

The University of Melbourne

**A_∞ -Categories and Matrix
Factorisations over Hypersurface
Singularities**

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Abstract

In this thesis we will give an exposition of the theory of matrix factorisations and A_∞ -categories, with applications to the study of hypersurface singularities and Landau-Ginzburg models in quantum field theory. The field suffers from a lack of non-trivial examples, and the primary aim of this thesis is to compute interesting new examples of A_∞ -algebras, A_∞ -modules, and A_∞ -bimodules arising naturally in the geometry and physics of hypersurface singularities and matrix factorisations.

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Chapter 1

Introduction

1.1 Hypersurface singularities and matrix factorisations

In this thesis we are interested in affine hypersurface singularities: singular affine schemes arising as the zero locus of a single polynomial W . In algebraic geometry, an important technique to study such singularities is to understand how they deform into other singularities. These deformations can be studied at the level of spaces, or at the level of categories, and it is the latter which is the point of view adopted in this thesis.

Just as we can study a group G by studying its category of representations, or a scheme Y by studying the category of quasi-coherent sheaves on Y , a category of interest in the study of isolated hypersurface singularities is the category of *matrix factorisations*. A matrix factorisation of a nonzerodivisor W in a commutative \mathbb{k} -algebra R is an odd R -linear operator d on a free $\mathbb{Z}/2\mathbb{Z}$ -graded R -module X satisfying the identity

$$d^2 = W \cdot \text{id}. \tag{1.1}$$

Equivalently, a matrix factorisation d of W is given by a square matrix

$$d = \begin{bmatrix} 0 & d_1 \\ d_0 & 0 \end{bmatrix},$$

such that $d_0 d_1 = d_1 d_0 = W$. We can understand a hypersurface singularity $\text{Spec}(W)$ by understanding the set of all matrix factorisations of W and how they deform.

The concepts previewed above have direct parallels in mathematical physics, namely in the study of Landau-Ginzburg models. In this context one considers a *superpotential*

$W \in \mathbb{C}[\mathbf{x}]$ which gives rise to a two-dimensional topological field theory [36, 46]. In particular, the bulk sector corresponding to the bulk chiral primaries in the associated conformal field theory is given by the Jacobi ring $\text{Jac}(W) = \mathbb{C}[\mathbf{x}]/(\partial W)$ [63], and the boundary conditions are described by matrix factorisations of W [5, 27, 36].

Moduli spaces are central to both the mathematical and physical setting. One example is the moduli space of D -branes in physics, which we model as the homotopy category of matrix factorisations over W [51]. It is therefore crucial to study deformations of matrix factorisations in order to understand the local structure of such moduli spaces.

1.2 Deformations and A_∞ -algebras

A deformation of a matrix factorisation, (X, d) , of W consists of a block off-diagonal matrix δ such that $(d + \delta)^2 = W \cdot \text{id}$. Since $d^2 = W \cdot \text{id}$, this is equivalent to

$$d\delta + \delta d + \delta^2 = 0. \quad (1.2)$$

Viewing d as an endomorphism of a graded R -module, we see that δ must be a degree +1 operator, so we can rewrite the above expression as

$$[d, \delta] + \delta^2 = 0, \quad (1.3)$$

where $[-, -]$ is the graded commutator. The space E of all matrices the same size as d forms a *differential graded algebra* (DG-algebra), with multiplication given by the matrix product, and differential $D = [d, -]$. For any DG-algebra with differential D , the equation $D(A) + A^2 = 0$ with $|A| = 1$ is called the *Maurer-Cartan equation*. Hence, the deformation problem for matrix factorisations is equivalent to the problem of finding solutions to the corresponding Maurer-Cartan equation [7].

Calculating the space of solutions to the Maurer-Cartan equation of this DG-algebra is difficult, because E is infinite dimensional as a \mathbb{k} -vector space. We are led to seek a finite-dimensional object which encodes the same deformation theoretic information as E . Such an object can be obtained using techniques from algebraic topology, and is called the A_∞ -*minimal model* of E .

