $\mathfrak{m}^m/\mathfrak{m}^{m+1}$. So φ agrees with the morphism of abelian groups defined in Proposition 63 (c). Thus $a_1, \dots, a_d$ is an A-quasiregular sequence. It then follows from Corollary 65 that $a_1, \dots, a_d$ is an A-regular seqence.

(4) We have $dim(A/\mathfrak{p}_i) = d - i$ for $1 \leq i \leq d$ by Proposition 51, and hence $ht.\mathfrak{p}_i = i$ by (2) and Theorem 90 (i). The ring $A/\mathfrak{p}_d$ is a field, and therefore trivially a regular local ring of the correct dimension. If $i < d$ then the maximal ideal $\mathfrak{m}/\mathfrak{p}_i$ of $A/\mathfrak{p}_i$ is generated by $d - i$ elements $\overline{x}_{i+1}, \dots, \overline{x}_d$. Therefore $A/\mathfrak{p}_i$ is regular, and hence $\mathfrak{p}_i$ is prime by (1).

(5) Let $\overline{A} = A/I$ and put $\overline{\mathfrak{m}} = \mathfrak{m}/I$. Then we can identify k with $\overline{A}/\overline{\mathfrak{m}}$ and there is clearly an isomorphism of k-modules

$$\mathfrak{m}^2/(\mathfrak{m}^2 + I) \cong \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$$

So we have

$$d - i = rank_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = rank_k \mathfrak{m}/(\mathfrak{m}^2 + I)$$

Since $I \subseteq \mathfrak{m}$ the A-module $(\mathfrak{m}^2 + I)/\mathfrak{m}^2$ is canonically a k-module, and we have a short exact sequence of k-modules

$$0 \longrightarrow (\mathfrak{m}^2 + I)/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/(\mathfrak{m}^2 + I) \longrightarrow 0$$

Consequently $d - i = rank_k \mathfrak{m}/\mathfrak{m}^2 - rank_k (\mathfrak{m}^2 + I)/\mathfrak{m}^2$, and therefore $rank_k(\mathfrak{m}^2 + I)/\mathfrak{m}^2 = i$. Thus we can choose i elements $y_1, \dots, y_i$ of I which span $\mathfrak{m}^2 + I$ mod $\mathfrak{m}^2$ over k, and $d - i$ elements $y_{i+1}, \dots, y_d$ of $\mathfrak{m}$ which, together with $y_1, \dots, y_i$, span $\mathfrak{m}$ mod $\mathfrak{m}^2$ over k (if $i = d$ then the original $y_1, \dots, y_i$ will do). Then $\{y_1, \dots, y_d\}$ is a regular system of parameters of A, so that $(y_1, \dots, y_i) = \mathfrak{p}$ is a prime ideal of height i by (4). Since $\mathfrak{p} \subseteq I$ and $dim(A/I) = dim(A/\mathfrak{p}) = d - i$, we must have $I = \mathfrak{p}$. □

Let A be an integral domain with quotient field K. A *fractional ideal* is an A-submodule of K. If M, N are two fractional ideals then so is $M \cdot N = \{\sum m_i n_i \,|\, m_i \in M, n_i \in N\}$. This product is associative, commutative and $M \cdot A = M$ for any fractional ideal M. For any nonzero ideal $\mathfrak{a}$ of A we put $\mathfrak{a}^{-1} = \{x \in K \,|\, x\mathfrak{a} \subseteq A\}$. Then $\mathfrak{a}^{-1}$ is a fractional ideal and we have $A \subseteq \mathfrak{a}^{-1}$. Moreover $\mathfrak{a} \cdot \mathfrak{a}^{-1} \subseteq A$ is an ideal of A.

Lemma 109. *Let* A *be a noetherian domain with quotient field* K*. Let* a *be a nonzero element of* A *and* $\mathfrak{p} \in Ass_A(A/(a))$*. Then* $\mathfrak{p}^{-1} \neq A$.

Proof. By definition of associated primes there is $b \notin (a)$ with $\mathfrak{p} = ((a) : b)$. Then $(b/a)\mathfrak{p} \subseteq A$ and $b/a \notin A$. □

Lemma 110. *Let* $(A, \mathfrak{m})$ *be a noetherian local domain such that* $\mathfrak{m} \neq 0$ *and* $\mathfrak{m}\mathfrak{m}^{-1} = A$*. Then* $\mathfrak{m}$ *is a principal ideal, and so* A *is regular of dimension* 1.

Proof. By assumption we have $dimA \geq 1$. By [AM69] Proposition 8.6 it follows that $\mathfrak{m} \neq \mathfrak{m}^2$. Take $a \in \mathfrak{m} - \mathfrak{m}^2$. Then $a\mathfrak{m}^{-1} \subseteq A$, and if $a\mathfrak{m}^{-1} \subseteq \mathfrak{m}$ then $(a) = a\mathfrak{m}^{-1}\mathfrak{m} \subseteq \mathfrak{m}^2$, contradicting the choice of a. Since $a\mathfrak{m}^{-1}$ is an ideal we must have $a\mathfrak{m}^{-1} = A$, that is, $(a) = a\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$. Using the dimension theory of noetherian local rings we see that $dimA \leq 1$ and therefore A is regular of dimension 1. □

Theorem 111. *Let* $(A, \mathfrak{m})$ *be a noetherian local ring of dimension* 1*. Then* A *is regular iff. it is normal.*

Proof. If A is regular then it is a normal domain by Theorem 108. Now suppose that A is normal (hence a domain since $A \cong A_{\mathfrak{m}}$). By Lemma 110 to show A is regular it suffices to show that $\mathfrak{m}\mathfrak{m}^{-1} = A$. Assume the contrary. Then $\mathfrak{m}\mathfrak{m}^{-1}$ is a proper ideal, and since $1 \in \mathfrak{m}^{-1}$ we have $\mathfrak{m} \subseteq \mathfrak{m}\mathfrak{m}^{-1}$, hence $\mathfrak{m}\mathfrak{m}^{-1} = \mathfrak{m}$. Let $a_1, \ldots, a_n$ be generators for $\mathfrak{m}$ (since $dimA \geq 1$ we can assume all $a_i \neq 0$) and let $a \in \mathfrak{m}^{-1}$. Since $aa_i \in \mathfrak{m}$ for all i, we have coefficients $r_{ij} \in A, 1 \leq i, j \leq n$ and equations $aa_i = r_{i1}a_1 + \cdots + r_{in}a_n$. Collecting terms we have:

$$\begin{aligned}
0 &= (r_{11} - a)a_1 + \cdots + r_{1n}a_n \\
0 &= r_{21}a_1 + (r_{21} - a)a_2 + \cdots + r_{2n}a_n \\
&\vdots \\
0 &= r_{n1}a_1 + \cdots + (r_{nn} - a)a_n
\end{aligned}$$

The determinant of the coefficient matrix $B = (r_{ij} - \delta ij \cdot a)$ must satisfy $detB \cdot a_i = 0$ and thus $detB = 0$ since A is a domain. This gives an equation of integral dependence of a over A, whence $\mathfrak{m}^{-1} = A$ since A is integrally closed. But since $dimA = 1$ we have $\mathfrak{m} \in Ass(A/(b))$ for any nonzero $b \in \mathfrak{m}$ so that $\mathfrak{m}^{-1} \neq A$ by Lemma 109. Thus $\mathfrak{m}\mathfrak{m}^{-1} = A$ cannot occur. □

Theorem 112. *Let A be a noetherian normal domain. Then as subrings of the quotient field K of A we have*

$$A = \bigcap_{ht\mathfrak{p}=1} A_{\mathfrak{p}}$$

Moreover any nonzero proper principal ideal in A is unmixed, and if $dim(A) \leq 2$ then A is Cohen-Macaulay.

Proof. Suppose $0 \neq a$ is a nonunit of A and $\mathfrak{p} \in Ass(A/(a))$. We claim that $ht\mathfrak{p} = 1$. Replacing A by $A_{\mathfrak{p}}$ we may assume that A is local with maximal ideal $\mathfrak{p}$ (since $\mathfrak{p}A_{\mathfrak{p}} = ((a/1) : (b/1))$. Then we have $\mathfrak{p}^{-1} \neq A$ by Lemma 109. If $ht\mathfrak{p} > 1$ then $\mathfrak{p}\mathfrak{p}^{-1} = A$, since otherwise we can run the proof of Theorem 111 and obtain a contradiction (in that proof we only use $dimA = 1$ to show that $\mathfrak{m} \neq 0$ and that $\mathfrak{m} \in Ass(A/(b))$ for some nonzero $b \in \mathfrak{m}$). But then Lemma 110 implies that A is regular of dimension 1, contradicting the fact that $ht\mathfrak{p} > 1$. Hence $ht\mathfrak{p} = 1$, which shows that the ideal (a) is unmixed.

Now suppose $x \in A_{\mathfrak{p}}$ for all primes of height 1 and write $x = a/b$. We need to show that $x \in A$, so we can assume that b is not a unit and $a \notin (b)$. The ideal $((b) : a)$ is the annihilator of the nonzero element $a + (b)$ of $A/(b)$. The set of annihilators of nonzero elements of $A/(b)$ containing $((b) : a)$ has a maximal element since A is noetherian, and by Lemma 47 this maximal element is a prime ideal $\mathfrak{p} = ((b) : h)$ for some $h \notin (b)$. By definition $\mathfrak{p} \in Ass(A/(b))$ and thus $ht\mathfrak{p} = 1$. Since $a/b \in A_{\mathfrak{p}}$ we have $a/b = c/s$ for some $s \notin \mathfrak{p}$. Then $sa = bc \in (b)$ so $s \in ((b) : a) \subseteq \mathfrak{p}$, which is a contradiction. Hence we must have had $a \in (b)$ and thus $x \in A$ to begin with.

Now suppose that A is a noetherian normal domain with $dim(A) \leq 2$. By Theorem 95 it is enough to show that the unmixedness theorem holds in A. Since A is a domain it is clear that 0 has no embedded primes, and we have just shown that every proper principal ideal of height 1 is unmixed. If $I = (a_1, a_2)$ is a proper ideal of height 2, then every associated prime $\mathfrak{p}$ of I has $ht.\mathfrak{p} \geq 2$, but also $ht.\mathfrak{p} \leq 2$ since $dim(A) = 2$. Therefore I is unmixed and A is Cohen-Macaulay. □

Definition 23. Let A be a nonzero noetherian ring. Consider the following conditions about A for $k \geq 0$:

(S_k) For every prime $\mathfrak{p}$ of A we have $depth(A_{\mathfrak{p}}) \geq inf\{k, ht.\mathfrak{p}\}$.

(R_k) For every prime $\mathfrak{p}$ of A, if $ht.\mathfrak{p} \leq k$ then $A_{\mathfrak{p}}$ is regular.

The condition (S_0) is trivial, and for every $k \geq 1$ we have $(S_k) \Rightarrow (S_{k-1})$ and $(R_k) \Rightarrow (R_{k-1})$.

For a nonzero noetherian ring A we can express (S_k) differently as follows: for every prime $\mathfrak{p}$, if $ht.\mathfrak{p} \leq k$ then $depth(A_{\mathfrak{p}}) \geq ht.\mathfrak{p}$ and otherwise $depth(A_{\mathfrak{p}}) \geq k$. We introduce the following auxiliary condition for $k \geq 1$

(T_k) For every prime $\mathfrak{p}$ of A, if $ht.\mathfrak{p} \geq k$ then $depth(A_{\mathfrak{p}}) \geq k$.

It is not hard to see that for $k \geq 1$, the condition (S_k) is equivalent to (T_i) being satisfied for all $1 \leq i \leq k$.

Proposition 113. *Let A be a nonzero noetherian ring. Then*

(S_1) $\Leftrightarrow$ $Ass(A)$ has no embedded primes $\Leftrightarrow$ every prime $\mathfrak{p}$ with $ht.\mathfrak{p} \geq 1$ contains a regular element.

