Matsumura: Commutative Algebra Part 2

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October 5, 2006

This note closely follows Matsumura's book [Mat80] on commutative algebra. Proofs are the ones given there, sometimes with slightly more detail. While Matsumura's treatment is very good, another useful reference for this material is EGA IV_1, which treats some of the topics in greater generality.

Contents

1 Extension of a Ring by a Module

Let C be a ring and N an ideal of C with $N^2 = 0$. If $C' = C/N$ then the C-module N has a canonical C'-module structure. In a sense analogous with the notion of extension for modules, the data C, N is an "extension" of the ring C'.

Definition 1. Let C' be a ring and N a C'-module. An *extension of C' by N* is a triple (C, ε, i) consisting of a ring C, a surjective ring morphism $\varepsilon : C \longrightarrow C'$ and a morphism of C-modules $i : N \longrightarrow C$ such that $Ker(\varepsilon)$ is an ideal whose square is zero and the following sequence of C-modules is exact

$$0 \longrightarrow N \xrightarrow{i} C \xrightarrow{\varepsilon} C' \longrightarrow 0$$

Note that $Ker(\varepsilon)$ has a canonical C'-module structure, and i gives an isomorphism of C'-modules $N \cong Ker(\varepsilon)$. Two extensions $(C, \varepsilon, i), (C_1, \varepsilon_1, i_1)$ are said to be *isomorphic* if there exists a ring morphism $f : C \longrightarrow C_1$ such that $\varepsilon_1 f = \varepsilon$ and $fi = i_1$. That is, the following diagram of abelian groups commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & C & \xrightarrow{\varepsilon} & C' & \longrightarrow & 0 \\ & & \Downarrow & & \downarrow f & & \Downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow[i_1]{} & C_1 & \xrightarrow[\varepsilon_1]{} & C' & \longrightarrow & 0 \end{array}$$

The morphism f is necessarily an isomorphism. The relation of being isomorphic is an equivalence relation on the class of extensions of C' by N.

Given C' and N we can always construct an extension as follows: take the abelian group $C' \oplus N$ and define a multiplication by

$$(a, x)(b, y) = (ab, ay + bx)$$

This is a commutative ring with identity $(1, 0)$, which we denote by $C' * N$. The map $\varepsilon : C' * N \longrightarrow C'$ defined by $(a, x) \mapsto a$ is a surjective morphism of rings. There is also a ring morphism

$\varphi : C' \longrightarrow C' * N$ defined by $c \mapsto (c, 0)$, and clearly $\varphi\varepsilon = 1$. The map $i : N \longrightarrow C' * N$ defined by $n \mapsto (0, n)$ is a morphism of $C' * N$-modules, whose image is the ideal $0 \oplus N$ which clearly has square zero. So $(C' * N, \varepsilon, i)$ is an extension of C' by N. We call this the *trivial extension*.

An extension (C, ε, i) of C' by N is said to be *split* if there is a ring morphism $s : C' \longrightarrow C$ such that $\varepsilon s = 1_C$. Clearly any extension isomorphic to the trivial extension is split.

Definition 2. Let A be a ring, and (C, ε, i) be an extension of a ring C' by a C'-module N such that C, C' are A-algebras and ε is a morphism of A-algebras. Then (C, ε, i) is called an *extension of the A-algebra C' by N*. The extension is said to be A-trivial, or to split over A, if there exists a morphism of A-algebras $s : C' \longrightarrow C$ with $\varepsilon s = 1_{C'}$.

Suppose we are given an extension $E = (C, \varepsilon, i)$ of C' by M and a morphism $g : M \longrightarrow N$ of C'-modules. We claim there exists an extension $g_*(E) = (D, \varepsilon', i')$ of C' by N and a ring morphism $f : C \longrightarrow D$ making a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & C & \xrightarrow{\varepsilon} & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow[i']{} & D & \xrightarrow[\varepsilon']{} & C' & \longrightarrow & 0 \end{array}$$

The ring D is obtained as follows: we view the C'-module N as a C-module and form the trivial extension $C * N$. Then $M' = \{(i(x), -g(x)) \,|\, x \in M\}$ is an ideal of $C * N$ and we put $D = (C * N)/M'$. The map $\varepsilon' : D \longrightarrow C'$ defined by $(c, n) + M' \mapsto \varepsilon(c)$ is a well-defined surjective ring morphism, and $i' : N \longrightarrow D$ defined by $n \mapsto (0, n) + M'$ is a morphism of D-modules. With these definitions, it is clear that the bottom row of the above diagram is exact, and the diagram commutes with $f : C \longrightarrow D$ defined by $c \mapsto (c, 0) + M'$. $Ker(\varepsilon')$ is the ideal $(Ker(\varepsilon) \oplus N)/M'$ of D, which clearly has square zero. Hence $g_*(E) = (D, \varepsilon', i')$ is an extension of C' by N with the required property.

2 Derivations and Differentials

Definition 3. Let A be a ring and M an A-module. A *deriviation* of A into M is a morphism of abelian groups $D : A \longrightarrow M$ such that $D(ab) = a \cdot D(b) + b \cdot D(a)$. The set of all derivations of A into M is denoted by $Der(A, M)$. It is an A-module in the natural way.

For any derivation D, $Ker(d)$ is a subring of A (in particular $D(1) = 0$). If A is a field, then $Ker(d)$ is a subfield. Let k be a ring and A a k-algebra. Then derivations $A \longrightarrow M$ which vanish on k are called *derivations over k*. Every derivation is a derivation over $\mathbb{Z}$. The set of such derivations form an A-submodule of $Der(A, M)$ which we denote by $Der_k(A, M)$. We write $Der_k(A)$ for $Der_k(A, A)$. Notice that any derivation $A \longrightarrow M$ over k is k-linear.