A priori A_∞ -algebras, which have their origins in Jim Stasheff's study of iterated loop spaces [59], have no immediate connection to the study of deformations in algebraic geometry. The link is provided by Kadeishvili's Minimal Model Theorem [25], which states that for any DG-algebra, A , one can construct a minimal A_∞ -algebra \mathcal{A} which is quasi-isomorphic to A . We call \mathcal{A} the A_∞ -minimal model of A . Since Maurer-Cartan solution spaces are invariant under quasi-isomorphism [33, 43], this result allows us to

use \mathcal{A} , which in many cases is finite dimensional, in place of E to solve our deformation problem. A major motivation of this thesis is to apply the techniques developed in [10, 11, 49] to compute examples of A_∞ -minimal models for some well known hypersurface singularities.

1.3 A_∞ -categories and non-commutative geometry

This appearance of A_∞ -algebras in physically relevant computations is not an isolated case, but an example of a series of such phenomena characterising the field of *non-commutative algebraic geometry* (NCAG), as developed in [28]. Broadly, NCAG is inspired by the observation that the derived category of complexes of coherent sheaves over a given scheme Y is able to detect an uncanny amount of information about Y [3]. Just as the Grothendieck school of algebraic geometry took the step of studying arbitrary commutative rings as generalised algebras of functions, non-commutative algebraic geometry studies certain kinds of abelian and triangulated categories as chain complexes of coherent sheaves on some hypothetical *non-commutative space* \mathcal{X} .

Bondal and Kapranov [2] introduced DG-categories into the subject as an enhancement of derived categories. Although this step had technical advantages, it was hindered by the mapping spaces in question being infinite-dimensional. As a solution, the notion of an A_∞ -category, a categorical generalisation of an A_∞ -algebra, was introduced [16], and a categorical analogue of the Minimal Model Theorem was developed [34, 38]. Although this step was justified for mathematical reasons, it was not until more recent work, including that of Fukaya and Lazaroiu [18, 19, 37], that the structure of A_∞ -categories found physical relevance. In subsequent years, A_∞ -categories have been indispensable in deformation quantisation [33], open string field theory [20, 22, 26], and, most famously, in the statement of the Homological Mirror Symmetry Conjecture [32].

Despite the pace of research in the last two decades centered on A_∞ -categories and A_∞ -algebras, there have only been a handful of explicit examples computed in the literature [12, 41, 45].

1.4 Plan for thesis

In recent years Murfet [11, 49], Dyckerhoff [10], and others have built on the work of Seidel [57] and Efimov [12] to develop the techniques and machinery needed to make explicit calculations feasible. This thesis aims to present a detailed and unified survey of this work, and to compute new explicit examples of A_∞ -structures arising from hypersurface singularities.

In Chapters 2 and 3 we recall the theory of A_∞ -algebras, A_∞ -modules, A_∞ -bimodules, and A_∞ -categories, incorporating basic examples. The main contribution of these two

chapters is a complete presentation of the generalised Minimal Model Theorem for A_∞ -categories, providing perhaps the first fully detailed proof in the literature which carefully accounts for signs.

In Chapter 4 we review the basic theory of matrix factorisations. We provide a new proof, due to Murfet, that \mathbb{k}_W^{stab} generates the DG-category $\text{mf}(W)$. We also present Orlov's theorem in the context of modules, showing that the derived category of singularities over $\text{Spec}(R/W)$ is equivalent to the homotopy category of matrix factorisations of W .

In Chapter 5 we construct a strong deformation retract of the DG-algebra $\text{End}(\mathbb{k}_W^{stab})$, to which we apply the Minimal Model Theorem. We then motivate and develop the formalism of configurations and Feynman calculus to compute concrete examples. This section culminates in a formula for the A_∞ -products on the A_∞ -minimal model of $\text{End}(\mathbb{k}_W^{stab})$ together with a basic example of the A_∞ -minimal model of the A_d singularity $W = x^d$. We end the chapter more involved example of the A_∞ -minimal model associated to $W = x^3 - y^3$.

In Chapter 6 we extend the construction in 5 to the context of A_∞ -modules, providing a method to compute the minimal model of $\text{End}(\mathbb{k}_W^{stab})$ -modules. We give an example of a family of A_∞ -modules over the A_∞ -algebra associated to $W = x^d$. We then apply this construction to give a method of computing the A_∞ -minimal model of an $\text{End}(\mathbb{k}_V^{stab})$ - $\text{End}(\mathbb{k}_W^{stab})$ -DG-bimodule by computing the equivalent $\text{End}(\mathbb{k}_{W-V}^{stab})$ -module minimal model. This provides a way to construct examples of A_∞ -bimodules, including examples arising from permutation defects. These are important examples in the physics literature [4, 15].