(S_2) $\Leftrightarrow$ (S_1) and $Ass(A/fA)$ has no embedded primes for any regular nonunit $f \in A$.

The ring A is Cohen-Macaulay iff it satisfies (S_k) for all $k \geq 0$.

Proof. For a noetherian ring A with prime ideal $\mathfrak{p}$, we have $depth(A_{\mathfrak{p}}) = 0$ iff. $\mathfrak{p}A_{\mathfrak{p}} \in Ass(A_{\mathfrak{p}})$ which by [Ash] Chapter 1, Lemma 1.4.2 is iff. $\mathfrak{p} \in Ass(A)$. So the associated primes are precisely those with $depth(A_{\mathfrak{p}}) = 0$. A prime $\mathfrak{p} \in Ass(A)$ is embedded iff. $ht.\mathfrak{p} \geq 1$, so saying that $Ass(A)$ has no embedded primes is equivalent to saying that if $\mathfrak{p} \in Spec(A)$ and $ht.\mathfrak{p} \geq 1$ then $depth(A_{\mathfrak{p}}) \geq 1$. Hence the first two statements are equivalent. If $Ass(A)$ has no embedded primes and $ht.\mathfrak{p} \geq 1$ then $\mathfrak{p}$ must contain a regular element, since otherwise by [Ash] Chapter 1, Theorem 1.3.6, $\mathfrak{p}$ is contained in an associated prime of A, and these all have height zero. Conversely, if every prime of height ≥ 1 contains a regular element, then certainly no prime of height ≥ 1 can be an associated prime of A, so $Ass(A)$ has no embedded primes.

To prove the second statement, we assume A is a nonzero noetherian ring satisfying (S_1), and show that (S_2) is equivalent to $Ass(A/fA)$ having no embedded primes. Suppose A satisfies (S_2) and let a regular nonunit f be given. If $\mathfrak{p} \in Ass(A/fA)$ then the following Lemma implies that $ht.\mathfrak{p} \geq depth(A_{\mathfrak{p}}) = 1$, and $\mathfrak{p}$ is a minimal prime iff. $ht.\mathfrak{p} = 1$. So the condition (T_2) shows that $Ass(A/fA)$ can have no embedded primes. Conversely, suppose $\mathfrak{p}$ is a prime ideal with $ht.\mathfrak{p} \geq 2$ not satisfying (T_2). Since A has no embedded primes, this can only happen if $depth(A_{\mathfrak{p}}) = 1$. But then by the following Lemma, $\mathfrak{p} \in Ass(A/fA)$ for some regular $f \in A$. Since $ht.\mathfrak{p} \geq 2$, this is an embedded prime, which is impossible.

If A is Cohen-Macaulay then $ht.\mathfrak{p} = depth(A_{\mathfrak{p}})$ for every prime $\mathfrak{p}$, so clearly (S_k) is satisfied for $k \geq 0$. Conversely if A satisfies every (S_k) then by choosing k large enough we see that $depth(A_{\mathfrak{p}}) \geq ht.\mathfrak{p}$ for every prime $\mathfrak{p}$, and hence A is Cohen-Macaulay. $\square$

Lemma 114. *Let A be a nonzero noetherian ring satisfying (S_1). Then for a prime $\mathfrak{p}$ the following are equivalent*

(i) $depth(A_{\mathfrak{p}}) = 1$;

(ii) There exists a regular element $f \in \mathfrak{p}$ with $\mathfrak{p} \in Ass(A/fA)$.

If $f \in \mathfrak{p}$ is regular and $\mathfrak{p} \in Ass(A/fA)$ then $\mathfrak{p}$ is a minimal prime of $Ass(A/fA)$ if and only if $ht.\mathfrak{p} = 1$.

Proof. Let $f \in \mathfrak{p}$ be a regular element. Then $f/1 \in A_{\mathfrak{p}}$ is regular, and it is not hard to see there is an isomorphism of $A_{\mathfrak{p}}$-modules $A_{\mathfrak{p}}/fA_{\mathfrak{p}} \cong (A/fA)_{\mathfrak{p}}$. Note also that

$$depth(A_{\mathfrak{p}}/fA_{\mathfrak{p}}) = depth(A_{\mathfrak{p}}) - 1 \tag{4}$$

$(i) \Rightarrow (ii)$ Since $ht.\mathfrak{p} = dim(A_{\mathfrak{p}}) \geq depth(A_{\mathfrak{p}})$ we have $ht.\mathfrak{p} \geq 1$, and therefore since A satisfies (S_1) there is a regular element $f \in \mathfrak{p}$. The above shows that $depth(A_{\mathfrak{p}}/fA_{\mathfrak{p}}) = depth((A/fA)_{\mathfrak{p}}) = 0$ and therefore by Lemma 71, $\mathfrak{p} \in Ass(A/fA)$, as required. $(ii) \Rightarrow (i)$ follows from Lemma 71 and (4). If (i) is satisfied, then the above proof shows that $\mathfrak{p}$ is an associated prime of A/fA for any regular $f \in \mathfrak{p}$.

Suppose $\mathfrak{p}$ is a minimal prime of $Ass(A/fA)$. Then by (i), $depth(A_{\mathfrak{p}}) = 1$, and since $\mathfrak{p}$ is a minimal prime over fA it follows from Krull's PID Theorem that $ht.\mathfrak{p} = 1$. Conversely if $depth(A_{\mathfrak{p}}) = ht.\mathfrak{p} = 1$ then clearly $\mathfrak{p}$ is minimal over fA. $\square$

Proposition 115. *Let A be a nonzero noetherian ring. Then A is reduced iff it satisfies (R_0) and (S_1).*

Proof. Suppose that A is reduced. Then Lemma 13 shows that A satisfies (R_0). Suppose that A does not satisfy (S_1). Let $\mathfrak{p}$ be an associated prime of A which is not minimal: so $ht.\mathfrak{p} \geq 1$ and $\mathfrak{p} = Ann(b)$ for some nonzero $b \in A$. Then $A_\mathfrak{p}$ is a reduced noetherian ring in which every element is either a unit or a zero-divisor, so by Lemma 12 we must have $dim(A) = 0$, which contradicts the fact that $ht.\mathfrak{p} \geq 1$. Therefore A must satisfy (S_1).

Now suppose that A satisfies (R_0) and (S_1). Let $a \in A$ be nonzero and nilpotent. By Lemma 47 there is an associated prime $\mathfrak{p} \in Ass(A)$ with $Ann(a) \subseteq \mathfrak{p}$. By (S_1) we have $ht.\mathfrak{p} = 0$ and therefore $A_\mathfrak{p}$ is a field by (R_0). Since $a/1 \in A_\mathfrak{p}$ is nilpotent we have $ta = 0$ for some $t \notin \mathfrak{p}$, which is a contradiction. Hence A is reduced. □

If A is a nonzero ring, the set S of all regular elements is a multiplicatively closed subset. Let ΦA denote the localisation $S^{-1}A$, which we call the *total quotient ring* of A. If A is a domain, this is clearly the quotient field.

Theorem 116 (Criterion of Normality). *A nonzero noetherian ring A is normal if and only if it satisfies (S_2) and (R_1).*

Proof. Let A be a nonzero noetherian ring. Suppose first that A is normal, and let $\mathfrak{p}$ be a prime ideal. Then $A_\mathfrak{p}$ is a field for $ht.\mathfrak{p} = 0$ and regular for $ht.\mathfrak{p} = 1$ by Theorem 111, hence the condition (R_1) is satisfied. Since A is normal it is reduced, so it satisfies (S_1) by Proposition 115. To show A satisfies (S_2) it suffices by Proposition 113 to show that $Ass(A/fA)$ has no embedded primes for any regular nonunit f. Let f be a regular nonunit with associated primes

$$Ass(A/fA) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

Suppose wlog that $\mathfrak{p} = \mathfrak{p}_1$ is an embedded prime, and that $\mathfrak{p}_1, \ldots, \mathfrak{p}_i$ are the associated primes contained in $\mathfrak{p}_1$. Since $A_\mathfrak{p}/fA_\mathfrak{p} \cong (A/fA)_\mathfrak{p}$ we have

$$Ass_{A_\mathfrak{p}}(A_\mathfrak{p}/fA_\mathfrak{p}) = \{\mathfrak{p}A_\mathfrak{p}, \mathfrak{p}_2 A_\mathfrak{p}, \ldots, \mathfrak{p}_i A_\mathfrak{p}\}$$

by [Ash] Chapter 1, Lemma 1.4.2. At least one of the $\mathfrak{p}_i$ is properly contained in $\mathfrak{p}$, so $\mathfrak{p}A_\mathfrak{p}$ is an embedded prime of $Ass(A_\mathfrak{p}/fA_\mathfrak{p})$. But since $A_\mathfrak{p}$ is a noetherian normal domain, this contradicts Theorem 112. Hence A satisfies (S_2).

Next, suppose that A satisfies (S_2) and (R_1). Then it also satisfies (R_0) and (S_1), so it is reduced. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of A. Then we have $0 = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. Let S be the set of all regular elements in A. Then by Proposition 113 the $\mathfrak{p}_i$ are precisely the prime ideals of A avoiding S. Therefore $\Phi A = S^{-1}A$ is an artinian ring, and since $(S^{-1}A)_{S^{-1}\mathfrak{p}_i} \cong A_{\mathfrak{p}_i}$, Proposition 14 gives an isomorphism of rings

$$\theta : \Phi A \longrightarrow \prod_{j=1}^{s} A_{\mathfrak{q}_j}, \qquad a/s \mapsto (a/s, \ldots, a/s)$$

where for each i, $\mathfrak{q}_i \in \{\mathfrak{p}_i, \ldots, \mathfrak{p}_r\}$ (some of the $\mathfrak{p}_i$ may more than once or not at all among the $\mathfrak{q}_j$). Also note that by Lemma 13 for each j the ring $K_j = A_{\mathfrak{q}_j}$ is a field. For each j let T_j be image of the ring morphism $A \longrightarrow K_j$. Then taking the product gives a subring $\prod_j T_j$ of $\prod_j K_j$ which contains the image of A under θ. Let $e_1, \ldots, e_s$ be the preimage in ΦA of the tuples $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. These clearly form a family of orthogonal idempotents in ΦA.

Suppose that we could show that A was integrally closed in ΦA. For each j the element e_j satisfies $e_j^2 - e_j = 0$, so $e_j \in A$. We claim that θ identifies the subrings A and $\prod_j T_j$. It is enough to show that θ maps the former subring onto the latter. If $a_1, \ldots, a_s \in A$ give a tuple $(a_1/1, \ldots, a_s/1)$ of $\prod_j T_j$, then since $e_j \in A$ we have $e_1 a_1 + \cdots + e_s a_s \in A$, and since $\theta(e_1) = (1, 0, \ldots, 0)$ and similarly for the other e_j, it is clear that

$$\theta(e_1 a_1 + \cdots + e_s a_s) = (a_1/1, \ldots, a_s/1)$$

as required. Since A is integrally closed in ΦA it is straightforward to check that each T_j is integrally closed in K_j, and is therefore a normal domain. Hence A is isomorphic to a direct product of normal domains, so A is a normal ring by Lemma 103.