If $\phi : M \longrightarrow N$ is a morphism of A-modules then composing with ϕ defines a morphism of A-modules $Der_k(A, M) \longrightarrow Der_k(A, N)$, so we have an additive covariant functor $Der_k(A, -) : A\mathbf{Mod} \longrightarrow A\mathbf{Mod}$. If $\psi : A \longrightarrow A'$ is a morphism of k-algebras and M an A'-module then composing with ψ defines a morphism of A-modules $Der_k(A', M) \longrightarrow Der_k(A, M)$.

If $\alpha : k \longrightarrow k'$ is a morphism of rings and A a k'-algebra then for any A-module M there is a trivial morphism of A-modules $Der_{k'}(A, M) \longrightarrow Der_k(A, M)$.

Remark 1. Let A be a ring, M an A-module and $D : A \longrightarrow M$ a derivation. We make some simple observations

- For $n > 1$ and $a \in A$ we have $D(a^n) = na^{n-1} \cdot D(a)$.
- Suppose A has prime characteristic p and let A^p denote the subring $\{a^p \,|\, a \in A\}$. Then D vanishes on A^p, since $D(a^p) = pa^{p-1}D(a) = 0$.
- If $u \in A$ is a unit then $D(u^{-1}) = -u^{-2} \cdot D(u)$.

Lemma 1. *Let A be a ring, M an A-module and $D : A \longrightarrow M$ a derivation. Then for $n \geq 2$ and $a_1, \dots, a_n \in A$ we have*

$$D(a_1 \cdots a_n) = \sum_{i=1}^{n} \left(\prod_{j \neq i} a_j \right) \cdot D(a_i)$$

Proof. The case $n = 2$ is trivial and the other cases follow by induction. □

Proposition 2. *Let k be a ring, A a k-algebra and M an A-module. Then there is a bijection between $Der_k(A, M)$ and morphisms of S-algebras $\varphi : A \longrightarrow A \oplus M$ with $\varepsilon\varphi = 1$.*

Proof. We make $A \oplus M$ into a ring as in Section 1, so $(a, m)(b, n) = (ab, a \cdot n + b \cdot m)$. There are canonical ring morphisms $A \longrightarrow A \oplus M, a \mapsto (a, 0)$ and $\varepsilon : A \oplus M \longrightarrow A, (a, m) \mapsto a$ and we make $A \oplus M$ a k-algebra using the former. Given a derivation $D : A \longrightarrow M$ over k we define $1 + D : A \longrightarrow A \oplus M, a \mapsto (a, D(a))$. It is easy to check this is a morphism of S-algebras with $\varepsilon(1 + D) = 1$. Conversely if φ is such a morphism of S-algebras, given $a \in A$ let $D(a) \in M$ be such that $\varphi(a) = (a, D(a))$. Then $D : A \longrightarrow M$ is a derivation over k. This defines the required bijection. □

Definition 4. Let k be a ring, A a k-algebra and set $B = A \otimes_k A$. Consider the morphisms of k-algebras $\varepsilon : B \longrightarrow A$ and $\lambda_1, \lambda_2 : A \longrightarrow B$ defined by $\varepsilon(a \otimes a') = aa'$ and $\lambda_1(a) = a \otimes 1, \lambda_2(a) = 1 \otimes a$. Once and for all, we make B into an A-algebra via λ_1. We denote the kernel of ε by $I_{A/k}$ or simply by I, and we put $\Omega_{A/k} = I/I^2$. The B-modules I, I^2 and $\Omega_{A/k}$ are also viewed as A-modules via $\lambda_1 : A \longrightarrow B$. The A-module $\Omega_{A/k}$ is called the *module of differentials* (or *of Kähler differentials*) of A over k.

We have $\varepsilon\lambda_1 = \varepsilon\lambda_2 = 1$ so if we denote the natural morphism $B \longrightarrow B/I^2$ by ν and if we put $d^* = \lambda_2 - \lambda_1$ then $d = \nu d^*$ is a derivation over k

$$\begin{aligned} d : A &\longrightarrow \Omega_{A/k} \\ b &\mapsto 1 \otimes b - b \otimes 1 + I^2 \end{aligned}$$

To see this is a derivation, note that since $(1 \otimes b - b \otimes 1)(1 \otimes a - a \otimes 1) \in I^2$ we have $1 \otimes ab + ab \otimes 1 = b \otimes a + a \otimes b \pmod{I^2}$ for any $a, b \in A$. This is called the *canonical derivation* and is denoted by $d_{A/k}$ if necessary. Any ring is canonically a $\mathbb{Z}$-algebra and we denote the A-module $\Omega_{A/\mathbb{Z}}$ by Ω_A.

Lemma 3. *Let k be a ring and A a k-algebra. Then there is a canonical isomorphism of A-modules $(A \otimes_k A)/I^2 \cong A \oplus \Omega_{A/k}$.*

Proof. Let B be the A-algebra $A \otimes_k A$. Then B is the internal direct sum of $\lambda_1(A)$ and I (it is clear that these A-submodules intersect trivially, and their sum is B since $x \otimes y = xy \otimes 1 + x(1 \otimes y - y \otimes 1)$). It follows that B/I^2 is the coproduct of the monomorphisms $A \longrightarrow B/I^2$ and $\Omega_{A/k} \longrightarrow B/I^2$. □

Definition 5. Let k be a ring and $\phi : A \longrightarrow A'$ a morphism of k-algebras. Then there is a canonical morphism of A-modules

$$\begin{aligned} \Omega_{\phi/k} : \Omega_{A/k} &\longrightarrow \Omega_{A'/k} \\ a \otimes b + I^2 &\mapsto \phi(a) \otimes \phi(b) + I'^2 \\ da &\mapsto d\phi(a) \end{aligned} \tag{1}$$

If $\psi : A' \longrightarrow A''$ is another morphism of k-algebras then $\Omega_{\psi/k}\Omega_{\phi/k} = \Omega_{\psi\phi/k}$ and $\Omega_{1/k} = 1$. In particular if $A \cong A'$ as k-algebras then $\Omega_{A/k} \cong \Omega_{A'/k}$.

