Differentiation, Division and the Bicategory of Landau-Ginzburg Models

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Contents

Notation

We take the following conventions throughout:

- $\mathbb{N}$ is the set of non-negative integers.
- $\mathbb{Z}_2$ is the integers modulo 2.
- δ_{ij} denotes the Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.
- For k a commutative ring, $k[x]$, $k[y]$, $k[z]$ etc. denote polynomial rings over k in some finite number of variables, possibly greater than one.
- If $f = (f_1, \cdots, f_n)$ is a sequence of polynomials and $u \in \mathbb{N}^n$ then $f^u = f_1^{u_1} \cdots f_n^{u_n}$. If we wish to specify the variable we may also write $f^u(x)$.
- If maps f and g are homotopic then we write $f \simeq g$, or $h : f \simeq g$ to specify a homotopy h.
- If a and b are elements of some graded ring (typically homogeneous morphisms between graded objects) then $|a|$ denotes the degree of a and $[a, b] = ab - (-1)^{|a||b|}ba$ is the graded commutator.
- If $u \in \mathbb{N}^n$ then $|u| = \sum_{i=1}^n u_i$.
- Tensor products of morphisms between graded objects are graded tensor products.
- When we do not wish to distinguish between chain complexes and cochain complexes we refer to any $\mathbb{Z}$-graded object with a degree ± 1 map which squares to zero as a *complex*.
- A complex is usually denoted as a pair (C, d_C) where C is the underlying graded object and d_C is the differential.
- If A is an object of any category then 1_A denotes its identity morphism. We may also write $1 = 1_A$ when this is not confusing.
- If $\mathcal{C}$ is a category $\mathcal{C}^\omega$ denotes its idempotent completion as defined in Definition B.2.1.

1 Introduction

The bicategory of Landau-Ginzburg models over a commutative ring k is a bicategory whose objects are polynomials with coefficients in k with a certain type of singularity and whose 1-morphisms are matrix factorisations. When $k = \mathbb{C}$ the objects of this bicategory are exactly the polynomials with isolated critical points. The applications of this bicategory, which we denote $\mathcal{LG}_k$, are fairly broad. In addition to algebraic geometry and the study of singularities, this bicategory is also used in mathematical physics in the study of certain topological field theories [CM16] and in the study of knot invariants [CM14].

An important concept from the theory of $\mathcal{LG}_k$ is the *cut operation* of [Mur18], which provides a method for computing compositions of 1-morphisms in $\mathcal{LG}_k$. It works by relating the composition of two 1-morphisms, which in a certain sense is an "infinite dimensional" object, to a finite dimensional representative. A key component of the cut operation is an explicit homotopy equivalence between a Koszul complex and its homology. Understanding precisely how the composition and its finite dimensional representative are related — and hence understanding how to compute the composition — requires knowing how to compute the maps in this homotopy equivalence.

Goals of this thesis

- To give a self-contained and accessible introduction to the bicategory of Landau-Ginzburg models and the cut operation. At the time of writing, most of the literature on $\mathcal{LG}_k$ is technical and inaccessible to non-experts. We give a detailed exposition in Section 5, including some difficult technical points which have not appeared before in print (for example, the proof of Lemma 5.5.13).

- To isolate and give a self-contained account of some key techniques that have been developed in the theory of $\mathcal{LG}_k$. This includes the construction of explicit homotopy equivalences between Koszul complexes and their homology, which is done in Section 3.

- To go beyond the existing theory by identifying a class of important examples in which a key technique can be developed more simply. In the current theory of [Mur18], constructing an explicit homotopy equivalence between the Koszul complex and its homology requires passing to a completion of the polynomial ring. In Section 3 we give conditions under which this construction can be done in the polynomial ring itself, rather than its completion. This gives a more elegant approach to the central idea in the construction of the cut operation — a system of generalised derivatives with respect to a quasi-regular sequence — based on polynomial division.

Summary of contributions by section

In Section 3 we show how to construct an explicit homotopy equivalence between a Koszul complex and its homology, paying particular attention to when the maps involved can actually be computed. This technique is used in [DM13; CM16; Mur18] in the context of $\mathcal{LG}_k$ but it need not be specific to $\mathcal{LG}_k$. In Section 3.2 we construct this homotopy equivalence in full generality which requires passing to a completion of the ring. This summarises results in [DM13; CM16; Mur18] in an isolated context and also provides additional details which are not present in the literature. The homotopy equivalence constructed in Section 3.2 requires knowing certain series expansions of elements in the completion and in Section 3.3 we consider how to compute these series expansions in the case of a polynomial ring. This expands upon ideas mentioned in [Mur19, Remark A.9]. The considerations of Section 3.3 lead to conditions under which it is not necessary to pass to the completion in order to construct the homotopy equivalence of Section 3.2. The situation where it is not necessary to pass to the completion exactly coincides with the situation where the homotopy equivalence of Section 3.2 can be computed at all elements.

To be more specific about the original contribution in Section 3, let k be a field and consider the polynomial ring $k[x] = k[x_1, \ldots, x_m]$. Let $t = (t_1, \ldots, t_n)$ be a sequence of elements in $k[x]$. The homotopy equivalence of Section 3.2 involving the Koszul complex of t relies on the existence of maps $\partial_{t_1}, \ldots, \partial_{t_n}$ where ∂_{t_i} acts like differentiating with respect to t_i. In [DM13; CM16; Mur18] these maps are defined using certain series expansions of polynomials in the I-adic topology on $k[x]$, where I is the ideal generated by the elements of t. The map ∂_{t_i} works on $f \in k[x]$ by modifying every term in this series expansion of f and so in general $\partial_{t_i}(f)$ is only an element of the I-adic completion of $k[x]$ rather than $k[x]$ itself. In Section 3.3 we consider how to compute the terms in this series expansion of f via polynomial division. This leads to conditions under which $\partial_{t_i}(f)$ is an element of the polynomial ring $k[x]$ and hence gives conditions where it is not necessary to pass to the completion to construct the homotopy equivalence in Section 3.2. These conditions encompass many important and natural examples and provide a setting where it is actually possible to compute things exactly, rather than as I-adic approximations. This is covered in Section 3.4. We refine this approach further in Section 3.5 by giving an alternative definition of the maps $\partial_{t_1}, \ldots, \partial_{t_n}$ under the conditions of Section 3.4. Under this definition we pass to the polynomial ring $k[x, y] = k[x_1, \ldots, x_m, y_1, \ldots, y_m]$ and, for $f \in k[x]$, consider the polynomial $f(x) - f(y)$ in $k[x, y]$. We compute $\partial_{t_i}(f)$ by first using polynomial division to obtain an expression of the form

$$f(x) - f(y) = r(x, y) + \sum_{i=1}^{n} q_i(x, y)(t_i(x) - t_i(y))$$

and then dividing $q_i(x, y)$ to obtain an expression of the form

$$q_i(x, y) = r_i(x, y) + \sum_{j=1}^{n} p_{ij}(x, y)(t_j(x) - t_j(y)) \ .$$

We then show that $\partial_{t_i}(f) = r_i(x, x)$. Practically speaking this perspective provides a much more efficient algorithm for computing the maps $\partial_{t_1}, \ldots, \partial_{t_n}$. Conceptually, it emphasises the relationship between polynomial division and the maps $\partial_{t_1}, \ldots, \partial_{t_n}$, and hence the role of polynomial division in computing the homotopy equivalence of Section 3.2. It also leads to a result which can be interpreted as a version of Taylor's Theorem for the maps $\partial_{t_1}, \ldots, \partial_{t_n}$.

In Section 4 we discuss *matrix factorisations*, which are the 1-morphisms in $\mathcal{LG}_k$. This section provides important background to Section 5. In particular, in Section 4.3 we show how perturbation techniques from homological algebra can be extended to the setting of matrix factorisations. The extension of these techniques is used frequently in literature on $\mathcal{LG}_k$ however the proofs which establish this extension are only sketched.

In Section 5 we discuss the bicategory of Landau-Ginzburg models. This bicategory is defined in Section 5.1 and is proved to form a bicategory in Sections 5.2 and 5.3. This follows the approach of [DM13; CM16; Mur18] but provides additional exposition. Finally in Sections 5.4, 5.5 and 5.6 we discuss the cut operation which was first defined in [Mur18]. Sections 5.4 and 5.5 include many additional details not found in [Mur18] in order to make the cut operation more accessible for non-experts. In Section 5.6 we give some examples of cuts.

In Appendix B we discuss idempotent completion of categories, paying particular attention to the case of preadditive categories. The results in Section B.2 are "well-known" but difficult to find in the literature and play a mundane but important role in the definition of $\mathcal{LG}_k$. Appendix C explains a way of interpreting matrix factorisations geometrically which is not found in the literature. This is somewhat tangential to the main focus of this thesis but can provide geometric intuition when working with matrix factorisations.

Thematic approach

Thematically, the approach of this thesis demonstrates an interplay between algebraic, differential and topological techniques. Polynomial singularities are algebro-geometric in nature and as such algebraic constructions, such as Koszul complexes and matrix factorisations, are used to study them. Polynomial division is a fundamental computational tool in algebraic geometry and it plays a central role in Section 3 by giving us a method of computing certain maps which are generalisations of partial differentiation. Moreover, these computational considerations lead to conditions under which these generalised partial derivative maps exist on the polynomial ring rather than on its completion. The partial derivative maps are used to define a *connection* on the Koszul complex, which is analogous to the concept from differential geometry of the same name. This connection is used to define the homotopy equivalence between the Koszul complex and its homology. While drawing a direct link is difficult, this construction echoes ideas used in relation to the de Rham complex from differential topology.

The homotopy equivalence we construct in Section 3 is a special type of homotopy equivalence called a *strong deformation retract*. This concept has its origins in topology and passes to the world of homological algebra via various homology theories including de Rham cohomology. Strong deformation retracts are useful because they are susceptible to the homological *perturbation techniques* described in Section 2.1. In Section 4.3 we extend these perturbation techniques to matrix factorisations. Complexes can be regarded as a kind of degenerate matrix factorisation and, using perturbation, we can convert a strong deformation retract of complexes into a strong deformation retract of non-trivial matrix factorisations. Via a different method, a Koszul complex can also be used to construct a certain type of matrix factorisation which we call a *Koszul matrix factorisation*. Using perturbation, we can translate a strong deformation retract involving a Koszul complex into a strong deformation retract involving the corresponding Koszul matrix factorisation under some circumstances.

The category of matrix factorisations relevant to singularities is the homotopy category and so homotopy equivalences of matrix factorisations tell us something about the singularities of the polynomial being factorised. The ideas described above are used in Section 5 when discussing the bicategory of Landau-Ginzburg models. In the case of the cut operation, formulae for these homotopy equivalences are given explicitly by modifying the strong deformation retract of Section 3. Hence, the homotopy equivalence associated to the cut operation, which is constructed using our algebraic analogue of a connection, is ultimately computed using polynomial division.

2 Background

In this section we introduce and contextualise some of the concepts central to this thesis. In Section 2.1 we discuss deformation retracts and perturbation techniques, beginning with the concept of a deformation retract as it arises in topology. In Section 2.2 we define Koszul complexes and establish our conventions for working with them. In Section 2.3 we discuss hypersurface singularities, matrix factorisations and the bicategory of Landau-Ginzburg models at a high level. Finally in Section 2.4 we give a straightforward existence result for the type of homotopy equivalence we explicitly construct in Section 3.

2.1 Deformation retracts

The notion of a deformation retract first arose in topology. Let X be a topological space and A a subspace of X.

Definition 2.1.1 ([Bre10, Chapter I Definition 14.8])**.** We say A is a *deformation retract* of X if there exists a homotopy $H : X \times [0,1] \to X$ which satisfies:

(1) $H(x,0) = x$ for all $x \in X$.

(2) $H(x,1) \in A$ for all $x \in X$.

(3) $H(a,1) = a$ for all $a \in A$.

We call A a *strong deformation retract* if in addition $H(a,t) = a$ for all $t \in [0,1]$ and $a \in A$.

Suppose A is a deformation retract with homotopy H. Let $i : A \to X$ denote the inclusion map and define $r : X \to A$ as $r(x) = H(x,1)$. The maps H, i and r are such that $ri = 1_A$ and $H : ir \simeq 1_X$, and so the maps i and r give a homotopy equivalence between A and X.

Example 2.1.2. Let $X = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$, which is the solid cylinder in $\mathbb{R}^3$ centred on the z-axis. Let A be the z-axis. Taking $H(x,y,z,t) = ((1-t)x, (1-t)y, z)$ shows that A is a strong deformation retract of X.

By considering the homology of a topological space we can extend the notion of deformation retracts to the setting of homological algebra. Many homology theories, such as de Rham cohomology and singular homology, will turn a (strong) deformation retract of topological spaces into a (strong) deformation retract of chain complexes in the sense of the following definition. Let $\mathcal{A}$ be an abelian category.

Definition 2.1.3. A *deformation retract* of complexes (L, d_L) and (M, d_M) of objects in $\mathcal{A}$ consists of morphisms

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h$$

where $pi = 1$ and $h : ip \simeq 1$. This deformation retract is called *strong* if in addition $h^2 = 0$, $hi = 0$ and $ph = 0$.

Strong deformation retracts are a special type of homotopy equivalence between chain complexes. They have the property that they can be modified using a technique called *perturbation*. Let

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h \tag{2.1}$$

be a deformation retract of chain complexes. A *perturbation* of (2.1) is a degree -1 map $\delta : M \to M$ such that $(d_M + \delta)^2 = 0$. Given such a perturbation δ we define the *perturbed data* as the morphisms

$$(L, d'_L) \underset{i'}{\overset{p'}{\leftrightarrows}} (M, d_M + \delta), \qquad h' \tag{2.2}$$

where $a = (1 - \delta h)^{-1}\delta$, $d'_L = d_L + pai$, $i' = i + hai$, $p' = p + pah$ and $h' = h + hah$. We call the perturbation δ *small* if $(1 - \delta h)$ is invertible.

Theorem 2.1.4 (Perturbation Lemma)**.** *If (2.1) is a strong deformation retract and δ is a small perturbation then the perturbed data (2.2) is also a strong deformation retract.*

The Perturbation Lemma is a powerful result with broad applications which are summarised in [Cra04]. The Perturbation Lemma as stated above is proved in [Cra04, Section 2.4]. In Theorem 4.3.2 we prove a version of this result for *matrix factorisations* which can easily be adapted into a proof of Theorem 2.1.4.[1] It is worth pointing out that the Perturbation Lemma for complexes holds somewhat more generally. We say that the morphisms

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h$$

are a *homotopy equivalence datum* if i and p are quasi-isomorphisms and $h : ip \simeq 1$. A homotopy equivalence datum is not necessarily a homotopy equivalence, but rather is a type of isomorphism in the derived category of $\mathcal{A}$. Small perturbations of homotopy equivalence data are defined as for deformation retracts, and a small perturbation of a homotopy equivalence datum is again a homotopy equivalence datum.

2.2 Koszul complexes

The Koszul complex is an important tool in homological algebra, commutative algebra and algebraic geometry which is particularly useful for doing calculations. For example, see [Wei94, Chapter 4.5] and [Eis94, Chapter 17]. The Koszul complex plays a significant role in this thesis. In Section 3 we construct a strong deformation retract between a Koszul complex and its homology. This is then used to compute compositions in the bicategory of Landau-Ginzburg models in Section 5. In Section 4.2 we also show how Koszul complexes can be used to construct a type of matrix factorisation called the *Koszul matrix factorisation*. This construction is used in Section 5 to define the unit 1-morphisms in the bicategory of Landau-Ginzburg models.

We define the Koszul complex as a chain complex following the conventions of [Wei94, Chapter 4.5]. Let R be a commutative ring and $t_i \in R$. Let $K(t_i)$ be the chain complex

$$0 \longrightarrow \underset{1}{R} \xrightarrow{t_i} \underset{0}{R} \longrightarrow 0$$

where the differential is multiplication by t_i and the degree of each object is as indicated on the diagram. Let $K_p(t_i)$ denote the degree p component of $K(t_i)$. We denote the canonical generator of $K_1(t_i)$ (the unit of the ring R) by θ_i. We now consider a sequence $t = (t_1, \dots, t_n)$ of elements of R.

[1]In the context of the proof of Theorem 4.3.2, setting $f = g = 0$ will yield exactly the calculations required to prove Theorem 2.1.4.

Definition 2.2.1. The *Koszul complex* of t is the tensor product

$$K(t) = K(t_1) \otimes_R \cdots \otimes_R K(t_n)$$

of chain complexes. We typically denote the differential of the Koszul complex by d_K.

The underlying graded module of the Koszul complex $(K(t), d_K)$ is the exterior algebra $\bigwedge(R^{\oplus n})$. This can be seen by induction on n. The base case is clear and in the inductive case note that the degree p component of $K(t)$ is

$$K_p(t) = \left(\bigwedge^{p-1}(R^{\oplus(n-1)}) \otimes_R K_1(t_n)\right) \oplus \left(\bigwedge^{p}(R^{\oplus(n-1)}) \otimes_R K_0(t_n)\right)$$

An isomorphism $K_p(t) \to \bigwedge^p(R^{\oplus n})$ is induced by mapping

$$(\theta_{i_1} \wedge \cdots \wedge \theta_{i_{p-1}}) \otimes \theta_n \mapsto \theta_{i_1} \wedge \cdots \wedge \theta_{i_{p-1}} \wedge \theta_n \qquad \text{and} \qquad (\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}) \otimes r \mapsto r(\theta_{i_1} \wedge \cdots \wedge \theta_{i_p})$$

where $i_1 < \cdots < i_p$ and $r \in K_0(t_n) = R$. In this presentation of $K(t)$ the differential works on the basis of $\bigwedge(R^{\oplus n})$ as

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_p} \longmapsto \sum_{j=1}^{p} (-1)^{j+1} t_{i_j} \theta_{i_1} \wedge \cdots \wedge \widehat{\theta}_{i_j} \wedge \cdots \wedge \theta_{i_p}$$

where $i_1 < \cdots < i_p$ and "$\widehat{\theta}_{i_j}$" indicates that θ_{i_j} is omitted from the wedge product. In future sections it will become cumbersome to use "$\wedge$" to denote products in $K(t)$ so from now on we omit the wedges and denote the wedge product by juxtaposition. That is we set $uv = u \wedge v$ for $u, v \in \bigwedge(R^{\oplus n})$.

Given $v \in R^{\oplus n}$ we define a degree $+1$ map $v \wedge (-) : \bigwedge(R^{\oplus n}) \to \bigwedge(R^{\oplus n})$ which is given by taking the wedge product with v. Typically we will consider this when $v = \theta_i$, and in this case we write $\theta_i = \theta_i \wedge (-)$. We can also define a degree -1 map associated to v as follows.

Definition 2.2.2. Let $v^* \in \operatorname{Hom}(R^{\oplus n}, R)$ denote the dual vector associated to $v \in R^{\oplus n}$. We define *contraction* by v^* as the degree -1 map $v^* \lrcorner (-) : \bigwedge(R^{\oplus n}) \to \bigwedge(R^{\oplus n})$ given on the basis by

$$\theta_{i_1} \cdots \theta_{i_p} \longmapsto \sum_{j=1}^{p} (-1)^{j+1} v^*(\theta_{i_j}) \theta_{i_1} \cdots \widehat{\theta}_{i_j} \cdots \theta_{i_p}$$

and extended linearly. When $v = \theta_i$ we denote $\theta_i^* = \theta_i^* \lrcorner (-)$.

Note that in the above definition we have switched to writing the wedge product using juxtaposition. The maps $\theta_1, \ldots, \theta_n$ and $\theta_1^*, \ldots, \theta_n^*$ on the exterior algebra satisfy certain relations called the *canonical anticommutation relations*, which we prove in Lemma 4.2.2. Using the contraction operator we can write $d_K = \sum_{i=1}^n t_i \theta_i^*$.

Another structure related to the exterior algebra is the *de Rham complex* of a manifold (see [Bre10, Chapter V.2]). Let M be a differentiable manifold, say a smooth $\mathbb{R}$-manifold of dimension n, and let Ω_p be the set of differential p-forms on M. The de Rham complex consists of the graded module $\bigoplus_{p=0}^n \Omega_p$ with differential given in a coordinate chart (U, φ) by

$$d(f\omega) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \omega \tag{2.3}$$

where $\varphi = (x_1, \ldots, x_n)$ are the coordinate functions, $f : M \to \mathbb{R}$ is a smooth function and $\omega = dx_{i_1} \cdots dx_{i_p}$ for some $i_1 < \cdots < i_p$. At a point $m \in M$ the p-form $f\omega$ is an element of the exterior algebra $\bigwedge^p (T_m M)^*$, where $(T_m M)^*$ is the dual of the tangent space of M at point m. The product $dx_i \omega$ in (2.3) is reminiscent of wedging $\theta_{i_1} \cdots \theta_{i_p}$ by θ_i in the exterior algebra $\bigwedge(R^{\oplus n})$. Although making this connection precise is somewhat challenging, the approach of Section 3 should be considered in this context; when the Koszul complex and "de Rham-like" structures exist on the exterior algebra they interact in useful ways. Given that the goal of Section 3 is to produce a strong deformation retract involving the Koszul complex, the analogy to the de Rham complex is more interesting considering that a strong deformation retract of manifolds induces a strong deformation retract of the correspond de Rham complexes.

2.3 Hypersurface singularities and matrix factorisations

Singularities are a topic of considerable interest in the field of algebraic geometry and their study has many applications to other areas of mathematics and physics. Let k be a commutative ring and consider a polynomial $f \in k[x] = k[x_1, \cdots, x_n]$. One way of studying the singularities of f is to consider the *matrix factorisations* of f. These are defined in Section 4, but concretely one can think of a matrix factorisation of f as a pair of $m \times m$ square matrices (P, Q) with entries in $k[x]$ such that $PQ = QP = fI$, where I denotes the identity matrix. Fixing a basis for $k[x]^{\oplus m}$ we can consider P and Q as morphisms

$$k[x]^{\oplus m} \xrightarrow{P} k[x]^{\oplus m} \xrightarrow{Q} k[x]^{\oplus m} \quad .$$

In a way that is made precise in Section 4.1, we can think of matrix factorisations as a similar kind of object to a chain complex. This object is $\mathbb{Z}_2$-graded (rather than $\mathbb{Z}$-graded) and has a differential which squares to f which is typically non-zero. We can define a morphism of matrix factorisations analogously to a chain map of complexes, and likewise for other concepts such as the notion of homotopy.

The connection between the matrix factorisations of f and the singularities of f can intuitively be understood by considering that factorisations of f correspond to algebraic subsets of the zero set of f, which we denote $V(f) = \{a \in k^n \mid f(a) = 0\}$. If we can write $f = gh$ for some polynomials $g, h \in k[x]$ then the inclusion of ideals $(g) \supseteq (f)$ gives us that $V(g) \subseteq V(f)$, and likewise for h. Typically polynomials with singularities have zero sets with more complicated algebraic subsets compared to polynomials without singularities. Matrix factorisations capture more nuanced information about f than factorising within $k[x]$, and a relationship between the matrix factorisations of f and algebraic subsets of $V(f)$ is made precise in Appendix C.

It turns out that the relevant category of matrix factorisations is one in which they are considered up to homotopy equivalence. We denote the homotopy category of finite rank[2] matrix factorisations of f by $\mathrm{hmf}(k[x], f)$. The following deep result from [Orl09] to some extent demonstrates the importance of $\mathrm{hmf}(k[x], f)$.

Theorem 2.3.1. *Suppose k is an algebraically closed field. Then* $\mathrm{hmf}(k[x], f)$ *is the zero category if and only if f has no singularities.*

Therefore, finding homotopy equivalences between matrix factorisations is an important task. In Section 4.3 we explain how the perturbation techniques of Section 2.1 can be extended to matrix factorisations.

[2]See Definition 4.1.2.

These ideas are relevant to the bicategory of Landau-Ginzburg models over k, denoted $\mathcal{LG}_k$. Broadly speaking, the objects of this bicategory are pairs $(k[y], U)$ where $k[y] = k[y_1, \cdots, y_m]$ and $U \in k[y]$ is a special kind of polynomial, called a *potential*, with a certain type of singularity. When $k = \mathbb{C}$ the objects of $\mathcal{LG}_k$ are exactly the polynomials which have isolated critical points. The category of 1-morphisms $(k[y], U) \to (k[z], V)$ is $\mathrm{hmf}(k[y, z], V(z) - U(y))$. Proving $\mathcal{LG}_k$ is a well-defined bicategory is a non-trivial task and requires producing homotopy equivalences between certain matrix factorisations. This is done in Sections 5.2 and 5.3.

In order to compute compositions of 1-morphisms in $\mathcal{LG}_k$ we need explicit formulae for the homotopy equivalences of Section 5.2 used to define composition. The results in Section 3 give the necessary formulae, and this approach leads to the definition of the *cut operation* on matrix factorisations, first given in [Mur18]. This operation provides an explicit homotopy equivalence between an infinite rank matrix factorisation and a finite rank matrix factorisation, both of which represent the composition of two 1-morphisms in $\mathcal{LG}_k$. In Section 5.4 and Section 5.5 we define the cut operation and develop tools for working with the cut. Finally in Section 5.6 we give examples of computing the composition of 1-morphisms in $\mathcal{LG}_k$ using the cut operation.

2.4 A preliminary existence result

Section 3 is devoted to producing formulae for strong deformation retracts between a Koszul complex and its homology. However, it is not too hard to prove that such homotopy equivalences exist, which we do in Proposition 2.4.2 below. Before presenting this result we show, as pointed out in [Cra04, Remark 2.3], that any deformation retract can be modified to produce a strong deformation retract at the cost of producing a more complicated formula for the homotopy. Let $\mathcal{A}$ be an abelian category and

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h \tag{2.4}$$

be a deformation retract in $\mathcal{A}$.

Lemma 2.4.1. *We can modify the homotopy h of (2.4) to produce a strong deformation retract involving (L, d_L), (M, d_M), i and p.*

Proof. First set $h_1 = -h(d_M h + h d_M)$. This gives us

$$h_1 i = -h d_M h i - h^2 d_M i = -h(ip - 1 - h d_M)i - h^2 d_M i = 0$$

and notice that

$$\begin{aligned} h_1 d_M + d_M h_1 &= -h(d_M h + h d_M) d_M - d_M h(d_M h + h d_M) \\ &= -h d_M h d_M - d_M h(d_M h + h d_M) \\ &= -h d_M(ip - 1 - d_M h) - (ip - 1 - h d_M)(ip - 1) \\ &= ip - 1 \end{aligned}$$

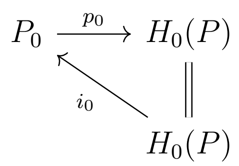
so we have $h_1 : ip \simeq 1$. Next we set $h_2 = -(d_M h_1 + h_1 d_M) h_1$ and likewise note that $p h_2 = 0$, $h_2 : ip \simeq 1$ and $h_2 i = 0$. Finally we set $h_3 = -h_2 d_M h_2$ and note that this gives us the desired strong deformation retract. □

Proposition 2.4.2. *Let (P, d) be a chain complex of projective objects of $\mathcal{A}$ such that $P_n = 0$ for all $n < 0$. Suppose that (P, d) is exact except in degree zero and that $H_0(P)$ is also projective. Then we have a strong deformation retract*

$$(H(P), 0) \leftrightarrows (P, d), \qquad h$$

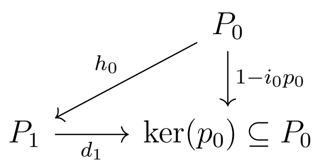
of chain complexes, where $(H(P), 0)$ *is the homology of* P *with zero differentials. Furthermore, the map* $P_0 \to H_0(P)$ *in this strong deformation retract is the quotient map.*

Proof. Recall that $H_0(P) = P_0 / \operatorname{im}(d_1)$. A chain map $p : (P, d) \to (H_0(P), 0)$ is obtained by considering the quotient morphism $p_0 : P_0 \to H_0(P)$. Since $H_0(P)$ is assumed to be projective and p_0 is an epimorphism we obtain a map $i_0 : H_0(P) \to P_0$ such that

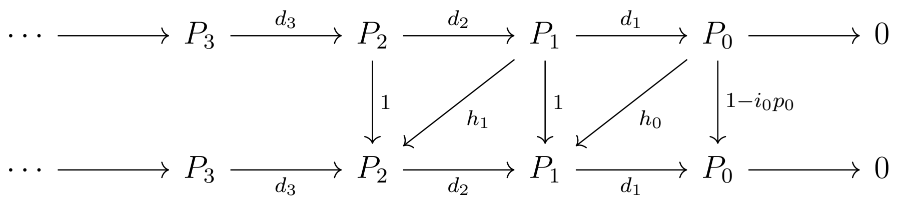


commutes. We extend i_0 to a chain map $i : (H_0(P), 0) \to (P, d)$, and we have $pi = 1$. We now construct a homotopy $h : 1 \simeq ip$.

We construct the maps $h_n : P_n \to P_{n+1}$ by induction on n. For h_0, note that $p_0(1 - i_0p_0) = 0$ and so $1 - i_0p_0$ factors through $\ker(p_0)$. Since $\ker(p_0) = \operatorname{im}(d_1)$ we have that $d_1 : P_1 \to \ker(p_0)$ is an epimorphism and so we can apply the lifting property of projective objects to obtain $h_0 : P_0 \to P_1$ such that

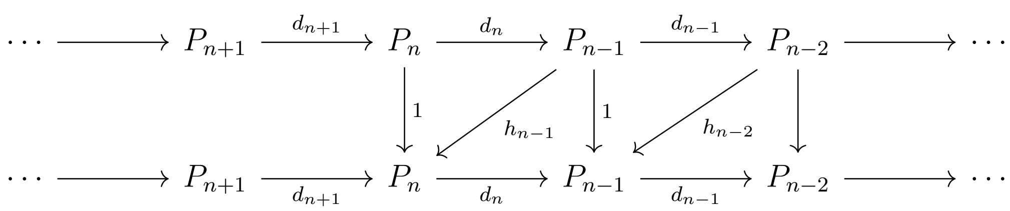


commutes. Next we note that $d_1(1 - h_0d_1) = d_1 - d_1h_0d_1 = d_1 - (1 - i_0p_0)d_1 = 0$ since $p_0d_1 = 0$, so $1 - h_0d_1$ factors through $\ker(d_1)$. Since $\operatorname{im}(d_2) = \ker(d_1)$ we have that $d_2 : P_2 \to \ker(d_1)$ is an epimorphism. Using the lifting property of projective objects we obtain $h_1 : P_1 \to P_2$ satisfying $d_2h_1 = 1 - h_0d_1$. Hence we have constructed the maps h_0 and h_1 in the following diagram



where $1 - i_0p_0 = d_1h_0$ and $1 = d_2h_1 + h_0d_1$.

Now let $n > 1$ and suppose we have constructed h_k for $k < n$. Then we are in the situation



where we have $1 = d_nh_{n-1} + h_{n-2}d_{n-1}$. Note that if $n = 1$ then $P_{-1} = 0$, $h_{-1} = 0$ and $d_0 = 0$. We now aim to construct $h_n : P_n \to P_{n+1}$. Note that $d_n(1 - h_{n-1}d_n) = d_n - d_nh_{n-1}d_n = d_n - (1 - h_{n-2}d_{n-1})d_n = 0$ so $1 - h_{n-1}d_n$ factors through $\ker(d_n)$. Since (P, d) is assumed to be exact in degree n the map $d_{n+1} : P_{n+1} \to \ker(d_n)$ is an epimorphism. Then, using the lifting property of projective objects we have $h_n : P_n \to P_{n+1}$ such that $1 = d_{n+1}h_n + h_{n-1}d_n$. The maps i, p and h form the desired deformation retract, which by Lemma 2.4.1 can be upgraded to a strong deformation retract. $\square$

Due to the use of the lifting property of projective objects, Proposition 2.4.2 does not provide insight into how to construct such strong deformation retracts. We can apply this result to Koszul complexes as follows. Let k be a commutative ring and R a commutative k-algebra.

Corollary 2.4.3. *Let $t = (t_1, \ldots, t_n)$ be a sequence in R such that the Koszul complex $(K(t), d_K)$ is exact except in degree zero. Let I be the ideal generated by the elements of t and suppose both R and R/I are projective k-modules. Then we have a strong deformation retract over k*

$$(R/I, 0) \leftrightarrows (K(t), d_K), \qquad h$$

where the map $K(t) \to R/I$ is the quotient map in degree zero.