So it only remains to show that A is integrally closed in ΦA. Suppose we have an equation of integral dependence in ΦA

$$(a/b)^n + c_1(a/b)^{n-1} + \cdots + c_n = 0$$

where a, b and the c_i are elements of A and b is A-regular. Then $a^n + \sum_{i=1}^n c_i a^{n-i} b^i = 0$. We want to prove that $a \in bA$, so we may assume b is a regular nonunit of A. To show that $a \in bA$ it suffices to show that $a_\mathfrak{p} \in b_\mathfrak{p} A_\mathfrak{p}$ for every associated prime $\mathfrak{p}$ of bA (here $a_\mathfrak{p}$ denotes $a/1 \in A_\mathfrak{p}$). Since bA is unmixed of height 1 by (S_2), it suffices to prove this for primes $\mathfrak{p}$ with $ht.\mathfrak{p} = 1$. By (R_1) if $ht.\mathfrak{p} = 1$ then $A_\mathfrak{p}$ is regular and therefore by Theorem 108 a normal domain. But in the quotient field of $A_\mathfrak{p}$ we have

$$a_\mathfrak{p}^n + \sum_{i=1}^n (c_i)_\mathfrak{p} a_\mathfrak{p}^{n-i} b_\mathfrak{p}^i = 0$$

If $b_\mathfrak{p} = 0$ then clearly $a_\mathfrak{p} = 0$. Otherwise $a_\mathfrak{p}/b_\mathfrak{p}$ is integral over $A_\mathfrak{p}$, and so $a_\mathfrak{p} \in b_\mathfrak{p} A_\mathfrak{p}$, as required. □

Corollary 117. *A nonzero normal noetherian ring A is isomorphic to a finite direct product of normal domains.*

Theorem 118. *Let A be a ring such that for every prime ideal $\mathfrak{p}$ the localisation $A_\mathfrak{p}$ is regular. Then the polynomial ring $A[x_1, \dots, x_n]$ over A has the same property.*

Proof. As in the proof of Theorem 98 we reduce to the case where $(A, \mathfrak{p})$ is a regular local ring, $n = 1$ and $\mathfrak{q}$ is a prime ideal of $B = A[x]$ lying over $\mathfrak{p}$. And we have to prove that $B_\mathfrak{q}$ is regular. We have $\mathfrak{q} \supseteq \mathfrak{p}B$ and $B/\mathfrak{p}B \cong k[x]$ where k is a field. Therefore either $\mathfrak{q} = \mathfrak{p}B$ or $\mathfrak{q} = \mathfrak{p}B + fB$ where $f \in B = A[x]$ is a monic polynomial of positive degree. Put $dim(A) = d \geq 0$. Then $\mathfrak{p}$ is generated by d elements, so if $\mathfrak{q} = \mathfrak{p}B$ then $\mathfrak{q}$ is generated by d elements, and by $d+1$ elements if $\mathfrak{q} = \mathfrak{p}B + fB$. From [Ash] Chapter 5 we know that $ht.\mathfrak{p}B = ht.\mathfrak{p} = d$ (use Propositions 5.6.3 and 5.4.3). On the other hand if $\mathfrak{q} = \mathfrak{p}B + fB$ then by Krull's Theorem $ht.\mathfrak{q} \leq d + 1$, and since $\mathfrak{q}$ contains $\mathfrak{p}$ properly, we must have $ht.\mathfrak{q} = d + 1$. This shows that $B_\mathfrak{q}$ is regular. □

Corollary 119. *If k is a field then $k[x_1, \dots, x_n]_\mathfrak{p}$ is a regular local ring for every prime ideal $\mathfrak{p}$ of $k[x_1, \dots, x_n]$.*

6.2 Homological Theory

The following results are proved in our Dimension notes.

Proposition 120. *Let A be a ring, M an A-module. Then*

(i) M is projective iff. $Ext^1_A(M, N) = 0$ for all A-modules N.

(ii) M is injective iff. $Ext^1_A(N, M) = 0$ for all A-modules N.

(iii) M is injective iff. $Ext^1_A(A/I, M) = 0$ for all ideals I of A.

(iv) M is flat iff. $Tor^A_1(A/I, M) = 0$ for all finitely generated ideals I.

(v) M is flat iff. $Tor^A_1(N, M) = 0$ for all finitely generated A-modules N.

So injectivity is characterised by vanishing of $Ext^1_A(-, M)$, and we can restrict consideration to ideal quotients in the first variable. Flatness is characterised by vanishing of $Tor^1_A(-, M)$ (or equivalently, $Tor^1_A(M, -)$) and we can restrict consideration to ideal quotients or finitely generated modules. The next result shows that the projectivity condition can also be restricted to a special class of modules:

Lemma 121. *Let A be a noetherian ring and M a finitely generated A-module. Then M is projective if and only if $Ext^1_A(M,N) = 0$ for all finitely generated A-modules N.*

Proof. Take an exact sequence $0 \longrightarrow R \longrightarrow F \longrightarrow M \longrightarrow 0$ with F finitely generated and free. Then R is finitely generated, so by assumption $Ext^1_A(M,R) = 0$. Thus the sequence $Hom(F,R) \longrightarrow Hom(R,R) \longrightarrow 0$ is exact. It follows that $R \longrightarrow F$ is a coretraction, so that M is a direct summand of a free module. $\square$

If A is a nonzero ring, then the *global dimension* of A, denoted $gl.dim(A)$, is the largest integer $n \geq 0$ for which there exists modules M, N with $Ext^n_A(M,N) \neq 0$. The *Tor dimension* of A, denoted $tor.dim(A)$, is the largest integer $n \geq 0$ for which there exists modules M, N with $Tor^A_n(M,N) \neq 0$. We know from our Dimension notes that

$$\begin{aligned} gl.dim(A) &= sup\{proj.dim.M \mid M \in A\mathbf{Mod}\} \\ &= sup\{inj.dim.M \mid M \in A\mathbf{Mod}\} \\ &= sup\{proj.dim.A/I \mid I \text{ a left ideal of } A\} \end{aligned}$$

and

$$\begin{aligned} tor.dim(A) &= sup\{flat.dim.M \mid M \in A\mathbf{Mod}\} \\ &= sup\{flat.dim.A/I \mid I \text{ is a left ideal of } A\} \end{aligned}$$

Proposition 122. *Let A be a noetherian ring. Then $tor.dim(A) = gl.dim(A)$ and for every finitely generated A-module M, $flat.dim.M = proj.dim.M$.*

Proof. See our Dimension notes. $\square$

Lemma 123. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, and let M be a finitely generated A-module. Then for $n \geq 0$*

$$proj.dim.M \leq n \iff Tor^A_{n+1}(M,k) = 0$$

In particular, if M is nonzero then $proj.dim.M$ is the largest $n \geq 0$ such that $Tor^A_n(M,k) \neq 0$.

Proof. This is trivial if $M = 0$, so assume M is nonzero. Since $flat.dim.M \leq proj.dim.M$ the implication $\Rightarrow$ is clear. We prove the converse by induction on n. Let $m = rank_k(M/\mathfrak{m}M)$. Then $m \geq 1$ since M is nonzero, and by Nakayama we can find elements $\{u_1, \ldots, u_m\}$ which generate M as an A-module and map to a k-basis in $M/\mathfrak{m}M$. Let $\varepsilon : A^m \longrightarrow M$ be induced by the elements u_i, and let K be the kernel of ε, which is finitely generated since A is noetherian. So we have an exact sequence

$$0 \longrightarrow K \longrightarrow A^m \longrightarrow M \longrightarrow 0$$

It follows that $proj.dim.M \leq proj.dim.K + 1$. If $n > 0$ then using the long exact Tor sequence we see that $Tor^A_{n+1}(M,k) \cong Tor^A_n(K,k)$, which proves the inductive step. So it only remains to consider the case $n = 0$. Then by assumption $Tor^A_1(M,k) = 0$ so the top row in the following commutative diagram of A-modules is exact

$$\begin{tikzcd} 0 \arrow[r] & K \otimes_A k \arrow[r] \arrow[d, Rightarrow] & A^m \otimes_A k \arrow[r, "\varepsilon\otimes 1"] \arrow[d, Rightarrow] & M \otimes_A k \arrow[r] \arrow[d, Rightarrow] & 0 \\ 0 \arrow[r] & K/\mathfrak{m}K \arrow[r] & k^m \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0 \end{tikzcd}$$

By construction $k^m \longrightarrow M/\mathfrak{m}M$ is the morphism of k-modules corresponding to the basis defined by the u_i, so it is an isomorphism. Hence $K/\mathfrak{m}K = 0$, so $K = 0$ by Nakayama's Lemma. Hence $M \cong A^m$ and so $proj.dim.M = 0$. $\square$

Remark 2. Let A be a ring and M an A-module. By localising any finite projective resolution of M, we deduce that $proj.dim_{A_\mathfrak{p}} M_\mathfrak{p} \le proj.dim_A M$ for any prime ideal $\mathfrak{p}$. Given an $A_\mathfrak{p}$-module N we have $N \cong N_\mathfrak{p}$ as $A_\mathfrak{p}$-modules and it follows that $gl.dim(A_\mathfrak{p}) \le gl.dim(A)$.

Lemma 124. *Let A be a nonzero noetherian ring and M a finitely generated A-module. Then*

(i) $proj.dim.M = sup\{proj.dim_{A_\mathfrak{m}} M_\mathfrak{m} \,|\, \mathfrak{m}$ a maximal ideal of $A\}$

(ii) For $n \ge 0$, $proj.dim.M \le n$ if and only if $Tor^A_{n+1}(M, A/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$.

(iii) For every maximal ideal $\mathfrak{m}$, $gl.dim(A_\mathfrak{m}) \le gl.dim(A)$. Moreover

$$gl.dim(A) = sup\{gl.dim(A_\mathfrak{m}) \,|\, \mathfrak{m} \text{ a maximal ideal of } A\}$$

Proof. (i) This is trivial if $M = 0$, so assume M is nonzero. For any module N and maximal ideal $\mathfrak{m}$ we know from Lemma 22 that there is an isomorphism of $A_\mathfrak{m}$-modules for $n \ge 0$

$$Ext^n_A(M, N)_\mathfrak{m} \cong Ext^n_{A_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$$

The module $Ext^n_A(M, N)$ is nonzero if and only if some $Ext^n_{A_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$ is nonzero, and $proj.dim.M$ is the largest integer $n \ge 0$ for which there exists a module N with $Ext^n_A(M, N) \ne 0$, so the claim is easily checked.

(ii) Let $n \ge 0$. Then by (i), $proj.dim.M \le n$ if and only if $proj.dim_{A_\mathfrak{m}} M_\mathfrak{m} \le n$ for every maximal ideal $\mathfrak{m}$. Since $A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} \cong (A/\mathfrak{m})_\mathfrak{m}$ as $A_\mathfrak{m}$-modules, we can use Lemma 22 and Lemma 123 to see that this if and only if for every maximal ideal $\mathfrak{m}$

$$0 = Tor^{A_\mathfrak{m}}_{n+1}(M_\mathfrak{m}, A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}) \cong Tor^A_{n+1}(M, A/\mathfrak{m})_\mathfrak{m}$$

If $\mathfrak{m}, \mathfrak{n}$ are distinct maximal ideals, then $(A/\mathfrak{m})_\mathfrak{n} = 0$, so $Tor^A_{n+1}(M, A/\mathfrak{m}) = 0$ if and only if $Tor^A_{n+1}(M, A/\mathfrak{m})_\mathfrak{m} = 0$, which completes the proof.

(iii) For any maximal ideal $\mathfrak{m}$ and $A_\mathfrak{m}$-module N, there is an isomorphism of $A_\mathfrak{m}$-modules $N \cong N_\mathfrak{m}$, so using (i) and the fact that $gl.dim(A) = sup\{proj.dim.M\}$ the various claims are easy to check. $\square$

Theorem 125. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Then for $n \ge 0$*

$$gl.dim(A) \le n \iff Tor^A_{n+1}(k, k) = 0$$

Consequently, we have $gl.dim(A) = proj.dim_A(k)$.