Now let $\alpha : k \longrightarrow k'$ be a morphism of rings and A a k'-algebra. Then there is a canonical morphism of A-modules

$$\begin{aligned} \Omega_{A/\alpha} : \Omega_{A/k} &\longrightarrow \Omega_{A/k'} \\ a \otimes b + I^2 &\mapsto a \otimes b + I'^2 \\ d_{A/k}(a) &\mapsto d_{A/k'}(a) \end{aligned} \tag{2}$$

Lemma 4. *Let k be a ring and A a k-algebra. Then $\Omega_{A/k}$ is generated as an A-module by the set $\{d(a) \mid a \in A\}$. In particular $\Omega_{k/k} = 0$.*

Proof. It suffices to show that I is generated as an A-module by the set $\{d^*(a) \mid a \in A\}$. For any $x, y \in A$ we have $x \otimes y = xy \otimes 1 + x \cdot d^*(y)$ in B and therefore

$$\sum_i x_i \otimes y_i = (\sum_i x_i y_i) \otimes 1 + \sum_i x_i \cdot d^*(y_i)$$

If $\sum_i x_i \otimes y_i \in I$ then $\sum_i x_i y_i = 0$ and therefore any element of I can be written in the form $\sum_i x_i \cdot d^*(y_i)$ for $x_i, y_i \in A$, which is what we wanted to show. □

Proposition 5. *Let k be a ring and A a k-algebra. Then the pair $(\Omega_{A/k}, d)$ has the following universal property: if D is a derivation of A over k into an A-module M, then there is a unique morphism of A-modules $f : \Omega_{A/k} \longrightarrow M$ making the following diagram commute*

$$\begin{array}{ccc} A & \xrightarrow{D} & M \\ {\scriptstyle d}\downarrow & \nearrow_{f} & \\ \Omega_{A/k} & & \end{array}$$

Proof. It follows from Lemma 4 that if f exists it is unique. For the existence, set $B = A \otimes_k A$, take the trivial extension $A * M$ and define a morphism of A-algebras $\phi : B \longrightarrow A * M$ by $\phi(x \otimes y) = (xy, x \cdot D(y))$. Since $I^2 \subseteq Ker\phi$ there is an induced morphism of A-algebras $\phi' : B/I^2 \longrightarrow A * M$ which maps $1 \otimes y - y \otimes 1 + I^2$ to $(0, D(y))$. Therefore the restriction to $\Omega_{A/k}$ induces a morphism of A-modules with the required property. □

Corollary 6. *Let k be a ring and A a k-algebra. Then for an A-module M there is a canonical isomorphism of A-modules natural in M, A and k*

$$Der_k(A, M) \longrightarrow Hom_A(\Omega_{A/k}, M)$$

Proof. The bijection given by Proposition 5 is clearly natural in M, A and k. □

Definition 6. Let k be a ring and A a k-algebra. For $r \geq 1$ we denote by $\Omega^r_{A/k}$ the r-th exterior product $\bigwedge^r \Omega_{A/k}$ (TES,Section 3). There is a canonical isomorphism of A-modules $\Omega_{A/k} \cong \Omega^1_{A/k}$ and we freely identify these modules.

Lemma 7. *Let k be a ring and A a k-algebra. If A is generated by a nonempty set $\{x_i\}_{i \in I}$ as a k-algebra then $\Omega_{A/k}$ is generated by $\{dx_i\}_{i \in I}$ as an A-module.*

Example 1. Let k be a ring and set $A = k[x_1, \ldots, x_n]$ for $n \geq 1$. For $1 \leq i \leq n$ we define a k-derivation $\partial/\partial x_i : A \longrightarrow A$ in the usual way

$$\partial/\partial x_i(f)(\alpha) = (\alpha_i + 1) f(\alpha + e_i)$$
$$\partial/\partial x_i(\sum_\alpha f(\alpha) x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \sum_{\alpha, \alpha_i \geq 1} \alpha_i f(\alpha) x_1^{\alpha} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n}$$

Since $\partial/\partial x_i \in Der_k(A)$ there exists a morphism of A-modules $f_i : \Omega_{A/k} \longrightarrow A$ with $f_i(dx_j) = \delta_{ij}$. If k is a ring and K a k-algebra then for $a \in K$ we have $d(f(a)) = f'(a) \cdot da$ in $\Omega_{K/k}$ where $d : K \longrightarrow \Omega_{K/k}$ is canonical.