Proof. Note that $K(t)$ is a free R-module, hence a projective k-module. The result is an application of Proposition 2.4.2. □

3 Differentiation, division and deformation retracts

In this section we construct an explicit strong deformation retract between a Koszul complex which is exact except in degree zero and its homology by generalising the notion of taking partial derivatives with respect to the variables in a polynomial ring. This approach uses the ideas of [DM13; CM16; Mur18] in which a similar idea is used to prove that composition in the bicategory of Landau-Ginzburg models is well-defined.

In Section 3.2 we do this in the most general setting following the method of [Mur18], providing additional exposition which is not present in the existing literature. This approach requires passing to a completion of the ring and computing the maps in the strong deformation retract involves knowing certain series expansions of ring elements. In Section 3.3 we show how to compute these series expansions in the case of a polynomial ring over a field. This suggests conditions under which it is not necessary to pass to completion in order to construct the strong deformation retract of Section 3.2. This is explained in Section 3.4 and is a new result. In Section 3.5 we build on the ideas of Section 3.4 and give a more efficient algorithm for computing the maps in the strong deformation retract under the same conditions. Later, beginning in Section 5.4, we use the ideas discussed in this section to describe a method for computing compositions in the bicategory of Landau-Ginzburg models.

The approach of Section 3.2 involves defining a generalisation of partial differentiation, where we differentiate with respect to elements of a sequence $t = (t_1, \ldots, t_n)$ of polynomials. In the most general case discussed in Section 3.2 it is necessary to make this definition over the I-adic completion of the polynomial ring, where I is the ideal generated by the elements of t. Under the conditions given in Section 3.4 this partial differentiation can be defined without passing to the completion. Hence another result implicit in this section are conditions under which it is possible to define generalised partial differentiation with respect to a sequence of polynomials.

3.1 Completions and regularity conditions on sequences

We begin with a brief discussion on formal completion and regularity conditions on sequences. The results discussed in this section are "well-known" but some proofs are difficult to find in the literature. Since they play an important role in subsequent sections a full discussion is warranted. When proofs cannot be found easily in the literature we give them here.

Let R be a commutative ring and $t = (t_1, \ldots, t_n)$ a sequence of elements in R. Let $I = (t_1, \cdots, t_n)$ be the ideal generated by the elements of t and $\pi : R \to R/I$ be the quotient map. Consider the polynomial ring $(R/I)[x] = (R/I)[x_1, \cdots, x_n]$ with coefficients in R/I. Define a map

$$\alpha : (R/I)[x] \longrightarrow \bigoplus_{m \geq 0} I^m/I^{m+1} \tag{3.1}$$

by setting $\alpha(x_i) = t_i + I^2$, where we denote $I^0 = R$. This map is always surjective. Indeed, consider $t^u + I^{m+1} \in I^m/I^{m+1}$ where $u \in \mathbb{N}^n$ is such that $\sum_{i=1}^n u_i = m$. It is straightforward to show that $\alpha(x^u) = t^u + I^{m+1}$. Note that any element of I^m/I^{m+1} can be written as a sum of elements of the form $at^u + I^{m+1}$ where $a \in R$ is not divisible by any of the t_i. Applying linearity proves that α is surjective.

Definition 3.1.1. We say the sequence t is:

(1) *regular* if each t_i is not a zero-divisor on $R/(t_1, \ldots, t_{i-1})$, and if the ring R/I is non-zero.

(2) *Koszul-regular* if the Koszul complex of t is exact except in degree zero.

(3) *quasi-regular* if the map α in (3.1) is an isomorphism.

The definition of Koszul-regular was first given in [Kab71, Definition 1] and the definition of quasi-regular was first given in [EGA, Volume IV Chapitre 0 15.1.7]. These regularity conditions and their relationships are also discussed in [Stacks, Sections 10.68, 10.69, 15.30] and quasi-regular sequences in particular are discussed in [Lip87, Chapter 3] and [Mat80, Section 15]. We have the following relationships between the regularity conditions, which is the main result of [Kab71].

Theorem 3.1.2. *For the sequence t we have:*

(1) If t is regular then t is Koszul-regular.

(2) If t is Koszul-regular then t is quasi-regular.

This is proved in [Kab71, Theorem 1.1] and also in [Stacks, Section 15.30]. In [Kab71] there is also a fourth regularity condition which is weaker than Koszul-regularity and stronger than quasi-regularity. Although it is not relevant for our purposes, it is worth pointing out that if R is a Noetherian local ring then any quasi-regular sequence of non-units is necessarily a regular sequence [Stacks, Lemma 10.69.6] and so by Theorem 3.1.2 the regularity conditions of Definition 3.1.1 are equivalent for such sequences in Noetherian local rings. Examples presented in [Kab71] show that the implications in Theorem 3.1.2 cannot be reversed in general, or even under some generous assumptions on the ring R.

We are mainly interested in quasi-regular sequences as they possess favourable properties with respect to completions of the ring R. We first recall the definition of the I-adic completion of R.

Definition 3.1.3. The *I-adic topology* on R is the smallest topology such that R is a topological group under addition and in which a neighbourhood U of zero is open if and only if $I^m \subseteq U$ for some $m \in \mathbb{N}$. The *I-adic completion* of R is the Cauchy completion of R with respect to the I-adic topology.

Let $\widehat{R}$ denote the I-adic completion of R. It is straightforward to show that R is a topological ring with respect to the I-adic topology and hence $\widehat{R}$ is also a ring, where Cauchy sequences are multiplied elementwise. One can show that any element $f \in \widehat{R}$ can be represented by a convergent series of the form

$$f = \sum_{u \in \mathbb{N}^n} s_u t^u$$

where $s_u \in R$. Another standard way of defining completions is as an inverse limit. For each integer $m \geq 2$ we have a canonical ring morphism $q_m : R/I^m \to R/I^{m-1}$. The I-adic completion coincides with the *inverse limit* of these morphisms, which is the ring

$$\varprojlim R/I^m = \left\{ (a_m) \in \prod_{m \geq 1} R/I^m \mid a_m = q_{m+1}(a_{m+1}) \right\} .$$

Elements of $\varprojlim R/I^m$ are called *coherent sequences*. One can see that $\widehat{R}$ is isomorphic to $\varprojlim R/I^m$ by noting that the sequence of partial sums of $f = \sum_{u \in \mathbb{N}^n} c_u t^u \in \widehat{R}$ can be thought of as a coherent sequence. When R is a Noetherian ring the I-adic completion has nice functorial properties.

Lemma 3.1.4 ([AM69, Proposition 10.14])**.** *If R is a Noetherian ring then the canonical ring morphism $R \to \widehat{R}$ is flat, meaning that the functor $\widehat{R} \otimes_R (-)$ on R-modules is exact.*

For more on the basic properties of completions see [AM69, Chapter 10]. We now proceed to discuss two lemmas which play a key role in the subsequent sections.

Lemma 3.1.5. *For I a finitely generated ideal we have $\widehat{R}/I\widehat{R} \cong R/I$.*

Proof. We define the I-adic completion of an R-module M in the same way as the completion of R, where we consider the submodules $I^m M$ in order to define the topology. We have a canonical map $\widehat{R} \otimes_R M \to \widehat{M}$ given by mapping $r \otimes a \mapsto ra$, where $\widehat{M}$ denotes the I-adic completion of M. In [AM69, Proposition 10.13] it is shown that if M is finitely generated then this canonical map is surjective.

Using the above, consider the map $\widehat{R} \otimes_R I \to \widehat{I}$. Since I is a finitely generated ideal this map is surjective. We can also consider the map $\widehat{R} \otimes_R I \to \widehat{R}$ which is again given by mapping $r \otimes i \mapsto ri$. Clearly the diagram

$$\begin{array}{ccc} & & \widehat{R} \\ & \nearrow & \uparrow \\ \widehat{R} \otimes_R I & & \\ & \searrow & \\ & & \widehat{I} \end{array}$$

commutes, and so since $\mathrm{im}(\widehat{R} \otimes_R I \to \widehat{R}) = I\widehat{R}$ and $\widehat{R} \otimes_R I \to \widehat{I}$ is surjective we have $I\widehat{R} = \mathrm{im}(\widehat{I} \to \widehat{R})$.

Next note that

$$\widehat{R} \otimes_R (R/I) \cong (\widehat{R} \otimes_R R)/I(\widehat{R} \otimes_R R) \cong \widehat{R}/I\widehat{R}$$

so it suffices to show $\widehat{R} \otimes_R (R/I)$ is isomorphic to R/I.

We represent elements of $\widehat{R}$ by coherent sequences. Let $r = (r_n)_{n \geq 1} \in \widehat{R}$ and $s \in R/I$. Define a map $\varphi : \widehat{R} \otimes_R (R/I) \to R/I$ by sending $r \otimes s \mapsto r_1 s$ and extending linearly. We also have a map $\psi : R/I \to \widehat{R} \otimes_R (R/I)$ given by mapping $s \mapsto (1) \otimes s$ where $(1) = (1, 1, \ldots)$. Clearly $\varphi\psi = 1$, so it remains to consider the composition $\psi\varphi$. This is given by $\psi\varphi(r \otimes s) = (r_1) \otimes s$ where $(r_1) = (r_1, r_1, \ldots)$. In $\widehat{R}$ we have

$$r - (r_1) = (0, r_2 - r_1, r_3 - r_1, \ldots) \in \mathrm{im}(\widehat{I} \to \widehat{R}) = I\widehat{R}$$

and so $r \otimes s - \psi\varphi(r \otimes s) = 0$, proving the claim. □

Let k be a commutative ring. Suppose R is a commutative k-algebra and notice that $\widehat{R}$ inherits a k-algebra structure from R. Now suppose that there exists a k-linear section $\sigma : R/I \to R$ of the quotient map π. That is, σ is such that $\pi\sigma = 1$. Such a map exists, for example, if R/I is projective over k. In this case the short exact sequence

$$0 \longrightarrow \ker(\pi) \longrightarrow R \xrightarrow{\ \pi\ } R/I \longrightarrow 0$$

splits over k, producing the required section of π. In particular, we always have a k-linear section of the quotient map when k is a field.

When a k-linear section of π exists, quasi-regular sequences generating I are sequences whose powers "independently generate $\widehat{R}$ over R/I" in the sense of the following lemma. This result is stated without proof in [Lip87, Lemma 3.1.1].

Lemma 3.1.6. *Suppose t is quasi-regular and that there exists a k-linear section $\sigma : R/I \to R$ of the quotient map $\pi : R \to R/I$. Then every $f \in \widehat{R}$ can be written uniquely as a convergent series of the form*

$$f = \sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u \tag{3.2}$$

where $r_u \in R/I$ and $t^u = t_1^{u_1} \cdots t_n^{u_n}$.

Proof. Let $\kappa : R \to \widehat{R}$ denote the canonical map into the completion. We have $\widehat{R}/I\widehat{R} \cong R/I$, so let $\widehat{\pi} : \widehat{R} \to R/I$ denote the quotient map. Note that this satisfies $\widehat{\pi}\kappa = \pi$, and so we can extend $\sigma : R/I \to R$ to a k-linear section $\widehat{\sigma}$ of $\widehat{\pi}$ by setting $\widehat{\sigma} = \kappa\sigma$.

Let $f \in \widehat{R}$. Then we have

$$\widehat{\pi}(f - \widehat{\sigma}\widehat{\pi}(f)) = 0$$

and so $f - \widehat{\sigma}\widehat{\pi}(f) \in I\widehat{R}$. Let $a_1, \ldots, a_n \in \widehat{R}$ be such that $f - \widehat{\sigma}\widehat{\pi}(f) = \sum_{i=1}^n a_i t_i$. In the same manner we obtain $a_{ij} \in \widehat{R}$ such that $a_i - \widehat{\sigma}\widehat{\pi}(a_i) = \sum_{j=1}^n a_{ij} t_j$ for all $i = 1, \ldots, n$. Hence we have

$$\begin{aligned} f &= \widehat{\sigma}\widehat{\pi}(f) + \sum_{i=1}^n a_i t_i \\ &= \widehat{\sigma}\widehat{\pi}(f) + \sum_{i=1}^n \widehat{\sigma}\widehat{\pi}(a_i) t_i + \sum_{i,j} a_{ij} t_i t_j \ . \end{aligned}$$

Let $e_i \in \mathbb{N}^n$ have a one in the i^{th} coordinate and zeros elsewhere. By setting $r_0 = \widehat{\pi}(f)$ and $r_{e_i} = \widehat{\pi}(a_i)$ we obtain the first few terms in the series expansion (3.2). Further terms can be produced by continuing in the same way.

For uniqueness, suppose for a contradiction that we have

$$\sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u = 0 \tag{3.3}$$

but at least one $r_u \neq 0$. Let $U = \min\{|u| \mid r_u \neq 0\}$, where $|u| = \sum_{i=1}^n u_i$ for $u \in \mathbb{N}^n$. The series (3.3) converges to zero in the I-adic topology, which means that there exists an M such that for all $m \geq M$ we have

$$\sum_{|u| \leq m} \sigma(r_u) t^u \in I^{U+1} \ .$$

By rearranging the equation

$$\sum_{|u| \leq m} \sigma(r_u) t^u = \sum_{|u| = U} \sigma(r_u) t^u + \sum_{U < |u| \leq m} \sigma(r_u) t^u$$

we find that $\sum_{|u|=U} \sigma(r_u) t^u \in I^{U+1}$, or in other words

$$\sum_{|u|=U} \sigma(r_u) t^u = 0 \qquad \text{in the ring } I^U/I^{U+1} \ .$$

Since t is quasi-regular this implies $\sigma(r_u) \in I$ for all $|u| = U$. Indeed, if this were not the case then this would give us a non-zero element of $(R/I)[x]$ which is sent to zero by the map α of (3.1). Applying π gives $r_u = 0$ for all $u \in \mathbb{N}^n$ such that $|u| = U$, which is a contradiction. □

3.2 Derivative systems and the strong deformation retract

Let k be a commutative ring and R a commutative k-algebra. Consider a quasi-regular sequence $t = (t_1, \dots, t_n)$ of elements of R and let $I = (t_1, \dots, t_n)$ be the ideal generated by the elements of t. Let $\widehat{R}$ be the I-adic completion of R and suppose we have a k-linear section $\sigma : R/I \to R$ of the quotient map such that $\sigma(1) = 1$.

For each t_i we define a map $\partial_{t_i} : \widehat{R} \to \widehat{R}$ as follows. Given $f \in \widehat{R}$, by Lemma 3.1.6 we can write f uniquely in the form $f = \sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u$. We define

$$\partial_{t_i}(f) = \sum_{u \in \mathbb{N}^n \setminus \{0\}} u_i \sigma(r_u) t^{u - e_i}$$

where $e_i = (0, \dots, 1, \dots, 0)$ has a one in its i^{th} entry and zeros elsewhere. In the following lemma we collect some basic facts about these maps, all of which follow directly from the definition.

Lemma 3.2.1. *Let $k[|t|]$ denote the k-algebra which consists of elements of $\widehat{R}$ of the form $\sum_{u \in \mathbb{N}^n} c_u t^u$ where $c_u \in k$.*

(1) $\partial_1, \dots, \partial_n$ are k-linear.

(2) $\partial_{t_i} \partial_{t_j} = \partial_{t_j} \partial_{t_i}$ for all $i, j = 1, \dots, n$.

(3) $\partial_{t_i}(t^v) = v_i t^{v - e_i}$ for all $i = 1, \dots, n$ and $v \in \mathbb{N}^n$, where we understand $0 t_j^{-1} = 0$.

(4) $\operatorname{im}(\sigma) = \bigcap_i \ker(\partial_{t_i})$.

(5) For $f \in k[|t|]$ and $r \in \bigcap_i \ker(\partial_{t_i})$ we have $\partial_{t_j}(rf) = r \partial_{t_j}(f)$ for all $j = 1, \dots, n$.

(6) (Leibniz rule) For $f \in k[|t|]$ and $g \in \widehat{R}$ we have $\partial_{t_j}(fg) = \partial_{t_j}(f) g + f \partial_{t_j}(g)$ for all $j = 1, \dots, n$.

Proof. Properties (1), (2), (4) and (5) follow directly from the definition. Property (3) also follows directly from the definition, and this is where we make use of the hypothesis that $\sigma(1) = 1$. To prove (6), let $g = \sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u \in \widehat{R}$ and $v \in \mathbb{N}^n$ and note that

$$\partial_{t_i}(t^v g) = \sum_{u \in \mathbb{N}^n} \sigma_{r_u} \partial_{t_i}(t^v t^u) = \sum_{u \in \mathbb{N}^n} \sigma_{r_u} (\partial_{t_i}(t^v) t^u + t^v \partial_{t_i}(t^u)) = \partial_{t_i}(t^v) g + t^v \partial_{t_i}(g)$$

where $t^v = t_1^{v_1} \cdots t_n^{v_n}$. Extending linearly proves the general statement of (6). □

As Lemma 3.2.1 suggests, the maps $\partial_{t_1}, \dots, \partial_{t_n}$ should be thought of as a type of partial differentiation with respect to the sequence $t = (t_1, \dots, t_n)$. Various combinations of the properties in Lemma 3.2.1 uniquely characterise these maps. For example one can show properties (1), (3), (4) and (5) uniquely determine $\partial_{t_1}, \dots, \partial_{t_n}$, and likewise for properties (1), (4) and (6). We refer to any sequence of maps satisfying all the properties of Lemma 3.2.1 as a *system of t-derivatives*. In later sections we will see that when R is a polynomial ring and t satisfies certain conditions that the system of t-derivatives exist as maps on R, rather than on its I-adic completion.

Next we consider the Koszul complex $(K(t), d_K)$ of t. We denote the graded module of the Koszul complex by $K(t) = \bigwedge(\bigoplus_{i=1}^n R dt_i)$ where $dt_1, \dots, dt_n$ are formal generators. This is in analogy with differential geometry: when $R = k[x_1, \dots, x_n]$ the underlying module of the Koszul complex is the exterior algebra of the module of *Kähler differentials* and we think of the degree p part of $K(t)$ as being the module of "p-forms on affine n-space".

We pass $(K(t), d_K)$ to the I-adic completion by applying the extension of scalars functor $\widehat{R} \otimes_R (-)$ to obtain the complex $(\widehat{K}(t), d_{\widehat{K}})$, which is the same as the Koszul complex of t regarded as a sequence in $\widehat{R}$. The underlying graded module of this complex is $\widehat{K}(t) = \bigwedge(\bigoplus_{i=1}^n \widehat{R}dt_i)$.

Definition 3.2.2. Given the system of t-derivatives $\partial_{t_1}, \ldots, \partial_{t_n}$ the corresponding *connection* is defined as the k-linear map $\nabla : \widehat{K}(t) \to \widehat{K}(t)$ given by

$$\nabla(f\omega) = \sum_{i=1}^n \partial_{t_i}(f) dt_i \omega$$

for $f \in R$ and $\omega = dt_{i_1} \cdots dt_{i_p}$.

The connection ∇ is the same as the connection of [Mur18, Section 3] and [DM13, Definition 2.8]. In [Mur18; DM13] a connection is first proved to exist and from this the maps $\partial_{t_1}, \ldots, \partial_{t_n}$ are extracted.

Lemma 3.2.3. $\nabla^2 = 0$.

Proof. Let $f\omega \in \widehat{K}(t)$ where $f \in \widehat{R}$ and $\omega = dt_{i_1} \cdots dt_{i_p}$. Then

$$\begin{aligned}
\nabla^2(f\omega) &= \sum_{i=1}^n \nabla(\partial_{t_i}(f) dt_i \omega) \\
&= \sum_{i=1}^n \sum_{j=1}^n \partial_{t_j}\partial_{t_i}(f) dt_j dt_i \omega \\
&= \sum_{i<j} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i \omega + \sum_{j<i} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i \omega \\
&= \sum_{i<j} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i \omega - \sum_{i<j} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i \omega \\
&= 0
\end{aligned}$$

using Lemma 3.2.1 and the fact that $dt_j dt_i = -dt_i dt_j$. □

In the terminology of [DM13; Mur18], Lemma 3.2.3 shows that ∇ is *flat* [DM13, Definition 2.8].

Lemma 3.2.4. *If $\mathbb{Q} \subseteq k$ then $d_{\widehat{K}}\nabla(k[|t|]) = I\widehat{R}$, where $k[|t|]$ is the k-algebra consisting of elements of $\widehat{R}$ the form $\sum_{u \in \mathbb{N}^n} c_u t^u$ where $c_u \in k$.*

Proof. For $t^u = t_1^{u_1} \cdots t_n^{u_n}$ we have

$$\begin{aligned}
d_{\widehat{K}}\nabla(t^u) &= \sum_{j=1}^n d_{\widehat{K}}(u_j t^{u-e_j} dt_j) \\
&= \sum_{j=1}^n \sum_{i=1}^n u_j t_i t^{u-e_j} dt_i^* \neg (dt_j) \\
&= |u| t^u
\end{aligned}$$

where $|u| = \sum_{i=1}^n u_i$. To avoid a more tedious argument in the case that some $u_j = 0$ we permit ourselves to write $0t_j^{-1} = 0$ and note that the above calculation is still correct. Hence we have $d_{\widehat{K}}\nabla(k[|t|]) \subseteq I\widehat{R}$. But the above calculation also shows that for any $t^u \in I\widehat{R}$ we have $t^u = d_{\widehat{K}}\nabla(|u|^{-1}t^u)$, where we know $|u|$ is invertible since $\mathbb{Q} \subseteq k$ and at least one $u_j \neq 0$. Applying linearity proves the claim. □

In the terminology of [DM13; Mur18], Lemma 3.2.4 shows that when $\mathbb{Q} \subseteq k$ the connection ∇ is *standard* [DM13, Definition 8.6].

Lemma 3.2.5. *The graded commutator of $d_{\widehat{K}}$ and ∇, denoted $[d_{\widehat{K}}, \nabla]$, is a morphism of complexes.*

Proof. First note that $[d_{\widehat{K}}, \nabla]$ has degree $\deg(d_{\widehat{K}}) + \deg(\nabla) = 0$. Then we have

$$[d_{\widehat{K}}, \nabla]d_{\widehat{K}} = d_{\widehat{K}}\nabla d_{\widehat{K}} + \nabla d_k^2 = d_{\widehat{K}}\nabla d_{\widehat{K}} + d_{\widehat{K}}^2\nabla = d_{\widehat{K}}[d_{\widehat{K}}, \nabla]$$

where we use that $d_{\widehat{K}}^2 = 0$. □

Lemma 3.2.6. *Let $\omega = dt_{i_1} \cdots dt_{i_p}$ and $f \in \widehat{R}$. Then $[d_{\widehat{K}}, \nabla](f\omega) = (p + d_{\widehat{K}}\nabla_0)(f) \cdot \omega$, where ∇_0 is the degree zero part of ∇.*

Proof. Without loss of generality suppose $i_1 < \cdots < i_p$. We first consider the case of $f = t^u = t_1^{u_1} \cdots t_n^{u_n}$ for $u \in \mathbb{N}^n$. Let $\delta_j = 0$ if $j \in \{i_1, \dots, i_p\}$ and $\delta_j = 1$ otherwise. Then we have

$$\begin{aligned} d_{\widehat{K}}\nabla(t^u\omega) &= \sum_{j=1}^n d_{\widehat{K}}(u_j t^{u-e_j} dt_j\omega) \\ &= \sum_{j=1}^n \delta_j \sum_{s=1}^n u_j t_s t^{u-e_j} dt_s^* \lrcorner (dt_j\omega) \\ &= \sum_{j=1}^n \delta_j u_j t^u \omega + \sum_{(j,l)\in A} \delta_j (-1)^l u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt}_{i_l} \cdots dt_{i_p} \end{aligned}$$

where $A = \{(j,l) \mid j = 1, \dots n,\ l = 1, \dots, p \text{ and } j \neq i_l\}$ and "$\widehat{dt}_{i_p}$" indicates that this factor is omitted from the above product. We also have

$$\begin{aligned} \nabla d_{\widehat{K}}(t^u\omega) &= \sum_{l=1}^p (-1)^{l+1} \nabla(t_{i_l} t^u dt_{i_1} \cdots \widehat{dt}_{i_l} \cdots dt_{i_p}) \\ &= \sum_{l=1}^p \sum_{j=1}^n (-1)^{l+1} \partial_{t_j}(t_{i_l} t^u) dt_j dt_{i_1} \cdots \widehat{dt}_{i_l} \cdots dt_{i_p} \\ &= \sum_{l=1}^p (-1)^{l+1} (u_{i_l} + 1) t^u dt_{i_l} dt_{i_1} \cdots \widehat{dt}_{i_l} \cdots dt_{i_p} \\ &\qquad + \sum_{(j,l)\in A} \delta_j (-1)^{l+1} u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt}_{i_l} \cdots dt_{i_p} \\ &= \sum_{j=1}^n (1-\delta_j)(u_j+1) t_u \omega - \sum_{(j,l)\in A} \delta_j (-1)^l u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt}_{i_l} \cdots dt_{i_p} \ . \end{aligned}$$

Hence we find

$$[d_{\widehat{K}}, \nabla](t^u\omega) = \left(\sum_{i=1}^n (1-\delta_j) + |u| \right) t^u \omega$$

where $|u| = \sum_{i=1}^n u_i$. Notice that $\sum_{i=1}^n (1-\delta_j) = p$ and by inspecting the proof of Lemma 3.2.4 we have $d_{\widehat{K}}\nabla(t^u) = |u| t^u$. Therefore we have shown

$$[d_{\widehat{K}}, \nabla](t^u\omega) = (p + d_{\widehat{K}}\nabla_0)(t^u) \cdot \omega \ .$$

For $f \in \widehat{R}$ write f in the form $f = \sum_{u\in\mathbb{N}^n} \sigma(r_u)t^u$ where each $r_u \in R/I$. Then by definition of the system of t-derivatives we have

$$\begin{aligned}[d_{\widehat{K}}, \nabla](f\omega) &= \sum_{u\in\mathbb{N}^n} \sigma(r_u)[d_{\widehat{K}}, \nabla](t^u\omega)\\ &= \sum_{u\in\mathbb{N}^n} \sigma(r_u)(p + d_{\widehat{K}}\nabla_0)(t^u)\cdot\omega\\ &= (p + d_{\widehat{K}}\nabla_0)(f)\cdot\omega\end{aligned}$$

which proves the general case. □

Lemma 3.2.7. *If $\mathbb{Q} \subseteq k$ then $[d_{\widehat{K}}, \nabla]$ is invertible away from degree zero.*

Proof. Let $p > 0$ and consider $\omega = dt_{i_1}\cdots dt_{t_p}$. By Lemma 3.2.6 we have

$$[d_{\widehat{K}}, \nabla](t^u\omega) = c\cdot t^u\omega$$

where $c = p + |u|$. Note that $c \neq 0$, and so since $\mathbb{Q} \subseteq k$ we know c^{-1} exists. This proves both injectivity and surjectivity, where for injectivity we also appeal to the uniqueness of Lemma 3.1.6. □

Now suppose that $\mathbb{Q} \subseteq k$ and also that $(\widehat{K}(t), d_{\widehat{K}})$ is exact except in degree zero. That is, we assume that t is a Koszul-regular sequence in $\widehat{R}$. This is true if t is Koszul-regular in R and R is Noetherian since in this case the canonical map $R \to \widehat{R}$ is flat, meaning that the functor $\widehat{R}\otimes_R(-)$ preserves exact sequences. In [Mur19, Appendix C] it is shown that if we instead assume R and R/I are projective k-modules, then t being Koszul-regular in R also implies t is Koszul-regular in $\widehat{R}$. The point is that $(\widehat{K}(t), d_{\widehat{K}})$ being exact in degree zero follows from t being Koszul-regular in R under many natural conditions on R and k.

We define $H : K(t) \to K(t)$ by $H = [d_{\widehat{K}}, \nabla]^{-1}\nabla$ in all degrees where $K(t)$ is non-zero and by $H = 0$ elsewhere. Note that since ∇ is a degree $+1$ map $[d_{\widehat{K}}, \nabla]^{-1}$ exists by Lemma 3.2.7 in all degrees needed to define H.

Lemma 3.2.8. *Under the preceding assumptions, the degree zero part of $1 - d_{\widehat{K}}H$ factors through $\widehat{R}/I\widehat{R}$.*

Proof. Let $f \in I\widehat{R}$. In Lemma 3.2.4 we showed that $d_{\widehat{K}}\nabla(k[t]) = (t)$ so let $g \in k[t]$ be such that $d_{\widehat{K}}\nabla(g) = f$. Then we have

$$\begin{aligned}(1 - d_{\widehat{K}}H)(f) &= (1 - d_{\widehat{K}}H)(d_{\widehat{K}}\nabla)(g)\\ &= d_{\widehat{K}}\nabla(g) - d_{\widehat{K}}[d_{\widehat{K}}, \nabla]^{-1}\nabla d_{\widehat{K}}\nabla(g)\\ &= d_{\widehat{K}}\nabla(g) - d_{\widehat{K}}[d_{\widehat{K}}, \nabla]^{-1}([d_{\widehat{K}}, \nabla] - d_{\widehat{K}}\nabla)\nabla(g)\\ &= 0\end{aligned}$$

since $\nabla^2 = 0$ by Lemma 3.2.3. □

From Lemma 3.1.5 we have that $R/I \cong \widehat{R}/I\widehat{R}$. Let $\pi : \widehat{K}_0(t) \to R/I$ be the quotient map. By Lemma 3.2.8 we obtain a map $\sigma' : R/I \to \widehat{K}_0(t)$ satisfying $1 - d_{\widehat{K}}H_0 = \pi\sigma'$. Let $(R/I, 0)$ denote the chain complex which has R/I in degree zero and is zero elsewhere. We extend both π and σ' to chain maps $\pi : (\widehat{K}(t), d_{\widehat{K}}) \to (R/I, 0)$ and $\sigma' : (R/I, 0) \to (\widehat{K}(t), d_{\widehat{K}})$ by setting $\pi = 0$ and $\sigma' = 0$ away from degree zero.

Proposition 3.2.9. *Under the preceding assumptions, the maps π, σ' and H form a strong deformation retract*

$$(R/I, 0) \underset{\sigma'}{\overset{\pi}{\rightleftarrows}} (\widehat{K}(t), d_{\widehat{K}}), \qquad H$$

over k.

Proof. We need to show that:

(1) $\pi\sigma' = 1$.

(2) $Hd_{\widehat{K}} + d_{\widehat{K}}H = 1 - \sigma'\pi$.

(3) $H^2 = 0$, $\pi H = 0$ and $H\sigma' = 0$.

For (1), first note that away from degree zero this is clear. In degree zero note that by considering the k-linear section of π we can represent any element of R/I by $\pi(f)$ where $f \in \widehat{R}$ is such that $\partial_{t_i}(f) = 0$ for all $i = 1, \dots, n$. Then we have

$$\pi\sigma'\pi(f) = \pi(f - d_{\widehat{K}}H(f)) = \pi(f) - \pi d_{\widehat{K}}[d_{\widehat{K}}, \nabla]^{-1}\nabla(f) = \pi(f)$$

since $\nabla(f) = 0$. Hence we have $\pi\sigma' = 1$ since π is an epimorphism.

For (2), in degree zero we have $1 - \sigma'\pi_0 = d_{\widehat{K}}H_0$ by our construction of σ'. For degree $p > 0$ we have

$$\begin{aligned} Hd_{\widehat{K}} + d_{\widehat{K}}H &= [d_{\widehat{K}}, \nabla]^{-1}\nabla d^p_{\widehat{K}} + d_{\widehat{K}}[d_{\widehat{K}}, \nabla]^{-1}\nabla_p \\ &= [d_{\widehat{K}}, \nabla]^{-1}\nabla d^p_{\widehat{K}} + [d_{\widehat{K}}, \nabla]^{-1}d_{\widehat{K}}\nabla_p \\ &= [d_{\widehat{K}}, \nabla]^{-1}[d_{\widehat{K}}, \nabla] \\ &= 1 \end{aligned}$$

where $d^p_{\widehat{K}}$ and ∇_p are the degree p parts of $d_{\widehat{K}}$ and ∇ respectively. Note we only had $d_{\widehat{K}}[d_{\widehat{K}}, \nabla]^{-1}\nabla_p = [d_{\widehat{K}}, \nabla]^{-1}d_{\widehat{K}}\nabla_p$ because $p > 0$. If $p = 0$ then the right-hand-side becomes $[d_{\widehat{K}}, \nabla]^{-1}d^1_{\widehat{K}}\nabla_0$, and $[d_{\widehat{K}}, \nabla]^{-1}$ does not exist in degree zero.