Proof. Since $tor.dim(A) = gl.dim(A)$ the implication $\Rightarrow$ is immediate. If $Tor^A_{n+1}(k, k) = 0$ then $proj.dim_A(k) \le n$ by Lemma 123. Hence $Tor^A_{n+1}(M, k) = 0$ for all modules M, so by Lemma 123, $proj.dim.M \le n$ for every finitely generated module M. Hence $gl.dim(A) \le n$. Using Lemma 123 again we see that $gl.dim(A) = proj.dim_A(k)$. $\square$

Proposition 126. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and M a nonzero finitely generated A-module. If $proj.dim.M = r < \infty$ and if $x \in \mathfrak{m}$ is M-regular, then $proj.dim(M/xM) = r + 1$.*

Proof. By assumption the following sequence of A-modules is exact

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

Therefore the sequence $0 \longrightarrow Tor^A_i(M/xM, k) \longrightarrow 0$ is exact and so $Tor^A_i(M/xM, k) = 0$ for $i > r + 1$. The following sequence of A-modules is also exact

$$0 = Tor^A_{r+1}(M, k) \longrightarrow Tor^A_{r+1}(M/xM, k) \longrightarrow Tor^A_r(M, k) \xrightarrow{x} Tor^A_r(M, k)$$

where x denotes left multiplication by x, which is equal to $Tor^A_r(x, k)$ and also $Tor^A_r(M, x)$ (see our Tor notes). Since $k = A/\mathfrak{m}$ is annihilated by x, so is $Tor^A_r(M, k)$. Therefore $Tor^A_{r+1}(M/xM, k) \cong Tor^A_r(M, k) \ne 0$ and hence $proj.dim(M/xM) = r + 1$ by Lemma 123. $\square$

Corollary 127. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, M a nonzero finitely generated A-module and $a_1, \ldots, a_s$ an M-regular sequence. If $proj.dim.M = r < \infty$ then $proj.dim(M/(a_1, \ldots, a_s)) = r + s$.*

Proof. Since A is local and $(a_1, \ldots, a_s)M \neq M$ we have $a_i \in \mathfrak{m}$ for each i. We proceed by induction on s. The case $s = 1$ was handled by Proposition 126. If $s > 1$ then set $N = M/(a_1, \ldots, a_{s-1})M$. Then $a_s \in \mathfrak{m}$ is N-regular, and by the inductive hypothesis $proj.dim.N = r + s - 1 < \infty$. So by the case $s = 1$, $proj.dim(N/a_sN) = r + s$, and $N/a_sN \cong M/(a_1, \ldots, a_s)M$, so we are done. □

Theorem 128. *Let $(A, \mathfrak{m}, k)$ be a regular local ring of dimension d. Then $gl.dim(A) = d$.*

Proof. If $d = 0$ then A is a field, and trivially $gl.dim(A) = 0$. Otherwise let $\{a_1, \ldots, a_d\}$ be a regular system of parameters. Then the sequence $a_1, \ldots, a_d$ is A-regular by Theorem 108 and $k = A/(a_1, \ldots, a_d)$ so $proj.dim.k = d$ by Corollary 127. Theorem 125 implies that $gl.dim(A) = d$. □

Among many other things, Theorem 128 allows us to give a much stronger version of Lemma 121 for regular local rings.

Corollary 129. *Let $(A, \mathfrak{m}, k)$ be a regular local ring of dimension d and M a finitely generated A-module. Then*

(i) M is projective if and only if $Ext^i(M, A) = 0$ for $i > 0$.

(ii) For $n \geq 0$ we have $proj.dim.M \leq n$ if and only if $Ext^i(M, A) = 0$ for $i > n$.

Proof. If $M = 0$ the result is trivial, so assume otherwise. (i) Suppose that $Ext^i(M, A) = 0$ for all $i > 0$. Since $Ext^i(M, -)$ is additive, it follows that $Ext^i(M, -)$ vanishes on finite free A-modules for $i > 0$. We show for $1 \leq j \leq d + 1$ that $Ext^j(M, N) = 0$ for every finitely generated A-module N (we may assume $d \geq 1$ since otherwise M is trivially projective).

Theorem 128 implies that $proj.dim.M \leq d$ and therefore $Ext^{d+1}(M, -) = 0$, so this is at least true for $j = d + 1$. Suppose that $Ext^j(M, -)$ vanishes on finitely generated modules, and let N be a finitely generated A-module. We can find a short exact sequence of finitely generated A-modules $0 \longrightarrow R \longrightarrow F \longrightarrow N \longrightarrow 0$ with F a finite free A-module. Since $Ext^{j-1}(M, F) = 0$ and $Ext^j(M, R) = 0$ by the inductive hypothesis, it follows from the long exact sequence that $Ext^{j-1}(M, N) = 0$, as required. The case $j = 1$ implies that M is projective, using Lemma 121.

(ii) The case $n = 0$ is (i), so assume $n \geq 1$. If $proj.dim.M \leq n$ then by definition $Ext^i(M, -) = 0$ for $i > n$, so this direction is trivial. For the converse, suppose that $Ext^i(M, A) = 0$ for $i > n$. We can construct an exact sequence

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with K finitely generated and the P_i finitely generated projectives. It suffices to show that K is projective. But by dimension shifting we have $Ext^i(K, A) \cong Ext^{i+n}(M, A) = 0$ for $i > 0$. Therefore by (i), K is projective and the proof is complete. □

Corollary 130 (Hilbert's Syzygy Theorem). *Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Then $gl.dim(A) = n$.*

Proof. See our Dimension notes for another proof. By Theorem 118 every local ring of A is regular. So if $\mathfrak{m}$ is a maximal ideal then $A_{\mathfrak{m}}$ is regular of global dimension $ht.\mathfrak{m}$ by Theorem 128. So by Lemma 124 (iii), $gl.dim(A)$ is the supremum of the heights of the maximal ideals in A, which is clearly $dim(A) = n$. □

Theorem 131. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, and M a nonzero finitely generated A-module. If $proj.dim(M) < \infty$ then*

$$proj.dim(M) + depth(M) = depth(A)$$

Proof. By induction on $depth(A)$. Let $proj.dim(M) = n \geq 0$. If $depth(A) = 0$ then $\mathfrak{m} \in Ass(A)$. This implies that there is a short exact sequence of A-modules

$$0 \longrightarrow k \longrightarrow A \longrightarrow C \longrightarrow 0$$

Thus we have an exact sequence

$$0 \longrightarrow Tor^A_{n+1}(M, C) \longrightarrow Tor^A_n(M, k) \longrightarrow Tor^A_n(M, A)$$

By Proposition 122, $flat.dim(M) = n$, so $Tor^A_{n+1}(M, C) = 0$. But if $n \geq 1$ then $Tor^A_n(M, A) = 0$ and Lemma 123 yields $Tor^A_{n+1}(M, C) \cong Tor^A_n(M, k) \neq 0$, which is a contradiction. Hence $proj.dim(M) = 0$. This means that M is projective and hence free by Proposition 24. Thus also $depth(M) = 0$ by Lemma 70, which completes the proof in the case $depth(A) = 0$.

Now we fix a ring A with $depth(A) > 0$ and proceed by induction on $depth(M)$. First suppose that $depth(M) = 0$. Then $\mathfrak{m} \in Ass(M)$, say $\mathfrak{m} = Ann(y)$ with $0 \neq y \in M$. Since $depth(A) > 0$ we can find a regular element $x \in \mathfrak{m}$. Find an exact sequence

$$0 \longrightarrow K \longrightarrow A^m \xrightarrow{\varepsilon} M \longrightarrow 0$$

It follows from Lemma 70 that M cannot be free, and hence by Proposition 24 cannot be projective either. Thus $proj.dim(M) = proj.dim(K)+1$. Choose $u \in A^m$ with $\varepsilon(u) = y$. Clearly $\mathfrak{m} \subseteq (K : u)$ and therefore $xu \in K$. Since x is regular on A^m and $u \notin K$ it follows that $xu \notin xK$. But $\mathfrak{m} \subseteq (xK : xu)$, so $\mathfrak{m} \in Ass(K/xK)$ and consequently $depth(K/xK) = 0$. Since K is a submodule of a free module, x is regular on K. By the third Change of Rings theorem for projective dimension (see our Dimension notes)

$$proj.dim_{A/x}(K/xK) = proj.dim_A(K) = proj.dim_A(M) - 1$$

By Lemma 83, $depth_{A/x}(A/x) = depth_A(A) - 1$, so using the inductive hypothesis (on A)

$$\begin{aligned} depth_A(A) &= 1 + depth_{A/x}(A/x) \\ &= 1 + depth_{A/x}(K/xK) + proj.dim_{A/x}(K/xK) \\ &= proj.dim_A(M) \end{aligned}$$

Finally, we consider the case $depth(M) > 0$. Let $x \in \mathfrak{m}$ be regular on M. By Lemma 82 we have $depth(M/xM) = depth(M) - 1$ and by Proposition 126, $proj.dim(M/xM) = proj.dim(M) + 1$. Using the inductive hypothesis (for M) we have

$$\begin{aligned} depth(A) &= depth(M/xM) + proj.dim(M/xM) \\ &= depth(M) - 1 + proj.dim(M) + 1 \\ &= depth(M) + proj.dim(M) \end{aligned}$$

which completes the proof. □

Remark 3. If A is a regular local ring of dimension d, then by Theorem 128 the global dimension of A is d, and for any A-module M we have $proj.dim.M \leq d$. We can now answer the question: how big is the difference $d - proj.dim.M$?

Corollary 132. *Let A be a regular local ring of dimension d, and M a nonzero finitely generated A-module. Then $proj.dim(M) + depth(M) = d$.*

Remark 4. With the notation of Corollary 132 the integer $proj.dim(M)$ measures "how projective" the module M is. To be precise, the closer $proj.dim(M)$ is to zero the more projective M is. Using the Corollary, we can rephrase this by saying that the projectivity of M is measured by the largest number of "independent variables" in M. The module M admits d independent variables if and only if it is projective.

6.3 Koszul Complexes

Throughout this section let A be a nonzero ring. In this section a *complex* will mean a positive chain complex in A**Mod** (notation of our Derived Functor notes). This is a sequence of A-modules and module morphisms $\{M_n, d_n : M_n \longrightarrow M_{n-1}\}_{n\in\mathbb{Z}}$ with $M_n = 0$ for $n < 0$ and $d_{n-1}d_n = 0$ for all n. Visually

$$\cdots \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0 \longrightarrow \cdots$$

We denote the complex by M and differentials d_n by d where no confusion is likely. Let $\mathcal{C}$ denote the abelian category of all positive chain complexes in A**Mod** (this is an abelian subcategory of the category **Ch**A**Mod** of all chain complexes). If L is a complex then for $k \geq 0$ let $L[-1]$ denote the complex obtained by shifting the objects and differentials one position left. That is, $L[-1]_n = L_{n-1}$. Clearly if $\varphi :\longrightarrow L'$ is a morphism of complexes then $\varphi[-1]_n = \varphi_{n-1}$ defines a morphism of complexes $\varphi[-1] : L[-1] \longrightarrow L'[-1]$. This defines an exact functor $T : \mathcal{C} \longrightarrow \mathcal{C}$, and clearly T^k shifts k positions left for $k \geq 1$. If M is an A-module, then we consider it as a complex concentrated in degree 0 and denote this complex also by M.