Lemma 8. *Let k be a ring and $A = k[x_1, \ldots, x_n]$ for $n \geq 1$. The differential $d : A \longrightarrow \Omega_{A/k}$ is defined by $df = \sum_{i=1}^n \partial f/\partial x_i \cdot dx_i$.*

Proof. One reduces easily to the case $f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for nonzero α, which is Lemma 1. □

Lemma 9. *Let k be a ring and $A = k[x_1, \ldots, x_n]$ for $n \geq 1$. Then $\Omega_{A/k}$ is a free A-module with basis $dx_1, \ldots, dx_n$.*

Proof. We already know that the dx_i generate $\Omega_{A/k}$ as an A-module. Suppose that $\sum_i a_i \cdot dx_i = 0$ with $a_i \in A$. Applying the morphism f_i of Example 1 we see that $a_i = 0$ for each i, as required. $\square$

Lemma 10. *Let k be a field and K a separable algebraic extension field of k. Then $\Omega_{K/k} = 0$.*

Proof. In fact, for any $\alpha \in K$ there is a polynomial $f(x) \in k[x]$ such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Since $d : K \longrightarrow \Omega_{K/k}$ is a derivation we have $0 = d(f(\alpha)) = f'(\alpha) \cdot d\alpha$, whence $d\alpha = 0$. Since $\Omega_{K/k}$ is generated by the $d\alpha$, this shows that $\Omega_{K/k} = 0$. $\square$

Definition 7. Suppose we have a commutative diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \uparrow & & \uparrow \\ k & \xrightarrow[\alpha]{} & k' \end{array} \tag{3}$$

Then there is a canonical morphism of A-modules

$$\begin{aligned} u_{A/k,A'/k'} : \Omega_{A/k} &\longrightarrow \Omega_{A'/k'} \\ a \otimes b + I^2 &\mapsto \phi(a) \otimes \phi(b) + I'^2 \\ d_{A/k}(a) &\mapsto d_{A'/k'}(\phi(a)) \end{aligned} \tag{4}$$

and a canonical morphism of A'-modules

$$\begin{aligned} v_{A/k,A'/k'} : \Omega_{A/k} \otimes_A A' &\longrightarrow \Omega_{A'/k'} \\ (a \otimes b + I^2) \otimes c &\mapsto c\phi(a) \otimes \phi(b) + I'^2 \\ d_{A/k}(a) \otimes c &\mapsto c \cdot d_{A'/k'}(\phi(a)) \end{aligned} \tag{5}$$

Proposition 11. *Let A, k' be k-algebras and let A' be the k-algebra $A \otimes_k k'$. Then the canonical morphism of A'-modules (5) $\Omega_{A/k} \otimes_A A' \longrightarrow \Omega_{A'/k'}$ is an isomorphism. More generally this is true whenever the diagram of rings (3) is a pushout.*

Proof. Set $B = A \otimes_k A, B' = A' \otimes_{k'} A'$, let I be the kernel of the canonical morphism of k-algebras $\varepsilon : A \otimes_k A \longrightarrow A$ and let I' be the kernel of $\varepsilon' : A' \otimes_{k'} A' \longrightarrow A'$. We have an isomorphism of abelian groups

$$\Omega_{A/k} \otimes_A A' = \Omega_{A/k} \otimes_A (A \otimes_k k') \cong (\Omega_{A/k} \otimes_A A) \otimes_k k' \cong \Omega_{A/k} \otimes_k k' \tag{6}$$

and an isomorphism of k-algebras

$$B' = (A \otimes_k k') \otimes_{k'} (A \otimes_k k') \cong A \otimes_k (k' \otimes_{k'} (A \otimes_k k')) \cong B \otimes_k k' \tag{7}$$

Since ε is surjective, the kernel of the morphism of k-algebras $\varepsilon \otimes 1 : B \otimes_k k' \longrightarrow A \otimes_k k'$ is $I \otimes_k k'$ (this denotes the image of $I \otimes_k k' \longrightarrow B \otimes_k k'$). We have a commutative diagram

$$\begin{array}{ccc} B \otimes_k k' & \Longrightarrow & B' \\ & {\scriptstyle \varepsilon\otimes 1}\searrow \quad \swarrow{\scriptstyle \varepsilon'} & \\ & A \otimes_k k' & \end{array}$$

Therefore (7) identifies the ideals $I \otimes_k A$ and I' and therefore also the ideals $I^2 \otimes_k A$ and I'^2. Taking quotients we have an isomorphism of k-algebras $B/I^2 \otimes_k k' \cong B'/I'^2$. But using Lemma 3 we have an isomorphism of abelian groups

$$B/I^2 \otimes_k k' \cong (A \oplus \Omega_{A/k}) \otimes_k k' \cong A' \oplus (\Omega_{A/k} \otimes_k k')$$

together with $B'/I^2 \cong A' \oplus \Omega_{A'/k'}$ this yields an isomorphism of abelian groups

$$\begin{aligned} \Omega_{A'/k'} &\longrightarrow \Omega_{A/k} \otimes_k k' \\ (a \otimes l) \otimes (b \otimes k) + I'^2 &\mapsto (a \otimes b - ab \otimes 1 + I^2) \otimes kl \end{aligned}$$

composed with (6) this shows that (5) is an isomorphism. $\square$

Lemma 12. *Let k be a ring, A a k-algebra and S a multiplicatively closed subset of A. Let T denote the multiplicatively closed subset $\{s \otimes t \,|\, s,t \in S\}$ of $A \otimes_k A$. Then there is a canonical isomorphism of k-algebras*

$$\tau : S^{-1}A \otimes_k S^{-1}A \longrightarrow T^{-1}(A \otimes_k A)$$
$$a/s \otimes b/t \mapsto (a \otimes b)/(s \otimes t)$$

Proof. We begin by defining a k-bilinear map

$$S^{-1}A \times S^{-1}A \longrightarrow T^{-1}(A \otimes_k A)$$
$$(a/s, b/t) \mapsto (a \otimes b)/(s \otimes t)$$

This induces the morphism of k-algebras τ. To define the inverse to τ, consider the morphism of k-algebras $A \otimes_k A \longrightarrow S^{-1}A \otimes_k S^{-1}A$ induced by the morphism of k-algebras $A \longrightarrow S^{-1}A$. This clearly maps T to units, and it is straightforward to check the induced morphism $T^{-1}(A \otimes_k A) \longrightarrow S^{-1}A \otimes_k S^{-1}A$ is inverse to τ. □