For (3), first note that $H^2 = [d_{\widehat{K}}, \nabla]^{-1}\nabla[d_{\widehat{K}}, \nabla]^{-1}\nabla$ so for $H^2 = 0$ it suffices to show $\nabla[d_{\widehat{K}}, \nabla]^{-1}\nabla = 0$. We showed in Lemma 3.2.3 that $\nabla^2 = 0$, so we have

$$\nabla[d_{\widehat{K}}, \nabla] = \nabla d_{\widehat{K}}\nabla + \nabla^2 d_{\widehat{K}} = \nabla d_{\widehat{K}}\nabla + d_{\widehat{K}}\nabla^2 = [d_{\widehat{K}}, \nabla]\nabla\ .$$

Then $\nabla = [d_{\widehat{K}}, \nabla]\nabla[d_{\widehat{K}}, \nabla]^{-1}$. Multiplying on the right by ∇ gives $0 = [d_{\widehat{K}}, \nabla]\nabla[d_{\widehat{K}}, \nabla]^{-1}\nabla$ which implies that $\nabla[d_{\widehat{K}}, \nabla]^{-1}\nabla = 0$ since $[d_{\widehat{K}}, \nabla]$ is invertible in this degree. Hence $H^2 = 0$. Next, we have $\pi_p H_{p-1} = 0$ in all degrees p since $\pi_p = 0$ when $p \neq 0$, and when $p = 0$ we have $H_{-1} = 0$. Finally, we show $H\sigma' = 0$. Away from degree zero this is clear and in degree zero we have

$$\begin{aligned} H\sigma'\pi &= H - Hd_{\widehat{K}}H \\ &= [d_{\widehat{K}}, \nabla]^{-1}\nabla - [d_{\widehat{K}}, \nabla]^{-1}\nabla d_{\widehat{K}}[d_{\widehat{K}}, \nabla]^{-1}\nabla \\ &= [d_{\widehat{K}}, \nabla]^{-1}\nabla - [d_{\widehat{K}}, \nabla]^{-1}([d_{\widehat{K}}, \nabla] - d_{\widehat{K}}\nabla)[d_{\widehat{K}}, \nabla]^{-1}\nabla \\ &= [d_{\widehat{K}}, \nabla]^{-1}d_{\widehat{K}}\nabla[d_{\widehat{K}}, \nabla]^{-1}\nabla \\ &= 0 \end{aligned}$$

where we again use that $\nabla[d_{\widehat{K}}, \nabla]^{-1}\nabla = 0$. ∎

Compare Proposition 3.2.9 to Corollary 2.4.3, where we also showed that there is a strong deformation retract between the Koszul complex of a Koszul-regular sequence and its homology. Proposition 3.2.9 provides explicit formulae for the maps in the strong deformation retract at the cost of passing to the I-adic completion of R, while Corollary 2.4.3 does not give any information about how to compute these maps. In Corollary 2.4.3 we had the hypothesis that R and R/I were projective over k, whereas in Proposition 3.2.9 we only needed to assume that the quotient map $\pi : R \to R/I$ has a k-linear section. On the other hand, in Proposition 3.2.9 we need to assume $\mathbb{Q} \subseteq k$ while no such assumption is needed for Corollary 2.4.3.

While Proposition 3.2.9 does provide formulae for the maps in the strong deformation retract it is not clear how to compute these maps elementwise. In order to compute the connection ∇ at an element $f \in \widehat{R}$ we need to know the series expansion

$$f = \sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u$$

where $r_u \in R/I$. In this general setting it is not clear how to compute the coefficients $\sigma(r_u)$. In the next section we will show how these coefficients can be generated when $f \in R$ and R is a polynomial ring over a field.

3.3 The case of a polynomial ring over a field

In the previous section we gave formulae for a strong deformation between a Koszul complex and its homology over a completion of the ring. Our ability to actually compute these maps, however, relies on knowing certain series expansions of ring elements. In this section and the next we address this issue in the case of a polynomial ring over a field.

Let k be a field and consider the polynomial ring $k[x] = k[x_1, \dots, x_m]$. Let $t = (t_1, \dots, t_n)$ be a quasi-regular sequence in $k[x]$ and $I = (t_1, \dots, t_n)$ the ideal generated by the elements of t. Since k is a field there certainly exists a k-linear section of the quotient map $k[x] \to k[x]/I$. Over a polynomial ring we can do much better and choose the section so that we have an algorithm for computing coefficients in the series expansion from Lemma 3.1.6.

This algorithm uses Euclidean division of polynomials, so we begin by recalling some related concepts following the conventions of [CLO15, Chapter 2]. A *monomial* in $k[x]$ is any polynomial in the set $\{x^u\}_{u \in \mathbb{N}^m}$. A *monomial ordering* on $k[x]$ is a well-founded, total order relation $<$ on $\mathbb{N}^m$ with the property that $u < v \implies u + w < v + w$ for all $u, v, w \in \mathbb{N}^m$. A typical example of a monomial ordering is the lexicographic ordering on $\mathbb{N}^m$. Other examples are given in [CLO15, Section 2.2].

Let $f = \sum_{u \in \mathbb{N}^m} c_u x^u \in k[x]$ where $c_u \in k$ and finitely many $c_u \neq 0$. Any $c_u x^u$ for which $c_u \neq 0$ is called a *term* of f. Given a monomial ordering on $k[x]$, if $f \neq 0$ we define the *multi-degree* of f as

$$\operatorname{multideg}(f) = \max\{u \in \mathbb{N}^m \mid c_u \neq 0\}$$

where the maximum is taken with respect to the monomial ordering. Setting $u^* = \operatorname{multideg}(f)$ we define the *leading term* of f with respect to the given monomial ordering to be $\operatorname{LT}(f) = c_{u^*} x^{u^*}$. The coefficient c_{u^*} is called the *leading coefficient* and is denoted $\operatorname{LC}(f)$. We recall the properties of the division algorithm on $k[x]$ given in [CLO15, Theorem 2.3.3].

Theorem 3.3.1 (Division Algorithm)**.** *Given a monomial ordering on $k[x]$ and polynomials $f, s_1, \dots, s_n \in k[x]$ the division algorithm produces $r, q_1, \dots, q_n \in k[x]$ which satisfy*

(1) $f = r + \sum_{i=1}^{n} q_i s_i$.

(2) None of the terms of r *are divisible by* $\mathrm{LT}(s_i)$ *for any* $i = 1, \ldots, n$.

(3) For all $i = 1, \ldots, n$ *with* $q_i \neq 0$ *we have* $\mathrm{multideg}(f) \geq \mathrm{multideg}(q_i s_i)$.

We call the polynomial r in Theorem 3.3.1 a *remainder* of f divided by $s_1, \ldots, s_n \in k[x]$. The division algorithm is a fundamental computational tool in polynomial rings. For example, it can be used to compute whether an element f belongs to an ideal $J = (s_1, \ldots, s_n)$. If the remainder of dividing f by $s_1, \ldots, s_n$ is zero then we know $f \in J$. For general $s_1, \ldots, s_n$ the converse is unfortunately false, however for certain generating sets called *Gröbner bases* we do have that $f \in J$ if and only if the remainder found by the division algorithm is zero.

Definition 3.3.2. Fix a monomial ordering on $k[x]$ and let J be an ideal. Consider the set of leading terms of elements of J:

$$\mathrm{LT}(J) = \{\mathrm{LT}(f) \mid f \in J \setminus \{0\}\} \ .$$

We say that $g = (g_1, \ldots, g_n)$ is a *Gröbner basis for* J if g generates J and if the ideal generated by $\mathrm{LT}(J)$ is equal to $(\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_n))$. Given a sequence $g = (g_1, \ldots, g_n)$ we say that g is a *Gröbner basis* to mean that g is a Gröbner basis for the ideal generated by its elements.

Lemma 3.3.3 ([CLO15, Corollary 2.6.2])**.** *Fix a monomial ordering on* $k[x]$ *and let* $f \in k[x]$*. If* g *is a Gröbner basis for an ideal* J *then, when we apply the division algorithm to divide* f *by* g*, the remainder is zero if and only if* $f \in J$*.*

Whether or not a sequence is a Gröbner basis depends on the monomial ordering on $k[x]$. Given a finite generating set for an ideal and a monomial ordering we can compute a Gröbner basis for that ideal using an algorithm called *Buchberger's Algorithm.* This is given in [CLO15, Theorem 2.7.2].

Theorem 3.3.4 (Buchberger's Algorithm)**.** *Let* $J = (s_1, \ldots, s_n)$ *be an ideal of* $k[x]$*. Given a monomial ordering on* $k[x]$ *there is an algorithm computing polynomials* $g = (g_1, \ldots, g_{n'})$ *and* $\{a_{ij}\}_{i,j}$ *such that* g *is a Gröbner basis for* J *and* $g_i = \sum_{j=1}^{n} a_{ij} s_j$*.*

We now return to consider the quasi-regular sequence t and the ideal I generated by the elements of t. A Gröbner basis for I can be used to produce a k-linear section of the quotient map $\pi : k[x] \to k[x]/I$. Fix a monomial ordering on $k[x]$ and let $g = (g_1, \ldots, g_{n'})$ be a Gröbner basis for I. Define the k-vector space

$$C = \{r \in k[x] \mid \text{no term of } r \text{ is divisible by any of the } \mathrm{LT}(g_i)\} \ . \tag{3.4}$$

Lemma 3.3.5. *The quotient map* $\pi : k[x] \to k[x]/I$ *restricts to an isomorphism* $C \to k[x]/I$*.*

Proof. For injectivity suppose $r \in C$ is such that $\pi(r) = 0$. Applying the division algorithm to divide r by g yields the remainder r, since none of the terms in r are divisible by any of the $\mathrm{LT}(g_i)$. Since $r \in I$, by Lemma 3.3.3 we have $r = 0$.

For surjectivity, consider $f \in k[x]$. Via the division algorithm we obtain an expression for f of the form

$$f = r + \sum_{i=1}^{n} q_i g_i$$

where $r \in C$. Then $\pi(f) = \pi(r)$. Noting that $\pi : k[x] \to k[x]/I$ is surjective proves the claim. □

Note that if g is not a Gröbner basis then the restriction $\pi|_C$ will fail to be injective. Lemma 3.3.5 gives us a k-linear section $\sigma : k[x]/I \to k[x]$ of π by letting σ be the inverse of $\pi|_C$. Hence by Lemma 3.1.6 we obtain the following result. Let $\widehat{k[x]}$ denote the I-adic completion of $k[x]$.

Lemma 3.3.6. *Any element $f \in \widehat{k[x]}$ can be uniquely expressed as a series of the form*

$$f = \sum_{u \in \mathbb{N}^n} r_u t^u$$

where $r_u \in C$.

Proof. Immediate from Lemma 3.1.6 and Lemma 3.3.5. □

We now consider an algorithm to generate the coefficients in the series expansion of Lemma 3.3.6 for an element of the polynomial ring $f \in k[x]$. The idea is as follows. Let $\{a_{ij}\}_{i,j}$ be the polynomials arising from Buchberger's Algorithm which satisfy $g_i = \sum_{j=1}^n a_{ij} t_j$. Given $f \in k[x]$ we can divide f by g to obtain polynomials $r_0 \in C$ and $q_1, \ldots, q_{n'} \in k[x]$ satisfying

$$f = r_0 + \sum_{i=1}^{n'} q_i g_i = r_0 + \sum_{j=1}^{n} \left(\sum_{i=1}^{n'} a_{ij} q_i \right) t_j \ .$$

Setting $p_j = \sum_{i=1}^{n'} a_{ij} g_i$, we can then divide each of the $p_1, \ldots, p_n$ by g to obtain polynomials $r_j \in C$ and $q_{1j}, \ldots, q_{n'j} \in k[x]$ for $j = 1, \ldots, n$ satisfying

$$f = r_0 + \sum_{j=1}^{n} r_j t_j + \sum_{j=1}^{n} \sum_{i=1}^{n'} q_{ij} g_i t_j = r_0 + \sum_{j=1}^{n} r_j t_j + \sum_{j,l=1}^{n} \left(\sum_{i=1}^{n'} q_{ij} a_{il} \right) t_i t_l \ .$$

The polynomials $r_0, r_1, \ldots, r_n \in C$ are the coefficients of the zeroth and first order terms in the series expansion for f of Lemma 3.3.6, and we can continue to generate higher order coefficients in this manner. This algorithm is formalised in Algorithm 3.3.7, which generates tuples (u, r) where $r \in C$ and $u \in \mathbb{N}^n$. Such an output indicates that r is the coefficient of t^u in the unique series expansion of f given in Lemma 3.3.6. In general this algorithm will not terminate, but rather can be viewed as generating a sequence converging to f in the I-adic topology by considering the partial sums of the series expansion for f. Hence, Algorithm 3.3.7 can be used in conjuncton with the results in Section 3.2 to produce I-adic approximations of the maps in the strong deformation retract of Proposition 3.2.9.

Before continuing we comment on the assumption that k is a field. This was needed for two reasons:

- We needed the concept of a Gröbner basis for the ideal I of $k[x]$.
- We needed an algorithm to divide any element of $k[x]$ by a Gröbner basis for I.

In [AL94, Corollary 4.1.15, Corollary 4.1.16] it is shown that when k is a commutative Noetherian ring the notion of a Gröbner basis can be appropriately extended to $k[x]$, and that every ideal of $k[x]$ has a Gröbner basis in this sense. With additional assumptions on k detailed in [AL94, Chapter 4], one can show that there exist generalisations of Buchberger's Algorithm which compute Gröbner bases for ideals.

By inspecting the proof of the division algorithm in [CLO15, Theorem 2.3.3] we can see that that the assumption that k is a field is only needed to ensure that the leading

Algorithm 3.3.7 Generate series coefficients

Require: A polynomial $f \in k[x]$, a quasi-regular sequence $t = (t_1, \ldots, t_n)$, a Gröbner basis $g = (g_1, \ldots, g_{n'})$ for $I = (t_1, \ldots, t_n)$ and polynomials $\{a_{ij}\}_{i,j}$ such that $g_i = \sum_{j=1}^{n} a_{ij} t_j$. Let C be defined from g as in (3.4).

$Q \leftarrow \{(\vec{0}, f)\}$

▷ *The elements of Q are tuples (u, q) where $u \in \mathbb{N}^n$ and $q \in k[x]$. If $(u, q) \in Q$ then we know q is the coefficient of t^u in an intermediate expression for f. The polynomials appearing in Q are exactly the coefficients which still need to be divided. That is, they are all non-zero and are not known to be in C.* ◁

while $Q \neq \emptyset$ **do**

  $Q_{\text{new}} \leftarrow \emptyset$

  for all $(u, q) \in Q$ **do**

    Apply the division algorithm to obtain $r \in C$ and $p_1, \ldots, p_{n'} \in k[x]$ satisfying

$$q = r + \sum_{i=1}^{n'} p_i g_i$$

    along with the other conditions in Theorem 3.3.1.

    output (u, r) ▷ *Final coefficent of t^u*

    $q'_j \leftarrow \sum_{i=1}^{n'} a_{ij} p_i$ for each $j = 1, \ldots, n$ ▷ *Intermediate coefficient of t^{u+e_j}*

    $Q_{\text{new}} \leftarrow Q_{\text{new}} \cup \{(u + e_j, q'_j) \mid j = 1, \ldots, n \text{ where } q'_j \neq 0\}$

  $Q \leftarrow$ COLLECTCOEFFICIENTS(Q_{new}) ▷ *Adds together coefficients of the same t^u*

function COLLECTCOEFFICIENTS(Q)

  $Q_{\text{collected}} \leftarrow \emptyset$

  for all u where $(u, p) \in Q$ for some p **do**

    Let $p_1, \ldots, p_s$ be all the polynomials such that $(u, p_i) \in Q$

    if $\sum_{i=1}^{s} p_i \neq 0$ **then**

      $Q_{\text{collected}} \leftarrow \{(u, \sum_{i=1}^{s} p_i)\} \cup Q_{\text{collected}}$

  return $Q_{\text{collected}}$

coefficients of the divisors are invertible. Since we only ever divide polynomials by a Gröbner basis for I, all that we need is for the leading coefficients of this Gröbner basis to be invertible in k. Since we need to assume $\mathbb{Q} \subseteq k$ anyway in order to construct the strong deformation retract of Proposition 3.2.9 this will be true for many examples. More generally, [AL94, Chapter 4] gives conditions on k under which there exists a version of the division algorithm on $k[x]$.

3.4 When passing to the completion is not necessary

Again let k be a field and consider the polynomial ring $k[x] = k[x_1, \ldots, x_m]$ with a fixed monomial ordering. In this section we find conditions which mean it is not necessary to pass to a completion of $k[x]$ in order to define a strong deformation retract between a Koszul complex and its homology. We retain the context of the previous section: let $t = (t_1, \ldots, t_n)$ be quasi-regular, $I = (t_1, \ldots, t_n)$ the ideal generated by elements of t and $g = (g_1, \ldots, g_{n'})$ a Gröbner basis for I. Let $\{a_{ij}\}_{i,j}$ be polynomials satisfying $g_i = \sum_{j=1}^{n} a_{ij} t_j$ and $C \subseteq k[x]$ be the k-linear subspace defined using g in (3.4).

Passing to the I-adic completion of $k[x]$ is necessary in Section 3.2 in order to define the connection of Definition 3.2.2. By Lemma 3.1.6 we know that we can express elements of this completion uniquely as a series of powers of elements of t. This lets us define the maps $\partial_{t_1}, \dots, \partial_{t_n}$, which act like differentiation with respect to the $t_1, \dots, t_n$, and hence the connection. It is fairly easy, however, to come up with examples where passing to the completion is not necessary. For example, if $t = (x_1, \dots, x_m)$ then we can take ∂_{x_i} to be the usual differentiation with respect to x_i and hence we can define the connection as a map $\nabla : K(t) \to K(t)$ where $K(t)$ is the Koszul complex of t as a sequence in $k[x]$ rather than in the I-adic completion.

The idea is to consider the circumstances under which Algorithm 3.3.7 terminates. If it terminates for all $f \in k[x]$ then, via Lemma 3.3.6, we know that any $f \in k[x]$ has a unique expression of the form

$$f = \sum_{u \in \mathbb{N}^n} r_u t^u$$

where $r_u \in C$ and finitely many $r_u \neq 0$. That is, the series of Lemma 3.3.6 is now a sum in $k[x]$ and so we can define the maps $\partial_{t_1}, \dots, \partial_{t_n}$ on $k[x]$ rather than its I-adic completion. The remainder of Section 3.2 proceeds identically with $k[x]$ replacing $\widehat{k[x]}$, and so we obtain the desired strong deformation retract.

Proposition 3.4.1. *Algorithm 3.3.7 terminates for all $f \in k[x]$ if $I \neq k[x]$ and the polynomials $\{a_{ij}\}_{i,j}$ are all constants.*

Proof. Let Q_N be the value of Q in Algorithm 3.3.7 at the end of the N^{th} repetition of the loop on line 3. For a polynomial $q \in k[x]$ we abuse notation and write $q \in Q_N$ to mean $(u, q) \in Q_N$ for some $u \in \mathbb{N}^n$. If $Q_N \neq \emptyset$ define

$$B_N = \max\{\text{multideg}(q) \mid q \in Q_N\}$$

and

$$A = \max\{\text{multideg}(a_{ij}) \mid i = 1, \dots, n',\ j = 1, \dots, n,\ a_{ij} \neq 0\}$$

where each maximum is taken with respect to the chosen monomial ordering. Note that at least one $a_{ij} \neq 0$ since otherwise $I = 0$, contradicting t being quasi-regular.

Consider $q \in Q_{N-1}$ and let $p_1, \dots, p_{n'}$ be the polynomials computed from q on line 6. By Theorem 3.3.1 we have that $\text{multideg}(q) \geq \text{multideg}(p_i g_i)$. Since $I \neq k[x]$ the polynomials $g_1, \dots, g_{n'}$ are not constants. This implies $\text{multideg}(q) > \text{multideg}(p_i)$ and in particular $B_{N-1} > \text{multideg}(p_i)$ for $p_i \neq 0$. Next consider the q'_j defined on line 8 from the $p_1, \dots, p_{n'}$ above. It is easy to see that for non-zero polynomials $h, h' \in k[x]$ we have $\text{multideg}(hh') = \text{multideg}(h) + \text{multideg}(h')$ and, if $h + h' \neq 0$, that $\text{multideg}(h + h') = \max\{\text{multideg}(h), \text{multideg}(h')\}$ [CLO15, Lemma 2.2.8]. Hence we have

$$\begin{aligned} \text{multideg}(q'_j) &\leq \max\{\text{multideg}(a_{ij} p_i) \mid i = 1, \dots, n\} \\ &\leq \max\{\text{multideg}(a_{ij}) + \text{multideg}(p_i) \mid i = 1, \dots, n\} \\ &\leq A + \max\{\text{multideg}(p_i) \mid i = 1, \dots, n\} \\ &< A + B_{N-1}\ . \end{aligned}$$

Note that the elements of Q_N are non-zero sums of various non-zero q'_j defined on line 8, and so if $Q_N \neq \emptyset$ we have

$$B_N < A + B_{N-1}\ .$$

Hence, if $\{a_{ij}\}_{i,j}$ are constant polynomials then $A = (0, \dots, 0)$. In this case the sequence $B_1, B_2, \dots$ is strictly decreasing and, since monomial orderings are well-founded, it follows that it must be finite. Hence for sufficiently large N we have $Q_N = \emptyset$ and so Algorithm 3.3.7 terminates. □

Corollary 3.4.2. *If $I \neq k[x]$ and $\{a_{ij}\}_{i,j}$ are constant polynomials. Then every $f \in k[x]$ can be written uniquely in the form*

$$f = \sum_{u \in \mathbb{N}^n} r_u t^u$$

where $r_u \in C$ and finitely many $r_u \neq 0$.

Proof. Such an expression exists since under these hypotheses Algorithm 3.3.7 terminates. For uniqueness, in order to apply Lemma 3.3.6 we need to know that the canonical map $\kappa : k[x] \to \widehat{k[x]}$ is injective. Since $k[x]$ is a Noetherian integral domain this follows from Krull's Intersection Theorem: see [AM69, p.105 and Corollary 10.18]. Hence Lemma 3.3.6 gives uniqueness. □

Now suppose $I \neq k[x]$ and $\{a_{ij}\}_{i,j}$ are constant polynomials. Then for each $i = 1, \ldots, n$ we define $\partial_{t_i} : k[x] \to k[x]$ as

$$\partial_{t_i}(f) = \sum_{u \in \mathbb{N}^n \setminus \{0\}} u_i r_u t^{u-e_i} \qquad \text{where } f = \sum_{u \in \mathbb{N}^n} r_u t^u \in k[x] \tag{3.5}$$

and $r_u \in C$ with finitely many $r_u \neq 0$. This is well-defined by Corollary 3.4.2. We now consider the Koszul complex $(K(t), d_K)$ of t, where $K(t) = \bigwedge(\bigoplus_{i=1}^n k[x]dt_i)$ and $dt_1, \ldots, dt_n$ are formal generators. Analogous to Definition 3.2.2 we define the *connection* $\nabla : K(t) \to K(t)$ as

$$\nabla(f\omega) = \sum_{i=1}^n \partial_{t_i}(f) dt_i \omega \qquad \text{where } f \in k[x] \text{ and } \omega = dt_{i_1} \cdots dt_{i_p} \, .$$

Corollary 3.4.3. *Given the preceding assumptions, if t is Koszul-regular and k is characteristic zero then we have a strong deformation retract over k*

$$(k[x]/I, 0) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (K(t), d_K), \qquad H$$

where $(k[x]/I, 0)$ is concentrated in degree zero, $H = [d_K, \nabla]^{-1}\nabla$, π is the quotient map and σ is uniquely determined by π and $1 - d_K\nabla$.

Proof. Note that since k is a field, k having characteristic zero is equivalent to $\mathbb{Q} \subseteq k$. Strictly speaking, in order to prove this corollary one would need to reproduce all the proofs in Section 3.2 with $k[x]$ in place of its I-adic completion. By inspecting the proofs in Section 3.2 we note that they are all essentially identical with $k[x]$ in place of its I-adic completion. □

Before continuing to the next section we discuss the case that $\{a_{ij}\}_{i,j}$ are not all constants. Using examples, we illustrate that there is a lot of scope for variability in the behaviour of Algorithm 3.3.7 in this case. We first consider a simple example in which things go very "wrong" in the sense that the series expansion of many elements in $k[x]$ have infinitely many non-zero terms.

Example 3.4.4. We take $k = \mathbb{Q}$ and consider the two variable polynomial ring $\mathbb{Q}[x, y]$ with the lexicographic monomial ordering in which $x > y$. Let $t = (y^2 + 1, xy + 1)$. The sequence t is regular since

$$\mathbb{Q}[x, y]/(y^2 + 1) \cong K[x]$$

where $K = \mathbb{Q}(\sqrt{-1}) \subseteq \mathbb{C}$ is the smallest field containing $\sqrt{-1}$ with $\mathbb{Q}$ as a subfield. In particular $K[x]$ is an integral domain, so $xy+1$ is certainly not a zero divisor on $\mathbb{Q}[x,y]/(y^2+1)$. Hence by Theorem 3.1.2 we have that t is both Koszul regular and quasi-regular.

A Gröbner basis for the ideal $I = (y^2+1, xy+1)$ is $g = (x-y, y^2+1)$, and we have

$$x - y = x(y^2+1) - y(xy+1) = xt_1 - yt_2 \ .$$

We consider the polynomial x. Applying the division algorithm to divide the polynomial x by g we obtain

$$x = y + (x-y) = y + xt_1 - yt_2 \ .$$

The monomial y is not divisible by any of the $\mathrm{LT}(g_i)$, so the terms y and yt_2 are the first two terms we have found in the series expansion of Lemma 3.3.6 for the polynomial x. However, we need to continue to divide the coefficient x of t_1 in the above expression. In this way we obtain

$$\begin{aligned} x &= y + (y + (x-y))t_1 - yt_2 \\ &= y + yt_1 - yt_2 + (xt_1 - yt_2)t_1 \\ &= y + yt_1 - yt_2 + (y + (x-y))t_1^2 - yt_2t_1 \\ &= y + yt_1 - yt_2 + yt_1^2 - yt_2t_1 + (xt_1 - yt_2)t_1^2 \\ &= y + yt_1 - yt_2 + yt_1^2 - yt_2t_1 + (y + (x-y))t_1^3 - yt_1^2t_2 \\ &= y + yt_1 - yt_2 + yt_1^2 - yt_2t_1 + yt_1^3 - yt_1^2t_2 + (xt_1 - yt_2)t_1^3 \end{aligned}$$

and so on. Hence the series expansion for the polynomial x in the I-adic topology is

$$x = \sum_{i=0}^{\infty} y(t_1^i - t_1^i t_2) \ .$$

Example 3.4.4 is an instance of a more general family of examples. Consider the first step in the series expansion of one of the coefficient polynomials $a_{i_0 j_0}$:

$$a_{i_0 j_0} = r + \sum_{i=1}^{n'} q_i g_i = r + \sum_{j=1}^{n} \left(\sum_{i=1}^{n'} a_{ij} q_i \right) t_j \ .$$

If $q_{i_0} \neq 0$ then under most circumstances $a_{i_0 j_0}$ will have an infinite series expansion. For the sake of simplicity, if we assume $q_{i_0} = 1$ and $q_i = 0$ for $i \neq i_0$ then we have

$$a_{i_0 j_0} = r + a_{i_0 j_0} t_{j_0} = r + rt_{j_0} + a_{i_0 j_0} t_{j_0}^2 = r + rt_{j_0} + rt_{j_0}^2 + a_{i_0 j_0} t_{j_0}^3 = \sum_{i=0}^{\infty} rt_{j_0}^i \ .$$

Example 3.4.5. We take $k = \mathbb{C}$ and consider the three variable polynomial ring $\mathbb{C}[x,y,z]$ with the lexicographic monomial ordering in which $x > y > z$. Consider the ADE singularity of type E_7 which is represented by the polynomial $U = xy^3z^2 + x^3 + z^2$. Let $t = (2x^2 + y^2, 3xy^2, 2z)$ be the sequence of partial derivatives of U. Since U is a *potential* (Definition 5.1.1) the sequence t is Koszul-regular and hence quasi-regular by Theorem 3.1.2. A Gröbner basis for the ideal $I = (2x^2 + y^2, 3xy^2, 2z)$ is $g = (3x^2 + y^3, 3xy^2, y^5, 2z)$, and we have

$$y^5 = y^2(2x^2 + y^3) - x(3xy^2) = y^2 t_1 - xt_2 \ .$$

We have not yet been able to find a polynomial in $k[x]$ for which Algorithm 3.3.7 terminates.[3] We conjecture that Algorithm 3.3.7 terminates for all $f \in k[x]$ with t and g as above, despite the coefficient polynomials relating t to g not all being constants. It is a similar story for the singularities of type D_n, which are represented by the polynomials $V_n = z^2 + yx^2 + y^{n-1}$ for $n \geq 4$. For the remaining ADE singularities of type A_n, E_6 and E_8 the sequence of partial derivatives is already a Gröbner basis, so Corollary 3.4.2 shows that Algorithm 3.3.7 terminates for all inputs in these cases.

3.5 An efficient algorithm for computing the connection

Let k be a field and consider the polynomial ring $k[x] = k[x_1, \ldots, x_m]$ with a fixed monomial order. We retain the context of the previous section: let $t = (t_1, \ldots, t_n)$ be quasi-regular, $I = (t_1, \ldots, t_n)$ the ideal generated by elements of t and $g = (g_1, \ldots, g_{n'})$ a Gröbner basis for I. Let $\{a_{ij}\}_{i,j}$ be polynomials satisfying $g_i = \sum_{j=1}^{n} a_{ij} t_j$ and $C \subseteq k[x]$ be the k-linear subspace defined using g at (3.4).

In this section we describe a more efficient algorithm for computing the connection in the case that $\{a_{ij}\}_{i,j}$ are constant polynomials. Computing the connection amounts to computing the maps $\partial_{t_1}, \ldots, \partial_{t_n}$. The naive approach for computing these maps on $f \in k[x]$ is to compute the series expansion of f of Corollary 3.4.2 using Algorithm 3.3.7. For $u \in \mathbb{N}^n$ let $|u| = \sum_{i=1}^{n} u_i$ and let

$$\deg_t(f) = \max\{|u| \mid r_u \neq 0\}$$

where $f = \sum_{u \in \mathbb{N}^n} r_u t^u$ is the expansion of f in Corollary 3.4.2, so $r_u \in C$ and finitely many $r_u \neq 0$. In the worst case, Algorithm 3.3.7 will require in the order of $2^{\deg_t(f)}$ uses of the division algorithm. In the following we show that each map ∂_{t_i} can be computed with exactly two uses of the division algorithm, and if one is computing all maps $\partial_{t_1}, \ldots, \partial_{t_n}$ at the same time then this can be done with $n + 1$ uses of the division algorithm.

This approach uses an alternative way of defining the maps $\partial_{t_1}, \ldots, \partial_{t_n}$ in the case that $\{a_{ij}\}_{i,j}$ are constant polynomials. This alternative definition shows that the maps $\partial_{t_1}, \ldots, \partial_{t_n}$ satisfy a property which is analogous to Taylor's Theorem. Consider the one variable case $k[x] = k[x_1]$ and let $f \in k[x]$. For another formal variable y one can show that in the polynomial ring $k[x, y]$ we have

$$f(x) = \sum_{p=0}^{\infty} \frac{1}{p!} f^{(p)}(y)(x - y)^p$$

analogously to the analytic Taylor's Theorem, where $f^{(p)} = \frac{d^p}{dx^p}(f)$ is the p^{th} partial derivative of f. Rearranging this we have

$$f(x) - f(y) = f'(y)(x - y) + (x - y)^2 \sum_{p=2}^{\infty} \frac{1}{p!} f^{(p)}(y)(x - y)^{p-2}$$

or in other words, $f'(y)$ is the remainder of $f(x) - f(y)$ divided by $(x - y)^2$. A similar result can be shown in the multivariate case.