If L and M are two complexes, we define a chain complex $L \otimes M$ by

$$\begin{aligned}(L \otimes M)_n &= \bigoplus_{p+q=n} L_p \otimes_A M_q \\ &= (L_0 \otimes_A M_n) \oplus (L_1 \otimes_A M_{n-1}) \oplus \cdots \oplus (L_n \otimes_A M_0)\end{aligned}$$

If $x \otimes y$ is an element of one of these summands, then by abuse of notation we also use $x \otimes y$ to denote the image in $(L \otimes M)_n$. For $n \geq 1$ and integers $p, q \geq 0$ with $p + q = n$ we induce a morphism $\kappa_{p,q} : L_p \otimes_A M_q \longrightarrow (L \otimes M)_{n-1}$ of A-modules out of the tensor product using the following formula

$$\kappa_{p,q}(x \otimes y) = \begin{cases} d_L(x) \otimes y + (-1)^p x \otimes d_M(y) & p > 0, q > 0 \\ d_L(x) \otimes y & q = 0 \\ (-1)^p x \otimes d_M(y) & p = 0 \end{cases}$$

Together these define a morphism of A-modules $d : (L \otimes M)_n \longrightarrow (L \otimes M)_{n-1}$. It is easy to check that this makes $L \otimes M$ into a complex of A-modules. Given morphisms of complexes $\varphi : L \longrightarrow L'$ and $\psi : M \longrightarrow M'$ we obtain for each pair of integers $p, q \geq 0$ a morphism of A-modules $\varphi_p \otimes \psi_q : L_p \otimes_A M_q \longrightarrow L'_p \otimes_A M'_q$, and these give rise to a morphism of complexes

$$\varphi \otimes \psi : L \otimes M \longrightarrow L' \otimes M'$$
$$(\varphi \otimes \psi)_n = (\varphi_0 \otimes \psi_0) \oplus (\varphi_1 \otimes \psi_1) \oplus \cdots \oplus (\varphi_n \otimes \psi_n)$$

So the tensor product defines a covariant functor $- \otimes - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ which is additive in each variable. That is, for any complex L the partial functors $L \otimes -$ and $- \otimes L$ are additive.

Proposition 133. *For complexes L, M, N there is a canonical isomorphism*

$$\lambda_{L,M,N} : (L \otimes M) \otimes N \longrightarrow L \otimes (M \otimes N)$$

which is natural in all three variables.

Proof. For $n \geq 0$ we have an isomorphism of A-modules

$$\begin{aligned}
((L \otimes M) \otimes N)_n &= \bigoplus_{p+q=n} (L \otimes M)_p \otimes_A N_q \\
&= \bigoplus_{p+q=n} \left(\bigoplus_{r+s=p} L_r \otimes_A M_s \right) \otimes_A N_q \\
&\cong \bigoplus_{r+s+q=n} (L_r \otimes_A M_s) \otimes_A N_q \\
&\cong \bigoplus_{r+s+q=n} L_r \otimes_A (M_s \otimes_A N_q) \\
&\cong \bigoplus_{p+q=n} L_p \otimes_A \left(\bigoplus_{r+s=q} M_r \otimes_A N_s \right) \\
&= (L \otimes (M \otimes N))_n
\end{aligned}$$

Given integers with $r+s+q=n$ and elements $x \in L_r, y \in M_s, z \in N_q$ we have $x \otimes y \in (L \otimes M)_p$ and this isomorphism sends $(x \otimes y) \otimes z \in ((L \otimes M) \otimes N)_n$ to $x \otimes (y \otimes z)$ in $(L \otimes (M \otimes N))_n$. It is straightforward to check that this is an isomorphism of complexes natural in all three variables. □

Proposition 134. *For any complex L the functors $L \otimes -$ and $- \otimes L$ are naturally equivalent and both are right exact. The functor $A \otimes -$ is naturally equivalent to the identity functor, and $A[-1] \otimes -$ is naturally equivalent to T.*

Proof. To show that $L \otimes -$ and $- \otimes L$ are naturally equivalent, the only subtle point is that for $p, q \geq 0$ if $\varphi : L_p \otimes M_q \cong M_q \otimes L_p$ is the canonical isomorphism, then we use the isomorphism $(-1)^{pq}\varphi$ in defining $(L \otimes M)_n \cong (M \otimes L)_n$. Suppose we have a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{C}$. Then for every $j \geq 0$ the sequence of A-modules $0 \longrightarrow A_j \longrightarrow B_j \longrightarrow C_j \longrightarrow 0$ is exact, and therefore

$$L_i \otimes A_j \longrightarrow L_i \otimes B_j \longrightarrow L_i \otimes C_j \longrightarrow 0$$

is also exact for any $i \geq 0$. Coproducts are exact in $A\mathbf{Mod}$ so for any $n \geq 0$ the following sequence is also exact

$$\bigoplus_{i+j=n} L_i \otimes A_j \longrightarrow \bigoplus_{i+j=n} L_i \otimes B_j \longrightarrow \bigoplus_{i+j=n} L_i \otimes C_j \longrightarrow 0$$

But this is $(L \otimes A)_n \longrightarrow (L \otimes B)_n \longrightarrow (L \otimes C)_n \longrightarrow 0$, so the sequence $L \otimes A \longrightarrow L \otimes B \longrightarrow L \otimes C \longrightarrow 0$ is pointwise exact and therefore exact. Consider A as a complex concentrated in degree 0. For a complex M the natural isomorphism $M \cong A \otimes M$ is given pointwise by the isomorphism $M_n \cong A \otimes M_n$. There is also a natural isomorphism $A \otimes M \cong M$ given pointwise by $A \otimes M_n \cong M_n$. It is not hard to check that this is the same as $M \cong A \otimes M$ followed by the twist $A \otimes M \cong M \otimes A$. The complex $A[-1] \otimes M$ is isomorphic to $M[-1]$ but we have to be careful, since the signs of the differentials in $A[-1] \otimes M$ are the opposite of those in $M[-1]$, so we use the isomorphism $M[-1]_n = M_{n-1} \cong A \otimes M_{n-1}$ given by $(-1)^{n+1}\psi$ where $\psi : M_{n-1} \cong A \otimes M_{n-1}$ is canonical. This isomorphism is clearly natural in M.

On the other hand, there is a natural isomorphism $M[-1] \cong M \otimes A[-1]$ given pointwise by $M[-1]_n \cong M_{n-1} \otimes A$, with no sign problems. In fact this isomorphism is $M[-1] \cong A[-1] \otimes M$ followed by the twist $A[-1] \otimes M \cong M \otimes A[-1]$. □

In our Module Theory notes we define the *exterior algebra* $\wedge M$ associated to any A-module M. It is a graded A-algebra, and if M is free of rank $n \geq 1$ with basis $\{x_1, \ldots, x_n\}$ then for $0 \leq p \leq n$, $\wedge^p M$ is free of rank $\binom{n}{p}$ with basis $x_{i_1} \wedge \cdots \wedge x_{i_p}$ indexed by strictly ascending sequences $i_1 < \cdots < i_p$ in the set $\{1, \ldots, n\}$. For $p > n$ we have $\wedge^p M = 0$.

Definition 24. Fix $n \geq 1$ and let $F = A^n$ be the canonical free A-module of rank n, with canonical basis $x_1, \ldots, x_n$. Suppose we are given elements $a_1, \ldots, a_n \in A$. We define a complex of A-modules called the *Koszul complex*, and denoted $K(a_1, \ldots, a_n)$

$$\cdots \xrightarrow{d_{p+1}} \wedge^p F \xrightarrow{d_p} \cdots \xrightarrow{d_3} \wedge^2 F \xrightarrow{d_2} \wedge^1 F \xrightarrow{d_1} \wedge^0 F \longrightarrow 0$$

We identify $\wedge^1 F$ with F and $\wedge^0 F$ with A. These modules become zero beyond $\wedge^n F$. The map d_1 is defined by $d_1(x_i) = a_i$. For $p \geq 2$ with $\wedge^p F \neq 0$ we define

$$d_p(x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sum_{r=1}^{p} (-1)^{r-1} a_{i_r} (x_{i_1} \wedge \cdots \wedge \widehat{x}_{i_r} \wedge \cdots \wedge x_{i_p})$$

where $\widehat{x}_{i_r}$ indicates that we have omitted x_{i_r}. All other morphisms are zero. It is not hard to check that $d_p d_{p+1} = 0$ for all $p \geq 1$, so this is actually a complex.

Definition 25. Let $a_1, \ldots, a_n \in A$. If C is a chain complex, then we denote by $C(a_1, \ldots, x_n)$ the tensor product $C \otimes K(a_1, \ldots, x_n)$. If M is an A-module then we consider it is as a complex concentrated in degree 0 and denote by $K(a_1, \ldots, a_n, M)$ the complex $M \otimes K(a_1, \ldots, a_n)$. This is isomorphic to the complex

$$\cdots \longrightarrow M \otimes \wedge^p F \longrightarrow \cdots \longrightarrow M \otimes \wedge^2 F \longrightarrow M \otimes \wedge^1 F \longrightarrow M \otimes \wedge^0 F \longrightarrow 0$$

Example 6. If $a_1 \in A$ then $K(a_1)$ is isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{a_1} A \longrightarrow 0$$

concentrated in degrees 0 and 1, where the morphism $A \longrightarrow A$ is left multiplication by a_1. Then $H_0(K(a_1)) = A/a_1 A$ and $H_1(K(a_1)) = Ann(a_1)$ as A-modules.

Proposition 135. *Let $a_1, \ldots, a_n \in A$ and a multiplicatively closed set $S \subseteq A$ be given. Then there is a canonical isomorphism of complexes of $S^{-1}A$-modules $S^{-1}K(a_1, \ldots, a_n) \cong K(a_1/1, \ldots, a_n/1)$.*

Proof. There is a canonical isomorphism of $S^{-1}A$-modules $S^{-1}F \cong (S^{-1}A)^n$ identifying $x_i/1$ with the canonical ith basis element. Using (TES,Corollary 16) we have for each $p \geq 0$ a canonical isomorphism of $S^{-1}A$-modules

$$S^{-1}K(a_1, \ldots, a_n)_p = S^{-1}(\bigwedge_A^p F) \cong \bigwedge_{S^{-1}A}^p S^{-1}F \cong \bigwedge_{S^{-1}A}^p G^n = K(a_1/1, \ldots, a_n/1)_p$$

where $G = (S^{-1}A)^n$. Together these isomorphisms form an isomorphism of complexes of $S^{-1}A$-modules, as required. □

Proposition 136. *Let $a_1, \ldots, a_{n+1} \in A$ with $n \geq 1$. Then there is a canonical isomorphism*

$$K(a_1, \ldots, a_n) \otimes K(a_{n+1}) \cong K(a_1, \ldots, a_{n+1})$$

Proof. Let $T = K(a_1, \ldots, a_n) \otimes K(a_{n+1})$. Write $F = A^n$ and let $\{x_1, \ldots, x_n\}$ be the canonical basis. Let $G = A$ with canonical basis $\{x_{n+1}\}$. Then $T_0 = \wedge^0 F \otimes \wedge^0 G \cong A \otimes A \cong A$ and

$$T_1 = (\wedge^0 F \otimes \wedge^1 G) \oplus (\wedge^1 F \otimes \wedge^0 G) \cong \wedge^1 G \oplus \wedge^1 F \cong A^{n+1}$$

For $p \geq 2$ we have

$$T_p = \bigoplus_{i+j=p} \wedge^i F \otimes \wedge^j G \cong (\wedge^{p-1} F \otimes \wedge^1 G) \oplus (\wedge^p F \otimes \wedge^0 G) \cong \wedge^{p-1} F \oplus \wedge^p F$$

So for $p > n+1$ we have $T_p = 0$, and for $p \leq n+1$ the A-module T_p is free of rank $\binom{n+1}{p}$. So at least the modules T_p are free of the same rank as $K_p(a_1, \ldots, a_{n+1})$. Let $H = A^{n+1}$ have canonical basis $e_1, \dots, e_{n+1}$. The isomorphism $\wedge^0 H \cong T_0$ sends 1 to $1 \otimes 1$. The isomorphism $\wedge^1 H \cong T_1$ sends $e_1, \dots, e_n$ to $x_i \otimes 1$ and e_{n+1} to $1 \otimes 1$. For $p \geq 2$ the action of isomorphism $\wedge^p H \cong T_p$ on a basis element $e_{i_1} \wedge \dots \wedge e_{i_p}$ is described in two cases: if $i_p \leq n$ then use the basis element $(x_{i_1} \wedge \dots \wedge x_{i_p}) \otimes 1$ of $\wedge^p F \otimes \wedge^0 G$, and otherwise if $i_p = n+1$ use the basis element $(x_{i_1} \wedge \dots \wedge x_{i_{p-1}}) \otimes 1$ of $\wedge^{p-1} F \otimes \wedge^1 G$. One checks that these isomorphisms are compatible with the differentials. □