Theorem 13. *Suppose we have ring morphisms $\phi : k \longrightarrow A, \psi : A \longrightarrow B$. Then there is a canonical exact sequence of B-modules*

$$\Omega_{A/k} \otimes_A B \xrightarrow{v} \Omega_{B/k} \xrightarrow{u} \Omega_{B/A} \longrightarrow 0 \qquad (8)$$

The morphism v is a coretraction if and only if the canonical map $Der_k(B,T) \longrightarrow Der_k(A,T)$ is surjective for every B-module T. In that case we have a split exact sequence of B-modules

$$0 \longrightarrow \Omega_{A/k} \otimes_A B \xrightarrow{v} \Omega_{B/k} \xrightarrow{u} \Omega_{B/A} \longrightarrow 0$$

Proof. Let v be the morphism of (5) and u the morphism of (2). Therefore

$$v(d_{A/k}(a) \otimes b) = b \cdot d_{B/k}(\psi(a))$$
$$u(b \cdot d_{B/k}(b')) = b \cdot d_{B/A}(b')$$

It is clear that u is surjective. Since $d_{B/A}(\psi(a)) = 0$ we have $uv = 0$. Now showing that u is the cokernel of v is equivalent to checking that the following sequence of abelian groups is exact for any B-module T

$$Hom_B(\Omega_{B/A}, T) \longrightarrow Hom_B(\Omega_{B/k}, T) \longrightarrow Hom_B(\Omega_{A/k} \otimes_A B, T)$$

Using the canonical isomorphism $Hom_B(\Omega_{A/k} \otimes_A B, T) \cong Hom_A(\Omega_{A/k}, T)$ and Corollary 6 we reduce to checking that the following sequence of abelian groups is exact

$$Der_A(B,T) \longrightarrow Der_k(B,T) \longrightarrow Der_k(A,T) \qquad (9)$$

where the second map is $D \mapsto D \circ \psi$ (observe that the first map is trivially injective). This is clearly exact, so we have proved that (8) is an exact sequence of B-modules. A morphism of B-modules $M' \longrightarrow M$ has a left inverse if and only if the induced map $Hom_B(M,T) \longrightarrow Hom_B(M',T)$ is surjective for any B-module T. Thus, v has a left inverse if and only if the natural map $Der_k(B,T) \longrightarrow Der_k(A,T)$ is surjective for any B-module T. □

Corollary 14. *Suppose we have ring morphisms $\phi : k \longrightarrow A, \psi : A \longrightarrow B$. Then the canonical morphism of B-modules $v : \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k}$ is an isomorphism if and only if every derivation of A over k into any B-module T can be extended uniquely to a k-derivation $B \longrightarrow T$.*

Proof. The morphism v is an isomorphism if and only if it is a coretraction and an epimorphism, so if and only if $\Omega_{B/A} = 0$ and $Der_k(B,T) \longrightarrow Der_k(A,T)$ is surjective for every B-module T. Since $Der_A(B,T)$ is isomorphic to the kernel of this morphism and $\Omega_{B/A} = 0$ if and only if $Der_A(B,T) \cong Hom_B(\Omega_{B/A}, T) = 0$ for every B-module T, we see that v is an isomorphism if and only if $Der_k(B,T) \longrightarrow Der_k(A,T)$ is an isomorphism for every B-module T, which is what we wanted to show. □

Corollary 15. *Let k be a ring and A a k-algebra. If S is a multiplicatively closed subset of A, then there is a canonical isomorphism of $S^{-1}A$-modules*

$$\xi : S^{-1}\Omega_{A/k} \longrightarrow \Omega_{S^{-1}A/k}$$
$$(a \otimes b + I^2)/s \mapsto a/s \otimes b/1 + I'^2$$
$$d_{A/k}(a)/s \mapsto 1/s \cdot d_{S^{-1}A/k}(a)$$

The inverse $\Omega_{S^{-1}A/k} \longrightarrow S^{-1}\Omega_{A/k}$ maps $d(1/s)$ to $-1/s^2 \cdot ds$ for any $s \in S$.

Proof. Set $B = S^{-1}A$ and let $\psi : A \longrightarrow B$ be the canonical ring morphism. By Corollary 14 it suffices to show that $Der_k(B,T) \longrightarrow Der_k(A,T)$ is bijective for any B-module T. Let $D : A \longrightarrow T$ be a derivation of A over k into a B-module T and define $E : B \longrightarrow T$ by $E(a/s) = 1/s \cdot D(a) - a/s^2 \cdot D(s)$. It is not hard to check that this is a well-defined derivation of B over k which extends D, and is unique with this property. Therefore ξ is an isomorphism of $S^{-1}A$-modules, as required. □

Corollary 16. *Let k be a ring and B any localisation of a finitely generated k-algebra. Then $\Omega_{B/k}$ is a finitely generated B-module.*

Proof. Suppose that A is a finitely generated k-algebra and $B = S^{-1}A$ for some multiplicatively closed set S of A. By Lemma 7, $\Omega_{A/k}$ is a finitely generated A-module and therefore $\Omega_{S^{-1}A/k} \cong S^{-1}\Omega_{A/k}$ is a finitely generated $S^{-1}A$-module, as required. □

Let k be a ring, A a k-algebra and $\mathfrak{a}$ an ideal of A. Set $B = A/\mathfrak{a}$ and define a map $\mathfrak{a} \longrightarrow \Omega_{A/k} \otimes_A B$ by $x \mapsto d_{A/k}(x) \otimes 1$. It sends $\mathfrak{a}^2$ to zero, hence induces a morphism of B-modules

$$\delta : \mathfrak{a}/\mathfrak{a}^2 \longrightarrow \Omega_{A/k} \otimes_A B$$
$$x + \mathfrak{a}^2 \mapsto d_{A/k}(x) \otimes 1 \tag{10}$$

Theorem 17. *Let k be a ring, A a k-algebra and $\mathfrak{a}$ an ideal of A. If $B = A/\mathfrak{a}$ then there is a canonical exact sequence of B-modules*

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{v} \Omega_{B/k} \longrightarrow 0 \tag{11}$$

Moreover

(i) Put $A_1 = A/\mathfrak{a}^2$. Then $\Omega_{A/k} \otimes_A B \cong \Omega_{A_1/k} \otimes_{A_1} B$ as B-modules.