Our first task is to extend the monomial ordering on $k[x]$, which we denote by $>_x$, to a compatible monomial ordering on $k[x, y] = k[x_1, \ldots, x_m, y_1, \ldots, y_m]$. We define

[3] Code for computing such examples can be found at `https://github.com/rohan-hitchcock/msc-thesis-examples`.

a monomial ordering $>_{x,y}$ on $k[x,y]$ as follows. Let $(a,b),(a',b') \in \mathbb{N}^m \times \mathbb{N}^m$ where $a,b,a',b' \in \mathbb{N}^m$. We say

$$(a,b) >_{x,y} (a',b') \qquad \text{if and only if} \qquad a >_x a' \text{ or } (a = a' \text{ and } b >_x b') \ .$$

That is, $>_{x,y}$ is the lexicographic ordering on $\mathbb{N}^m \times \mathbb{N}^m$ given by considering $>_x$ on each factor. This is clearly a monomial ordering on $k[x,y]$ which agrees with the monomial order on $k[x]$ when restricted to monomials involving only x variables, and for which $x_i > y_j$ for all $i,j = 1,\ldots,m$. In particular $\mathrm{LT}_x(f(x)) = \mathrm{LT}_{x,y}(f(x)+f(y))$ for all $f \in k[x]$. From now on we dispense with distinguishing between $>_x$ and $>_{x,y}$ and simply use $>$ and LT to refer to both monomial orderings.

Next we define $T_i = t_i(x) - t_i(y)$ and $G_i = g_i(x) - g_i(y)$ and consider $T = (T_1,\ldots,T_n)$ and $G = (G_1,\ldots,G_{n'})$. One can show that T is quasi-regular. Suppose that G is a Gröbner basis for (T).[4]

Lemma 3.5.1. *Any $F \in k[x,y]$ can be written uniquely in the form*

$$F = \sum_{u \in \mathbb{N}^n} r_u T^u$$

where $T^u = T_1^{u_1} \cdots T_n^{u_n}$, we have finitely many $r_u \neq 0$ and if $r_u \neq 0$ then no term of r_u is divisible by any of the $\mathrm{LT}(G_i) = \mathrm{LT}(g_i(x))$.

Proof. This is an application of Corollary 3.4.2. □

Given $f \in k[x]$ write

$$f(x) - f(y) = \sum_{u \in \mathbb{N}^n} r_u T^u$$

where the $r_u \in k[x,y]$ are the unique polynomials satisfying the conditions in Lemma 3.5.1. For each $u \in \mathbb{N}^n$ define a map $\rho_u : k[x] \to k[x,y]$ by setting $\rho_u(f) = r_u$. We now prove some facts about these maps. For $u,v \in \mathbb{N}^n$ define $u! = u_1!u_2!\cdots u_n!$ and

$$\binom{v}{u} = \begin{cases} 0 & \text{if any } v_i - u_i < 0 \\ \frac{v!}{u!(v-u)!} & \text{otherwise} \end{cases} .$$

Lemma 3.5.2. *ρ_u is k-linear.*

Proof. Let $f,g \in k[x]$. Then we can write

$$(f+g)(x) - (f+g)(y) = \sum_{u \in \mathbb{N}^n} (\rho_u(f) + \rho_u(g)) T^u \ .$$

Now, if $\rho_u(f) + \rho_u(g) \neq 0$ then no term of $\rho_u(f) + \rho_u(g)$ is divisible by any of the $\mathrm{LT}(G_i)$. Hence the right-hand-side satisfies the conditions in Lemma 3.5.1 and so by uniqueness $\rho_u(f+g) = \rho_u(f) + \rho_u(g)$. Likewise $\rho_u(cf) = c\rho_u(f)$ for $c \in k$. □

Lemma 3.5.3. *$\rho_u(t^v) = \binom{v}{u} t^{v-u}(y)$ for all $v \in \mathbb{N}^n$ and $u \neq 0$.*

[4] Under these hypotheses it is clear that $(G) = (T)$. We conjecture that g being a Gröbner basis implies that G is a Gröbner basis for any sequence g.

Proof. It suffices to prove that

$$t^v(x) = \sum_u \binom{v}{u} t^{v-u}(y) T^u . \tag{3.6}$$

Indeed, having shown (3.6) holds we have

$$t^v(x) - t^v(y) = \sum_{u \neq 0} \binom{v}{u} t^{v-u}(y) T^u$$

where we note that no term of $t^{v-u}(y)$ is divisible by any of the $\mathrm{LT}(G_i) = \mathrm{LT}(g_i(x))$, since $t^{v-u}(y)$ is a polynomial in y.

We proceed by induction on $|v| = \sum_i v_i$. If $v = 0$ then both sides of (3.6) are equal to 1. Now suppose that $|v| \geq 1$. Let i be such that $v_i > 0$. Then, using the induction hypothesis, we have

$$\begin{aligned}
t^v(x) &= t_i(x) t^{v-e_i}(x) \\
&= t_i(x) \sum_u \binom{v-e_i}{u} t^{v-e_i-u}(y) T^u \\
&= (t_i(y) + T_i) \sum_u \binom{v-e_i}{u} t^{v-e_i-u}(y) T^u \\
&= \sum_u \binom{v-e_i}{u} t^{v-u}(y) T^u + \sum_u \binom{v-e_i}{u} t^{v-e_i-u}(y) T^{u+e_i} \\
&= \sum_u \binom{v-e_i}{u} t^{v-u}(y) T^u + \sum_{u \neq 0} \binom{v-e_i}{u-e_i} t^{v-u}(y) T^u \\
&= t^v(y) + \sum_{u \neq 0} \left(\binom{v-e_i}{u} + \binom{v-e_i}{u-e_i} \right) t^{v-u}(y) T^u \\
&= t^v(y) + \sum_{u \neq 0} \binom{v}{u} t^{v-u}(y) T^u \\
&= \sum_u \binom{v}{u} t^{v-u}(y) T^u
\end{aligned}$$

which proves the claim. □

Lemma 3.5.4. *For $r \in C \subseteq k[x]$ and $u \neq 0$ we have $\rho_u(rt^v) = r(x)\rho_u(t^v)$ for all $v \in \mathbb{N}^n$.*

Proof. First note that if $r = 0$ the result is immediate. Supposing $r \neq 0$, using Lemma 3.5.3 we have

$$\begin{aligned}
r(x)t^v(x) - r(y)t^v(y) &= r(x)t^v(x) - r(x)t^v(y) + r(x)t^v(y) - r(y)t^v(y) \\
&= r(x)(t^v(x) - t^v(y)) + (r(x) - r(y))t^v(y) \\
&= r(x) \sum_{u \neq 0} \binom{v}{u} t^{v-u}(y) T^u + (r(x) - r(y))t^v(y) \\
&= (r(x) - r(y))t^v(y) + \sum_{u \neq 0} \binom{v}{u} r(x) t^{v-u}(y) T^u .
\end{aligned}$$

That $r \in C$ exactly means that $\mathrm{LT}(g_i) = \mathrm{LT}(G_i)$ does not divide any term of r for all $i = 1, \ldots, n'$. Hence $\mathrm{LT}(G_i)$ does not divide any term of $(r(x)-r(y))t^v(y)$ or $\binom{v}{u} r(x) t^{v-u}(y)$ for all $i = 1, \ldots, n$ and $u \in \mathbb{N}^n$. Hence by Lemma 3.5.1 this proves the claim. □

Algorithm 3.5.7 Computing ∂_{t_j}

procedure DIFFERENTIATE$(f, j, t_1, \ldots, t_n)$
  Use the division algorithm in $k[x, y]$ to obtain $r(x, y), q_1(x, y), \ldots, q_n(x, y)$ satisfying

$$f(x) - f(y) = r(x, y) + \sum_{i=1}^{n} q_i(x, y)(t_i(x) - t_i(y))$$

  where the terms of $r(x, y)$ are not divisible by any of the $\mathrm{LT}(g_i)$.
  Use the division algorithm in $k[x, y]$ to obtain $r'(x, y), p_1(x, y), \ldots, p_n(x, y)$ satisfying

$$q_j(x, y) = r'(x, y) + \sum_{i=1}^{n} p_i(x, y)(t_i(x) - t_i(y))$$

  where the terms of $r'(x, y)$ are not divisible by any of the $\mathrm{LT}(g_i)$.
  return $\varphi(r'(x, y))$

Now let $e_i \in \mathbb{N}^n$ have a one in the i^{th} coordinate and zeros elsewhere and let $\varphi : k[x, y] \to k[x]$ be the k-algebra morphism identifying x and y.

Proposition 3.5.5. $\partial_{t_i} = \varphi\rho_{e_i}$.

Proof. Lemma 3.5.2, Lemma 3.5.3, and Lemma 3.5.4 give us a method for computing $\varphi\rho_{e_i}(f)$ for any $f \in k[x]$ via Corollary 3.4.2. This exactly agrees with the definition of ∂_{t_i}. □

Corollary 3.5.6. *From Algorithm 3.5.7 we have* $\partial_{t_j}(f) = \text{DIFFERENTIATE}(f, j, t_1, \ldots, t_n)$.

Proof. This is immediate from Proposition 3.5.5. □

Proposition 3.5.5 and Corollary 3.5.6 are the main results of this section, but before continuing we will make explicit the connection between the maps $\partial_{t_1}, \ldots, \partial_{t_n}$ and Taylor's Theorem. Noting that the maps $\partial_{t_1}, \ldots, \partial_{t_n}$ commute, for $v \in \mathbb{N}^n$ we define $\partial_t^v = \partial_{t_1}^{v_1} \cdots \partial_{t_n}^{v_n}$. The next result is analogous to Taylor's Theorem.

Proposition 3.5.8. $\partial_t^v = v!\varphi\rho_v$ *for all* $v \in \mathbb{N}^n$ *with* $v \neq 0$.

Proof. Let $f \in k[x]$ and write

$$f(x) = \sum_u r_u(x) t^u(x)$$

where finitely many $r_u \neq 0$ and if $r_u \neq 0$ then no term of r_u is divisible by any of the $\mathrm{LT}(g_i)$. By Lemma 3.5.3 we have $\rho_v(t^u) = \binom{u}{v} t^{u-v}(y)$ and so

$$\begin{aligned}
\partial_t^v(f) &= \sum_u v! \binom{u}{v} r_u(x) t^{u-v}(x) \\
&= v! \sum_u r_u(x) \varphi\rho_v(t^u) \\
&= v! \sum_u \varphi\rho_v(r_u t^u) \\
&= v!\varphi\rho_v(f)
\end{aligned}$$

where we have that $r_u(x)\rho_v(t^u) = \rho_v(r_u t^u)$ by Lemma 3.5.4. □

4 Matrix factorisations

In this section we discuss matrix factorisations, which provide a means of studying singularities of polynomials. In Section 4.1 we define matrix factorisations, and more generally *linear factorisations*, and make some other preliminary definitions. In Section 4.2 we define a particular type of matrix factorisation, called a *Koszul matrix factorisation*, which is constructed from a Koszul complex. In Section 4.3 we show how the perturbation techniques discussed in Section 2.1 can be extended to linear factorisations. The extension of these techniques means that the results of Section 3 can be used to construct explicit homotopy equivalences between matrix factorisations. Examples of how these ideas can be used together are demonstrated Corollary 4.3.4 and Corollary 4.3.5.

The ideas discussed in this section are central to the definition of bicategory of Landau-Ginzburg models, which is the focus of Section 5. The categories of 1-morphisms in this bicategory are categories of matrix factorisations, and the perturbation techniques of Section 4.3 play an important role in showing that composition and unit 1-morphisms in the bicategory work as promised. The application of the results of Section 3 to matrix factorisations provides a means of computing compositions of 1-morphisms in this bicategory.

4.1 Definition and basic results

Let R be a commutative ring and $f \in R$. The most concrete way to think about a matrix factorisation of f is as a pair of $n \times n$ square matrices (P, Q) with entries in R satisfying the relations

$$PQ = QP = f \cdot I$$

where I is the $n \times n$ identity matrix.

Example 4.1.1. Let k be a commutative ring, $R = k[x, y]$ and $f = x^2 - y^2$. Then the following is a matrix factorisation of f:

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} x & -y \\ -y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ -y & x \end{pmatrix} \begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} x^2 - y^2 & 0 \\ 0 & x^2 - y^2 \end{pmatrix} .$$

By choosing n generators for $R^{\oplus n}$ we can associate to P and Q morphisms $p, q : R^{\oplus n} \to R^{\oplus n}$ respectively. Hence, the data of a matrix factorisation can be expressed like so:

$$\underset{0}{R^{\oplus n}} \xrightarrow{\quad p \quad} \underset{1}{R^{\oplus n}} \xrightarrow{\quad q \quad} \underset{0}{R^{\oplus n}} .$$

If we consider the $\mathbb{Z}_2$-grading indicated on the diagram we can view a matrix factorisation as a $\mathbb{Z}_2$-graded R-module $X = X_0 \oplus X_1$ where $X_i \cong R^{\oplus n}$, together with an odd endomorphism $d_X : X \to X$ given by $p : X_0 \to X_1$ and $q : X_1 \to X_0$ on each graded component of X. That is, $d_X = \left(\begin{smallmatrix} 0 & q \\ p & 0 \end{smallmatrix}\right)$. This endomorphism has the property that $d_X^2 = f \cdot 1_X$. This point of view suggests some generalisations: allowing 'infinite rank' matrix factorisations by not requiring the X_0 and X_1 to be finitely generated, or more generally allowing X_0 and X_1 to be R-modules which are not necessarily free or finitely generated. Hence we arrive at the following definition.

Definition 4.1.2. A *linear factorisation* of $f \in R$ is a pair (X, d_X) where X is a $\mathbb{Z}_2$-graded R-module and $d_X : X \to X$ is an odd endomorphism such that $d_X^2 = f \cdot 1_X$. A *matrix factorisation* is a linear factorisation in which the underlying module is free. A matrix factorisation is *finite rank* if the underlying module is finitely generated.

A complex of R-modules is a $\mathbb{Z}$-graded R-module C equipped with a degree ± 1 endomorphism $d_C : C \to C$ which squares to zero. Many definitions from the world of complexes transfer to the world of linear factorisations. Before we state these definitions, note that any complex (C, d_C) can be regarded as a linear factorisation of zero by considering the $\mathbb{Z}_2$-grading on C given by taking $\bigoplus_{n \in \mathbb{Z}} C_{2n}$ to be the even part and $\bigoplus_{n \in \mathbb{Z}} C_{2n+1}$ to be the odd part.

Definition 4.1.3. Let (X, d_X) and (Y, d_Y) be linear factorisations of $f \in R$. We make the following definitions:

(1) The map d_X is called the *differential*.

(2) The *shift* of (X, d_X) is the linear factorisation obtained by interchanging the odd and even parts of X. It is denoted $(X[1], d_X)$.

(3) A *morphism of linear factorisations* $(X, d_X) \to (Y, d_Y)$ is a degree zero R-linear map $\alpha : X \to Y$ which commutes with the differential, meaning that both squares of the diagram

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_0 \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_0 \end{array}$$

commute.

(4) A *homotopy* of the morphism $\alpha : (X, d_X) \to (Y, d_Y)$ is an odd R-linear map $h : X \to Y$ such that $\alpha = h d_X + d_Y h$.

(5) Two morphisms $\alpha, \beta : (X, d_X) \to (Y, d_Y)$ are *homotopic* if there is a homotopy of $\alpha - \beta$. We denote this by $\alpha \simeq \beta$ or $h : \alpha \simeq \beta$ if we wish to specify the homotopy h.

(6) The linear factorisations (X, d_X) and (Y, d_Y) are *homotopy equivalent* if there are morphisms $\gamma : (X, d_X) \to (Y, d_Y)$ and $\gamma' : (Y, d_Y) \to (X, d_X)$ such that $\gamma\gamma' \simeq 1_Y$ and $\gamma'\gamma \simeq 1_X$.

The relevant category of matrix factorisations is the homotopy category in which homotopic morphisms are identified. We denote the homotopy category of matrix factorisations of $f \in R$ by $\mathrm{HMF}(R, f)$. The full subcategory of objects which are homotopy equivalent to a finite rank matrix factorisation is denoted by $\mathrm{hmf}(R, f)$.

Definition 4.1.4. The *tensor product* over R of linear factorisations (X, d_X) and (Y, d_Y) is defined to be

$$(X, d_X) \otimes_R (Y, d_Y) = (X \otimes_R Y, d_X \otimes 1 + 1 \otimes d_Y)$$

where the tensor product of morphisms is the graded tensor product. We may denote the differential of the tensor product by $d_{X \otimes Y}$.

The tensor product of matrix factorisations was first defined in [Yos98, Definition 1.2]. If (X, d_X) is a linear factorisation of $f \in R$ and (Y, d_Y) a linear factorisation of $g \in R$ then their tensor product is a linear factorisation of $f + g$. Indeed we have

$$\begin{aligned} d_{X \otimes Y}^2 &= (d_X \otimes 1)^2 + (d_X \otimes 1)(1 \otimes d_Y) + (1 \otimes d_Y)(d_X \otimes 1) + (1 \otimes d_Y)^2 \\ &= d_X^2 \otimes 1 + d_X \otimes d_Y - d_X \otimes d_Y + 1 \otimes d_Y^2 \\ &= (f + g) \cdot 1 \end{aligned}$$

where we use that the graded tensor product satisfies

$$(\nu_1 \otimes \mu_1)(\nu_2 \otimes \mu_2) = (-1)^{|\mu_1||\nu_2|} \nu_1\nu_2 \otimes \mu_1\mu_2$$

for appropriate homogeneous maps $\nu_1, \nu_2, \mu_1, \mu_2$.

4.2 The Koszul matrix factorisation

In this section we describe a way of making the Koszul complex into a matrix factorisation. This gives us a way of producing non-trivial matrix factorisations and is also used to define the unit 1-morphisms in the bicategory of Landau-Ginzburg models. Let R be a commutative ring and $f \in R$. Suppose $a_1, \dots, a_n, b_1, \dots, b_n \in R$ are such that $f = \sum_{i=1}^n a_i b_i$. Before continuing we introduce some terminology.

Definition 4.2.1. Let E be a $\mathbb{Z}_2$-graded ring, not assumed to be commutative, and consider odd elements $\theta_1, \dots, \theta_n, \theta_1^*, \dots, \theta_n^* \in E$. We say these elements satisfy the *canonical anticommutation relations* if

(1) $\theta_i \theta_j + \theta_j \theta_i = 0$

(2) $\theta_i^* \theta_j^* + \theta_j^* \theta_i^* = 0$

(3) $\theta_i \theta_j^* + \theta_j^* \theta_i = \delta_{ij}$

for all $i, j = 1, \dots, n$, where δ_{ij} is the Kronecker delta.

We generally consider the canonical anticommutation relations in the case that $E = \operatorname{End}(A)$ where A is some graded object of a category. In the case that E has odd elements satisfying the canonical anticommutation relations, A admits a Clifford algebra representation (this is essentially proved in Lemma 5.6.2). When A is an exterior algebra we always have endomorphisms satisfying the canonical anticommutation relations.

Lemma 4.2.2. *Consider the exterior algebra $\bigwedge(\bigoplus_{i=1}^n Re_i)$ where $e_1, \dots, e_n$ are formal generators. Let $\theta_i, \theta_i^* : \bigwedge(\bigoplus_{i=1}^n Re_i) \to \bigwedge(\bigoplus_{i=1}^n Re_i)$ denote the R-linear maps given by wedge multiplication $\theta_i = e_i \wedge (-)$ and contraction $\theta_i^* = e_i^* \lrcorner (-)$ for $i = 1, \dots, n$. Then the θ_i and θ_i^* are odd maps which satisfy the canonical anticommutation relations.*

Proof. Recall that our convention is to omit the wedge symbol "$\wedge$" and denote multiplication in the exterior algebra by juxtaposition. In the following we refer to conditions (1), (2) and (3) of Definition 4.2.1. Since the wedge product is anticommutative, (1) is clear. For (2) and (3) we compute the various maps on a basis element $\omega = e_{i_1} \cdots e_{i_p}$ where $i_1 < \cdots < i_p$.

Consider the contraction operators θ_a^* and θ_b^*. First note that if $a \notin \{i_1, \dots, i_p\}$ then $\theta_a^* \theta_b^*(\omega) = \theta_b^* \theta_a^*(\omega) = 0$, and likewise if $b \notin \{i_1, \dots, i_p\}$ or if $a = b$. Hence we consider the case that $a = i_{j_a}$ and $b = i_{j_b}$ for some indices $j_a \neq j_b$. Without loss of generality suppose $j_a > j_b$. We have

$$\begin{aligned}
\theta_a^* \theta_b^*(\omega) &= (-1)^{j_b+1} \theta_a^*(e_{i_1} \cdots \widehat{e}_{i_{j_b}} \cdots e_{i_p}) \\
&= (-1)^{j_a} (-1)^{j_b+1} e_{i_1} \cdots \widehat{e}_{i_{j_b}} \cdots \widehat{e}_{i_{j_a}} \cdots e_{i_p} \\
&= (-1)^{j_a} \theta_b^*(e_{i_1} \cdots \widehat{e}_{i_{j_a}} \cdots e_{i_p}) \\
&= -\theta_b^* \theta_a^*(\omega)
\end{aligned}$$

and so we have shown $\theta_a^* \theta_b^* + \theta_b^* \theta_a^* = 0$.

Now consider the operators θ_a and θ_b^*. We first study the case in which $a \neq b$. First note that if $b \notin \{i_1, \dots, i_p\}$ then $\theta_a \theta_b^*(\omega) = \theta_b^* \theta_a(\omega) = 0$, and likewise if $a \in \{i_1, \dots, i_p\}$. Hence let $b = i_{j_b}$ for some index j_b and let j_a be the largest index such that $i_{j_a} < a$ or $j_a = 0$ if no such index exists. We immediately have that

$$\theta_a \theta_b^*(\omega) = (-1)^{j_b+1} e_a e_{i_1} \cdots \widehat{e}_{j_b} \cdots e_{i_p} \; .$$

When $j_a < j_b$ this gives

$$\begin{aligned}\theta_a\theta_b^*(\omega) &= (-1)^{j_b+1}(-1)^{j_a}e_{i_1}\cdots e_a\cdots \widehat{e}_{i_{j_b}}\cdots e_{i_p}\\ &= (-1)^{j_a+1}\theta_b^*(e_{i_1}\cdots e_a\cdots e_{i_p})\\ &= -\theta_b^*\theta_a(\omega)\end{aligned}$$

where we understand the indices in the above products to be ascending. Otherwise if $j_a > j_b$ we have

$$\begin{aligned}\theta_a\theta_b^*(\omega) &= (-1)^{j_b+1}(-1)^{j_a-1}e_{i_1}\cdots \widehat{e}_{i_{j_b}}\cdots e_a\cdots e_{i_p}\\ &= (-1)^{j_a-1}\theta_b^*(e_{i_1}\cdots e_a\cdots e_{i_p})\\ &= -\theta_b^*\theta_a(\omega)\end{aligned}$$

where again we understand the indices in the above product to be ascending.

It remains to consider the case in which $a = b$. First suppose $a \notin \{i_1, \dots, i_p\}$ and let j_0 be the largest index such that $i_{j_0} < a$ or $j_0 = 0$ if no such index exists. Then we have $\theta_a\theta_a^*(\omega) = 0$ and

$$\theta_a^*\theta_a(\omega) = (-1)^{j_0}\theta_b^*(e_{i_1}\cdots e_a\cdots e_{i_p}) = \omega$$

where we understand the indices in the above product to be ascending. In the case that $a \in \{i_1, \dots, i_p\}$ let $a = i_{j_0}$ for some index j_0. Then we have $\theta_a^*\theta_a(\omega) = 0$ and

$$\theta_a\theta_a^*(\omega) = (-1)^{j_0+1}\theta_a(e_{i_1}\cdots \widehat{e}_{i_{j_0}}\cdots e_{i_p}) = \omega$$

as required. Hence we have shown that $\theta_a\theta_b^* + \theta_b^*\theta_a = \delta_{ab}$ which completes the proof. □

Lemma 4.2.3. *Suppose M is a $\mathbb{Z}_2$-graded R-module with odd R-linear maps $\theta_i, \theta_i^* : M \to M$, $i = 1, \dots, n$ satisfying the canonical anticommutation relations. Then, setting $\delta_+ = \sum_{i=1}^n a_i\theta_i$ and $\delta_- = \sum_{i=1}^n b_i\theta_i^*$, we have that $(M, \delta_- + \delta_+)$ is a linear factorisation of f.*

Proof. Set $\delta = \delta_- + \delta_+$. We need to show that $\delta^2 = f \cdot 1_M$ which we do by induction on n. By noting that $\theta_i^2 = 0$ and $\theta_i^{*2} = 0$ we see the case of $n = 1$ is clear. For the inductive case set $f' = \sum_{i=1}^{n-1} a_ib_i$ and $\delta' = \sum_{i=1}^{n-1} a_i\theta_i + \sum_{i=1}^{n-1} b_i\theta_i^*$. By the induction hypothesis we have $\delta'^2 = f' \cdot 1_M$ and so

$$\begin{aligned}\delta^2 &= (\delta' + a_n\theta_n + b_n\theta_n^*)^2\\ &= f' \cdot 1_M + a_n(\delta'\theta_n + \theta_n\delta') + b_n(\delta'\theta_n^* + \theta_n^*\delta') + a_nb_n \cdot 1_M\\ &= f \cdot 1_M + a_n(\delta'\theta_n + \theta_n\delta') + b_n(\delta'\theta_n^* + \theta_n^*\delta')\ .\end{aligned}$$

Noting that $\delta'\theta_n + \theta_n\delta' = \delta'\theta_n^* + \theta_n^*\delta' = 0$ gives the desired result. □

Therefore $(\bigwedge(R^{\oplus n}), \delta_- + \delta_+)$ is a matrix factorisation of f, where $\delta_+ = \sum_{i=1}^n a_i\theta_i$ and $\delta_- = \sum_{i=1}^n b_i\theta_i^*$, and θ_i and θ_i^* are as in Lemma 4.2.2. Furthermore, if we consider the Koszul complex $K(b)$ of $b = (b_1, \dots, b_n)$, then δ_- will be the usual differential of the Koszul complex.

Definition 4.2.4. The matrix factorisation $(K(b), \delta_- + \delta_+)$ of f is called a *Koszul matrix factorisation* of f.

Koszul matrix factorisations were first defined (but not named as such) in [BGS87, Section 2] where they were used to construct examples of maximal Cohen-Macaulay modules. The properties of the Koszul matrix factorisation are also discussed in [Dyc11, Section 2.3] and [CM14, Appendix D].

4.3 Perturbation of matrix factorisations

In Section 2.1 we discussed strong deformation retracts and perturbation of complexes in an abelian category. In this section we adapt these ideas to the setting of linear factorisations. Let R be a commutative ring.

Definition 4.3.1. Let (L, d_L) and (M, d_M) be linear factorisations of $f \in R$. A *deformation retract* of (L, d_L) and (M, d_M) over R consists of morphisms

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h$$

where $pi = 1$ and $h : ip \simeq 1$. This deformation retract is called *strong* if in addition $h^2 = 0$, $hi = 0$ and $ph = 0$.

As for complexes, a strong deformation retract is a special type of homotopy equivalence of linear factorisations which can be modified by perturbation. Let

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h \tag{4.1}$$

be a deformation retract of linear factorisations of $f \in R$. A *perturbation* of (4.1) is an odd R-linear map $\delta : M \to M$ such that $(d_M + \delta)^2 = g \cdot 1$ for some $g \in R$, where possibly $f \neq g$. The perturbation δ is called *small* if $(1 - \delta h)$ is invertible. Given a small perturbation δ of (4.1) the *perturbed data* is

$$(L, d'_L) \underset{i'}{\overset{p'}{\leftrightarrows}} (M, d_M + \delta), \qquad h' \tag{4.2}$$

where $a = (1 - \delta h)^{-1}\delta$, $d'_L = d_L + pai$, $i' = i + hai$, $p' = p + pah$ and $h' = h + hah$. In Section 4.1 we discussed how a complex can be regarded as a linear factorisation of zero by taking the obvious $\mathbb{Z}_2$-grading on the complex. When doing so, any (strong) deformation retract of complexes in the sense of Definition 2.1.3 yields a (strong) deformation retract of linear factorisations in the sense of Definition 4.3.1. A version of the perturbation lemma also holds for linear factorisations.

Theorem 4.3.2 (Perturbation Lemma for linear factorisations)**.** *If (4.1) is a strong deformation retract and δ is a small perturbation of (4.1) then the perturbed data (4.2) is also a strong deformation retract.*

Proof. The proof of this theorem closely follows the proof in [Cra04, Section 2.4] of the analogous statement for complexes. We begin by proving the following statements:

(1) $\delta h a = a h \delta = a - \delta$.

(2) $(1 - \delta h)^{-1} = 1 + ah$ and $(1 - h\delta)^{-1} = 1 + ha$.

(3) $aipa + ad_M + d_M a = (g \cdot 1 - f \cdot 1)(1 + ah + ha)$.

For (1), note that by definition of a we have $(1 - \delta h)a = \delta$, proving $a - \delta = \delta h a$. For the other equality, we can write $\delta h \delta = \delta - (1 - \delta h)\delta$ and multiply on the left by $(1 - \delta h)^{-1}$ to get $ah\delta = a - \delta$. Statement (2) is proved by observing

$$\begin{aligned}(1 + ah)(1 - \delta h) &= 1 + ah - \delta h - ah\delta h \\ &= 1 + ah - \delta h - (a - \delta)h \\ &= 1\end{aligned}$$

and similarly that $(1+ha)(1-h\delta)=1$, $(1-h\delta)(1+ha)=1$ and $(1+ha)(1-h\delta)=1$. For (3) we compute directly. Using (1) and (2) above, and the fact that $h^2=0$ we have

$$\begin{aligned}
ad_M + d_M a + aipa &= ad_M + d_M a + a(1 + d_M h + hd_M)a \\
&= ad_M(1+ha) + (1+ah)d_M a + a^2 \\
&= ad_M(1-h\delta)^{-1} + (1-\delta h)^{-1} d_M a + a^2 \\
&= (1-\delta h)^{-1}\big[(1-\delta h)ad_M + d_M a(1-h\delta) \\
&\qquad + (1-\delta h)a^2(1-h\delta)\big](1-h\delta)^{-1} \\
&= (1+ah)\big[(a-\delta ha)d_M + d_M(a-ah\delta) \\
&\qquad + (a-\delta ha)(a-ah\delta)\big](1+ha) \\
&= (1+ah)\big[(a-a+\delta)d_M + d_M(a-a+\delta) \\
&\qquad + (a-a+\delta)(a-a+\delta)\big](1+ha) \\
&= (1+ah)\big[\delta d_M + d_M\delta + \delta^2\big](1+ha) \\
&= (1+ah)\big[(d_M+\delta)^2 - d_M^2\big](1+ha) \\
&= (g\cdot 1 - f\cdot 1)(1+ah)(1+ha) \\
&= (g\cdot 1 - f\cdot 1)(1+ah+ha)
\end{aligned}$$

proving (3).

<u>*(L, d'_L) is a linear factorisation of g.*</u> We need to show that ${d'_L}^2 = g\cdot 1$. We have

$$\begin{aligned}
{d'_L}^2 &= (d_L + pai)^2 \\
&= f\cdot 1 + d_L pai + paid_L + paipai \\
&= f\cdot 1 + d_L pai + paid_L + p\left((g\cdot 1 - f\cdot 1)(1+ah+ha) - ad_M - d_M a\right)i \\
&= f\cdot 1 + d_L pai + paid_L + g\cdot 1 - f\cdot 1 - p\left(ad_M + d_M a\right)i \\
&= g\cdot 1 + d_L pai + paid_L - pad_M i - pd_M ai \\
&= g\cdot 1 + pa\big(id_L - d_M i\big) + \big(d_L p - pd_M\big)ai \\
&= g\cdot 1
\end{aligned}$$

where we use that $pi=1$, $pd_M = d_L p$, $id_L = d_M i$, $hi=0$, $ph=0$, $h^2=0$, and equation (3) above.

<u>*i' is a morphism.*</u> We need to show that $i'd'_L = (d_M+\delta)i'$. We have

$$\begin{aligned}
i'd'_L - (d_M+\delta)i' &= (i+hai)(d_L+pai) - (d_M+\delta)(i+hai) \\
&= id_L + haid_L + ipai + haipai - d_M i - \delta i - d_M hai - \delta hai \\
&= haid_L + ipai + h\big((g\cdot 1 - f\cdot 1)(1+ah+ha) - ad_M - d_M a\big)i \\
&\qquad - \delta i - d_M hai - (a-\delta)i \\
&= haid_L + ipai - h(ad_M + d_M a)i - d_M hai - ai \\
&= ha(id_L - d_M i) + (ip - hd_M - d_M h - 1)ai \\
&= 0
\end{aligned}$$

where we use (1), (3), $hi=0$, $h^2=0$, $id_L = d_M i$ and $ip-1 = hd_M + d_M h$.