For any $a \in A$ we have an exact sequence of complexes

$$0 \longrightarrow A \longrightarrow K(a) \longrightarrow A[-1] \longrightarrow 0$$

Let C be any complex. Tensoring with C and using the natural isomorphisms $C \otimes A \cong C$ and $C \otimes A[-1] \cong C[-1]$ we have an exact sequence

$$0 \longrightarrow C \longrightarrow C(a) \longrightarrow C[-1] \longrightarrow 0$$

For $p \in \mathbb{Z}$ we have $H_p(C[-1]) = H_{p-1}(C)$, so the long exact homology sequence is

$$\cdots \longrightarrow H_{p+1}(C) \longrightarrow H_{p+1}(C(a)) \longrightarrow H_p(C) \xrightarrow{\delta_p} H_p(C) \longrightarrow \cdots$$

$$\cdots \xrightarrow{\delta_1} H_1(C) \longrightarrow H_1(C(a)) \longrightarrow H_0(C) \xrightarrow{\delta_0} H_0(C) \longrightarrow H_0(C(a)) \longrightarrow 0$$

It is not difficult to check that the connecting morphism δ_p is multiplication by $(-1)^p a$. Therefore

Lemma 137. *If C is a complex with $H_p(C) = 0$ for $p > 0$ then $H_p(C(a)) = 0$ for $p > 1$ and there is an exact sequence*

$$0 \longrightarrow H_1(C(a)) \longrightarrow H_0(C) \xrightarrow{a} H_0(C) \longrightarrow H_0(C(a)) \longrightarrow 0$$

If a is $H_0(C)$-regular, then we have $H_p(C(a)) = 0$ for all $p > 0$ and $H_0(C(a)) \cong H_0(C)/aH_0(C)$.

Theorem 138. *Let A be a ring, M an A-module and $a_1, \dots, a_n$ an M-regular sequence in A. Then we have*

$$H_p(K(a_1, \dots, a_n, M)) = 0 \quad (p > 0)$$
$$H_0(K(a_1, \dots, a_n, M)) \cong M/(a_1, \dots, a_n)M$$

Proof. The last piece of Koszul complex $K(a_1, \dots, a_n, M)$ is isomorphic to

$$\cdots \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

where the last map is $(m_1, \dots, m_n) \mapsto (a_1 m_1, \dots, a_n m_n)$. So clearly there is an isomorphism of A-modules $H_0(K(a_1, \dots, a_n, M)) \cong M/(a_1, \dots, a_n)M$. We prove the other claim by induction on n, having already proven the case $n = 1$ in Lemma 137. Let C be the complex $K(a_1, \dots, a_{n-1}, M)$. Then $H_0(C) \cong M/(a_1, \dots, a_{n-1})M$ so that a_n is $H_0(C)$-regular. By the inductive hypothesis $H_p(C) = 0$ for $p > 0$ and therefore by Lemma 137, $H_p(C \otimes K(a_n)) = 0$ for $p > 0$. But by Lemma 136 and Proposition 133 there is an isomorphism $C \otimes K(a_n) \cong K(a_1, \dots, a_n, M)$, which completes the proof. □

Remark 5. In other words, for an M-regular sequence $a_1, \dots, a_n$ the corresponding Koszul complex $K(a_1, \dots, a_n, M)$ gives a canonical resolution of the A-module $M/(a_1, \dots, a_n)M$. That is, the following sequence is exact

$$\cdots \longrightarrow M \otimes \wedge^2 F \longrightarrow M \otimes \wedge^1 F \longrightarrow M \otimes \wedge^0 F \longrightarrow M/(a_1, \dots, a_n)M \longrightarrow 0$$

Taking $M = A$ we see that the Koszul complex $K(a_1, \dots, a_n)$ gives a free resolution of the A-module $A/(a_1, \dots, a_n)$. That is, the following sequence is exact

$$0 \longrightarrow \wedge^n F \longrightarrow \cdots \longrightarrow \wedge^2 F \longrightarrow \wedge^1 F \longrightarrow \wedge^0 F \longrightarrow A/(a_1, \dots, a_n) \longrightarrow 0 \tag{5}$$

In particular we observe that $proj.dim_A(A/(a_1, \dots, a_n)) \leq n$.

Lemma 139. *Let A be a ring and $a_1, \dots, a_n$ an A-regular sequence. Then for any A-module M there is a canonical isomorphism of A-modules $Ext_A^n(A/(a_1, \dots, a_n), M) \cong M/(a_1, \dots, a_n)M$.*

Proof. We have already observed that (5) is a projective resolution of $A/(a_1, \dots, a_n)$. Taking $Hom_A(-, M)$ the end of the complex we are interested in is

$$\cdots \longrightarrow Hom_A(\wedge^{n-1}F, M) \longrightarrow Hom_A(\wedge^n F, M) \longrightarrow 0$$

Use the canonical bases to define isomorphisms $\wedge^{n-1}F \cong A^n$ and $\wedge^n F \cong A$. Then we have a commutative diagram

$$\begin{tikzcd} Hom_A(\wedge^{n-1}F, M) \arrow[r] \arrow[d, Rightarrow] & Hom_A(\wedge^n F, M) \arrow[d, Rightarrow] \\ M^n \arrow[r, "\psi"'] & M \end{tikzcd}$$

where $\psi(m_1, \dots, m_n) = \sum_{r=1}^n (-1)^{r-1} a_r m_r$. It is clear that $Im\psi = (a_1, \dots, a_n)M$, so we have an isomorphism of A-modules $Ext_A^n(A/(a_1, \dots, a_n), M) \cong M/(a_1, \dots, a_n)M$. $\square$

Definition 26. Let $(A, \mathfrak{m}, k)$ be a local ring and $u : M \longrightarrow N$ a morphism of A-modules. We say that u is *minimal* if $u \otimes 1 : M \otimes k \longrightarrow N \otimes k$ is an isomorphism. Clearly any isomorphism $M \cong N$ is minimal.

Lemma 140. *Let $(A, \mathfrak{m}, k)$ be a local ring. Then*

(i) Let $u : M \longrightarrow N$ be a morphism of finitely generated A-modules. Then u is minimal if and only if it is surjective and $Ker(u) \subseteq \mathfrak{m}M$.

(ii) If M is a finitely generated A-module then there is a minimal morphism $u : F \longrightarrow M$ with F finite free and $rank_A F = rank_k(M \otimes k)$.

Proof. (i) Suppose that u is minimal. Let N' be the image of M. Then since $M/\mathfrak{m}M \cong N/\mathfrak{m}N$ we have $N' + \mathfrak{m}N = N$ and therefore $N' = N$ by Nakayama, so u is surjective. It is clear that $Ker(u) \subseteq \mathfrak{m}M$. Conversely suppose that u is surjective and $Ker(u) \subseteq \mathfrak{m}M$. Since u is surjective it follows that $u(\mathfrak{m}M) = \mathfrak{m}N$. Therefore the morphism of A-modules $M \longrightarrow N \longrightarrow N/\mathfrak{m}N$ has kernel $\mathfrak{m}M$ and so $M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N$ is an isomorphism, as required. (ii) If $M = 0$ then this is trivial, since we can take $F = 0$. Otherwise let $m_1, \dots, m_n$ be a minimal basis of M and $u : A^n \longrightarrow M$ the corresponding morphism. This is clearly minimal. $\square$

Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated A-module. A free resolution

$$L : \cdots \longrightarrow L_i \xrightarrow{d_i} L_{i-1} \longrightarrow \cdots \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

is called a *minimal resolution* if $L_0 \longrightarrow M$ is minimal, and $L_i \longrightarrow Ker(d_{i-1})$ is minimal for each $i \geq 1$. Since $L_{i+1} \longrightarrow L_i \longrightarrow Ker(d_{i-1}) = 0$ for all $i \geq 1$ it follows that in the complex of A-modules $L \otimes k$

$$\cdots \longrightarrow L_i \otimes k \longrightarrow L_{i-1} \otimes k \longrightarrow \cdots \longrightarrow L_0 \otimes k \longrightarrow 0$$

the differentials are all zero. Therefore we have $Tor_i^A(M, k) \cong L_i \otimes k$ as k-modules for all $i \geq 0$. Since M, k are finitely generated, for $i \geq 0$ the A-modules $Tor_i^A(M, k)$ and $L_i \otimes k$ are finitely generated. Hence $L_i \otimes k$ is a finitely generated free k-module, which shows that L_i is a finitely generated A-module.

Proposition 141. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated A-module. Then a minimal free resolution of M exists, and is unique up to a (non-canonical) isomorphism.*

Proof. By Lemma 140 (ii) we can find a minimal morphism $d_0 : L_0 \longrightarrow M$ with L_0 finite free of rank $rank_k(M \otimes k)$. Let $K \longrightarrow L_0$ be the kernel of d_0. Find a minimal morphism $L_1 \longrightarrow K$ with L_1 finite free, and so on. This defines a minimal free resolution of M. To prove the uniqueness, let $\varepsilon : L \longrightarrow M$ and $\varepsilon' : L' \longrightarrow M$ be two minimal free resolutions of M. We can lift the identity 1_M to a morphism of chain complexes $\varphi : L \longrightarrow L'$, so we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & L_1 \arrow[r] \arrow[d, "\varphi_1"'] & L_0 \arrow[r, "\varepsilon"] \arrow[d, "\varphi_0"'] & M \arrow[d, Rightarrow, no head] \\
\cdots \arrow[r] & L'_1 \arrow[r] & L'_0 \arrow[r, "\varepsilon'"'] & M
\end{tikzcd}$$

Since $\varepsilon, \varepsilon'$ are minimal, the map $\varphi_0 \otimes 1 : L_0 \otimes k \longrightarrow L'_0 \otimes k$ is an isomorphism of k-modules. In particular we have

$$rank_A L_0 = rank_k(L_0 \otimes k) = rank_k(L'_0 \otimes k) = rank_A L'_0$$

So L_0, L'_0 are free of the same finite rank. We claim that φ_0 is an isomorphism. This is trivial if $L_0 = L'_0 = 0$, so assume they are both nonzero. Then φ_0 is described by a square matrix $T \in M_n(A)$. If you take residues you get the matrix $T' \in M_n(k)$ of $\varphi_0 \otimes 1$, which has nonzero determinant since it is an isomorphism. But it is clear that $det(T) + \mathfrak{m} = det(T')$, so $det(T) \notin \mathfrak{m}$. Therefore φ_0 itself is an isomorphism.

Since φ_0 is an isomorphism, so the induced morphism on the kernels $Ker(\varepsilon) \longrightarrow Ker(\varepsilon')$, and we can repeat the same argument to see that φ_1 is an isomorphism, and similarly to show that all the φ_i are isomorphisms. ☐

Lemma 142. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and $u : F \longrightarrow G$ a morphism of finitely generated free A-modules. Then u is minimal if and only if it is an isomorphism.*

Definition 27. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated A-module. Choose a minimal free resolution of M. Then for $i \geq 0$ the integer $b_i = rank_A L_i \geq 0$ is called the i-th *Betti number* of M. It is independent of the chosen resolution, and moreover $rank_k Tor_i^A(M, k) = b_i$.

Example 7. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and let M be a finitely generated A-module. Then

(i) Proposition 141 shows that $b_0 = rank_k(M \otimes k)$.

(ii) If $M = 0$ then the zero complex is a minimal free resolution of M, so $b_i = 0$ for $i \geq 0$.

(iii) If M is flat then $Tor_i^A(M, k) = 0$ for all $i \geq 1$, so $b_i = 0$ for $i \geq 1$. In particular this is true if M is free or projective.