We also include an appendix covering the basic theory of rooted trees, fixing the notation for the thesis.

1.5 Notation and Terminology

Throughout the thesis \mathbb{k} denotes a field of characteristic 0. When we refer to a ring R mean a commutative, unital ring. Throughout we will write $\otimes = \otimes_{\mathbb{k}}$ unless otherwise stated. We use the following notation:

| | |
|---------------------------|--|
| $\text{mf}(W)$ | The DG-category of matrix factorisations. |
| $\text{hmf}(W)$ | The homotopy category of matrix factorisations |
| $\text{Perf}(S)$ | The category of perfect complexes of S -modules. |
| $\text{Perf}_{\infty}(A)$ | The category of perfect complexes of A_{∞} -modules over A . |
| F_{θ} | The $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{k} -vector space $\mathbb{k}\theta_1 \oplus \cdots \oplus \mathbb{k}\theta_n$ |
| F_{ψ} | The $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{k} -vector space $\mathbb{k}\psi_1 \oplus \cdots \oplus \mathbb{k}\psi_n$ |
| F_{ψ^*} | The $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{k} -vector space $\mathbb{k}\psi_1^* \oplus \cdots \oplus \mathbb{k}\psi_n^*$ |

We also work with a number of different A_{∞} -structures in this thesis, so to avoid confusion we will outline the different notations below:

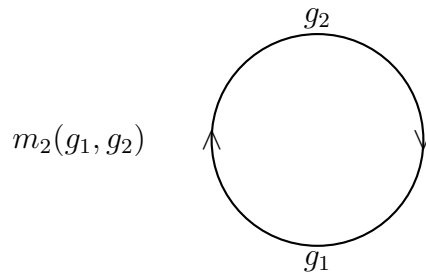
| | |
|------------|--|
| (A, m) | A_{∞} -algebra with standard higher products. |
| (A, b) | A_{∞} -algebra with shifted higher products. |
| (M, ν) | A_{∞} -module or A_{∞} -bimodule with shifted higher products. |
| μ | Standard A_{∞} -category composition products. |
| m | Forward A_{∞} -category composition products. |
| r | Forward suspended A_{∞} -category composition products. |

Chapter 2

A_∞ -algebras

2.1 Introduction

Consider a pointed topological space $(X, *)$, and let ΩX denote the *loop space* of X . A point of ΩX is a continuous map $g : [0, 1] \rightarrow X$ such that $g(0) = g(1) = *$. There is a composition map $m_2 : \Omega X \times \Omega X \rightarrow \Omega X$ sending two loops, g_1 and g_2 , to the loop $m_2(g_1, g_2)$, which runs through g_1 on the first half of the interval and g_2 on the second half:



This almost gives the loop space a group structure, with the exception being that m_2 is not associative. For three loops, g_1, g_2 , and g_3 , the composition $m_2(m_2(g_1, g_2), g_3)$ runs through g_3 on the second half of the interval, whereas $m_2(g_1, m_2(g_2, g_3))$ runs through g_3 on the last quarter of the interval.

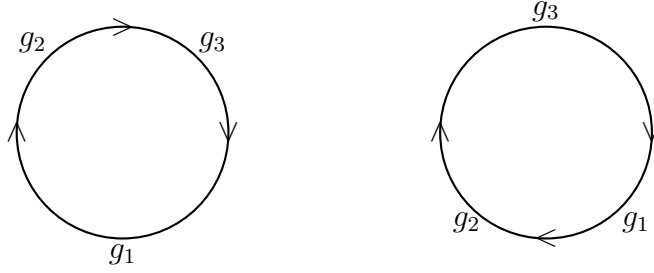


Figure 2.1: The two ways of composing three loops are not equal.

However, the two ways of composing three loops are equal *up to homotopy*

$$m_3 : [0, 1] \times (\Omega X)^3 \rightarrow \Omega X. \quad (2.1)$$

Observe now that the number of ways of composing n loops is in bijection with the number of binary rooted trees with n leaves, so there are five ways of composing four loops, 14 ways to compose five loops, and so on. In the case of four loops, there are two paths of homotopies joining the nested composition $m_2(m_2(m_2(g_1, g_2), g_3)g_4)$ to $m_2(g_1, m_2(g_2, m_2(g_3, g_4)))$, and these two paths are related by a homotopy

$$m_4 : K_4 \times (\Omega X)^4 \rightarrow \Omega X, \quad (2.2)$$

where K_4 denotes the pentagon bounded by the two paths. More generally, in [59] Stasheff defined polytopes K_n of dimension $n - 2$ for all $n \geq 2$, and showed that there are higher homotopies $m_n : K_n \times (\Omega X)^n \rightarrow \Omega X$ of all orders. This is the first example of a structure which is associative *up to higher homotopies*, called an A_∞ -space.