$\underline{p' \textit{ is a morphism.}}$ We need to show that $d'_L p' = p'(d_M + \delta)$. We have

$$
\begin{aligned}
d'_L p' - p'(d_M + \delta) &= (d_L + pai)(p + pah) - (p + pah)(d_M + \delta) \\
&= d_L p + paip + d_L pah + paipah - pd_M - pahd_M - pah\delta - p\delta \\
&= paip + d_L pah + p\big((g \cdot 1 - f \cdot 1)(1 + ah + ha) - ad_M - d_M a\big)h \\
&\qquad - pahd_M - p(a - \delta) - p\delta \\
&= paip + d_L pah - p(ad_M + d_M a)h - pahd_M - pa \\
&= pa(ip - d_M h - hd_M - 1) + (d_L p - pd_M)ah \\
&= 0
\end{aligned}
$$

where we use (1), (3), $ph = 0$, $h^2 = 0$, $d_L p = pd_M$ and $ip - 1 = d_M h + hd_M$.

$\underline{p'i' = 1.}$ This is straightforward:

$$p'i' = (p + pah)(i + hai) = 1$$

since $ph = 0$, $hi = 0$ and $h^2 = 0$.

$\underline{h' \textit{ is a homotopy from } i'p' \textit{ to } 1.}$ We need to show $i'p' - 1 = h'(d_M + \delta) + (d_M + \delta)h'$. Setting $d'_M = d_M + \delta$ we have

$$
\begin{aligned}
1 + h'd'_M + d'_M h' - i'p' &= 1 + (h + hah)(d_M + \delta) + (d_M + \delta)(h + hah) \\
&\qquad - (i + hai)(p + pah) \\
&= 1 + hd_M + hahd_M + h\delta + hah\delta + d_M h + d_M hah + \delta h + \delta hah \\
&\qquad - ip - ipah - haip - haipah \\
&= hahd_M + h\delta + hah\delta + d_M hah + \delta h + \delta hah \\
&\qquad - ipah - haip - haipah \\
&= hahd_M + h\delta + h(a - \delta) + d_M hah + \delta h + (a - \delta)h - ipah \\
&\qquad - haip - h\big((g \cdot 1 - f \cdot 1)(1 + ah + ha) - ad_M - d_M a\big)h \\
&= hahd_M + ha + d_M hah + ah - ipah - haip + h\big(ad_M + d_M a\big)h \\
&= ha(hd_M + 1 - ip + d_M h) + (d_M h + 1 - ip + hd_M)ah \\
&= 0
\end{aligned}
$$

where we use (1), (3), $h^2 = 0$ and $ip - 1 = hd_M + d_M h$.

This has shown that the maps i', p', d'_L, d'_M and h' form a deformation retract. It is clearly also a strong deformation retract, and so this completes the proof. □

One can show that we can replace the condition that the initial deformation retract in Theorem 4.3.2 be strong with the following conditions on the deformation retract and small perturbation:

(1) $p\delta = 0$ and $ph = 0$.

(2) $(d_M + \delta)^2 = d_M^2$.

Before discussing some corollaries of the Perturbation Lemma we note a useful sufficient condition for a perturbation to be small. In fact, as explained in [Cra04, Remark 2.3.iii] a perturbation δ of the deformation retract (4.1) is small if and only if δh is *locally nilpotent*, meaning for all $x \in M$ we have $(\delta h)^n(x) = 0$ for some n. For our purposes we only need to show that if δh is nilpotent then δ is small.

Lemma 4.3.3. *A perturbation δ of the deformation retract (4.1) is small if $(\delta h)^n = 0$ for sufficiently large n, in which case $(1 - \delta h)^{-1} = \sum_{k\geq 0}(\delta h)^k$.*

Proof. If $(\delta h)^n = 0$ then $\sum_{k\geq 0}(\delta h)^k = \sum_{k=0}^{n-1}(\delta h)^k$, so this sum is well-defined. We then compute

$$(1-\delta h)\sum_{k=0}^{n-1}(\delta h) = \sum_{k=0}^{n-1}(\delta h)^k - \sum_{k=0}^{n-1}(\delta h)^{k+1} = \sum_{k=0}^{n-1}(\delta h)^k - \sum_{k=1}^{n}(\delta h)^k = 1 - (\delta h)^n = 1$$

and likewise $\sum_{k=0}^{n-1}(\delta h)^k(1-\delta h) = 1$. Hence we have $(1-\delta h)^{-1} = \sum_{k=0}^{n-1}(\delta h)^k$. □

Corollary 4.3.4. *Consider the deformation retract of linear factorisations of f given in (4.1) and suppose it is a strong deformation retract. Then for any linear factorisation (Z, d_Z) of $g \in R$ the following is a strong deformation retract of linear factorisations of $f + g$:*

$$(L \otimes_R Z, d_L \otimes 1 + 1 \otimes d_Z) \underset{i\otimes 1}{\overset{p\otimes 1}{\leftrightarrows}} (M \otimes_R Z, d_M \otimes 1 + 1 \otimes d_Z), \qquad h \otimes 1 \quad .$$

Proof. Tensoring the modules in (4.1) by Z we obtain

$$(L \otimes_R Z, d_L \otimes 1) \underset{p\otimes 1}{\overset{i\otimes 1}{\leftrightarrows}} (M \otimes_R Z, d_M \otimes 1), \qquad h \otimes 1 \quad .$$

which is also a strong deformation retract. Note that $1 \otimes d_Z$ is a small perturbation since $(1 - h \otimes d_Z)^{-1} = (1 + h \otimes d_Z)$. Set $a = (1 + h \otimes d_Z)(1 \otimes d_Z) = 1 \otimes d_Z + h \otimes d_Z^2$ and apply the Perturbation Lemma to the strong deformation retract above. Note that we have

$$(p \otimes 1)a(i \otimes 1) = pi \otimes d_Z + phi \otimes d_Z^2 = 1 \otimes d_Z$$

so the new differential on the left is $d_L \otimes 1 + 1 \otimes d_Z$. We also have

$$(p \otimes 1)a(h \otimes 1) = ph \otimes d_Z + ph^2 \otimes d_Z^2 = 0$$

and likewise $(h \otimes 1)a(i \otimes 1) = 0$ and $(h \otimes 1)a(h \otimes 1) = 0$. Hence we obtain the claimed strong deformation retract from the Perturbation Lemma. □

The Perturbation Lemma can also be used to produce a strong deformation retract involving the Koszul matrix factorisation of some element $f \in R$. Suppose $a_1, \ldots, a_n \in R$ and $t_1, \ldots, t_n \in R$ are sequences such that $f = \sum_{i=1}^n a_i t_i$ and consider the Koszul complex $(K(t), d_K)$ of $t = (t_1, \ldots, t_n)$. Set $\delta_+ = \sum_{i=1}^n a_i\theta_i$ and $\delta_- = d_K = \sum_{i=1}^n t_i\theta_i^*$, where θ_i and θ_i^* are wedging and contraction operators on $K(t)$ respectively as in Lemma 4.2.2. Suppose we have a strong deformation retract of complexes over R

$$(R/I, 0) \underset{i}{\overset{p}{\leftrightarrows}} (K(t), \delta_-), \qquad h \tag{4.3}$$

where $I = (t_1, \ldots, t_n)$ is the ideal generated by the elements of t. We produced similar strong deformation retracts in Corollary 2.4.3, Proposition 3.2.9 and Corollary 3.4.3. Under these hypotheses we have a strong deformation retract involving a Koszul matrix factorisation of f.

Corollary 4.3.5. *Given the preceding assumptions, we have a strong deformation retract over R*

$$(R/(t), 0) \leftrightarrows (K(t), \delta_- + \delta_+), \qquad h'$$

of linear factorisations of f.

Proof. First note that f acts by zero on R/I so $(R/I, 0)$ is indeed a linear factorisation of f.

Next note that we can regard the complexes in (4.3) as linear factorisations of zero and so view (4.3) as a strong deformation retract of linear factorisations. With respect to the usual $\mathbb{Z}$-grading on $K(t)$, both h and δ_+ are degree $+1$ maps and so $\delta_+ h$ has degree $+2$. Since $K(t)$ is a bounded complex there exists a sufficiently large m such that $(\delta_+ h)^m = 0$. Then by Lemma 4.3.3 we have that δ_+ is a small perturbation of (4.3). The desired strong deformation retract is obtained using the Perturbation Lemma, noting that δ_+ is the zero map on R/I. □

Corollary 4.3.4 and Corollary 4.3.5 are not actually used in the following section (the Perturbation Lemma is used directly) however these corollaries demonstrate *how* we use the Perturbation Lemma. In each subsequent application of the Perturbation Lemma we either use it to tensor a strong deformation retract of linear factorisations with a third linear factorisation, or to introduce a perturbation which converts a Koszul complex into a Koszul matrix factorisation. Nuances of these applications mean that the hypotheses of Corollary 4.3.4 and Corollary 4.3.5 are not satisfied exactly. In Corollary 2.4.3, Proposition 3.2.9 and Corollary 3.4.3 R is a k-algebra for some commutative ring k and the strong deformation retract produced by these results is over k not R, meaning that Corollary 4.3.4 and Corollary 4.3.5 cannot be applied in these settings. These two results, however, capture the essence of how we use the Perturbation Lemma for linear factorisations.

5 The bicategory of Landau-Ginzburg Models

In this section we describe the bicategory of Landau-Ginzburg models over a commutative ring k, which we denote $\mathcal{LG}_k$, and prove that it is a bicategory. This bicategory was first described in [BR07; McN09; CR10]. Here we take the approach of [Mur18; DM13; CM16] and provide additional details and exposition which are not present in the literature.

We begin by defining the data of $\mathcal{LG}_k$ in Section 5.1, and in Sections 5.2 and 5.3 we prove that this data does indeed form a bicategory. Sections 5.4 and 5.5 address the cut operation of [Mur18], which provides a means of computing compositions in $\mathcal{LG}_k$. Examples of such computations are given in Section 5.6. The other sections of this thesis provide important background to this section. Matrix factorisations, which were introduced in Section 4, are the 1-morphisms in $\mathcal{LG}_k$ and perturbation techniques for matrix factorisations are used throughout this section. The explicit strong deformation retract between the Koszul complex and its homology discussed in Section 3 is key to constructing the cut operation. In Appendix A we discuss bicategories, and definition of a bicategory is given in Definition A.2.1. Idempotent completion of categories also plays a subtle but important role in defining $\mathcal{LG}_k$, and this is discussed in Appendix B.

5.1 The data of the bicategory

Let k be a commutative ring. The objects of $\mathcal{LG}_k$ are certain types of polynomials called potentials. Theorem 3.1.2 shows that the following definition of a potential is equivalent to the definition given in [CM16, Definition 2.4].

Definition 5.1.1. Let k be a commutative ring. A polynomial $U \in k[x] = k[x_1, \dots, x_n]$ is called a *potential* if:

(1) The sequence of partial derivatives $(\partial_{x_1} U, \dots, \partial_{x_n} U)$ is Koszul-regular.

(2) The Jacobi ring $k[x]/(\partial_{x_1} U, \dots, \partial_{x_n} U)$ is a finitely generated free k-module.

Given potentials $U \in k[x]$ and $V \in k[y]$ we denote the homotopy category of finite rank matrix factorisations of $V(y) - U(x)$ over $k[x, y]$ by

$$h(U, V) = \mathrm{hmf}(k[x, y], V(y) - U(x)) \ .$$

In order to define $\mathcal{LG}_k$ we need to use the *idempotent completion* of $h(U, V)$ (Definition B.2.1) which we denote by $h(U, V)^\omega$. Background on idempotent completion is given in Appendix B. In particular, Corollary B.3.3 shows that we can regard $h(U, V)^\omega$ as the full subcategory of $\mathrm{HMF}(k[x, y], V(y) - U(x))$ consisting of all direct summands of finite rank matrix factorisations.

Definition 5.1.2. The *bicategory of Landau-Ginzburg models* over k, denoted $\mathcal{LG}_k$, consists of the following data:

(1) The objects of $\mathcal{LG}_k$ are pairs $(k[x], U)$ where $k[x] = k[x_1, \dots, x_n]$ is a polynomial ring and $U \in k[x]$ is a potential.

(2) The category of 1-morphisms $(k[x], U) \to (k[y], V)$ is $h(U, V)^\omega$.

(3) Composition of the 1-morphisms

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

is given by taking the tensor product of linear factorisations over $k[y]$:

$$(X, d_X) \otimes_{k[y]} (Y, d_Y) = (X \otimes_{k[y]} Y, d_X \otimes 1 + 1 \otimes d_Y) \ .$$

(4) Consider an object $(k[x], U)$ where $k[x] = k[x_1, \dots, x_n]$ and the polynomial ring $k[x, x'] = k[x_1, \dots, x_n, x'_1, \dots, x'_n]$. The unit 1-morphism (I_U, d_{I_U}) of $(k[x], U)$ is a Koszul matrix factorisation (Definition 4.2.4) of $U(x') - U(x) \in k[x, x']$ arising from the Koszul complex of the sequence $(x_1 - x'_1, \dots, x_n - x'_n)$ in $k[x, x']$.

According to the definition of a bicategory we also need to specify the associator, left unitor and right unitor natural isomorphisms as part of the data of $\mathcal{LG}_k$. However, doing so now is awkward so we delay this until Sections 5.2 and 5.3 where we show that such natural isomorphisms exist and, together with the data given above, form a bicategory.

Note that it is not clear that the tensor product is a well-defined composition functor. Consider the 1-morphisms

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

in $\mathcal{LG}_k$. Writing $X = k[x, y]^{\oplus m}$ and $Y = k[y, z]^{\oplus m'}$ for some $m, m' \in \mathbb{N}$ we have that

$$X \otimes_{k[y]} Y = (k[x, y] \otimes_{k[y]} k[y, z])^{mm'} = k[x, y, z]^{mm'}$$

which is free, but not finitely generated over $k[x, z]$. Hence it is only clear that the composition of (X, d_X) and (Y, d_Y) belongs to $\mathrm{HMF}(k[x, z], W(z) - U(x))$, rather than $h(U, W)^{\omega}$.

We adopt similar approaches, both based on the strategy of [Mur18, Strategy 4.2], for showing that composition and unit 1-morphisms work as expected. In each case we begin with a strong deformation retract between the Koszul complex of a Koszul-regular sequence. We then modify this strong deformation retract using the perturbation techniques described in Section 4.3 to produce a homotopy equivalence between appropriate objects. In the case of composition this means showing that $(X \otimes_{k[y]} Y, d_{X \otimes Y})$ is, up to homotopy, the direct summand of a matrix factorisation which is finite rank over $k[x, z]$.

5.2 Composition

We begin with composition. The sequence of partial derivatives of a potential is Koszul-regular by hypothesis so we will begin with a strong deformation retract between the Jacobi ring of a potential and the Koszul complex of its sequence of partial derivatives. Before proceeding to the main result in Proposition 5.2.5 we will need to develop tools which allow us to remove the Koszul differential of this initial Koszul complex. The key observation from [Mur18, Lemma 2.12] which allows us to do so is as follows.

Lemma 5.2.1. *Let (Z, d_Z) be a finite rank matrix factorisation of a polynomial $U \in k[x] = k[x_1, \dots, x_n]$. Multiplication by a partial derivative $\partial_{x_i} U$ is a null-homotopic map on (Z, d_Z).*

Proof. Since Z is free over $k[x]$ we can fix a $k[x]$-basis for Z and write d_Z as a matrix with respect to this basis. We then differentiate the matrix of d_Z entrywise with respect to x_i to obtain a matrix $\lambda_i = \partial_{x_i}(d_Z)$. By the Leibniz rule we have $\partial_{x_i} U \cdot 1_Z = \partial_{x_i}(d_Z^2) = d_Z \lambda_i + \lambda_i d_Z$. □

Lemma 5.2.2. *Let $\alpha, \beta : (C, d_C) \to (D, d_D)$ be morphisms of either chain complexes or linear factorisations, and suppose they are homotopic via $h : \alpha \simeq \beta$. Then $\mathrm{cone}(\alpha) \cong \mathrm{cone}(\beta)$.*

Proof. We define cone(α) as

$$\cdots \longrightarrow C_{n+1} \oplus D_{n+2} \xrightarrow{d_\alpha} C_n \oplus D_{n+1} \xrightarrow{d_\alpha} C_{n-1} \oplus D_n \longrightarrow \cdots$$

where $d_\alpha = \left(\begin{smallmatrix} d_C & 0 \\ \alpha & -d_D \end{smallmatrix}\right)$, and if (C, d_C) is a linear factorisation then addition in the indices is modulo 2. Likewise cone(β) is the graded object $C \oplus D[1]$ with differential $d_\beta = \left(\begin{smallmatrix} d_C & 0 \\ \beta & -d_D \end{smallmatrix}\right)$. Define $\varphi : C \oplus D[1] \to C \oplus D[1]$ as $\varphi = \left(\begin{smallmatrix} 1 & 0 \\ -h & 1 \end{smallmatrix}\right)$. We have

$$\varphi d_\alpha = \begin{pmatrix} d_C & 0 \\ -hd_C + \alpha & -d_D \end{pmatrix}$$

and

$$d_\beta \varphi = \begin{pmatrix} d_C & 0 \\ \beta + d_D h & -d_D \end{pmatrix} .$$

These agree since $\alpha - \beta = hd_C + d_D h$ and so $\varphi : \mathrm{cone}(\alpha) \to \mathrm{cone}(\beta)$ is a morphism. Likewise we define $\psi : \mathrm{cone}(\beta) \to \mathrm{cone}(\alpha)$ as $\psi = \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)$, and we have $\varphi\psi = 1$ and $\psi\varphi = 1$. □

Let R be a commutative k-algebra, (Z, d_Z) a linear factorisation in which Z is an R-module and let $t = (t_1, \cdots, t_n)$ be a sequence of elements of R which all act null-homotopically on (Z, d_Z). Consider the Koszul complex $(K(t), d_K)$ of t. We now aim to construct an isomorphism of linear factorisations

$$(K(t) \otimes_R Z, d_K \otimes 1 + 1 \otimes d_Z) \xrightarrow{\cong} (\textstyle\bigwedge(\bigoplus_{i=1}^n R\theta_i) \otimes_R Z, 1 \otimes d_Z)$$

where $\theta_1, \ldots, \theta_n$ are formal generators. We begin by sketching why we expect such an isomorphism to exist, and then in Lemma 5.2.4 we provide an explicit isomorphism.

For each t_i consider the Koszul complex $(K(t_i), d_{K_i})$ of t_i regarded as a one element sequence. It is straightforward to check that $(K(t_i), d_{K_i}) \otimes_R (Z, d_Z) \cong \mathrm{cone}(t_i)$, where $\mathrm{cone}(t_i)$ is the cone of the map $t_i : (Z, d_Z) \to (Z, d_Z)$ given by multiplication by t_i. By Lemma 5.2.2 we have $\mathrm{cone}(t_i) \cong \mathrm{cone}(0)$. But cone(0) is just $(Z \oplus Z[1], d_Z \oplus (-d_Z))$, and $Z \oplus Z[1] \cong (R \oplus R[1]) \otimes_R Z$ as R-modules. Noting that we have $K(t) = K(t_1) \otimes_R \cdots \otimes_R K(t_n)$ we can inductively show $(K(t) \otimes_R Z, 1 \otimes d_Z) \cong (K(t) \otimes_R Z, d_K \otimes 1 + 1 \otimes d_Z)$ as linear factorisations over R.

We now write down this isomorphism explicitly. First note that we have a canonical isomorphism $\bigwedge(\bigoplus_{i=1}^n R\theta_i) \otimes_R Z \cong \bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z$ so it suffices to find an isomorphism

$$\varphi : (\textstyle\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z, d_K + d_Z) \longrightarrow (\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z, d_Z) \quad .$$

Let $\theta_1^*, \ldots, \theta_n^*$ be the contraction operators on $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ and for each $i = 1, \ldots, n$ let $\lambda_i : t_i \simeq 0$ be the homotopy on Z arising in Lemma 5.2.1. We consider both θ_i^* and λ_i^* acting on $\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z$ as $\theta_i^* = \theta_i^* \otimes 1$ and $\lambda_i = 1 \otimes \lambda_i$ respectively, recalling that this tensor product is graded with respect to the $\mathbb{Z}_2$-grading. Following [Mur18, Section 4.2] we define

$$\exp(\delta) = \sum_{m \geq 0} \frac{1}{m!} \delta^m \qquad \text{and} \qquad \exp(-\delta) = \sum_{m \geq 0} \frac{(-1)^m}{m!} \delta^m$$

where $\delta = \sum_{i=1}^n \lambda_i \theta_i^*$. This definition makes sense because δ is nilpotent: δ has degree -1 with respect to the $\mathbb{Z}$-grading on $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ and this graded module is zero in negative degree.

Lemma 5.2.3. *$\delta\theta_j^* = \theta_j^*\delta$ for all $j = 1, \ldots, n$.*

Proof. Recall that for the graded tensor product

$$(\nu_1 \otimes \mu_1)(\nu_2 \otimes \mu_2) = (-1)^{|\mu_1||\nu_2|}\nu_1\nu_2 \otimes \mu_1\mu_2$$

where $\nu_1, \nu_2, \mu_1, \mu_2$ are appropriate homogeneous maps. Hence since $\lambda_i = 1 \otimes \lambda_i$ and $\theta_j^* = \theta_j^* \otimes 1$ we have $\lambda_i\theta_j^* = -\theta_j^*\lambda_i$ and so

$$\delta\theta_j^* = \sum_{i=1}^{n} \lambda_i\theta_i^*\theta_j^* = -\sum_{i=1}^{n} \lambda_i\theta_j^*\theta_i^* = \theta_j^*\delta$$

as claimed. □

Lemma 5.2.4 ([Mur18, Proposition 4.12])**.** *The map*

$$\exp(\delta) : (\textstyle\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z, d_K + d_Z) \longrightarrow (\textstyle\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z, d_Z)$$

is an isomorphism with inverse $\exp(-\delta)$*.*

Proof. Clearly $\exp(\delta)$ and $\exp(-\delta)$ are mutually inverse isomorphisms of modules so it suffices to show that they commute with the differentials. We first show that $[d_Z, \delta^m] = m\delta^{m-1}d_K$ for $m \geq 1$, where $[d_Z, \delta^m]$ is the graded commutator with respect to the $\mathbb{Z}_2$-grading. When $m = 1$ we have

$$[d_Z, \delta] = \sum_{i=1}^{n}[d_Z, \lambda_i]\theta_i^* = \sum_{i=1}^{n} t_i\theta_i^* = d_K$$

where we recall that $d_K = \sum_{i=1}^n t_i\theta_i^*$. Now consider $m > 1$. First note that

$$\begin{aligned}\sum_{i=0}^{m-1} \delta^i[d_Z, \delta]\delta^{m-i-1} &= \sum_{i=0}^{m-1} \delta^i d_Z\delta^{m-i} - \sum_{i=0}^{m-1} \delta^{i+1}d_Z\delta^{m-i-1}\\ &= \sum_{i=0}^{m-1} \delta^i d_Z\delta^{m-i} - \sum_{i=1}^{m} \delta^i d_Z\delta^{m-i}\\ &= [d_Z, \delta^m]\end{aligned}$$

and so we have

$$[d_Z, \delta^m] = \sum_{i=0}^{m-1} \delta^i[d_Z, \delta]\delta^{m-i-1} = \sum_{i=0}^{m-1} \delta^i d_K\delta^{m-i-1} = \sum_{i=0}^{m-1} \delta^{m-1}d_K = m\delta^{m-1}d_K$$

since by Lemma 5.2.3 we have $d_K\delta = \delta d_K$. Next we compute $[d_Z, \exp(\delta)]$. We have

$$[d_Z, \exp(\delta)] = \sum_{m\geq 0} \frac{1}{m!}[d_Z, \delta^m] = \sum_{m\geq 1} \frac{1}{(m-1)!}\delta^{m-1}d_K = \exp(\delta)d_K\ .$$

Since $[d_Z, \exp(\delta)] = d_Z \exp(\delta) - \exp(\delta)d_Z$, rearranging this expression gives

$$\exp(\delta)(d_Z + d_K) = d_Z \exp(\delta)$$

which shows $\exp(\delta)$ is a morphism of linear factorisations as required. □

Now consider the following 1-morphisms in $\mathcal{LG}_k$

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

where $k[y] = k[y_1, \cdots, y_n]$. We now show that the composition of (X, d_X) and (Y, d_Y) is well-defined. This is done in several places in the literature however these papers often have other goals which can complicate the situation considerably and do not include exposition which would be helpful to non-experts. Here we give a proof which presents this result in isolation and includes details which are alluded to in the literature. There are several ways to present this result, and here we most closely follow the approach of [Mur18].

Proposition 5.2.5. *The composition of (X, d_X) and (Y, d_Y), which is $(X \otimes_{k[y]} Y, d_{X \otimes Y})$, is a direct summand of a matrix factorisation which is finite rank over $k[x, z]$.*

Proof. Let $t = (\partial_{y_1} V, \ldots, \partial_{y_n} V)$ be the sequence of partial derivatives in $k[y]$. Consider the Jacobi ring $J_V = k[y]/(t)$ and the Kozul complex $(K(t), d_K)$ of t. Since V is a potential J_V is a free k-module and by Corollary 2.4.3 we obtain a strong deformation retract

$$(J_V, 0) \underset{\sigma}{\overset{\pi}{\leftrightarrows}} (K(t), d_K), \qquad h$$

over k. We would like to tensor both sides of this strong deformation retract by $X \otimes_{k[y]} Y$ and mix in the differential $d_{X \otimes Y}$ along the lines of Corollary 4.3.4. However, while all the modules in the above strong deformation retract are $k[y]$-modules, the maps σ and h are only k-linear *a priori*.

The solution is to fix a $k[x, z]$-basis of the form $\{e_a \otimes f_b\}_{a,b}$ for $X \otimes_{k[y]} Y$ and define $k[x, z]$-linear maps

$$(X \otimes_{k[y]} J_V \otimes_{k[y]} Y, 0) \underset{\tilde{\sigma}}{\overset{\pi \otimes 1}{\leftrightarrows}} (K(t) \otimes_{k[y]} X \otimes_{k[y]} Y, d_K \otimes 1), \qquad \tilde{h} \tag{5.1}$$

as follows. Since π and d_K in the original strong deformation retract are $k[y]$-linear we obtain $\pi \otimes 1$ and $d_K \otimes 1$ by applying the functor $(-) \otimes_{k[y]} X \otimes_{k[y]} Y$. The maps $\tilde{\sigma}$ and $\tilde{h}$ are defined on the basis $\{e_a \otimes f_b\}_{a,b}$ as

$$\tilde{\sigma}(e_a \otimes r \otimes f_b) = \sigma(r) \otimes e_a \otimes f_b$$

for $r \in J_V$, and for $g \in K(t)$

$$\tilde{h}(g \otimes e_a \otimes f_b) = h(g) \otimes e_a \otimes f_b$$

where we extend $k[x, z]$-linearly. Notice that the maps $\tilde{\sigma}$ and $\tilde{h}$ depend on the choice of basis for $X \otimes_{k[y]} Y$. It is straightforward to see that (5.1) is a strong deformation retract over $k[x, z]$.

Following Section 4.3 we now view $d = 1 \otimes d_{X \otimes Y}$ as a perturbation of (5.1). We aim to show d is a small perturbation. By Lemma 4.3.3 it suffices to show that $(d\tilde{h})^m = 0$ for sufficiently large m. With respect to the $\mathbb{Z}$-grading arising from $K(t)$ (and ignoring the grading on $X \otimes_{k[y]} Y$) we note that $d\tilde{h}$ is a degree -1 operator. Since $K(t)$ is a bounded complex we have that $(d\tilde{h})^m = 0$ for some sufficiently large m.

Hence by the Perturbation Lemma (Theorem 4.3.2) we obtain a strong deformation retract over $k[x, z]$

$$(Y|X, d_{Y|X}) \underset{\sigma'}{\overset{\pi'}{\leftrightarrows}} (K(t) \otimes_k X \otimes_{k[y]} Y, d_K \otimes 1 + 1 \otimes d_{X \otimes Y}), \qquad h'$$

where $\sigma' = \tilde{\sigma} + \tilde{h}a\tilde{\sigma}$, $\pi' = \pi \otimes 1 + (\pi \otimes 1)a\tilde{h}$, $h' = \tilde{h} + \tilde{h}a\tilde{h}$ and $a = (1 - d\tilde{h})^{-1}d$, and we define $Y|X = X \otimes_{k[y]} J_V \otimes_{k[y]} Y$ and $d_{Y|X} = d_X \otimes 1 + 1 \otimes d_Y$.

Lemma 5.2.1 shows that the partial derivatives of V act null-homotopically on $X \otimes_{k[y]} Y$ and so we have an isomorphism of linear factorisations

$$\varphi : (K(t) \otimes_{k[y]} X \otimes_{k[y]} Y, d_K \otimes 1 + 1 \otimes d_{X\otimes Y}) \longrightarrow (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X\otimes Y})$$

by Lemma 5.2.4. Hence we obtain a strong deformation retract

$$(Y|X, d_{Y|X}) \underset{\varphi\sigma'}{\overset{\pi'\varphi^{-1}}{\rightleftarrows}} (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X\otimes Y}), \qquad \varphi h' \varphi^{-1} \tag{5.2}$$

over $k[x, y]$. Since V is a potential $J_V \cong k^{\oplus m}$ for some m and so

$$Y|X = X \otimes_{k[y]} J_V \otimes_{k[y]} Y \cong k[x, z]^{\oplus m} \ .$$

So, the left-hand-side of (5.2) is a finite rank matrix factorisation and the right-hand-side has $(X \otimes_{k[y]} Y, d_{Y\otimes X})$ as a direct summand. □

It is worth comparing the perturbation step in the proof of Proposition 5.2.5 to the proof of Corollary 4.3.4, where we showed that tensoring a strong deformation retract of linear factorisations by a third linear factorisation does not disturb the strong deformation retract. In Corollary 4.3.4 the strong deformation retract was over a commutative ring R, and the tensor product was taken over the same ring. In Proposition 5.2.5 our strong deformation retract was over k, but we wanted to take tensor products over $k[y]$. This required us to introduce a basis in order to extend the maps in the strong deformation retract to the new modules. Then, when showing the differential $1 \otimes d_{X\otimes Y}$ is a small perturbation of (5.1) we need to use the $\mathbb{Z}$-grading on $K(t)$ since the homotopy $\tilde{h}$ in (5.1) does not commute with $1 \otimes d_{X\otimes Y}$.

Proposition 5.2.5 and Corollary B.3.3 show that the tensor product defines a functor

$$h(U, V) \times h(V, W) \to h(U, W)^{\omega} \ .$$

By the property of idempotent completion given in Lemma B.3.4 this extends to a functor

$$h(U, V)^{\omega} \times h(V, W)^{\omega} \to h(U, W)^{\omega}$$

which is composition in the bicategory $\mathcal{LG}_k$. This shows that composition in $\mathcal{LG}_k$ is well-defined. Since tensor products are associative up to natural isomorphism this composition functor is associative in the sense of Definition A.2.1. It is also straightforward to show that it satisfies the relevant coherence condition given in Definition A.2.1.

5.3 Unit 1-morphisms

We now consider unit 1-morphisms in $\mathcal{LG}_k$. Let $U \in k[x] = k[x_1, \dots, x_n]$ be a potential and $k[x, x'] = k[x_1, \dots, x_n, x'_1, \dots, x'_n]$. Let $t = x'_i - x_i$ and consider the regular sequence $t = (t_1, \dots, t_n)$ in $k[x, x']$. There is clearly a sequence $a = (a_1, \dots, a_n)$ in $k[x, x']$ such that $U(x') - U(x) = \sum_{i=1}^{n} a_i t_i$. Let (I, d) denote the Koszul matrix factorisation of $U(x') - U(x)$ arising from this expression, as specified in Definition 4.2.4.

Proposition 5.3.1. *(I, d) satisfies the properties of the unit 1-morphism of $(k[x], U)$ in $\mathcal{LG}_k$.*

Proof. We need to show given a 1-morphism $(X, d_X) : (k[x'], U) \to (k[y], V)$ that (X, d_X) is isomorphic to $(I, d) \otimes_{k[x']} (X, d_X)$ in the category $h(U, V)$, and similarly given a 1-morphism $(Y, d_Y) : (k[z], W) \to (k[x], U)$ that (Y, d_Y) is isomorphic to $(Y, d_Y) \otimes_{k[x]} (I, d)$ in the category $h(W, U)$.