(ii) The morphism δ is a coretraction if and only if the extension $0 \longrightarrow \mathfrak{a}/\mathfrak{a}^2 \longrightarrow A/\mathfrak{a}^2 \longrightarrow B \longrightarrow 0$ of the k-algebra B by $\mathfrak{a}/\mathfrak{a}^2$ is trivial over k. That is, δ is a coretraction if and only if $A/\mathfrak{a}^2 \longrightarrow B$ is a retraction of k-algebras.

Proof. The surjecitivity of v follows from that of $A \longrightarrow B$. Clearly $v\delta = 0$, so as in the proof of Theorem 13 to show that (11) is exact it suffices to show that for any B-module T the following sequence is exact

$$Der_k(A/\mathfrak{m}, T) \longrightarrow Der_k(A, T) \longrightarrow Hom_A(\mathfrak{a}, T)$$

where we have used the canonical isomorphism of abelian groups $Hom_B(\mathfrak{a}/\mathfrak{a}^2, T) \cong Hom_A(\mathfrak{a}, T)$ and the second map is restriction to $\mathfrak{a}$. Exactness of this sequence is obvious, so we have shown that (11) is an exact sequence of B-modules.

(i) The canonical morphism of k-algebras $A \longrightarrow A_1$ induces a morphism of A-modules $\Omega_{A/k} \longrightarrow \Omega_{A_1/k}$, which induces a morphism of B-modules $\Omega_{A/k} \otimes_A B \longrightarrow \Omega_{A_1/k} \otimes_{A_1} B$. To show that this is an isomorphism, it suffices to show that the induced map $Hom_B(\Omega_{A_1/k} \otimes_{A_1} B, T) \longrightarrow Hom_B(\Omega_{A/k} \otimes_A B, T)$ is an isomorphism of abelian groups for every B-module T, so we have to show that the natural map $Der_k(A/\mathfrak{a}^2, T) \longrightarrow Der_k(A, T)$ is bijective for every $A/\mathfrak{a}$-module T, which is obvious.

(ii) By (i) we may replace A by A_1 in (11), so we assume $\mathfrak{a}^2 = 0$. Suppose that δ has a left inverse $w : \Omega_{A/k} \otimes_A B \longrightarrow \mathfrak{a}$. Putting $D(a) = w(d_{A/k}(a) \otimes 1)$ for $a \in A$ we obtain a derivation

$D : A \longrightarrow \mathfrak{a}$ over k such that $D(x) = x$ for $x \in \mathfrak{a}$. Then the map $f : A \longrightarrow A$ given by $f(a) = a - D(a)$ is a morphism of k-algebras (use $\mathfrak{a}^2 = 0$) which satisfies $f(\mathfrak{a}) = 0$, and therefore induces a morphism of k-algebras $f' : B \longrightarrow A$, which is clearly a right inverse to $A \longrightarrow B$.

For the converse, suppose that $f' : B \longrightarrow A$ is a k-algebra morphism right inverse to $A \longrightarrow B$, and define $D(a) = a - f'(a + \mathfrak{a})$. This is a derivation $D : A \longrightarrow \mathfrak{a}$ over k, which induces a morphism of A-modules $h : \Omega_{A/k} \longrightarrow \mathfrak{a}$. Pairing this with the canonical B-action on $\mathfrak{a}$ gives the desired left inverse to δ. □

Example 2. Let k be a ring, $A = k[x_1, \ldots, x_n]$ for $n \geq 1$ and let $B = A/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of A. Then $\Omega_{A/k} \otimes_A B$ is a free B-module with basis $z_i = d_{A/k}(x_i) \otimes 1$ and by Theorem 17 we have an exact sequence of B-modules

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{v} \Omega_{B/k} \longrightarrow 0$$

$$\delta(f + \mathfrak{a}^2) = \sum_{i=1}^{n} \partial f/\partial x_i \cdot z_i$$

where $f \in \mathfrak{a}$ and we use Lemma 8. For example if k is a field of characteristic zero and $\mathfrak{a}$ is the principal ideal $(y^2 - x^3)$ in the polynomial ring $A = k[x, y]$ then $B = k[x, y]/(y^2 - x^3)$ is the affine coordinate ring of the plane curve $y^2 = x^3$, which has a cusp at the origin. $Im\delta$ is the B-submodule generated by $-3x^2(dx \otimes 1) + 2y(dy \otimes 1)$. Therefore $\Omega_{B/k}$ is isomorphic as a B-module to the quotient of $B \oplus B$ by the submodule generated by $(-3x^2 + \mathfrak{a}, 2y + \mathfrak{a})$. Therefore $-3xy \cdot dx + 2x^2 \cdot dy$ is a nonzero torsion element of $\Omega_{B/k}$ (apply x).