(iv) If M is free of finite rank $s \geq 1$ then $M \otimes k$ is a free k-module of rank s, so $b_0 = s$.

Lemma 143. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated A-module. Suppose that we have two complexes L, F together with morphisms $\varepsilon, \varepsilon'$ such that the following sequences are exact in the last two nonzero positions*

$$L : \cdots \longrightarrow L_i \xrightarrow{d_i} L_{i-1} \longrightarrow \cdots \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

$$F : \cdots \longrightarrow F_i \xrightarrow{d'_i} F_{i-1} \longrightarrow \cdots \xrightarrow{d'_1} F_0 \xrightarrow{\varepsilon'} M \longrightarrow 0$$

Assume the following

(i) L is a minimal free resolution of M;

(ii) Each F_i is a finitely generated free A-module;

(iii) $\varepsilon' \otimes 1 : F_0 \otimes k \longrightarrow M \otimes k$ *is injective;*

(iv) *For each* $i \geq 0$, $d'_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$ *and the induced morphism* $F_{i+1}/\mathfrak{m}F_{i+1} \longrightarrow \mathfrak{m}F_i/\mathfrak{m}^2F_i$ *is an injection.*

Then there exists a morphism of complexes $f : F \longrightarrow L$ *lifting the identity* 1_M *such that for* f_i *maps* F_i *isomorphically onto a direct summand of* L_i*. Consequently we have*

$$rank_A F_i \leq rank_A L_i = rank_k Tor_i^A(M, k)$$

Proof. Both L, F are positive chain complexes, with F projective and L acyclic, so by our Derived Functor notes there is a morphism $f : F \longrightarrow L$ of chain complexes giving a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_0 \arrow[r, "\varepsilon'"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, Rightarrow, no head] & 0 \\
\cdots \arrow[r] & L_0 \arrow[r, "\varepsilon"'] & M \arrow[r] & 0
\end{tikzcd}$$

We have to prove that for each $i \geq 0$ the morphism $f_i : F_i \longrightarrow L_i$ is a coretraction.We claim that f_i is a coretraction iff. $f_i \otimes 1 : F_i \otimes k \longrightarrow L_i \otimes k$ is an injective morphism of k-modules. One implication is clear. So assume that $f_i \otimes 1$ is injective. The claim is trivial if either of F_i, L_i are zero, so assume they are both of nonzero finite rank. Pick bases for F_i, L_i (which are obviously minimal bases), and use the fact that $f_i \otimes 1$ is a coretraction to define a morphism $\varphi : L_i \longrightarrow F_i$ such that $(\varphi f_i) \otimes 1 : F_i \otimes k \longrightarrow F_i \otimes k$ is the identity. By Lemma 142 it follows that φf_i is an isomorphism, and therefore clearly f_i is a coretraction.

We prove by induction that $f_i \otimes 1$ is injective for all $i \geq 0$. By assumptions $(i), (iii)$ it is clear that $f_0 \otimes 1$ is injective. We have the following commutative diagram

$$\begin{tikzcd}
F_1 \arrow[rr, "d'_1"] \arrow[dr, "\beta"'] \arrow[ddd, "f_1"'] & & F_0 \arrow[r, "\varepsilon'"] \arrow[ddd, "f_0"'] & M \arrow[ddd, Rightarrow, no head] \\
& Ker\varepsilon' \arrow[ur] \arrow[d, "\alpha"'] & & \\
& Ker\varepsilon \arrow[dr] & & \\
L_1 \arrow[rr, "d_1"'] \arrow[ur, "\gamma"] & & L_0 \arrow[r, "\varepsilon"'] & M
\end{tikzcd}$$

By assumption $\gamma \otimes 1$ is an isomorphism. So to show $f_1 \otimes 1$ is injective, it suffices to show that $\alpha \otimes 1, \beta \otimes 1$ are injective, or equivalently that $\alpha^{-1}(\mathfrak{m}Ker\varepsilon) = \mathfrak{m}Ker\varepsilon'$ and $\beta^{-1}(\mathfrak{m}Ker\varepsilon') = \mathfrak{m}F_1$. Suppose that $a \in F_1$ and $d'_1(a) \in \mathfrak{m}Ker\varepsilon'$. Since $\varepsilon' \otimes 1$ is injective, we have $\mathfrak{m}Ker\varepsilon' \subseteq \mathfrak{m}^2F_0$. Hence $d'_1(a) \in \mathfrak{m}^2F_0$ and therefore by (iv) $a \in \mathfrak{m}F_1$, as required.

Now suppose that $a \in Ker\varepsilon'$ and $f_0(a) \in \mathfrak{m}Ker\varepsilon$. Let g be such that $gf_0 = 1$. Then $f_0(a) \in \mathfrak{m}^2L_0$ and therefore $a = gf_0(a) \in g(\mathfrak{m}^2L_0) \subseteq \mathfrak{m}^2F_0$. Let $b \in F_1$ be such that $d'_1(b) = a \in \mathfrak{m}^2F_0$. Then (iv) implies that $b \in \mathfrak{m}F_1$ and therefore $a = \beta(b) \in \mathfrak{m}Ker\varepsilon'$, as required. This shows that $f_1 \otimes 1$ is injective.

Suppose that $f_i \otimes 1$ is injective for some $i \geq 1$. Then we show $f_{i+1} \otimes 1$ is injective using a similar setup. We replace $Ker\varepsilon'$ by Imd'_{i+1} (in the case $i = 0$ they are equal) and use (iv) to show that $Kerd'_i \subseteq \mathfrak{m}F_i$ and (i) to show that $Kerd_i \subseteq \mathfrak{m}L_i$. The proof that $\beta \otimes 1$ is injective is straightforward. For $\alpha \otimes 1$, let $a \in Imd'_{i+1}$ be such that $f_i(a) \in \mathfrak{m}Kerd_i$. As before we find that $f_i(a) \in \mathfrak{m}^2L_i$, and hence $a = gf_i(a) \in \mathfrak{m}^2F_i$. Let $b \in F_{i+1}$ be such that $a = d'_{i+1}(b)$. Then by (iv), $b \in \mathfrak{m}F_{i+1}$ and therefore $a \in \mathfrak{m}Imd'_{i+1}$, as required. This proves that f_i is a coretraction for $i \geq 0$, and the rank claim follows from the fact that $rank_A F_i = rank_k(F_i \otimes k) \leq rank_k(L_i \otimes k)$. $\square$

Lemma 144. *Let A be a ring with maximal ideal $\mathfrak{m}$. If $s \notin \mathfrak{m}$ and $a \in A$, then $sa \in \mathfrak{m}^k$ implies $a \in \mathfrak{m}^k$ for any $k \geq 1$.*

Theorem 145. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and let $s = rank_k \mathfrak{m}/\mathfrak{m}^2$. Then we have*

$$rank_k Tor_i^A(k,k) \geq \binom{s}{i} \qquad 0 \leq i \leq s$$

Here $rank_k Tor_i^A(k,k)$ is the i-th Betti number of the A-module k.

Proof. We have $Tor_0^A(k,k) \cong k$ as k-modules, so $rank_k Tor_0^A(k,k) = rank_k k = 1$, which takes care of the case $s = 0$. So assume that $s \geq 1$ and let $\{a_1, \ldots, a_s\}$ be a minimal basis of $\mathfrak{m}$, with associated Koszul complex $F = K(a_1, \ldots, a_s)$. The canonical morphism $\varepsilon' : F_0 \cong A \longrightarrow k$ gives a complex exact in the last two nonzero places

$$\cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow k \longrightarrow 0$$

We claim this complex satisfies the conditions of Lemma 143. It clearly satisfies (ii) and (iii). It only remains to check condition (iii). By the definition of $d_{p+1} : F_{p+1} \longrightarrow F_p$ it is clear that $d_{p+1}(F_{p+1}) \subseteq \mathfrak{m}F_p$ for $p \geq 0$. We also have to show that $d_{p+1}^{-1}(\mathfrak{m}^2 F_p) \subseteq \mathfrak{m}F_{p+1}$. . This is trivial if $p + 1 > s$, and also if $p = 0$ since $\{a_1, \ldots, a_s\}$ is a minimal basis. So assume $0 < p \leq s - 1$. Assume that

$$\begin{aligned} & d_{p+1}\left(\sum_{i_1 < \cdots < i_{p+1}} m_{i_1 \cdots i_{p+1}}(x_{i_1} \wedge \cdots \wedge x_{i_{p+1}})\right) \\ &= \sum_{i_1 < \cdots < i_{p+1}} \sum_{r=1}^{p} (-1)^{r-1} a_{i_r} m_{i_1 \cdots i_{p+1}}(x_{i_1} \wedge \cdots \wedge \widehat{x}_{i_r} \wedge \cdots \wedge x_{i_{p+1}}) \in \mathfrak{m}^2 F_p \end{aligned}$$

Then collecting terms, we obtain a number of equations of the form $\sum (-1)^{e_t} a_t m_t \in \mathfrak{m}^2$ where a_t is one of the a_{i_r} and m_t one of the $m_{i_1 \cdots i_{p+1}}$. Since the residues of the a_i give a basis of $\mathfrak{m}/\mathfrak{m}^2$ over k, it follows that $m_t \in \mathfrak{m}$, which completes the proof that F satisfies all the conditions of Lemma 143. Choosing any minimal free resolution L of M, and applying Lemma 143 we see that for $0 \leq i \leq s$

$$\binom{s}{i} = rank_A F_i \leq rank_k Tor_i^A(k,k)$$

as required. □

Theorem 146 (Serre). *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Then A is regular if and only if the global dimension of A is finite.*

Proof. We have already proved one part in Theorem 128. So suppose that $gl.dim(A) < \infty$. Then $Tor_s^A(k,k) \neq 0$ by Theorem 145, hence $gl.dim(A) \geq rank_k \mathfrak{m}/\mathfrak{m}^2$ since by Proposition 122 we have $tor.dim(A) = gl.dim(A)$. On the other hand, it follows from Theorem 125 that $proj.dim(k) = gl.dim(A) < \infty$, so by Theorem 131 we have $gl.dim(A) = proj.dim(k) = depth(A)$. Therefore we get

$$dim(A) \leq rank_k \mathfrak{m}/\mathfrak{m}^2 \leq gl.dim(A) = depth(A) \leq dim(A)$$

Whence $dim(A) = rank_k \mathfrak{m}/\mathfrak{m}^2$, and A is regular. □

Corollary 147. *If A is a regular local ring then $A_\mathfrak{p}$ is regular for any $\mathfrak{p} \in Spec(A)$.*

Proof. Let M be a nonzero $A_\mathfrak{p}$-module. Then considering M as an A-module, there is an exact sequence of finite length $n \leq gl.dim(A)$ with all P_i projective

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Since $A_{\mathfrak{p}}$ is flat the following sequence is also exact

$$0 \longrightarrow (P_n)_{\mathfrak{p}} \longrightarrow \cdots \longrightarrow (P_0)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow 0$$

The modules $(P_i)_{\mathfrak{p}}$ are projective $A_{\mathfrak{p}}$-modules, and $M \cong M_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$-modules, so it follows that $gl.dim(A_{\mathfrak{p}}) \leq gl.dim(A) < \infty$. □

Definition 28. A ring A is called a *regular ring* if $A_{\mathfrak{p}}$ is a regular local ring for every prime ideal $\mathfrak{p}$ of A. Note that A is not required to be noetherian. Regularity is stable under ring isomorphism. A noetherian local ring A is regular in this sense if and only if it is regular in the normal sense.

It follows from Theorem 108 that any regular ring is normal, and a noetherian regular ring is Cohen-Macaulay. It follows from Lemma 8 and Theorem 90 that a regular ring is catenary.