We show the first isomorphism involving the matrix factorisation (X, d_X). The second proceeds similarly. Theorem 3.1.2 tells us that t is a Koszul-regular sequence, and so using Corollary 2.4.3 we obtain a strong deformation retract over k

$$(k[x'], 0) \rightleftarrows (K(t), d_K), \qquad h$$

where $(K(t), d_K)$ is the Koszul complex of t. As in the proof of Proposition 5.2.5 we fix a $k[x, y]$-basis for X in order to define a strong deformation retract

$$(X, 0) \rightleftarrows (K(t) \otimes_{k[x']} X, d_K \otimes 1), \qquad \tilde{h} \tag{5.3}$$

over $k[x, y]$. Now set $\delta_+ = \sum_{i=1}^{n} a_i \theta_i$ where $\theta_1, \cdots, \theta_n$ are the wedging operators on $K(t)$. Note that $(\delta_+ \otimes 1)\tilde{h}$ is a degree $+2$ operator with respect to the $\mathbb{Z}$-grading on $K(t)$. Since $K(t)$ is bounded this means $(\delta_+ \otimes 1)\tilde{h}$ is nilpotent. Likewise $(1 \otimes d_X)\tilde{h}$ is a degree $+1$ operator with respect to the $\mathbb{Z}$-grading on $K(t)$ and so $\delta_+ \otimes 1 + 1 \otimes d_X$ is a small perturbation of (5.3) by Lemma 4.3.3. Since $d = d_K + \delta_+$ this gives a strong deformation retract

$$(X, d_X) \rightleftarrows (I \otimes_{k[x']} X, d \otimes 1 + 1 \otimes d_X), \qquad h'$$

over $k[x, y]$ as required. That a 1-morphism $(Y, d_Y) : (k[y], V) \to (k[x], U)$ is homotopy equivalent to $(Y \otimes_{k[x]} I, d_{Y \otimes I})$ is shown similarly. □

Proposition 5.3.1 should be compared to Corollary 4.3.5, where we showed how to produce a strong deformation retract involving a Koszul matrix factorisation. In Proposition 5.3.1 we used a combination of the ideas Corollary 4.3.4 and Corollary 4.3.5 to simultaneously produce the Koszul matrix factorisation and tensor both sides of the deformation retract by another matrix factorisation.

5.4 The cut operation

Let k be a commutative ring and consider following morphisms

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

in $\mathcal{LG}_k$. Let $k[y] = k[y_1, \cdots, y_n]$, $t = (\partial_{y_1} V, \ldots, \partial_{y_n} V)$ be the sequence of partial derivatives of V, $I = (\partial_{y_1} V, \ldots, \partial_{y_n} V)$ the ideal generated by the elements of t and $J_V = k[y]/I$ be the Jacobi ring of V.

Definition 5.4.1. The *cut* of the matrix factorisations (X, d_X) and (Y, d_Y) is the matrix factorisation $(Y|X, d_{Y|X})$ where

$$Y|X = X \otimes_{k[y]} J_V \otimes_{k[y]} Y \qquad \text{and} \qquad d_{Y|X} = d_X \otimes 1 + 1 \otimes d_Y \ .$$

The cut operation on matrix factorisations was first defined in [Mur18]. The cut $(Y|X, d_{Y|X})$ arose in the proof of Proposition 5.2.5 which showed that $(X \otimes_{k[y]} Y, d_{X \otimes Y})$ is the direct summand of a finite rank matrix factorisation. The key idea of this proof was to show that there is a strong deformation retract

$$(Y|X, d_{Y|X}) \rightleftarrows (\textstyle\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X \otimes Y}), \qquad h \tag{5.4}$$

over $k[x,z]$. The cut $(Y|X, d_{Y|X})$ is finite rank, and the module on the right-hand-side of (5.4) is a direct sum of copies of $X \otimes_{k[y]} Y$, some of which are shifted in degree.

The goal of this section and the next is to better understand how $(X \otimes_{k[y]} Y, d_{X\otimes Y})$ sits inside the cut $(Y|X, d_{Y|X})$. This will involve producing explicit formulae for the maps in (5.4). Let $(K(t), d_K)$ denote the Koszul complex of t. The key steps in producing the strong deformation retract in (5.4) were as follows:

(1) We began with a strong deformation retract over k

$$(J_V, 0) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (K(t), d_K), \qquad h \quad .$$

(2) We tensored both sides of this strong deformation retract by $X \otimes_{k[y]} Y$ and used the Perturbation Lemma to mix in the differential $d_{X\otimes Y}$.

(3) We showed that there is an isomorphism

$$\varphi : (K(t) \otimes_{k[y]} X \otimes_{k[y]} Y, d_K \otimes 1 + 1 \otimes d_{X\otimes Y}) \longrightarrow (\textstyle\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X\otimes Y}) \; .$$

The effect of the perturbation step on morphisms is explained in Theorem 4.3.2 and the isomorphism in (3) is explicitly given in Lemma 5.2.4. So in order to produce formulae for (5.4) all that remains is to give formulae for the strong deformation retract in (1). For this we can either use Proposition 3.2.9 or Corollary 3.4.3. Proposition 3.2.9 is most general but requires passing to the I-adic completion $\widehat{k[y]}$ of $k[y]$, while the Corollary 3.4.3 requires us to assume that the sequence of partial derivatives t is a Gröbner basis. For the sake of working in the most general setting we will use Proposition 3.2.9, however when coming up with examples of cuts it may be easier to avoid the completion and use Corollary 3.4.3. Indeed, this is the approach we take in Section 5.6 when giving examples.

Let $(X\widehat{\otimes}Y, d_{X\widehat{\otimes}Y})$ denote the matrix factorisation obtained by applying the extension of scalars functor $\widehat{k[y]} \otimes_{k[y]} (-)$. Our first goal is to produce an explicit strong deformation retract over $k[x,z]$

$$(Y|X, d_{Y|X}) \rightleftarrows (\textstyle\bigwedge(k^{\oplus n}) \otimes_k X\widehat{\otimes}Y, 1 \otimes d_{X\widehat{\otimes}Y}), \qquad h$$

following the strategy outlined above. Having done so we will then address the difference between $(X\widehat{\otimes}Y, d_{X\widehat{\otimes}Y})$ and $(X \otimes_{k[y]} Y, d_{X\otimes Y})$ by showing that the canonical map $\kappa : X \otimes_{k[y]} Y \to X\widehat{\otimes}Y$ is an isomorphism in the homotopy category.

We now ensure the conditions of Proposition 3.2.9 are satisfied. First of all suppose $\mathbb{Q} \subseteq k$. Since V is a potential, J_V is free over k and t is Koszul-regular in $k[y]$. Since J_V is free we have a k-linear section $J_V \to k[y]$ of the quotient map. This section can be chosen so that $1 \mapsto 1$ by choosing k-bases for J_V and $k[y]$. For Proposition 3.2.9 we also need t to be Koszul-regular in $\widehat{k[y]}$. This is clearly true if we assume that k is Noetherian, however as shown in [Mur19, Appendix C] this assumption can be removed with some thought.

Let $\partial_{t_1}, \dots, \partial_{t_n} : \widehat{k[y]} \to \widehat{k[y]}$ be the system of t-derivatives defined in Section 3.2. We consider the Koszul complex $(\widehat{K}(t), d_{\widehat{K}})$ of t over $\widehat{k[y]}$ and denote its underlying module as in Section 3.2 by

$$\widehat{K}(t) = \textstyle\bigwedge(\bigoplus_{i=1}^{n} \widehat{k[y]} dt_i)$$

where $dt_1, \dots, dt_n$ are formal generators. Let $\nabla : \widehat{K}(t) \to \widehat{K}(t)$ be the connection arising from $\partial_{t_1}, \dots, \partial_{t_n}$ of Definition 3.2.2. Given the above assumptions, Proposition 3.2.9 yields a strong deformation retract over k

$$(J_V, 0) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (\widehat{K}(t), d_{\widehat{K}}), \qquad h \tag{5.5}$$

where π is the quotient map, $h = [d_{\widehat{K}}, \nabla]^{-1}\nabla$ and σ is as given in Proposition 3.2.9. A formula for $[d_K, \nabla]^{-1}$ can be determined by inspecting the proof of Lemma 3.2.7.

Next, we tensor (5.5) by $X\widehat{\otimes}Y$ and mix the differential $d_{X\widehat{\otimes}Y}$ using the Perturbation Lemma as in the proof of Proposition 5.2.5. We fix a $k[x,z]$-basis for $X\widehat{\otimes}Y$ of the form $\{e_a \otimes f_b\}_{a,b}$ and define a strong deformation retract over $k[x,z]$

$$(J_V \otimes_{\widehat{k[y]}} X\widehat{\otimes}Y, 0) \underset{\tilde{\sigma}}{\overset{\pi\otimes 1}{\rightleftarrows}} (\widehat{K}(t) \otimes_{\widehat{k[y]}} X\widehat{\otimes}Y, d_{\widehat{K}} \otimes 1), \qquad \tilde{h} \tag{5.6}$$

where $\tilde{\sigma}(r \otimes e_a \otimes f_b) = \sigma(r) \otimes e_a \otimes f_b$ and $\tilde{h}(g \otimes e_a \otimes f_b) = h(g) \otimes e_a \otimes f_b$. Note that on the left-hand-side of (5.6) we have

$$J_V \otimes_{\widehat{k[y]}} X\widehat{\otimes}Y \cong \left(X\widehat{\otimes}Y\right)/I\left(X\widehat{\otimes}Y\right) \cong X \otimes_{k[y]} (\widehat{k[y]}/I\widehat{k[y]}) \otimes_{k[y]} Y \cong Y|X$$

where we have that $J_V \cong \widehat{k[y]}/I\widehat{k[y]}$ by Lemma 3.1.5.

Now set $d = 1 \otimes d_{X\widehat{\otimes}Y}$ and view d as a perturbation of (5.6). Let $a = (1 - d\tilde{h})^{-1}d$ and, since $d\tilde{h}$ is nilpotent (see the proof of Proposition 5.2.5), we have that $(1 - d\tilde{h})^{-1} = \sum_{l\geq 0}(d\tilde{h})^l$ by Lemma 4.3.3. By the Perturbation Lemma (Theorem 4.3.2) we have a strong deformation retract over $k[x,y]$

$$(Y|X, d_{Y|X}) \underset{\sigma_\infty}{\overset{\pi_\infty}{\rightleftarrows}} (\widehat{K}(t) \otimes_{\widehat{k[y]}} X\widehat{\otimes}Y, d_{\widehat{K}} \otimes 1 + 1 \otimes d), \qquad h_\infty$$

where $\sigma_\infty = \tilde{\sigma} + \tilde{h}a\tilde{\sigma}$, $\pi_\infty = \pi \otimes 1 + (\pi \otimes 1)a\tilde{h}$ and $h_\infty = \tilde{h} + \tilde{h}a\tilde{h}$. One can show via a direct calculation that $(\pi \otimes 1)a\tilde{h} = 0$ and so $\pi_\infty = \pi \otimes 1$. The maps σ_∞ and h_∞ can be written more conveniently as

$$\sigma_\infty = \tilde{\sigma} + \tilde{h}\sum_{l\geq 0}(d\tilde{h})^l d\tilde{\sigma} = \sum_{l\geq 0}(\tilde{h}d)\tilde{\sigma} \tag{5.7}$$

and

$$h_\infty = \tilde{h} + \tilde{h}\sum_{l\geq 0}(d\tilde{h})^l d\tilde{h} = \sum_{l\geq 0}(\tilde{h}d)\tilde{h}\ . \tag{5.8}$$

We now consider the isomorphism constructed in Lemma 5.2.4 which removes the Koszul differential $d_{\widehat{K}} \otimes 1$. Let $\alpha : \widehat{K}(t) \otimes_{\widehat{k[y]}} X\widehat{\otimes}Y \to \bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k X\widehat{\otimes}Y$ be the canonical $k[x,z]$-module isomorphism where $\theta_1, \ldots, \theta_n$ are formal generators. Using Lemma 5.2.1 we construct homotopies $\lambda_i : t_i \simeq 0$ on $X\widehat{\otimes}Y$ where t_i acts by multiplication. Following Lemma 5.2.4 we set $\delta = \sum_{i=1}^n \lambda_i\theta_i^*$ and define

$$\exp(\delta) = \sum_{m\geq 0}\frac{1}{m!}\delta^m \qquad \text{and} \qquad \exp(-\delta) = \sum_{m\geq 0}\frac{(-1)^m}{m!}\delta^m\ .$$

Taking $\varphi = \exp(\delta)\alpha$ and $\varphi^{-1} = \alpha^{-1}\exp(-\delta)$ gives us the isomorphism in (3). Putting all of this together we have constructed a strong deformation retract over $k[x,z]$

$$(Y|X, d_{Y|X}) \underset{\Phi}{\overset{\Phi'}{\rightleftarrows}} (\bigwedge(\textstyle\bigoplus_{i=1}^n k\theta_i) \otimes_k X\widehat{\otimes}Y, 1 \otimes d_{X\widehat{\otimes}Y}), \qquad H \tag{5.9}$$

where $\Phi = \exp(\delta)\alpha\sigma_\infty$, $\Phi' = (\pi \otimes 1)\alpha^{-1}\exp(-\delta)$ and $H = \exp(\delta)\alpha h_\infty\alpha^{-1}\exp(-\delta)$.

It remains to address the difference between $(X\widehat{\otimes}Y, d_{X\widehat{\otimes}Y})$ and $(X \otimes_{k[y]} Y, d_{X\otimes Y})$. The canonical completion map $k[y] \to \widehat{k[y]}$ induces a $k[x,y,z]$-linear map $\kappa : X \otimes_{k[y]} Y \to X\widehat{\otimes}Y$. We now show that κ is an isomorphism in the homotopy category.

Lemma 5.4.2. *The canonical map $\kappa : X \otimes_{k[y]} Y \to X \widehat{\otimes} Y$ is a homotopy equivalence of linear factorisations.*

Proof. Let $Z = X \otimes_{k[y]} Y$ and $\widehat{Z} = X \widehat{\otimes} Y$. We have the following commutative diagram

$$\begin{array}{ccccc}
\widehat{Z}/I\widehat{Z} & \leftrightarrows & \bigwedge(k^{\oplus n}) \otimes_k \widehat{Z} & \xrightarrow{p'} & \widehat{Z} \\
\cong \Big| & & \Big\uparrow & & \Big\uparrow \kappa \\
Z/IZ & \leftrightarrows & \bigwedge(k^{\oplus n}) \otimes_k Z & \xrightarrow{p} & Z
\end{array}$$

of $k[x, z]$-linear maps. That $\widehat{Z}/I\widehat{Z} \cong Z/IZ$ follows directly from the fact that $k[y]/I \cong \widehat{k[y]}/I\widehat{k[y]}$ and was also shown above. The maps $\widehat{Z}/I\widehat{Z} \leftrightarrows \bigwedge(k^{\oplus n}) \otimes_k \widehat{Z}$ and $Z/IZ \leftrightarrows \bigwedge(k^{\oplus n}) \otimes_k Z$ are each obtained in the same way using the idea of the proof of Proposition 5.2.5. That is, the former is the strong deformation retract (5.9) and the latter is the strong deformation retract (5.4). In particular, each of these pairs of maps are homotopy equivalences over $k[x, z]$. The maps p and p' are both given by projecting on to the degree zero part of $\bigwedge(k^{\oplus n})$, and the map $\bigwedge(k^{\oplus n}) \otimes_k Z \to \bigwedge(k^{\oplus n}) \otimes_k \widehat{Z}$ is induced from $k[y] \to \widehat{k[y]}$.

Therefore $\bigwedge(k^{\oplus n}) \otimes_k Z$ and $\bigwedge(k^{\oplus n}) \otimes_k \widehat{Z}$ are homotopy equivalent over $k[x, z]$ and hence so are Z and $\widehat{Z}$. As well as being $k[x, z]$-linear, all maps involved are morphisms of linear factorisations so this completes the proof. ☐

A version of Lemma 5.4.2, which is proved in essentially the same way, holds for pushing forward linear factorisations along any flat morphism of rings.[5] This is discussed in [DM13, Remark 7.7]. If k is Noetherian then $k[y] \to \widehat{k[y]}$ is flat, so in the Noetherian setting Lemma 5.4.2 can be viewed as a special case of this result. However, as we have previously stated, a Noetherian hypothesis on k is not required to construct the cut operation. A proof of Lemma 5.4.2 without the Noetherian hypothesis is discussed in [Mur19, Appendix C].

[5] Recall that a morphism of rings $R \to R'$ is *flat* if the extension of scalars functor $R' \otimes_R (-)$ preserves exact sequences. By *pushing forward* we mean applying the extension of scalars functor to the linear factorisation.

5.5 The Clifford action on the cut

We retain the context from the previous section. In particular we consider the morphisms

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

in $\mathcal{LG}_k$ and the strong deformation retract given in (5.9). For simplicity we will focus on the matrix factorisation $(X \widehat{\otimes} Y, d_{X\widehat{\otimes}Y})$, rather than $(X \otimes_{k[y]} Y, d_{X\otimes Y})$. As shown in Lemma 5.4.2 these are equivalent in the homotopy category.

We now consider the action (up to homotopy) of a Clifford algebra on the cut of (X, d_X) and (Y, d_Y). Given the strong deformation retract (5.9) it is not surprising that such an action exists. As shown in Lemma 4.2.2 the wedging and contraction operators $\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*$ on $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ satisfy the canonical anticommutation relations (Definition 4.2.1), and any object with endomorphisms satisfying these relations admits a Clifford algebra representation (this is essentially shown in Lemma 5.6.2).

The strong deformation retract (5.9) transfers this Clifford action to the cut, at least up to homotopy. This Clifford action is interesting because it tells us how $(X \widehat{\otimes} Y, d_{X\widehat{\otimes}Y})$ — and hence $(X \otimes_{k[y]} Y, d_{X\otimes Y})$ — sits inside the cut $(Y|X, d_{Y|X})$. Consider the operator

$$e = \theta_1^* \cdots \theta_n^* \theta_n \cdots \theta_1 \ .$$

This is idempotent and projects onto the degree zero part of $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$. If we consider e acting on $\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k X \widehat{\otimes} Y$ then e splits as

$$e : \bigwedge(\textstyle\bigoplus_{i=1}^n k\theta_i) \otimes_k X \widehat{\otimes} Y \longrightarrow X \widehat{\otimes} Y \longrightarrow \bigwedge(\textstyle\bigoplus_{i=1}^n k\theta_i) \otimes_k X \widehat{\otimes} Y \quad .$$

We now define a Clifford action on the cut $(Y|X, d_{Y|X})$ and show that it is equal to the Clifford action arising as described above. This is done closely following the approach of [Mur18] but with additional exposition which is not found in the literature. In the following, if α and β are endomorphisms of the same graded object then $|\alpha|$ denotes the degree of the morphism α and $[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha$ is the graded commutator.

Lemma 5.5.1 (Jacobi identity)**.** *Let E be a graded ring, not assumed to be commutative. For $a, b \in E$ let $|a|$ denote the degree of a and $[a, b] = ab - (-1)^{|a||b|}ba$ the graded commutator on E. Then for all $a, b, c \in E$ we have*

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0 \ .$$

Proof. First note that the degree of $[a, b]$ is $|a| + |b|$. Then

$$\begin{aligned}
&(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] \\
&\quad = (-1)^{|a||c|}a(bc - (-1)^{|b||c|}cb) - (-1)^{|a||b|+2|a||c|}(bc - (-1)^{|b||c|}cb)a \\
&\qquad + (-1)^{|b||a|}b(ca - (-1)^{|c||a|}ac) - (-1)^{|b||c|+2|b||a|}(ca - (-1)^{|c||a|}ac)b \\
&\qquad + (-1)^{|c||b|}c(ab - (-1)^{|a||b|}ba) - (-1)^{|c||a|+2|c||b|}(ab - (-1)^{|a||b|}ba)c \\
&\quad = (-1)^{|a||c|}abc - (-1)^{|a||c|+|b||c|}acb - (-1)^{|a||b|}bca + (-1)^{|b||c|+|a||b|}cba \\
&\qquad + (-1)^{|b||a|}bca - (-1)^{|b||a|+|c||a|}bac - (-1)^{|b||c|}cab + (-1)^{|b||c|+|c||a|}acb \\
&\qquad + (-1)^{|c||b|}cab - (-1)^{|c||b|+|a||b|}cba - (-1)^{|c||a|}abc + (-1)^{|c||a|+|a||b|}bac \\
&\quad = 0
\end{aligned}$$

proving the claim. ∎

Our first step is to extend $\partial_{t_1}, \ldots, \partial_{t_n}$ to $k[x,z]$-linear maps on $X \widehat{\otimes} Y$. We do this using the $k[x,z]$-basis $\{e_a \otimes f_b\}_{a,b}$ we chose for $X \widehat{\otimes} Y$ in Section 5.4. Consider the map $X \widehat{\otimes} Y \to X \widehat{\otimes} Y$ defined by sending

$$e_a \otimes h \otimes f_b \longmapsto e_a \otimes \partial_{t_i}(h) \otimes f_b$$

for all basis elements $e_a \otimes f_b$ and $h \in \widehat{k[y]}$, and extending $k[x,z]$-linearly. We also denote this map by ∂_{t_i}.

Lemma 5.5.2. $[\partial_{t_i}, t_j] = \delta_{ij}$.

Proof. We first consider $\partial_{t_i} : \widehat{k[y]} \to \widehat{k[y]}$. By the Leibniz rule in Lemma 3.2.1 we have

$$\partial_{t_i}(t_j h) = \partial_{t_i}(t_j)h + t_j \partial_{t_i}(h) = \delta_{ij} h + t_j \partial_{t_i}(h)$$

for all $h \in \widehat{k[y]}$. Rearranging the above gives $[\partial_{t_i}, t_j](h) = \delta_{ij} h$. For the extension of ∂_{t_i} to $X \widehat{\otimes} Y$, first note that t_j and ∂_{t_i} are degree zero maps on $X \widehat{\otimes} Y$. Then we have

$$[\partial_{t_i}, t_j](e_a \otimes h \otimes f_b) = e_a \otimes [\partial_{t_i}, t_j](h) \otimes f_b = e_a \otimes \delta_{ij} h \otimes f_b$$

which proves the claim. □

Lemma 5.5.3. *The map $[d_{X \widehat{\otimes} Y}, \partial_{t_i}]$ induces a $k[x,z]$-linear map on $Y|X$.*

Proof. Using Lemma 5.5.1 and Lemma 5.5.2 we have

$$[[d_{X \widehat{\otimes} Y}, \partial_{t_i}], t_j] = -[d_{X \widehat{\otimes} Y}, [\partial_{t_i}, t_j]] - [\partial_{t_i}, [t_j, d_{X \widehat{\otimes} Y}]] = -[d_{X \widehat{\otimes} Y}, \delta_{ij}] = 0$$

where we have that $[t_j, d_{X \widehat{\otimes} Y}] = 0$ since $d_{X \widehat{\otimes} Y}$ is $k[y]$-linear. This shows that $[d_{X \widehat{\otimes} Y}, \partial_{t_i}]$ commutes with multiplication by t_j. In particular this means that

$$[d_{X \widehat{\otimes} Y}, \partial_{t_i}](I(X \widehat{\otimes} Y)) \subseteq I(X \widehat{\otimes} Y)$$

which, recalling that $Y|X \cong X \widehat{\otimes} Y \big/ I(X \widehat{\otimes} Y)$, implies that $[d_{X \widehat{\otimes} Y}, \partial_{t_i}]$ induces a map on the cut. □

Definition 5.5.4. For each $i = 1, \ldots, n$, let

$$\mathrm{At}_i : Y|X \longrightarrow Y|X$$

denote the map induced by $[d_{X \widehat{\otimes} Y}, \partial_{t_i}]$. We call At_i the i^{th} *Atiyah class.*

The maps At_i are called "Atiyah classes" because they arise from a concept in algebraic geometry of the same name (see [Mur18, Definition 3.8] and [DM13, Section 9]). Now denote $d = d_{Y|X}$ and let λ_i be a homotopy $\lambda_i : t_i \simeq 0$ on the cut $Y|X$ from Lemma 5.2.1.

Lemma 5.5.5. $[d, \mathrm{At}_i] = 0$

Proof. Noting that $\deg(\mathrm{At}_i) = \deg(d) = 1$ we have

$$\begin{aligned}
[d, \mathrm{At}_i] &= d\,\mathrm{At}_i + \mathrm{At}_i\, d \\
&= d[d, \partial_{t_i}] + [d, \partial_{t_i}]d \\
&= d(d\partial_{t_i} - \partial_{t_i} d) + (d\partial_{t_i} - \partial_{t_i} d)d \\
&= d^2 \partial_{t_i} - \partial_{t_i} d^2 \\
&= 0
\end{aligned}$$

where we use that d^2 is multiplication by $W(z) - U(x)$ and that this commutes with ∂_{t_i}. □

Lemma 5.5.6. *There is a homotopy* $\mu_{ij} : [\lambda_i, \lambda_j] \simeq \partial_{y_i y_j}(V)$.

Proof. Fix a basis for $Y|X$ and let $\partial_{y_i}(d)$ denote the matrix of d differentiated entrywise. Recall from the proof of Lemma 5.2.1 that $\lambda_j = \partial_{y_j}(d)$. Then, applying ∂_{y_i} to the equation $d\lambda_j + \lambda_j d = \partial_{y_j}(V)$ gives

$$\lambda_j \lambda_i + d\partial_{y_j y_i}(d) + \partial_{y_j y_i}(d)d + \lambda_i \lambda_j = \partial_{y_j y_i}(V) \ .$$

Hence setting $\mu_{ij} = -\partial_{y_j y_i}(d)$ proves the claim. □

On the cut $Y|X$ we define $k[x, z]$-linear maps

$$\gamma_i = \mathrm{At}_i \qquad \text{and} \qquad \gamma_i^\dagger = -\lambda_i - \frac{1}{2}\sum_{p=1}^{n} \partial_{y_p}(t_i) \,\mathrm{At}_p \tag{5.10}$$

for $i = 1, \ldots, n$. Note that these are odd. The next result is [Mur18, Theorem 3.11].

Proposition 5.5.7. *The operators* γ_i, $\gamma_i^\dagger : Y|X \to Y|X$ *for* $i = 1, \ldots, n$ *satisfy the canonical anticommutation relations of Definition 4.2.1 up to homotopy.*

Proof. We need to show

$$[\gamma_i, \gamma_j] \simeq 0 \ , \qquad [\gamma_i^\dagger, \gamma_j^\dagger] \simeq 0 \ , \qquad [\gamma_i, \gamma_j^\dagger] \simeq \delta_{ij}$$

for all $i, j = 1, \ldots, n$. We have

$$\begin{aligned} [\gamma_i, \gamma_j] &= [\mathrm{At}_i, \mathrm{At}_j] \\ &= [\mathrm{At}_i, [d, \partial_{t_j}]] \\ &= -[\partial_{t_j}, [\mathrm{At}_i, d]] - [d, [\partial_{t_j}, \mathrm{At}_i]] \\ &= -[d, [\partial_{t_j}, \mathrm{At}_i]] \end{aligned}$$

using Lemma 5.5.1 and Lemma 5.5.5. Setting $h_{ij} = -[\partial_{t_j}, \mathrm{At}_i]$ we have $h_{ij} : [\gamma_i, \gamma_j] \simeq 0$. Next we compute

$$\begin{aligned} [\lambda_i, \mathrm{At}_j] &= [\lambda_i, [d, \partial_{t_j}]] \\ &= -[\partial_{t_j}, [\lambda_i, d]] + [d, [\partial_{t_j}, \lambda_i]] \\ &= -[\partial_{t_j}, t_i] + [d, [\partial_{t_j}, \lambda_i]] \\ &= -\delta_{ij} + [d, [\partial_{t_j}, \lambda_i]] \end{aligned}$$

where we use Lemma 5.5.1, Lemma 5.5.2 and that $[\lambda_i, d] = t_i$ by definition. Set $g_{ij} = [\partial_{t_j}, \lambda_i]$ so we have $[\lambda_i, \mathrm{At}_j] = -\delta_{ij} + [d, g_{ij}]$. Then

$$\begin{aligned} [\gamma_i^\dagger, \gamma_j] &= \left[-\lambda_i - \frac{1}{2}\sum_{p=1}^{n} \partial_{y_p}(t_i)\,\mathrm{At}_p, \mathrm{At}_j \right] \\ &= \delta_{ij} - [d, g_{ij}] - \frac{1}{2}\sum_{p=1}^{n} \partial_{y_p}(t_i)[\mathrm{At}_p, \mathrm{At}_j] \\ &= \delta_{ij} - [d, g_{ij}] - \frac{1}{2}\sum_{p=1}^{n} \partial_{y_p}(t_i)[d, h_{pj}] \\ &= \delta_{ij} - \left[d, g_{ij} + \frac{1}{2}\sum_{p=1}^{n} \partial_{y_p}(t_i) h_{pj} \right] \end{aligned}$$

which shows $[\gamma_j^\dagger, \gamma_j] \simeq \delta_{ij}$. Finally we have

$$\begin{aligned}
[\gamma_i^\dagger, \gamma_j^\dagger] &= \left[\lambda_i + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_i)\,\mathrm{At}_p, \lambda_j + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_j)\,\mathrm{At}_p\right] \\
&= [\lambda_i, \lambda_j] + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_j)[\lambda_i, \mathrm{At}_p] \\
&\quad + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_i)[\mathrm{At}_p, \lambda_j] + \frac{1}{4}\sum_{p,q} \partial_{y_p}(t_i)\partial_{y_q}(t_j)[\mathrm{At}_p, \mathrm{At}_q] \\
&= \partial_{y_i y_j}(V) + [d, \mu_{ij}] + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_j)(-\delta_{ip} + [d, g_{ip}]) \\
&\quad + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_i)(-\delta_{jp} + [d, g_{jp}]) + \frac{1}{4}\sum_{p,q} \partial_{y_p}(t_i)\partial_{y_q}(t_j)[d, h_{pq}] \\
&= \left[d, \mu_{ij} + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_j) g_{ip} + \frac{1}{2}\sum_{p=1}^n \partial_{y_p}(t_i) g_{jp} + \frac{1}{4}\sum_{p,q} \partial_{y_p}(t_i)\partial_{y_q}(t_j) h_{pq}\right]
\end{aligned}$$

where $\mu_{ij} : [\lambda_i, \lambda_j] \simeq \partial_{y_i y_j}(V)$ is the homotopy arising in Lemma 5.5.6. □

We now prove that the operators $\gamma_1^\dagger, \dots, \gamma_n^\dagger, \gamma_1, \dots, \gamma_n$ on the cut arise via the strong deformation retract from the wedging and contraction operators $\theta_1, \dots, \theta_n, \theta_1^*, \dots, \theta_n^*$ on $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$. This is done in [Mur18, Section 4.2], however the following contains many details which are not given there. Consider the strong deformation retract in (5.9). It consists of maps

$$(Y|X, d_{Y|X}) \underset{\sigma_\infty}{\overset{\pi\otimes 1}{\leftrightarrows}} (\widehat{K}(t) \otimes_{\widehat{k[y]}} X \widehat{\otimes} Y, d_1) \underset{\varphi}{\overset{\varphi^{-1}}{\leftrightarrows}} (\textstyle\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k X \widehat{\otimes} Y, d_2)$$

where $d_1 = d_K \otimes 1 + 1 \otimes d_{X \widehat{\otimes} Y}$, $d_2 = 1 \otimes d_{X \widehat{\otimes} Y}$ and $\varphi = \exp(\delta)\alpha$. The definitions of α, δ, $\exp(\delta)$ and $\exp(-\delta)$ are given in just prior to (5.9). Recall that $\varphi^{-1} = \alpha^{-1}\exp(-\delta)$.

We define $T(\theta_j^*) = \exp(-\delta)\theta_j^* \exp(\delta)$ and $T(\theta_j) = \exp(-\delta)\theta_j \exp(\delta)$. These are the actions of θ_j^* and θ_j respectively under the module automorphism $\exp(-\delta)$.

Lemma 5.5.8. $T(\theta_i^*) = \theta_i^*$.