Example 3. More generally let k be a ring, A a k-algebra and $B = A[x_1, \ldots, x_n]$. Let T be a B-module and let $D \in Der_k(A, T)$. We define a derivation $E : B \longrightarrow T$ over k in the following way

$$E(f) = \sum_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot D(f(\alpha))$$

That is, apply D to the coefficients and act with the variables. This shows that the canonical map $Der_k(B, T) \longrightarrow Der_k(A, T)$ is surjective. It follows from Theorem 13 that we have a split exact sequence of B-modules

$$0 \longrightarrow \Omega_{A/k} \otimes_A B \xrightarrow{v} \Omega_{B/k} \xrightarrow{u} \Omega_{B/A} \longrightarrow 0$$

The left inverse w to v is defined by $w(d_{B/k}(f)) = \sum_\alpha d_{A/k}(f(\alpha)) \otimes x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The corresponding right inverse to p to u is defined by $p(d_{B/A}(f)) = d_{B/k}(f) - \sum_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot d_{B/k}(f(\alpha))$ and in particular $p(d_{B/A}(x_i)) = d_{B/k}(x_i)$. Therefore by Lemma 9 we have a canonical isomorphism of B-modules

$$\Omega_{B/k} \cong (\Omega_{A/k} \otimes_A B) \oplus Bd_{B/k}(x_1) \oplus \cdots \oplus Bd_{B/k}(x_n) \tag{12}$$

Let $\mathfrak{a}$ be an ideal of B and put $C = A/\mathfrak{a}$. Denote by z_i the element $d_{B/k}(x_i) \otimes 1$ of $\Omega_{B/k} \otimes_B C$ and let $y_i = x_i + \mathfrak{a}$. Tensoring (12) with C we have an isomorphism of C-modules

$$\Omega_{B/k} \otimes_B C \cong (\Omega_{A/k} \otimes_A C) \oplus Cz_1 \oplus \cdots \oplus Cz_n$$

Finally, Theorem 17 gives an exact sequence of C-modules

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B C \xrightarrow{v} \Omega_{C/k} \longrightarrow 0$$

$$\delta(f + \mathfrak{a}^2) = \sum_{\alpha} d_{A/k}(f(\alpha)) \otimes y_1^{\alpha_1} \cdots y_n^{\alpha_n} + \sum_{i=1}^{n} \partial f/\partial x_i \cdot z_i$$

where $f \in \mathfrak{a}$ and we use Lemma 8.

3 Separability

Definition 8. Let k be a field and K an extension field of k. A transcendence basis $\{x_\lambda\}_{\lambda\in\Lambda}$ of K/k is called a *separating transcendence basis* if K is separably algebraic over the field $k(\{x_\lambda\}_{\lambda\in\Lambda})$. We say that K is *separably generated* over k if it has a separating transcendence basis.

Proposition 18. *Let $k \subseteq K \subseteq L$ be finitely generated field extensions.*

(i) If L/K is pure transcendental then $rank_L\Omega_{L/k} = rank_K\Omega_{K/k} + tr.deg.L/K$.

(ii) If L/K is separable algebraic then $rank_L\Omega_{L/k} = rank_K\Omega_{K/k}$.

(iii) If $L = K(t)$ where t is purely inseparable over K then $rank_L\Omega_{L/k} = rank_K\Omega_{K/k}$ or $rank_L\Omega_{L/k} = rank_K\Omega_{K/k} + 1$.

(iv) If L/K is purely inseparable algebraic then $rank_L\Omega_{L/k} \geq rank_K\Omega_{K/k}$.

Proof. Notice that if K/k is a finitely generated field extension then $rank_K\Omega_{K/k}$ is finite by Corollary 16, so the statement of the result makes sense. (i) Since L/K is a finitely generated field extension, $r = tr.deg.L/K$ is finite. If $r = 0$ then $K = L$ and the result is trivial, so assume $r \geq 1$ and let $\{t_1,\dots,t_r\}$ be a transcendence basis for L/K with $L = K(t_1,\dots,t_r)$. Therefore L is K-isomorphic to the quotient field of the polynomial ring $B = K[x_1,\dots,x_r]$. By Example (3) we have an isomorphism of B-modules

$$\Omega_{B/k} \cong (\Omega_{K/k} \otimes_K B) \oplus Bdx_1 \oplus \cdots \oplus Bdx_r$$

Localising and using Corollary 15 we get an isomorphism of L-modules

$$\Omega_{L/k} \cong (\Omega_{K/k} \otimes_K L) \oplus Ldt_1 \oplus \cdots \oplus Ldt_r$$

Therefore $rank_L\Omega_{L/k} = rank_K\Omega_{K/k} + r$, as required.

(ii) If $L = K$ this is trivial, so assume otherwise. Since L/K is finitely generated and algebraic, it is finite. Therefore L/K is a finite separable extension, which has a primitive element $t \in L$ by the primitive element theorem (so $t \notin K$ and $L = K(t)$). Let f be the minimum polynomial of t, and set $n = deg(f) \geq 2$. Then $1, t, \dots, t^{n-1}$ is a K-basis for L. To show that $rank_L\Omega_{L/k} = rank_K\Omega_{K/k}$ it suffices to show that the canonical morphism of L-modules $\Omega_{K/k} \otimes_K L \longrightarrow \Omega_{L/k}$ is an isomorphism. By Corollary 14 it is enough to show that the canonical map $Der_k(L,T) \longrightarrow Der_k(K,T)$ is bijective for any L-module T. Injectivity follows from Lemma 10, so let $D : K \longrightarrow T$ be a derivation of K into T over k. We define $E : L \longrightarrow T$ by

$$E(a_0 + a_1t + \cdots + a_{n-1}t^{n-1}) = D(a_0) + t \cdot D(a_1) + \cdots + t^{n-1} \cdot D(a_{n-1})$$

It is not hard to check that this is a derivation of L into T over k extending D, so the proof is complete.