Lemma 148. *A ring A is regular if and only if $A_{\mathfrak{m}}$ is regular for all maximal ideals $\mathfrak{m}$.*

Proof. One implication is clear. For the other, given a prime ideal $\mathfrak{p}$, find a maximal ideal $\mathfrak{m}$ with $\mathfrak{p} \subseteq \mathfrak{m}$. Then $A_{\mathfrak{p}} \cong (A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}}$, so $A_{\mathfrak{p}}$ is a regular local ring. □

Lemma 149. *If A is a regular ring and $S \subseteq A$ is multiplicatively closed, then $S^{-1}A$ is a regular ring.*

Lemma 150. *If A is a regular ring, then so is $A[x_1, \ldots, x_n]$. In particular $k[x_1, \ldots, x_n]$ is a regular ring for any field k.*

Proof. This follows immediately from Theorem 118. □

Theorem 151. *Let A be a regular local ring which is a subring of a domain B, and suppose that B is a finitely generated A-module. Then B is flat (equivalently free) over A if and only if B is Cohen-Macaulay. In particular, if B is regular then it is a free A-module.*

Proof. Since B is a finitely generated A-module it is integral over A, and so by Lemma 91 if B is flat it is Cohen-Macaulay. Conversely, suppose that B is Cohen-Macaulay. If $dim(A) = 0$ then A is a field so B is trivially flat, so throughout we may assume $dim(A) \geq 1$. Since A is normal the going-down theorem holds between A and B by Theorem 42, so by Theorem 55 (3) for any proper ideal I of A, IB is proper and $ht.I = ht.IB$. We claim that $depth_A(A) = depth_A(B)$. Notice that $depth_A(B)$ is finite, since otherwise $\mathfrak{m} \in Ass_A(B)$ and hence $dim(A) = 0$.

Firstly we prove the inequality $\leq$. Since A is regular it is Cohen-Macaulay, so $depth_A(A) = dim(A)$. Set $s = dim(A)$ and let $\{a_1, \ldots, a_s\}$ be a regular system of parameters. Then $ht.(a_1, \ldots, a_i)A = i$ and therefore $ht.(a_1, \ldots, a_i)B = i$ for all $1 \leq i \leq s$ by Theorem 108. It follows from Corollary 97 that $a_1, \ldots, a_s$ is a B-regular sequence, and therefore $depth_A(B) \geq s$.

To prove the reverse inequality, set $d = depth_A(B)$ and let $a_1, \ldots, a_d \in \mathfrak{m}$ be a maximal B-regular sequence. Then as elements of B the sequence $a_1, \ldots, a_d$ is B-regular, so by Lemma 74 we have $ht.(a_1, \ldots, a_d) = d$. But $(a_1, \ldots, a_d) \subseteq \mathfrak{m}B$ and $ht.\mathfrak{m}B = ht.\mathfrak{m} = dim(A)$, so $d \leq dim(A)$, as required.

Since $gl.dim(A) < \infty$ we have $proj.dim_A(B) < \infty$, so we can apply Theorem 131 to see that $proj.dim_A(B) + depth_A(B) = depth_A(A)$, so $proj.dim_A(B) = 0$ and therefore B is projective. Since A is local and B finitely generated, projective $\Leftrightarrow$ free $\Leftrightarrow$ flat, so the proof is complete. □

6.4 Unique Factorisation

Recall that if A is a ring, two elements $p, q \in A$ are said to be *associates* if $p = uq$ for some unit $u \in A$. This is an equivalence relation on the elements of A.

Definition 29. Let A be an integral domain. An element of A is *irreducible* if it is a nonzero nonunit which cannot be written as the product of two nonunits. An element $p \in A$ is *prime* if it is a nonzero nonunit with the property that if $p|ab$ then $p|a$ or $p|b$. Equivalently p is prime iff. (p) is a nonzero prime ideal. We say A is a *unique factorisation domain* if every nonzero nonunit $a \in A$ can be written essentially uniquely as $a = up_1 \cdots p_r$ where u is a unit and each p_i is irreducible.

Essentially uniquely means that if $a = vq_1 \cdots q_s$ where v is a unit and the q_j irreducible, then $r = s$ and after reordering (if necessary) q_i is an associate of p_i. The property of being a UFD is stable under ring isomorphism.

Theorem 152. *A noetherian domain A is a UFD if and only if every prime ideal of height 1 is principal.*

Lemma 153. *Let A be a noetherian domain and let $x \in A$ be prime. Then A is a UFD if and only if A_x is.*

Proof. By assumption (x) is a prime ideal of height 1. If $\mathfrak{p}$ is a prime ideal of height 1 then either $x \in \mathfrak{p}$, in which case $\mathfrak{p} = (x)$, or $x \notin \mathfrak{p}$, and these primes are in bijection with the primes of A_x. So using Theorem 152 it is clear that if A is a UFD so is A_x. Suppose that A_x is a UFD and let $\mathfrak{p}$ be a prime ideal of height 1 in A. We can assume that $x \notin \mathfrak{p}$. Let $a \in \mathfrak{p}$ be such that $\mathfrak{p}A_x = a/1A_x$. By [AM69] Corollary 10.18 we have $\cap_i(x^i) = 0$, so if $x|a$ there is a largest integer $n \geq 1$ with $x^n|a$. Write $a = cx^n$. Since $x \notin \mathfrak{p}$ we have $c \in \mathfrak{p}$, so by replacing a with c we can assume $\mathfrak{p}A_x = a/1A_x$ with $a \notin (x)$. Then it is clear that $\mathfrak{p} = (a)$, as required. $\square$

Definition 30. Let R be an integral domain. If M is a torsion-free R-module then the *rank* of M is the maximum number of linearly independent elements in M, $rank(M) \in \{0, 1, \ldots, \infty\}$.

Proposition 154. *Let R be an integral domain and M a torsion-free R-module. If $T \subseteq R$ is multiplicatively closed, then $T^{-1}M$ is a torsion-free $T^{-1}R$-module and $rank_{T^{-1}R}(T^{-1}M) = rank_R(M)$.*

Proof. If $rank(M) = 0$ this is trivial, so assume M is nonzero. It is clear that $T^{-1}M$ is torsion-free. If $rank_R(M) = r$ and $x_1, \ldots, x_r \in M$ are linearly independent, then $x_1/1, \ldots, x_r/1 \in T^{-1}M$ are linearly independent over $T^{-1}R$. Similarly if $x_1/s_1, \ldots, x_n/s_n \in T^{-1}M$ are linearly independent in $T^{-1}M$, then $x_1, \ldots, x_n$ are linearly independent in M. So the result is clear. $\square$

Corollary 155. *Let R be an integral domain with quotient field K. Then*

(i) If M is a torsion-free R-module, $rank_R(M) = dim_K(M \otimes K)$.

(ii) If M, N are two torsion-free R-modules of finite rank, then $rank_R(M \oplus N) = rank_R(M) + rank_R(N)$.

In particular if M is a free R-module then the rank just defined is equal to the normal free rank, and we can write $rank(M)$ without confusion.

Let R be a noetherian domain and suppose $a_1, \ldots, a_n \in R$ are linearly independent elements which do not generate R. Then $a_1, \ldots, a_r$ is an R-regular sequence, so by Lemma 74 the ideals $(a_1, \ldots, a_i)$ have height i for $1 \leq i \leq n$. So it follows immediately that

Lemma 156. *Let R be a noetherian domain and I an ideal. Then $rank(I) \leq ht.I$.*

Lemma 157. *Let R be a domain and M a finitely generated projective R-module of rank 1. Then $\wedge^i M = 0$ for $i > 1$.*

Proof. By localisation. If $\mathfrak{p}$ is a prime ideal then $M_\mathfrak{p}$ is a finitely generated projective module over the local ring $R_\mathfrak{p}$, so $M_\mathfrak{p}$ is free of rank 1 and $M_\mathfrak{p} \cong R_\mathfrak{p}$. Hence for $i > 1$

$$(\wedge^i M)_\mathfrak{p} \cong \wedge^i M_\mathfrak{p} \cong \wedge^i R_\mathfrak{q} = 0$$

as required. $\square$

Theorem 158 (Auslander-Buchsbaum). *A regular local ring $(A, \mathfrak{m})$ is UFD.*

Proof. We use induction on $dimA$. If $dimA = 0$ then A is a field, and if $dimA = 1$ then A is a principal ideal domain. Suppose $dimA > 1$ and let $a_1, \ldots, a_d$ be a regular system of parameters. Then $x = a_1$ is prime by Theorem 108, so it suffices by Lemma 153 to show that A_x is UFD. Let $\mathfrak{q}$ be a prime ideal of height 1 in A_x and put $\mathfrak{p} = \mathfrak{q} \cap A$, so $\mathfrak{q} = \mathfrak{p}A_x$. By Theorem 128, $gl.dim.A = dimA < \infty$, so we can produce an exact sequence of A-modules with all F_i finitely generated free

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathfrak{p} \longrightarrow 0 \tag{6}$$

Maximal ideals of A_x correspond to primes of A maximal among those not containing x. These primes must all be properly contained in $\mathfrak{m}$, so if $\mathfrak{P}A_x$ is a maximal ideal then $ht.\mathfrak{P} < dimA$. Therefore $(A_x)_{\mathfrak{P}A_x} \cong A_{\mathfrak{P}}$ is UFD by the inductive assumption, and so $\mathfrak{q}(A_x)_{\mathfrak{n}}$ is either principal or zero for every maximal $\mathfrak{n}$ of A_x. Then by Lemma 124 we have $proj.dim_{A_x}(\mathfrak{q}) = 0$ and therefore $\mathfrak{q}$ is projective. Localising (6) with respect to $S = \{1, x, x^2, \ldots\}$ we see that the following sequence of A_x-modules is exact

$$0 \longrightarrow F'_n \longrightarrow F'_{n-1} \longrightarrow \cdots \longrightarrow F'_0 \longrightarrow \mathfrak{q} \longrightarrow 0 \tag{7}$$

where $F'_i = F_i \otimes A_x$ are finitely generated and free over A_x. If we decompose (7) into short exact sequences

$$\begin{gathered}
0 \longrightarrow K'_0 \longrightarrow F'_0 \longrightarrow \mathfrak{q} \longrightarrow 0 \\
0 \longrightarrow K'_1 \longrightarrow F'_1 \longrightarrow K'_0 \longrightarrow 0 \\
\vdots \\
0 \longrightarrow F'_n \longrightarrow F'_{n-1} \longrightarrow K'_{n-2} \longrightarrow 0
\end{gathered} \tag{8}$$

then the first sequence splits since $\mathfrak{q}$ is projective. Hence K'_0 must be projective, and in this way we show that all the sequences split, and all the K'_i are projective. It follows that

$$\bigoplus_{i \text{ even}} F'_i \cong \bigoplus_{i \text{ odd}} F'_i \oplus \mathfrak{q}$$

Thus, we have finite free A_x-modules F, G such that $F \cong G \oplus \mathfrak{q}$. Since A_x is a noetherian domain and $\mathfrak{q}$ a nonzero ideal of height 1, it follows from Lemma 156 that $rank(\mathfrak{q}) = 1$. If $rank(G) = r$ then $rank(F) = r + 1$.

So to show $\mathfrak{q}$ is principal and complete the proof, it suffices to show that $\mathfrak{q}$ is free. But by our notes on Tensor, Symmetric and Exterior algebras we have

$$A_x \cong \bigwedge^{r+1} F \cong \bigwedge^{r+1} (G \oplus \mathfrak{q}) \cong \bigoplus_{i+j=r+1} (\wedge^i G) \otimes (\wedge^j \mathfrak{q}) \cong (\wedge^{r+1} G \otimes \wedge^0 \mathfrak{q}) \oplus (\wedge^r G \otimes \wedge^1 \mathfrak{q}) \cong \mathfrak{q}$$

Since $\wedge^{r+1} G = 0, \wedge^r G \cong A_x$ and $\wedge^i \mathfrak{q} = 0$ for $i > 1$ by Lemma 157. ☐

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