Proof. Recall from Lemma 5.2.3 that $\delta\theta_j^* = \theta_j^*\delta$. From this the result is immediate. □

We now compute $T(\theta_j)$. Notice that

$$T(\theta_j) = \theta_j - [\theta_j, \exp(-\delta)]\exp(\delta)$$

so it suffices to compute the commutator $[\theta_j, \exp(-\delta)]$. We break this up over several lemmas.

Lemma 5.5.9. $[\theta_j, \delta] = -\lambda_j$

Proof. We compute this directly:

$$[\theta_j, \delta] = \sum_{i=1}^n [\theta_j, \lambda_i\theta_i^*] = \sum_{i=1}^n \theta_j\lambda_i\theta_i^* - \lambda_i\theta_i^*\theta_j = -\sum_{i=1}^n \lambda_i(\theta_j\theta_i^* + \theta_i^*\theta_j) = -\lambda_j$$

where we use that $\theta_j\theta_i^* + \theta_i^*\theta_j = \delta_{ij}$, and that $\theta_j\lambda_i = -\lambda_i\theta_j$. For the latter, see the proof of Lemma 5.2.3. □

Lemma 5.5.10. $[\theta_j, \delta^m] = \sum_{p=0}^{m-1} \delta^p[\theta_j, \delta]\delta^{m-p-1}$

Proof. We proved a similar result when proving Lemma 5.2.4. Notice that the right-hand-side is a telescoping sum:

$$\begin{aligned}\sum_{p=0}^{m-1} \delta^p[\theta_j, \delta]\delta^{m-p-1} &= \sum_{p=0}^{m-1} \delta^p\theta_j\delta^{m-p} - \sum_{p=0}^{m-1} \delta^{p+1}\theta_j\delta^{m-p-1} \\ &= \sum_{p=0}^{m-1} \delta^p\theta_j\delta^{m-p} - \sum_{p=1}^{m} \delta^p\theta_j\delta^{m-p} \\ &= [\theta_j, \delta^m]\end{aligned}$$

which proves the claim. □

Lemma 5.5.11. *We have a homotopy* $\delta^p\lambda_j \simeq \lambda_j\delta^p - p\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\delta^{p-1}$ *for all* $p \geq 0$.

Proof. First note that the $p = 0$ case is trivial. Recall from Lemma 5.5.6 that $[\lambda_i, \lambda_j] \simeq \partial_{y_iy_j}(V)$. Then

$$\delta\lambda_j = \sum_{i=1}^n \lambda_i\theta_i^*\lambda_j = -\sum_{i=1}^n \lambda_i\lambda_j\theta_i^* \simeq \sum_{i=1}^n(\lambda_j\lambda_i - \partial_{y_iy_j}(V))\theta_i^* \simeq \lambda_j\delta - \sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*$$

where we use Lemma 5.2.3. We now proceed by induction on p. Supposing the relation holds for $p-1$ and also using the $p = 1$ case shown above we have

$$\begin{aligned}\delta^p\lambda_j &\simeq \delta(\lambda_j\delta^{p-1} - (p-1)\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\delta^{p-2}) \\ &\simeq \left(\lambda_j\delta - \sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\right)\delta^{p-1} - (p-1)\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\delta^{p-1} \\ &\simeq \lambda_j\delta^p - p\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\delta^{p-1}\end{aligned}$$

where we also use Lemma 5.2.3. □

Lemma 5.5.12. *We have a homotopy* $T(\theta_j) \simeq \theta_j - \lambda_j - \frac{1}{2}\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*$.

Proof. Using Lemma 5.5.9, Lemma 5.5.10 and Lemma 5.5.11 we have

$$\begin{aligned}[\theta_j, \exp(-\delta)] &= \sum_{m\geq 0} \frac{(-1)^m}{m!}[\theta_j, \delta^m] \\ &= -\sum_{m\geq 0} \frac{(-1)^m}{m!}\sum_{p=0}^{m-1} \delta^p\lambda_j\delta^{m-p-1} \\ &\simeq \sum_{m\geq 0} \frac{(-1)^m}{m!}\sum_{p=0}^{m-1}\left(\lambda_j\delta^p - p\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\delta^{p-1}\right)\delta^{m-p-1} \\ &\simeq -\lambda_j\sum_{m\geq 0} \frac{(-1)^m}{m!}m\delta^{m-1} + \sum_{m\geq 0} \frac{(-1)^m}{m!}\left(\sum_{p=0}^{m-1} p\right)\left(\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\right)\delta^{m-2} \\ &\simeq \lambda_j\exp(-\delta) + \left(\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\right)\sum_{m\geq 0} \frac{(-1)^m}{m!}\frac{m(m-1)}{2}\delta^{m-2} \\ &\simeq \lambda_j\exp(-\delta) + \frac{1}{2}\sum_{i=1}^n \partial_{y_iy_j}(V)\theta_i^*\exp(-\delta)\end{aligned}$$

and so

$$T(\theta_j) = \theta_j - [\theta_j, \exp(-\delta)]\exp(\delta) = \theta_j - \lambda_j - \frac{1}{2}\sum_{i=1}^{n} \partial_{y_i y_j}(V)\theta_i^*$$

as claimed. □

The following result is [Mur18, Proposition 4.35]. The proof there is more terse and we fill in more details in the proof below.

Lemma 5.5.13. *We have a homotopy* $(\pi\otimes 1)\alpha^{-1}\theta_j^*\alpha\sigma_\infty \simeq \mathrm{At}_j$ *and* $(\pi\otimes 1)\alpha^{-1}\theta_j\alpha\sigma_\infty = 0$.

Proof. Recall that we denote $\widehat{K}(t) = \bigwedge(\bigoplus_{i=1}^n \widehat{k[y]}dt_i)$ where $dt_1, \dots, dt_n$ are formal generators. First note that $\alpha^{-1}\theta_j^*\alpha = dt_j^*$ and $\alpha^{-1}\theta_j\alpha = dt_j$. That $(\pi\otimes 1)dt_j\sigma_\infty = 0$ is clear since $\sigma_\infty = 0$ except in degree zero of $\widehat{K}(t)$, and likewise for $\pi\otimes 1$. Wedging by dt_j has degree $+1$ with respect to the grading on $\widehat{K}(t)$ so we have $(\pi\otimes 1)dt_j\sigma_\infty = 0$.

Proving the other relation is more involved. In (5.7) we showed

$$\sigma_\infty = \sum_{l\geq 0}(\tilde{h}d)^l\tilde{\sigma}$$

where the maps $\tilde{h}$ and $\tilde{\sigma}$ are defined as part of the strong deformation retract in (5.6). Let $\nabla = \sum_{i=1}^n \partial_{t_i}dt_i$ be the connection of Definition 3.2.2 associated to the system of t-derivatives $\partial_{t_1}, \dots, \partial_{t_n}$. Let $\tau = [d_K, \nabla]$. Then on the $k[x,z]$-basis for $X\widehat{\otimes}Y$ used to define $\tilde{h}$ and $\tilde{\sigma}$ we have $\tilde{h} = \tau^{-1}\nabla$.

We now work modulo the ideal I. By Lemma 3.2.6 we have that $\tau(r\omega) \equiv pr\omega \pmod I$ where $\omega = dt_{i_1}\cdots dt_{i_p}$ and $r \in k[y]$. Hence, since ∇ is k-linear, we have $\nabla\tau^{-1} \equiv \tau^{-1}\nabla \pmod I$. Then, for $l > 0$ we have

$$\begin{aligned}(\tilde{h}d)^l\tilde{\sigma} &= \tau^{-1}\nabla d\tau^{-1}\nabla d\tau^{-1}\cdots\nabla d\tau^{-1}\nabla d\tilde{\sigma}\\ &= \tau^{-1}([d,\nabla] - d\nabla)\tau^{-1}\nabla d\tau^{-1}\cdots\nabla d\tau^{-1}\nabla d\tilde{\sigma}\\ &\equiv \tau^{-1}[d,\nabla]\tau^{-1}\nabla d\tau^{-1}\cdots\nabla d\tau^{-1}\nabla d\tilde{\sigma} \qquad \bmod I\end{aligned}$$

where to obtain the last line we use $\nabla\tau^{-1} \equiv \tau^{-1}\nabla \pmod I$ and that $\nabla^2 = 0$ by Lemma 3.2.3. By repeatedly applying the relation $\nabla d\tau^{-1}\nabla \equiv [d,\nabla]\tau^{-1}\nabla \pmod I$ we arrive at the expression

$$(\tilde{h}d)^l\tilde{\sigma} \equiv (\tau^{-1}[d,\nabla])^{l-1}\tau^{-1}\nabla d\tilde{\sigma} \qquad \bmod I\ .$$

Next we note that $\nabla\tilde{\sigma} = 0$ since $\tilde{\sigma}$ is non-zero only in degree zero of $K(t)$ and $\nabla(K_0(t)) = 0$. From this we find

$$\begin{aligned}(\tilde{h}d)^l\tilde{\sigma} &\equiv (\tau^{-1}[d,\nabla])^{l-1}\tau^{-1}\nabla d\tilde{\sigma} && \bmod I\\ &\equiv (\tau^{-1}[d,\nabla])^{l-1}\tau^{-1}([d,\nabla] - d\nabla)\tilde{\sigma} && \bmod I\\ &\equiv (\tau^{-1}[d,\nabla])^{l-1}\tau^{-1}[d,\nabla]\tilde{\sigma} && \bmod I\\ &\equiv (\tau^{-1}[d,\nabla])^{l}\tilde{\sigma} && \bmod I\ .\end{aligned}$$

Next note that $[d,\nabla] = \sum_{i=1}^n t_i[d,\partial_{t_i}]$. Then for $r_{ab} = e_a\otimes r\otimes f_b \in Y|X$, where $r \in J_V$ and $\{e_a\otimes f_b\}_{a,b}$ is the $k[x,z]$-basis for $X\otimes_{k[y]} Y$ used to define $\tilde{\sigma}$, we have

$$\begin{aligned}\tilde{h}d\tilde{\sigma}(r_{ab}) &\equiv \sum_{i=1}^n \tau^{-1}[d,\partial_{t_i}](\tilde{\sigma}(r_{ab})dt_i) && \bmod I\\ &\equiv \sum_{i=1}^n [d,\partial_{t_i}](\tilde{\sigma}(r_{ab}))dt_i && \bmod I\end{aligned}$$

and likewise

$$\begin{aligned}
\tilde{h}d\tilde{h}d\tilde{\sigma}(r_{ab}) &\equiv \tilde{h}d\left(\sum_{i=1}^{n}[d,\partial_{t_i}](\tilde{\sigma}(r_{ab}))\right) && \mod I\\
&\equiv \sum_{i,j}\tau^{-1}\left([d,\partial_{t_i}]^2(\tilde{\sigma}(r_{ab}))dt_j dt_i\right) && \mod I\\
&\equiv \sum_{i,j}\frac{1}{2}[d,\partial_{t_i}]^2(\tilde{\sigma}(r_{ab}))dt_j dt_i && \mod I\ .
\end{aligned}$$

In the same way we can show that $(\tilde{h}d)^l\tilde{\sigma}(r_{ab})$ is in the degree l part of $\widehat{K}(t)$.

We now compute $(\pi\otimes 1)dt_j^*\sigma_\infty$. First note that $\pi\otimes 1$ sends anything involving an element of I to zero, so when precomposing with $\pi\otimes 1$ all equivalences modulo I become equalities. Next, we note that dt_j^* anti-commutes with $[d,\partial_{t_j}]$. Since $\pi\otimes 1$ is zero away from degree zero, the only term of $\sigma_\infty = \tilde{\sigma}+\tilde{h}d\tilde{\sigma}+(\tilde{h}d)^2\tilde{\sigma}+\cdots$ which contributes to the composition $(\pi\otimes 1)dt_j^*\sigma_\infty$ is $\tilde{h}d\tilde{\sigma}$. Then we have

$$\begin{aligned}
(\pi\otimes 1)dt_j^*\sigma_\infty(r) &= \sum_{i=1}^{n}(\pi\otimes 1)dt_j^*[d,\partial_{t_i}]\tilde{\sigma}(r)dt_i\\
&= -\sum_{i=1}^{n}\mathrm{At}_i\, dt_j^* dt_i\\
&\simeq \mathrm{At}_j
\end{aligned}$$

by definition of the Atiyah classes. □

Proposition 5.5.14. *Let*

$$(Y|X, d_{Y|X}) \underset{\Phi}{\overset{\Phi'}{\rightleftarrows}} \left(\bigwedge\left(\bigoplus_{i=1}^n k\theta_i\right)\otimes_k X\widehat{\otimes}Y, 1\otimes d_{X\widehat{\otimes}Y}\right), \qquad H$$

be the strong deformation retract given in (5.9). Then $\gamma_j^\dagger \simeq \Phi'\theta_j\Phi$ and $\gamma_j\simeq\Phi'\theta_j^*\Phi$.

Proof. Note that $\Phi'\theta_j\Phi = (\pi\otimes 1)\alpha^{-1}T(\theta_j)\alpha\sigma_\infty$ and likewise for $\Phi'\theta_j^*\Phi$. The result follows from Lemma 5.5.8, Lemma 5.5.12 and Lemma 5.5.13. □

5.6 Examples

Let k be a commutative ring. In this section we consider examples of morphisms in $\mathcal{LG}_k$ between potentials which are quadratic forms. These examples arise from a correspondence discussed in [BEH87] between matrix factorisations of a quadratic form and modules over the Clifford algebra of that quadratic form. While we do not prove so here, when $k=\mathbb{C}$ the examples of morphisms of quadratic forms discussed in this section are exhaustive.

Consider the polynomial rings $k[x]=k[x_1,\ldots,x_n]$ and $k[y]=k[y_1,\cdots,y_m]$ and the potentials $U(x)=\sum_{i=1}^n x_i^2\in k[x]$. and $V(y)=\sum_{j=1}^m y_j^2\in k[y]$. Let C_{UV} denote the $\mathbb{Z}_2$-graded k-algebra with odd generators $\mu_1,\ldots,\mu_n$ and $\nu_1,\ldots,\nu_m$ which satisfy the relations

$$[\mu_i,\mu_j]=-2\delta_{ij}\ , \qquad [\nu_i,\nu_j]=2\delta_{ij}\ , \qquad [\mu_i,\nu_j]=0 \tag{5.11}$$

for all $i=1,\ldots,n$ and $j=1,\ldots,m$, where $[a,b]$ is the graded commutator. Observe that C_{UV} is the Clifford algebra associated to the quadratic form $V(y)-U(x)$. We can use $\mathbb{Z}_2$-graded C_{UV}-modules which are free and finitely generated over k to construct morphisms $(k[x],U)\to(k[y],V)$ in $\mathcal{LG}_k$.

Lemma 5.6.1. *Let $\widetilde{X}$ be a $\mathbb{Z}_2$-graded C_{UV}-module which is free and finitely generated over k. Let $X = \widetilde{X} \otimes_k k[x,y]$ and define $d_X : X \to X$ as $d_X = \sum_{i=1}^n x_i\mu_i + \sum_{j=1}^m y_j\nu_j$. Then (X, d_X) is a matrix factorisation of $V(y) - U(x)$ over $k[x,y]$.*

Proof. The only thing to show is that $d_X^2 = V(y) - U(x)$. We have

$$
\begin{aligned}
d_X^2 &= \sum_{i=1}^n \sum_{i'=1}^n x_i x_{i'} \mu_i \mu_{i'} + \sum_{i=1}^n \sum_{j=1}^m x_i y_j \mu_i \nu_j + \sum_{i=1}^n \sum_{j=1}^m x_i y_j \nu_j \mu_i + \sum_{j=1}^m \sum_{j'=1}^m y_j y_{j'} \nu_j \nu_{j'} \\
&= \sum_{i=1}^n x_i^2 \mu_i^2 + \sum_{i<i'} x_i x_{i'} [\mu_i, \mu_{i'}] + \sum_{i=1}^n \sum_{j=1}^m x_i y_j [\mu_i, \nu_j] + \sum_{j<j'} [\nu_j, \nu_{j'}] y_j y_{j'} + \sum_{j=1}^m y_j^2 \nu_j^2 \\
&= \sum_{j=1}^m y_j^2 - \sum_{i=1}^n x_i^2 \\
&= V(y) - U(x)
\end{aligned}
$$

as required. Here we use that $\nu_j^2 = 1$ and $\mu_i^2 = -1$, which follow directly from the relations in (5.11). □

We now consider endomorphisms of $(k[x], U)$. For notational clarity we consider a second potential $(k[y], V)$ which is identical to $(k[x], U)$ but with differently named variables. That is, both $k[x]$ and $k[y]$ are polynomial rings in n variables and $U(x) = \sum_{i=1}^n x_i^2$ and $V(y) = \sum_{i=1}^n y_i^2$. We consider morphisms $(k[x], U) \to (k[y], V)$ which are the same thing as endomorphisms of $(k[x], U)$. As above we let C_{UV} denote the $\mathbb{Z}_2$-graded k-algebra with odd generators $\mu_1, \dots, \mu_n$ and $\nu_1, \dots, \nu_n$ satisfying the relations of (5.11) with $m = n$.

Denote the unit 1-morphism of $(k[x], U)$ by (I, d). This arises from a $\mathbb{Z}_2$-graded C_{UV}-module as follows. Regarding (I, d) as a morphism $(k[x], U) \to (k[y], V)$, recall from Definition 5.1.2 that the unit 1-morphism is defined to be the Koszul matrix factorisation of $V(y) - U(x)$ arising from the sequence $(y_1 - x_1, \dots, y_n - x_n)$. Concretely, this is the matrix factorisation

$$
(I, d) = \left(\bigwedge\left(\bigoplus\nolimits_{i=1}^n k[x,y]\theta_i\right), \sum\nolimits_{i=1}^n (y_i - x_i)\theta_i^* + \sum\nolimits_{i=1}^n (y_i + x_i)\theta_i\right)
$$

where $\theta_1, \dots, \theta_n$ are formal generators. Note that $\bigwedge(\bigoplus_{i=1}^n k[x,y]\theta_i) \cong \bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k k[x,y]$, and that we can rewrite the differential as

$$
\begin{aligned}
d &= \sum_{i=1}^n (y_i - x_i)\theta_i^* + \sum_{i=1}^n (y_i + x_i)\theta_i \\
&= \sum_{i=1}^n y_i(\theta_i + \theta_i^*) + \sum_{i=1}^n x_i(\theta_i - \theta_i^*) \ .
\end{aligned}
$$

Lemma 5.6.2. *Set $\alpha_i = \theta_i - \theta_i^*$ and $\beta_i = \theta_i + \theta_i^*$. Then $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ satisfy the relations of (5.11). That is,*

$$
[\alpha_i, \alpha_j] = -2\delta_{ij} \ , \qquad [\beta_i, \beta_j] = 2\delta_{ij} \ , \qquad [\alpha_i, \beta_j] = 0
$$

for all $i, j = 1, \dots, n$.

Proof. We have

$$
\begin{aligned}
[\alpha_i, \alpha_j] &= (\theta_i - \theta_i^*)(\theta_j - \theta_j^*) + (\theta_j - \theta_j^*)(\theta_i - \theta_i)^* \\
&= \theta_i\theta_j + \theta_j\theta_i - \theta_i^*\theta_j - \theta_j\theta_i^* - \theta_i\theta_j^* - \theta_j^*\theta_i + \theta_i^*\theta_j^* + \theta_j^*\theta_i^* \\
&= [\theta_i, \theta_j] - [\theta_j, \theta_i^*] - [\theta_i, \theta_j^*] + [\theta_i^*, \theta_j^*] \\
&= -2\delta_{ij}
\end{aligned}
$$

by Lemma 4.2.2. The relations $[\beta_i, \beta_j] = 2\delta_{ij}$ and $[\alpha_i, \beta_j] = 0$ proceed in a similar manner. $\square$

Lemma 5.6.2 means that $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ is a representation of C_{UV} over k, where we let μ_i act by $\theta_i - \theta_i^*$ and ν_i act by $\theta_i + \theta_i^*$. Hence the unit 1-morphism of $(k[x], U)$ arises from the C_{UV}-module $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ as described in Lemma 5.6.1. Lemma 5.6.2 also suggests other C_{UV}-representations on $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$; we can let ν_i act by $\theta_j + \theta_j^*$ with possibly $i \neq j$. We can describe all such representations of C_{UV} as follows. Let S_n denote the symmetric group on n integers and $\sigma \in S_n$. We define a representation of C_{UV} on $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ by mapping

$$\mu_i \mapsto \theta_i - \theta_i^* \qquad\qquad \nu_i \mapsto \theta_{\sigma^{-1}i} + \theta_{\sigma^{-1}i}^* \ .$$

Denote this representation of C_{UV} by $\tilde{I}_\sigma$. Setting $I_\sigma = \tilde{I}_\sigma \otimes_k k[x, y]$ and

$$\begin{aligned} d_\sigma &= \sum_{i=1}^n x_i\mu_i + \sum_{i=1}^n y_i\nu_i \\ &= \sum_{i=1}^n x_i(\theta_i - \theta_i^*) + \sum_{i=1}^n y_i(\theta_{\sigma^{-1}i} + \theta_{\sigma^{-1}i}^*) \\ &= \sum_{i=1}^n (y_{\sigma i} - x_i)\theta_i^* + \sum_{i=1}^n (y_{\sigma i} + x_i)\theta_i \end{aligned}$$

we obtain a matrix factorisation (I_σ, d_σ) of $V(y) - U(x)$ by Lemma 5.6.1.

We now study compositions of such endomorphisms of $(k[x], U)$. Let $(k[z], W)$ denote a third potential identical to $(k[x], U)$ and $(k[y], V)$. Let C_{VW} denote the $\mathbb{Z}_2$-graded k-algebra generated by odd generators $\overline{\nu}_1, \dots, \overline{\nu}_n$ and $\omega_1, \dots, \omega_n$, where $\overline{\nu}_i$ and ω_j satisfy relations in (5.11), substituting μ_i for $\overline{\nu}_i$ and ν_j for ω_j. Given $\sigma, \tau \in S_n$ we define morphisms

$$(k[x], U) \xrightarrow{(I_\sigma, d_\sigma)} (k[y], V) \xrightarrow{(J_\tau, d_\tau)} (k[z], W)$$

where (I_σ, d_σ) is as above and (J_τ, d_τ) arises as in Lemma 5.6.1 from the representation $\tilde{J}_\tau$ of C_{VW} given by letting $\overline{\nu}_i$ and ω_i act on $\bigwedge(\bigoplus_{i=1}^n k\psi_i)$ via

$$\overline{\nu}_i \mapsto \psi_i - \psi_i^* \qquad\qquad \omega_i \mapsto \psi_{\tau^{-1}i} + \psi_{\tau^{-1}i}^* \ .$$

Now consider the composition of (I_σ, d_σ) and (J_τ, d_τ), or more specifically the finite rank representative of this composition arising via the cut operation defined in Section 5.4. Note that the Jacobi ring of V is $J_V = k$. The cut of (I_σ, d_σ) and (J_τ, d_τ) is the matrix factorisation

$$J_\tau | I_\sigma = \tilde{I}_\sigma \otimes_k \tilde{J}_\tau \otimes_k k[x, z] \qquad\qquad d_{J_\tau|I_\sigma} = \sum\nolimits_{i=1}^n z_i\omega_i + \sum\nolimits_{i=1}^n x_i\mu_i \ .$$

Note that terms involving the variables $y_1, \dots, y_n$ do not appear in the differential $d_{J_\tau|I_\sigma}$ because elements of the ideal $(y_1, \dots, y_n)$ act trivially on $J_\tau|I_\sigma$. To extract the finite rank representative of the composition of (I_σ, d_σ) and (J_τ, d_τ) we consider the action of the operators $\gamma_1, \dots, \gamma_n$ and $\gamma_1^\dagger, \dots, \gamma_n^\dagger$ on $(J_\tau|I_\sigma, d_{J_\tau|I_\sigma})$ as defined at (5.10). By Proposition 5.5.7 these operators satisfy the canonical anticommutation relations. Matching the notation of (5.10) we set $t_i = \frac{\partial V}{\partial y_i} = 2y_i$ and so we have $\partial_{t_i} = \frac{1}{2}\frac{\partial}{\partial y_i}$. Then by the definition given in (5.10) we have

$$\begin{aligned}\gamma_j &= \mathrm{At}_j \\ &= [d_{I_\sigma \otimes I_\tau}, \partial_{t_j}] \\ &= \frac{1}{2}\left[\sum_{i=1}^n x_i\mu_i + \sum_{i=1}^n y_i\nu_i + \sum_{i=1}^n y_i\overline{\nu}_i + \sum_{i=1}^n z_i\omega_i, \partial_{t_j}\right] \\ &= \frac{1}{2}\sum_{i=1}^n [y_j, \partial_{t_j}]\nu_i + \frac{1}{2}\sum_{i=1}^n [y_j, \partial_{t_j}]\overline{\nu}_i \\ &= \frac{-1}{2}(\nu_j + \overline{\nu}_j)\end{aligned}$$

where we use that $[\partial_{t_j}, t_i] = \delta_{ij}$ from Lemma 5.5.2, and that ∂_{t_j} commutes with x_i and z_j (recall also that the commutator is graded and ∂_{t_j} is a degree zero map). For $\gamma_j^\dagger$ we have

$$\begin{aligned}\gamma_j^\dagger &= -\partial_{t_j}(d_\sigma) - \frac{1}{2}\sum_{l=1}^n \frac{\partial^2 V}{y_l y_j}\mathrm{At}_l \\ &= -\nu_j - \mathrm{At}_j \\ &= -\frac{1}{2}(\nu_j - \overline{\nu}_j) \ .\end{aligned}$$

By inspecting the proof of Proposition 5.5.7 we see that in this case $\gamma_1, \dots, \gamma_n$ and $\gamma_1^\dagger, \dots, \gamma_n^\dagger$ satisfy the canonical anticommutation relations *strictly*, not just up to homotopy. Hence by Proposition 5.5.14 the finite rank representative of the composition of (I_σ, d_σ) and (J_τ, d_τ) is given by splitting the idempotent

$$\begin{aligned}e &= \gamma_1 \cdots \gamma_n \gamma_n^\dagger \cdots \gamma_1^\dagger \\ &= \frac{1}{2^n}(\nu_1 + \overline{\nu}_1)\cdots(\nu_n + \overline{\nu}_n)(\nu_n - \overline{\nu}_n)\cdots(\nu_1 - \overline{\nu}_1) \ .\end{aligned}$$

This can be seen to be

$$\mathrm{im}(e) = \bigcap_i \ker(\gamma_i^\dagger) = \bigcap_i \ker(\nu_i - \overline{\nu}_i) \ .$$

A Bicategories

In this section we define and motivate the notion of a bicategory, closely following [Bor94, Chapter 7].

A.1 2-categories

Let $\mathcal{C}$ be a category. First observe that for any object A of $\mathcal{C}$, the identity morphism $1_A : A \to A$ can be viewed as a morphism of sets $u_A : \{*\} \to \mathcal{C}(A, A)$ which identifies 1_A (i.e. $u_A(*) = 1_A$). With this interpretation the identity axioms of the category ($1_A f = f$ for all $f : B \to A$ and $g 1_A = g$ for all $g : A \to B$) can be expressed as the assertion that the following diagrams in the category of sets

$$
\begin{array}{ccc}
\{*\} \times \mathcal{C}(A,B) & \xleftarrow{\;\cong\;} & \mathcal{C}(A,B) \\
\Big\downarrow{\scriptstyle u_A \times 1} & \nearrow{\scriptstyle c_{AAB}} & \\
\mathcal{C}(A,A) \times \mathcal{C}(A,B) & &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}(B,A) & \xrightarrow{\;\cong\;} & \mathcal{C}(B,A) \times \{*\} \\
 & \nwarrow{\scriptstyle c_{BAA}} & \Big\downarrow{\scriptstyle 1 \times u_A} \\
 & & \mathcal{C}(B,A) \times \mathcal{C}(A,A)
\end{array}
\tag{A.1}
$$

commute for all pairs of objects A and B in $\mathcal{C}$, where $c_{XYZ} : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ is composition. The associativity axiom can also be expressed as the assertion that the diagram of sets

$$
\begin{array}{ccc}
\mathcal{C}(A,B) \times \mathcal{C}(B,C) \times \mathcal{C}(C,D) & \xrightarrow{1 \times c_{BCD}} & \mathcal{C}(A,B) \times \mathcal{C}(B,D) \\
\Big\downarrow{\scriptstyle c_{ABC} \times 1} & & \Big\downarrow{\scriptstyle c_{ABD}} \\
\mathcal{C}(A,C) \times \mathcal{C}(C,D) & \xrightarrow[c_{ACD}]{} & \mathcal{C}(A,D)
\end{array}
\tag{A.2}
$$

commutes for all objects A, B, C and D of $\mathcal{C}$.

Informally, a 2-category is a category in which we also have higher order morphisms between the morphisms of objects, and everything “works”. To be more precise:

Definition A.1.1. A *2-category* $\mathcal{C}$ is a category in which $\mathcal{C}(A, B)$ is a category for all objects A and B, and:

(1) Composition $\mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$ is a functor.

(2) The map $u_A : \{*\} \to \mathcal{C}(A,A)$ which identifies 1_A is a functor, where $\{*\}$ is regarded as the category with one object and one morphism.

(3) The diagrams in (A.1) and (A.2) above commute as diagrams of categories.

The objects of $\mathcal{C}(A, B)$ are called *1-morphisms* and the morphisms of $\mathcal{C}(A, B)$ are called *2-morphisms*.

Following [Bor94], we denote objects in a 2-category by capital letters A, B, $C\dots$, 1-morphisms by lower-case letters a, b, $c\dots$, and 2-morphisms by Greek letters α, β, $\gamma\dots$. A 1-morphism is denoted with an arrow $A \to B$ as usual, and a 2-morphism is denoted with an arrow $a \Rightarrow b$.

Now let $\mathcal{C}$ be a 2-category, In $\mathcal{C}$ there are two notions of composition of 2-morphisms: applying the composition functor $c_{ABC} : \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$ to a pair of 2-morphisms, and composing 2-morphisms within $\mathcal{C}(A,B)$. For the first case, consider the following situation:

$$A \;\underset{g}{\overset{f}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}}\!\!\Downarrow\!\alpha\; B \;\underset{m}{\overset{\ell}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}}\!\!\Downarrow\!\beta\; C \; .$$

In the category $\mathcal{C}(A,B) \times \mathcal{C}(B,C)$ we have objects (f,ℓ) and (g,m) and a morphism $(\alpha,\beta) : (f,\ell) \Rightarrow (g,m)$. Via the composition functor c_{ABC} we obtain a 2-morphism $c_{ABC}(\alpha,\beta) : \ell f \Rightarrow mg$. Such composition is sometimes called *horizontal composition* or *composition along objects* and we denote this by $\beta * \alpha = c_{ABC}(\alpha,\beta)$. This is in contrast to *vertical composition* or *composition along morphisms*, where in the situation of the diagram

$$A \;\begin{array}{c} \xrightarrow{\quad f \quad} \\ \Downarrow\psi \\ \xrightarrow{\quad g \quad} \\ \Downarrow\varphi \\ \xrightarrow{\quad h \quad} \end{array}\; B$$

we obtain a morphism denoted $\varphi \circ \psi : f \Rightarrow h$. Functoriality of composition means that in the situation of

$$A \;\begin{array}{c} \xrightarrow{\quad f \quad} \\ \Downarrow\psi \\ \xrightarrow{\quad g \quad} \\ \Downarrow\varphi \\ \xrightarrow{\quad h \quad} \end{array}\; B \;\begin{array}{c} \xrightarrow{\quad \ell \quad} \\ \Downarrow\alpha \\ \xrightarrow{\quad m \quad} \\ \Downarrow\beta \\ \xrightarrow{\quad n \quad} \end{array}\; C$$

we have

$$\begin{aligned}
(\beta * \varphi) \circ (\alpha * \psi) &= c_{ABC}(\varphi,\beta) \circ c_{ABC}(\psi,\alpha) \\
&= c_{ABC}((\varphi,\beta) \circ (\psi,\alpha)) \\
&= c_{ABC}(\varphi \circ \psi, \beta \circ \alpha) \\
&= (\beta \circ \alpha) * (\varphi \circ \psi) \; .
\end{aligned}$$

The prototypical example of a 2-category is one in which the objects are categories, the 1-morphisms are functors and the 2-morphisms are natural transformations. Many familiar concepts in this setting transfer naturally to a general 2-category.