(iii) Suppose that $L = K(t)$ where t is algebraic and purely inseparable over K. If $t \in K$ the claim is trivial, so assume otherwise. Let $char(k) = p \neq 0$ and $e \geq 1$ be minimal with $t^{p^e} = a \in K$. We claim that

$$rank_L\Omega_{L/k} = \begin{cases} rank_K\Omega_{K/k} + 1 & d_{K/k}(a) = 0 \\ rank_K\Omega_{K/k} & d_{K/k}(a) \neq 0 \end{cases} \tag{13}$$

The minimal polynomial of t over K is $f = x^{p^e} - a$ and we have a K-isomorphism $K[x]/(f) \cong L$. Set $C = K[x]/(f)$ and observe that by Example 3 we have an isomorphism of C-modules

$$\Omega_{C/k} \cong \big((\Omega_{K/k} \otimes_K C) \oplus Cz\big)/C\delta f$$

where $\delta f = -d_{K/k}(a) \otimes 1$. If $d_{K/k}(a) = 0$ then $rank_L\Omega_{L/k} = rank_C\Omega_{C/k} = rank_K\Omega_{K/k} + 1$. Otherwise δf is nonzero and therefore generates a subspace of rank 1. Subtracting this from the rank of $(\Omega_{K/k} \otimes_K C) \oplus Cz$, which is $rank_K\Omega_{K/k} + 1$, we see that $rank_L\Omega_{L/k} = rank_C\Omega_{C/k} = rank_K\Omega_{K/k}$, as required.

(iv) Now assume more generally that L/K is purely inseparable algebraic, say $L = K(t_1, \ldots, t_n)$. Then we have a chain of purely inseparable field extensions

$$K \subseteq K(t_1) \subseteq K(t_1, t_2) \subseteq \cdots \subseteq K(t_1, \ldots, t_n) = L$$

so the result follows immediately from (iii). □

Theorem 19. *Let $k \subseteq K \subseteq L$ be finitely generated field extensions. Then*

$$rank_L \Omega_{L/k} \geq rank_K \Omega_{K/k} + tr.deg.L/K$$

with equality if L is separably generated over K.

Proof. First we establish the inequality. Since L/K is a finitely generated field extension we can write $L = K(t_1, \ldots, t_n)$ where the first r generators $\{t_1, \ldots, t_r\}$ are a transcendence basis for L over K. Let $Q = K(t_1, \ldots, t_r)$ so that Q/K is pure transcendental and L/Q is algebraic. By Proposition 18(i) we have $rank_Q \Omega_{Q/k} = rank_K \Omega_{K/k} + r$, so we can reduce to the case where L/K is algebraic.

If K_s denotes the set of elements of L separable over K, then K_s/K is a finite separable field extension and L/K_s is purely inseparable and finitely generated. By Proposition 18(ii) we have $rank_{K_s} \Omega_{K_s/K} = rank_K \Omega_{K/k}$ so we reduce to the case where L/K is purely inseparable algebraic, which is Proposition 18(iv).

If L is separably generated over K then find a separating transcendence basis $\{t_1, \ldots, t_r\}$ and set $Q = K(t_1, \ldots, t_r)$. Since Q/K is pure transcendental and L/Q is separable algebraic we have by Proposition 18(i) and (ii)

$$rank_L \Omega_{L/k} = rank_Q \Omega_{Q/k} = rank_K \Omega_{K/k} + r$$

as required. □

Corollary 20. *Let L be a finitely generated extension of a field k. Then $rank_L \Omega_{L/k} \geq tr.deg.L/k$ with equality if and only if L is separably generated over k. In particular $\Omega_{L/k} = 0$ if and only if L is separably algebraic over k.*

Proof. The inequality is a special case of Theorem 19. Let S be a transcendence basis of L/k and set $Q = k(S)$ (note that S may be empty), so that Q/k is pure transcendental and L/Q is finitely generated algebraic (therefore also finite). Let K be the set of elements of L separable over Q, so that we have L/K purely inseparable and K/Q separable. Therefore by Proposition 18 we have

$$k \subseteq Q \subseteq K \subseteq L$$
$$rank_L \Omega_{L/k} \geq rank_K \Omega_{K/k} = rank_Q \Omega_{Q/k} = tr.deg.L/k$$

If L is separably generated over k then we can take S to be a separating transcendence basis, whence $K = L$ and we have the desired equality.

For the converse, suppose that $rank_L \Omega_{L/k} = tr.deg.L/k = r$. If $r = 0$ then $\Omega_{L/k} = 0$ and L/k is a finitely generated algebraic extension. Therefore $k = Q$ so K/k is separable algebraic and $\Omega_{K/k} = 0$. We have to show that L/k is separable algebraic (that is, $K = L$). Suppose otherwise that $t \in L \setminus K$. Let $char(k) = p \neq 0$ and $e \geq 1$ be minimal with $t^{p^e} = a \in K$. Since $\Omega_{K/k} = 0$ we have $d_{K/k}(a) = 0$ and therefore by (13), $rank_{K(t)/k} \Omega_{K(t)/k} = 1$. But $L/K(t)$ is purely inseparable so $rank_L \Omega_{L/k} \geq 1$, which is a contradiction. Therefore the converse is true for $r = 0$. In particular $\Omega_{L/k} = 0$ if and only if L is separably algebraic over k.

Now assume that $r \geq 1$ and let $x_1, \ldots, x_r \in L$ be such that $\{dx_1, \ldots, dx_r\}$ is a basis of $\Omega_{L/k}$ over L. Then we have $\Omega_{L/k(x_1,\ldots,x_r)} = 0$ by Theorem 13 so L is separably algebraic over $k(x_1, \ldots, x_r)$. Since $r = tr.deg.L/k$ the elements x_i must form a transcendence basis of L over k, so the proof is complete. □

References

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