Definition A.1.2. A pair of 1-morphisms $f : A \to B$ and $g : B \to A$ in a 2-category are *adjoint* if there exist 2-morphisms $\eta : 1_B \Rightarrow fg$ and $\epsilon : gf \Rightarrow 1_A$ such that the diagrams

$$\begin{array}{ccc} f & \xrightarrow{\eta * \iota_f} & fgf \\ & {\scriptstyle \iota_f}\searrow & \downarrow{\scriptstyle \iota_f * \epsilon} \\ & & f \end{array} \qquad\qquad \begin{array}{ccc} g & \xrightarrow{\iota_g * \eta} & gfg \\ & {\scriptstyle \iota_g}\searrow & \downarrow{\scriptstyle \epsilon * \iota_g} \\ & & g \end{array}$$

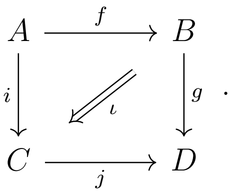
commute in $\mathcal{C}(B,A)$ and $\mathcal{C}(A,B)$ respectively, where ι_f denotes the identity 2-morphism on f and likewise for ι_g.

A.2 Bicategories

Consider any diagram of 1-morphisms in a 2-category $\mathcal{C}$, for example a square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[j]{} & D \end{array} \; . \qquad \text{(A.3)}$$

To say that (A.3) commutes is to say that $gf = ji$, or equivalently that we have the identity 2-morphism $\iota : gf \Rightarrow ji$. This may be indicated on (A.3) by filling in its face like so:



Of course, in a 2-category we may have 2-morphisms which are not the identity. It is therefore natural to consider diagrams which do not commute, but in which the faces are filled in with 2-morphisms. Of particular interest in defining bicategories are diagrams in which the faces are filled with 2-isomorphisms.

Suppose we have a collection of objects together with some candidate 1-morphisms and 2-morphisms. We would like to define a 2-category using this data, but the diagrams at (A.1) and (A.2) above do not commute. If these diagrams can instead be filled in with natural isomorphisms (2-isomorphisms in a category of categories) then this data describes a bicategory.

Definition A.2.1. A *bicategory* $\mathcal{B}$ consists of the following data:

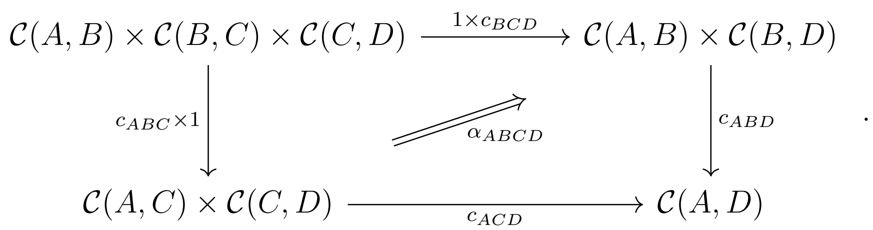
(1) A collection of objects.

(2) For every pair of objects A, B a category $\mathcal{B}(A, B)$ of 1-morphisms. An object f of this category is denoted $f : A \to B$.

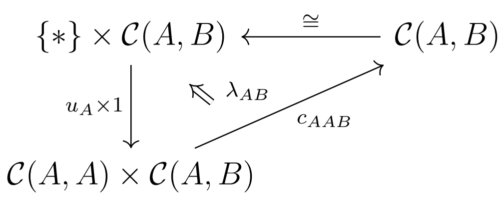
(3) For each object A a functor $u_A : \{*\} \to \mathcal{B}(A, A)$, where we denote $1_A = u_A(*)$.

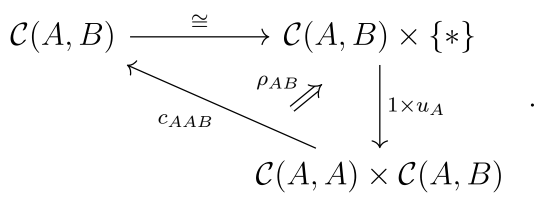
(4) For all objects A, B, C a composition functor $c_{ABC} : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C)$. For $f : A \to B$ and $g : B \to C$ we denote $gf = c_{ABC}(f, g)$.

(5) For all objects A, B, C, D a natural isomorphism α_{ABCD} called the *associator* satisfying:



(6) For all objects A, B natural isomorphisms λ_{AB} and ρ_{AB}, called the *left* and *right unitors* respectively, which satisfy:





This data is subject to two conditions, called *coherence conditions*, which are:

(I) Given 1-morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{j} E$ the following diagram in $\mathcal{B}(A, E)$ commutes

$$\begin{array}{ccccc}
((jh)g)f & \xrightarrow{\alpha_{g,h,j} * \iota_f} & (j(hg))f & \xrightarrow{\alpha_{f,gh,j}} & j((hg)f) \\
\downarrow{\scriptstyle \alpha_{f,g,jh}} & & & & \downarrow{\scriptstyle i_j * \alpha_{f,g,j}} \\
(jh)(gf) & & \xrightarrow{\qquad\qquad \alpha_{gf,h,j} \qquad\qquad} & & j(h(gf))
\end{array}$$

where we have written $\alpha_{f,g,h}$ for $\alpha_{ABCD}(f, g, h)$ and so on.

(II) Given 1-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ the following diagram in $\mathcal{B}(A, C)$ commutes

$$\begin{array}{ccc}
(g i_B) f & \xrightarrow{\alpha_{f,1_B,g}} & g(i_B f) \\
& {\scriptstyle \rho_g * \iota_f} \searrow \quad \swarrow {\scriptstyle \iota_g * \lambda_f} & \\
& gf &
\end{array}$$

where $i_B = u_B(* \to *)$ and again we write $\alpha_{f,1_B,h}$ for $\alpha_{ABBC}(f, 1_B, g)$ and so on.

Note that in general a bicategory is not a category. Due to the coherence conditions the definition of a bicategory appears very cumbersome in comparison to a 2-category. However, since many mathematical objects are defined only up to isomorphism, the bicategory is a more 'natural' concept.

Example A.2.2. We can define a bicategory in which the objects are rings (not necessarily commutative), and for rings R and S the category of 1-morphisms is the category of R-S-bimodules. Composition is given by taking the tensor product of bimodules. If we wanted to define a 2-category along these lines we would need to somehow arrange for the tensor products $(A \otimes_R B) \otimes_S C$ and $A \otimes_R (B \otimes_S C)$ to be equal — rather than naturally isomorphic — for associativity of composition to hold.

B Idempotents in Preadditive Categories

In this section we give a brief introduction to idempotent morphisms and idempotent completion of categories, focusing on categories which are preadditive. This topic is also covered in [Bor94, Section 6.5] (where "idempotent completion" is called "Cauchy completion") but no special attention is paid to preadditive categories. Throughout let $\mathcal{C}$ be a category, which for now we do not assume is preadditive, and let $\mathcal{C}^{\mathrm{op}}$ denote the opposite category.

B.1 Definitions and basic results

Definition B.1.1. An endomorphism $e : C \to C$ in $\mathcal{C}$ is an *idempotent* if $e^2 = e$.

Consider a pair of morphisms $s : R \to C$ and $r : C \to R$ such that $rs = 1_R$. Then $e = sr$ is an idempotent.

Definition B.1.2. We call an idempotent $e : C \to C$ *split* if there exist morphisms $s : R \to C$ and $r : C \to R$ such that $e = sr$ and $rs = 1_R$. We call the category $\mathcal{C}$ *idempotent complete* if all idempotents split.

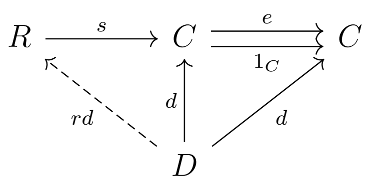
Lemma B.1.3 (Proposition 6.5.4 [Bor94])**.** *Let $e : C \to C$ be an idempotent. The following are equivalent:*

(1) $e = sr$ is split, where $s : R \to C$ and $r : C \to R$.

(2) The equaliser $\mathrm{eq}(e, 1_C)$ *exists and is equal to* (R, s).

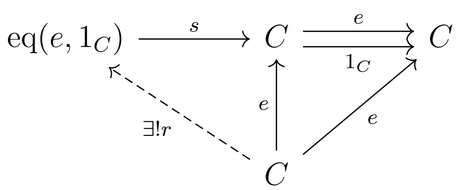
(3) The coequaliser $\mathrm{coeq}(e, 1_C)$ *exists and is equal to* (R, r).

Proof. Suppose e is split, so we have morphisms $s : R \to C$ and $r : C \to R$ such that $rs = 1_R$. We now prove that $\mathrm{eq}(e, 1_C) = (R, s)$ by showing (R, s) has the required universal property. We have $es = s$, and given another morphism $d : D \to C$ where $ed = d$, we have



where all four triangles commute. Indeed, setting $n = rd$ we have $sn = srd = ed = d$. Moreover if $n' : D \to R$ is another morphism which satisfies $sn' = d$ we have $sn = sn'$ and so $rsn = rsn' = n = n'$. This shows $\mathrm{eq}(e, 1_C) = (R, s)$ and so (1) $\implies$ (2). This also shows (1) $\implies$ (3), since this statement is equivalent to (1) $\implies$ (2) holding in $\mathcal{C}^{\mathrm{op}}$.

Supposing (2), there exists $r : C \to \mathrm{eq}(e, 1_C)$ such that



commutes. By applying the universal property to the morphism $s : \mathrm{eq}(e, 1_C) \to C$ and appealing to uniqueness we have $rs = 1$, which proves (2) $\implies$ (1). By making use of $\mathcal{C}^{\mathrm{op}}$ this also shows (3) $\implies$ (1). $\square$

Lemma B.1.4. *If $\mathcal{C}$ is a preadditive category then the following are equivalent:*

(1) $\mathcal{C}$ is idempotent complete.

(2) All idempotents have a kernel.

(3) All idempotents have a cokernel.

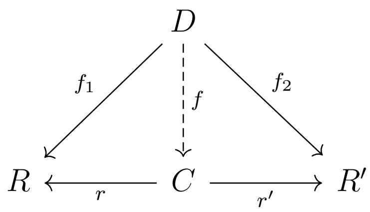
Proof. The equivalence (1) $\iff$ (2) can be proved using Lemma B.1.3 by observing that if $e : C \to C$ is an idempotent then so is $1 - e$, and that $\mathrm{eq}(1 - e, 1) = \ker(e)$. The equivalence (1) $\iff$ (3) can be proved in the same way in $\mathcal{C}^{\mathrm{op}}$. □

As a corollary note that any abelian category is idempotent complete. For an additive category, the property of "being idempotent complete" can be viewed as a weakening of "being abelian".

Lemma B.1.5. *Suppose $\mathcal{C}$ is preadditive. Let $e : C \to C$ be an idempotent such that e and $1 - e$ both split as $e = sr$ and $1 - e = s'r'$, where $s : R \to C$, $r : C \to R$, $s' : R' \to C$ and $r' : C \to R'$. Then $C \cong R \oplus R'$.*

Proof. Since $rs = 1_R$ and $r's' = 1_{R'}$ we have that s and s' are monomorphisms, and r and r' are epimorphisms. Also note that $rs' = 0$ since $rs'r' = r(1 - e) = 0$ and r' is an epimorphism. Likewise $r's = 0$.

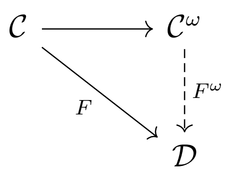
Suppose we have morphisms $f_1 : D \to R$ and $f_2 = D \to R'$. Then we have



where $f = sf_1 + s'f_2$. Clearly both triangles in this diagram commute. Suppose $g : D \to C$ also makes both triangles in the diagram above commute. Then $r'g = f_2 = r'sf_1 + r's'f_2 = r'f$. Since r' is an epimorphism this gives $g = f$, so f is unique and hence $C = R \times R'$. A similar argument shows that C is also the coproduct of R and R', which proves the lemma. □

B.2 Idempotent completion

Definition B.2.1. The *idempotent completion* of $\mathcal{C}$ is an idempotent complete category $\mathcal{C}^\omega$ together with a full and faithful functor $\mathcal{C} \to \mathcal{C}^\omega$ such that, given a functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is idempotent complete, there exists functor $F^\omega : \mathcal{C}^\omega \to \mathcal{D}$ such that



commutes, and moreover F^ω is unique up to isomorphism of functors.

Using the standard argument for objects defined via universal properties one can show that if $\mathcal{C}^\omega$ exists it is unique up to equivalence of categories. If $\mathcal{C}^\omega$ exists we can without loss of generality consider $\mathcal{C}$ to be a full subcategory of $\mathcal{C}^\omega$. In [Bor94, Proposition 6.5.9] it is proved that $\mathcal{C}^\omega$ exists when $\mathcal{C}$ is small.

Our goal now is to prove that when $\mathcal{C}$ is a subcategory of a preadditive, idempotent complete category $\mathcal{A}$, that $\mathcal{C}^\omega$ exists and is the full subcategory of $\mathcal{A}$ consisting of all objects which are a direct summand of some object of $\mathcal{C}$. Let A and B be objects of the same category. We say B is a *retract* of A if there exist morphisms

$$B \xrightarrow{\ s\ } A \xrightarrow{\ r\ } B$$

such that $rs = 1_B$.

Proposition B.2.2. *Let $\mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Suppose $\mathcal{D}$ is idempotent complete and that every object of $\mathcal{D}$ is a retract of an object of $\mathcal{C}$. Then $\mathcal{D}$ is the idempotent completion of $\mathcal{C}$.*

Proof. Without loss of generality suppose $\mathcal{C}$ is a full subcategory of $\mathcal{D}$. Let $F : \mathcal{C} \to \mathcal{E}$ be a functor to an idempotent complete category $\mathcal{E}$. We aim to construct a functor $\widetilde{F} : \mathcal{D} \to \mathcal{E}$ which fills in the diagram in Definition B.2.1.

Let D be an object of $\mathcal{D}$ and C an object of $\mathcal{C}$ such that D is a retract of C, so we have

$$D \xrightarrow{\ s\ } C \xrightarrow{\ r\ } D$$

where $rs = 1_D$. In order to ensure $\widetilde{F}$ is equal to F when restricted to objects of $\mathcal{C}$, if D happens to be an object of $\mathcal{C}$ then choose $C = D$ and $s = r = 1_D$. Consider the morphism $e = sr : C \to C$, which is an idempotent in $\mathcal{C}$. Since $\mathcal{E}$ is idempotent complete $F(e)$ splits and, using Lemma B.1.3, we can define $\widetilde{F}(D) = \mathrm{eq}(F(e), 1_{F(C)})$. In the case that D is an object of $\mathcal{C}$ choose the equaliser to be $(F(C), 1)$. We denote the associated equaliser morphism in $\mathcal{E}$ by $\sigma : \widetilde{F}(D) \to F(C)$. Also note that by the same argument as in Lemma B.1.3 we have a morphism $\rho : F(C) \to \widetilde{F}(D)$ in $\mathcal{E}$ such that $F(e) = \sigma\rho$ and $\rho\sigma = 1_{\widetilde{F}(D)}$.

Let $f : D_1 \to D_2$ be a morphism in $\mathcal{D}$. For $i = 1, 2$ let C_i be an object of $\mathcal{C}$ such that D_i is a retract of C_i. Let $s_i : D_i \to C_i$ and $r_i : C_i \to D_i$ be the morphisms in $\mathcal{D}$ in this retract and $e_i = s_i r_i$. Let $\rho_i : F(C_i) \to \widetilde{F}(D_i)$ and $\sigma_i : \widetilde{F}(D_i) \to F(C_i)$ be the morphisms in $\mathcal{E}$ which split $F(e_i)$. Note that the composition $s_2 f r_1 : C_1 \to C_2$ is a morphism in $\mathcal{C}$ and so in $\mathcal{E}$ we have the diagram

$$\begin{array}{ccccc}
\widetilde{F}(D_1) & \xrightarrow{\ \sigma_1\ } & F(C_1) & \underset{1}{\overset{F(e_1)}{\rightrightarrows}} & F(C_1) \\
 & \swarrow^{\exists! \tilde{f}} & \Big\downarrow{\scriptstyle F(s_2 f r_1)} & & \\
\widetilde{F}(D_2) & \xrightarrow[\ \sigma_2\ ]{} & F(C_2) & \underset{1}{\overset{F(e_2)}{\rightrightarrows}} & F(C_2)
\end{array}$$

where $\tilde{f}$ exists by the universal property of the equaliser. We define $\widetilde{F}(f) = \tilde{f}\sigma_1$.

To see that this defines a functor, first consider the case when $D_1 = D_2$ and $f = 1_{D_1}$. Then $\tilde{f}$ on the diagram above is a morphism such that $\sigma_1 \tilde{f} = F(e_1)$, so by uniqueness $\tilde{f} = \rho_1$. Therefore $\widetilde{F}(1_{C_1}) = \rho_1\sigma_1 = 1_{\widetilde{F}(D_1)}$ as required.

Consider another morphism $g : D_2 \to D_3$ in $\mathcal{D}$. We have the following diagram

$$\begin{array}{ccccc}
\widetilde{F}(D_2) & \xrightarrow{\ \sigma_2\ } & F(C_2) & \underset{1}{\overset{F(e_2)}{\rightrightarrows}} & F(C_2) \\
 & \swarrow^{\exists! \tilde{g}} & \Big\downarrow{\scriptstyle F(s_3 f r_2)} & & \\
\widetilde{F}(D_3) & \xrightarrow[\ \sigma_3\ ]{} & F(C_3) & \underset{1}{\overset{F(e_3)}{\rightrightarrows}} & F(C_3)
\end{array}$$

in $\mathcal{E}$. Let $\tilde{h} : F(C_1) \to \widetilde{F}(D_3)$ be the unique morphism in $\mathcal{E}$ satisfying $\sigma_3\tilde{h} = F(s_3gfr_1)$, so by definition we have $\widetilde{F}(gf) = \tilde{h}\sigma_1$. Note that

$$\sigma_3\tilde{g}\sigma_2\tilde{f} = F(s_3gr_2)F(s_2fr_1) = F(s_3gfr_1)$$

and so $\tilde{h} = \tilde{g}\sigma_3\tilde{f}$ by uniqueness. Therefore $\widetilde{F}(gf) = \widetilde{F}(g)\widetilde{F}(f)$ as required and hence we have shown that $\widetilde{F}$ defines a functor and by construction $\widetilde{F}|_{\mathcal{C}} = F$.

For uniqueness, suppose we have two functors $\widetilde{F}_1, \widetilde{F}_2 : \mathcal{D} \to \mathcal{E}$ such that $\widetilde{F}_1|_{\mathcal{C}} = \widetilde{F}_2|_{\mathcal{C}} = F$. Let D be an object in $\mathcal{D}$ and

$$D \xrightarrow{\ s\ } C \xrightarrow{\ r\ } D$$

be a retract with C an object of $\mathcal{C}$. Then for $i = 1, 2$ we have the retract

$$\widetilde{F}_i(D) \xrightarrow{\widetilde{F}_i(s)} F(C) \xrightarrow{\widetilde{F}_i(r)} \widetilde{F}_i(D)$$

in $\mathcal{E}$. Noting that $\widetilde{F}_i(s)\widetilde{F}_i(r) = F(e)$, we have $(\widetilde{F}_i(D), \widetilde{F}_i(s))$ is the equaliser $\mathrm{eq}(F(e), 1_{F(C)})$ by Lemma B.1.3. Therefore the functors are naturally isomorphic. □

Corollary B.2.3. *Let $\mathcal{C}$ be a subcategory of a preadditive, idempotent complete category $\mathcal{A}$. Then $\mathcal{C}^\omega$ is the full subcategory of $\mathcal{A}$ consisting of all objects which are direct summands of an object of $\mathcal{C}$.*

Proof. Let $\mathcal{D}$ be this subcategory. Clearly $\mathcal{C}$ is a subcategory of $\mathcal{D}$, and every object of $\mathcal{D}$ is a retract of some object of $\mathcal{C}$.

We now show that $\mathcal{D}$ is idempotent complete. If $e : D \to D$ is an idempotent in $\mathcal{D}$ then it splits in $\mathcal{A}$ as $e = sr$ where $s : R \to D$ and $r : D \to R$. By Lemma B.1.5 we have that R is a direct summand of D. Since D is a direct summand of an object of $\mathcal{C}$ we have that R is also a direct summand of the same object of $\mathcal{C}$ and hence R is in $\mathcal{D}$. Therefore the morphisms $s : R \to D$ and $r : D \to R$ are in $\mathcal{D}$ and e splits in $\mathcal{D}$. □

Corollary B.2.4. *The idempotent completion of the category of free modules over a commutative ring R is the category of projective modules over R.*

Proof. It is well-known that a module is projective if and only if it is a direct summand of a free module. □

B.3 Results used in Section 5

Let R be a commutative ring, $f \in R$. We quote the following result without proof.

Theorem B.3.1 (Proposition 1.6.8 [Nee01])**.** *Any triangulated category which admits all countable coproducts is idempotent complete.*

Corollary B.3.2. $\mathrm{HMF}(R, f)$ *is idempotent complete.*

Proof. $\mathrm{HMF}(R, f)$ admits all countable coproducts and the shift functor $X \mapsto X[1]$ induces a triangulated structure on $\mathrm{HMF}(R, f)$. □

Corollary B.3.3. $\mathrm{hmf}(R, f)^\omega$ *is the full subcategory of direct summands of objects of* $\mathrm{hmf}(R, f)$.

Proof. See Corollary B.2.3. □

Lemma B.3.4. *Let $\mathcal{C}$ be a preadditive category with a zero object and $\mathcal{C}^\omega$ its idempotent completion. A functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}^\omega$ extends uniquely to a functor $\mathcal{C}^\omega \times \mathcal{C}^\omega \to \mathcal{C}^\omega$.*

Proof. We can embed $\mathcal{C}$ into $\mathcal{C} \times \mathcal{C}$ via the functor $C \mapsto (C, 0)$ or via the functor $C \mapsto (0, C)$. Using this we can show $(\mathcal{C} \times \mathcal{C})^\omega$ is $\mathcal{C}^\omega \times \mathcal{C}^\omega$ directly from the definition. □

C A geometric interpretation of matrix factorisations

In this section we give a geometric interpretation of matrix factorisations of polynomials by relating them to algebraic subsets of the zero set of the polynomial being factorised. We do this with the aim of providing a geometric intuition for working with matrix factorisations.

Matrix factorisations were first introduced in [Eis80] and were initially used in the study of *maximal Cohen-Macaulay modules* [Yos90, Definition 1.1] over a local ring representing a hypersurface. Let S be a commutative ring and $f \in S$ be not a zero divisor. Set $R = S/(f)$ and let M be an R-module which, when regarded as an S-module, fits into a short exact sequence of the form

$$0 \longrightarrow S^{\oplus m} \longrightarrow S^{\oplus m} \longrightarrow M \longrightarrow 0$$

for some $m \in \mathbb{N}$. Denote the full subcategory of such R-modules by $\mathbf{C}(R)$. When S is a regular local ring $\mathbf{C}(R)$ is the category of maximal Cohen-Macaulay R-modules [Yos90, Chapter 7], although that is not how this category is usually defined.

Let $\mathrm{mf}(S, f)$ denote the category of finite rank matrix factorisations of f over S. We define a functor from $\mathrm{mf}(S, f)$ to $\mathbf{C}(R)$ as

$$\Gamma : \mathrm{mf}(S, f) \longrightarrow \mathbf{C}(R), \qquad (X, d) \longmapsto \mathrm{coker}(d_0)$$

on objects and on morphisms by applying the universal property of the cokernel.

Lemma C.1.1. *The map Γ is well-defined.*

Proof. Let (X, d) be a finite rank matrix factorisation and set $M = \mathrm{coker}(d_0)$. Since $d_1 d_0 = f \cdot 1_{X_1}$ and f is not a zero divisor we have that d_0 is a monomorphism, hence it is clear that M fits into a short exact sequence of S-modules of the required form. It remains to show that M is naturally an R-module. Denote the cokernel epimorphism by $e : X_1 \to M$. Then for $m = e(s) \in M$ we have $fm = e(fs) = ed_0 d_1(s) = 0$, so $fM = 0$. Hence M is naturally an R-module. □

Theorem C.1.2 ([Eis80, Corollary 6.3])**.** *The functor Γ induces an equivalence of categories from the homotopy category $\mathrm{hmf}(S, f)$ to the quotient category $\mathbf{C}(R)/R$.*

A proof of Theorem C.1.2 can be found in [Eis80] or [Yos90, Chapter 7]. The proofs in [Eis80; Yos90] are given in the context in which S is a regular local ring and so $\mathbf{C}(R)$ is the category of maximal Cohen-Macaulay R-modules, however they work in this broader context.

We now set $S = k[x] = k[x_1, \cdots, x_n]$. Let $V(f) = \{a \in k^n \mid f(a) = 0\}$ denote the zero set of f and (X, d) be a finite rank matrix factorisation of f. Since ΓX is naturally an R-module we have $\mathrm{Ann}_S(\Gamma X) \supseteq (f)$ and so the *support* of ΓX is the following algebraic subset of $V(f)$:

$$\mathrm{Supp}_S(\Gamma X) = \{a \in k^n \mid g(a) = 0 \text{ for all } g \in \mathrm{Ann}_S(\Gamma X)\} .$$

The map

$$\mathrm{mf}(S, f) \longrightarrow \{\text{algebraic subsets of } V(f)\} \qquad (X, d) \longmapsto \mathrm{Supp}_S(\Gamma X) \tag{C.1}$$

is well-defined and many algebraic subsets arise in its image.

Example C.1.3. Let $W \subseteq V(f)$ be an algebraic subset of the same dimension as $V(f)$. Then W is associated to a principal ideal $(g) \supseteq (f)$. That is $W = V(g)$. Writing $f = gh$ where $h \in k[x]$, we have the matrix factorisation

$$(X_g, d_g) = \quad S \xrightarrow{\ g\ } S \xrightarrow{\ h\ } S$$

of f. We have $\Gamma X_g = S/(g)$ and $\mathrm{Supp}_S(\Gamma X_g) = W$.

The issue with the map (C.1) is as follows. Let $g \in k[x]$ be such that $(g) \supseteq (f)$ and consider the matrix factorisation (X_g, d_g) defined in Example C.1.3. Since the unit element $1 \in k[x]$ is such that $(1) \supseteq (f)$ we can also consider the matrix factorisation (X_1, d_1) defined in the same way. Consider the matrix factorisation $(Y, d_Y) = (X_g, d_g) \oplus (X_1, d_1)$ which is

$$S \oplus S \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}} S \oplus S \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & h \end{pmatrix}} S \oplus S \quad .$$

In this case we have $\Gamma Y = R \oplus R/(g)$ and so $\mathrm{Supp}_S(\Gamma Y) = V(f)$. However there is a sense in which (Y, d_Y) ought to be associated to the subvariety $V(g)$ since it is obtained from (X_g, d_g) by adding a direct summand of the trivial matrix factorisation. It encodes no additional information about how f can be factorised, and indeed (Y, d_Y) and (X_g, d_g) are isomorphic in the homotopy category $\mathrm{hmf}(S, f)$. More generally we note that for any matrix factorisation (Y, d_Y) if ΓY has a direct summand of R then $\mathrm{Supp}_S(\Gamma Y) = V(f)$. In other words, direct summands of R in ΓY obscure geometric information about the matrix factorisation (Y, d_Y).

It is therefore natural to consider the quotient category $\mathbf{C}(R)/R$ in which objects that differ by only by direct summands of R are isomorphic. Theorem C.1.2 gives us that Γ is an equivalence of categories $\mathrm{hmf}(S, f) \to \mathbf{C}(R)/R$, so the task is now to associate objects of $\mathbf{C}(R)/R$ to subvarieties of $V(f)$ in a sensible way. The obvious approach is, for a module M in $\mathbf{C}(R)/R$, to take a representative M' of its isomorphism class which has no direct summands of R and map $M \mapsto \mathrm{Supp}_S(M')$. However, when $S = k[x]$ this module M' is not likely to be unique up to isomorphism. Whether or not isomorphism classes of $\mathbf{C}(R)/R$ have unique representatives is closely related to whether the category of R-modules satisfies the *Krull-Remak-Schmidt property*, for which we refer to [LW12, Chapter 1].

Since modules in $\mathbf{C}(R)$ are finitely generated, if the ring R has the property that $R \oplus M \cong R \oplus N$ implies $M \cong N$ for R-modules M and N then every isomorphism class of $\mathbf{C}(R)/R$ has a unique representative which does not have R as a direct summand.

Lemma C.1.4 ([LW12, Lemma 1.2])**.** *Suppose $\mathcal{A}$ is an idempotent complete category. Let M, X and Y be objects of $\mathcal{A}$, and suppose that* $\mathrm{End}(M)$ *is a local ring. Then $M \oplus X \cong M \oplus Y$ implies $X \cong Y$.*

By Lemma B.1.4 the category of R-modules is idempotent complete, and it is a standard exercise to show that that $\mathrm{End}_R(R) \cong R$. Hence this approach would work if $R = k[x]/(f)$ was a local ring, which is true only in trivial cases.

Instead we consider the power series ring $\widehat{S} = k[|x|]$ and define $\widehat{R} = \widehat{S}/(f)$. By extension of scalars we can embed $\mathrm{hmf}(S, f)$ as a full subcategory of $\mathrm{hmf}(\widehat{S}, f)$. Let $\widehat{\Gamma} : \mathrm{hmf}(\widehat{S}, f) \to \mathbf{C}(\widehat{R})/\widehat{R}$ denote the equivalence of categories of Theorem C.1.2. Given a matrix factorisation $(\widehat{X}, \widehat{d})$ of f over $\widehat{S}$ there is a unique $\widehat{R}$-module in the isomorphism class of $\widehat{\Gamma}\widehat{X}$ which has no direct summands of $\widehat{R}$ and so without loss of generality we can assume that $\widehat{\Gamma}\widehat{X}$ is this unique representative for all such matrix factorisations. Then given a matrix factorisation (X, d) in $\mathrm{hmf}(S, f)$ we can consider $\widehat{\Sigma}X = \mathrm{Supp}_{\widehat{S}}(\widehat{\Gamma}(X \otimes_S \widehat{S}))$. Since $\widehat{\Gamma}(X \otimes_S \widehat{S})$ is a finitely generated $\widehat{S}$-module we have

$$\widehat{\Sigma}X = \{\mathfrak{p} \in \mathrm{Spec}(\widehat{S}) \mid \mathfrak{p} \supseteq \mathrm{Ann}_{\widehat{S}}(\widehat{\Gamma}(X \otimes_S \widehat{S}))\} \ .$$

The next step is to pull back the primes in $\widehat{\Sigma}(X)$ along the ring morphism $S \to \widehat{S}$. Denote

$$\Sigma X = \{\mathfrak{p} \cap S \mid \mathfrak{p} \in \widehat{\Sigma}X\} \ .$$

We can do the same for the ideal $\mathrm{Ann}_{\widehat{S}}(\widehat{\Gamma}(X \otimes_S \widehat{S}))$. Set $A(X) = \mathrm{Ann}_{\widehat{S}}(\widehat{\Gamma}(X \otimes_S \widehat{S})) \cap S$. Since $S \to \widehat{S}$ has the going down property one can show

$$\Sigma X = \{\mathfrak{q} \in \mathrm{Spec}(S) \mid \mathfrak{q} \supseteq A(X)\}$$

and so ΣX corresponds to the zero set of the ideal $A(X)$. Hence we define

$$\Sigma : \mathrm{hmf}(S, f) \longrightarrow \{\text{algebraic subsets of } V(f)\}$$

as sending $(X, d) \mapsto \Sigma X$ as defined above. Notice that given a principal ideal $(g) \supseteq (f)$ in S we still have that $V(g)$ is in the image of this map.

The map Σ is compatible with direct sums in the following sense. Let (X, d_X) and (Y, d_Y) be objects of $\mathrm{hmf}(S, f)$ considered as objects of $\mathrm{hmf}(\widehat{S}, f)$. The functor $\widehat{\Gamma}$ is additive, so we have $\widehat{\Gamma}(X \oplus Y) = \widehat{\Gamma}X \oplus \widehat{\Gamma}Y$. For any ring T and T-modules M and N we have $\mathrm{Ann}_T(M \oplus N) = \mathrm{Ann}_T(M) \cap \mathrm{Ann}_T(N)$, so following this through gives $\Sigma(X \oplus Y) = \Sigma X \cup \Sigma Y$. A consequence of this is that the map Σ is not injective and as such Σ is not likely to be a useful way of studying $\mathrm{hmf}(S, f)$. Nonetheless, this map illustrates how one can think about matrix factorisations geometrically.

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