The notion of an A_∞ -algebra is the natural analogue of this structure on a graded vector space. In the decades following their construction, the use of A_∞ -algebras was mostly limited to abstract homotopy theory. It wasn't until the early 1990s that the relevance of A_∞ -algebras and related structures spread to the broader fields of algebra, geometry, and mathematical physics. In subsequent years the theory has demonstrated wide utility, being used by Kontsevich in his homological mirror symmetry program [32, 35], and Polishchuk in the study of the moduli space of curves [53, 54].

2.2 Differential graded algebras

We begin by recalling the definition of a differential graded algebra over a ring R . Let A be an associative R -algebra, and let I be either \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$. An I -grading on A is a decomposition $A = \bigoplus_{p \in I} A_p$ such that $A_p A_q \subset A_{p+q}$. In this section, when we say that an algebra or module is graded we mean it is I -graded. Given a homogeneous element

$a \in A_p$ we will write $|a| = p$.

Definition 2.2.1. A *differential graded algebra* (DG-algebra) A is a graded algebra equipped with a degree +1 map $d : A \rightarrow A$ which satisfies the following conditions:

1. $d \circ d = 0$.
2. (Graded Leibniz) $d(ab) = d(a)b + (-1)^{|a|}ad(b)$.

A morphism of DG-algebras A and B is a morphism of the underlying cochain complexes which is compatible with the multiplication and units.

Example 2.2.1. Let (C, d_C) be a chain complex. We define the endomorphism algebra of C as

$$\text{End}(C) := \text{Hom}(C, C),$$

with multiplication given by composition of functions, and grading

$$\text{End}(C) = \bigoplus_{p \in I} \text{Hom}(C^q, C^{q+p})$$

inherited from C . We define a differential d on $\text{End}(C)$ by

$$d(f) = d_C \circ f + (-1)^{|f|} f \circ d_C$$

which makes $\text{End}(C)$ into a DG-algebra.

Definition 2.2.2. Let (A, d_A) be a DG-algebra. A (left) *graded module* over A is an A -module M such that, for all $p, q \in I$, $A^q M^p \subset M^{p+q}$. A *DG-module* over A is a graded A -module M equipped with a degree +1 map $d : M \rightarrow M$ satisfying the following conditions:

1. $d \circ d = 0$,
2. If $a \in A$ and $m \in M$, $d(ma) = d(m)a + (-1)^{|m|}d_A a$.

2.3 A_∞ -algebras

In many areas of mathematics a natural problem is to transfer certain algebraic structures along appropriate equivalences of objects. This is a very straightforward process when we are dealing with isomorphisms of abelian groups and algebraic structures:

given an isomorphism $f : A \xrightarrow{\cong} A'$ of abelian groups we can transfer the multiplication on A to one on A' by defining $m_{A'}(a, b) = m(f^{-1}(a), f^{-1}(b))$.

The situation becomes more complicated when we attempt to transfer algebraic structures along homotopy equivalences rather than isomorphisms. Consider a homotopy equivalence $g : A \rightarrow B$ of complexes, where A is a DG-algebra. We could again try to transfer the multiplication from A to B , but we run into difficulties. In particular, any multiplication induced on B will fail to be associative in general.

It turns out that when working with a special kind of homotopy equivalence called a *deformation retract* we may transfer a multiplicative structure on A to one which is associative *up to higher homotopies* on B . It is this last notion which is encoded precisely in the definition of an A_∞ -algebra. There are several different conventions in the literature regarding signs, so for the sake of consistency we will follow the treatment of Keller [29].

Definition 2.3.1. Let \mathbb{k} be a field. An A_∞ -algebra over \mathbb{k} is a \mathbb{Z} (or \mathbb{Z}_2)-graded vector space A together with a collection of homogeneous \mathbb{k} -linear maps

$$m_n : A^{\otimes n} \rightarrow A, \quad n \geq 1,$$

of degree $2 - n$ satisfying the following A_∞ -relations

$$\sum_{n=r+s+t} (-1)^{r+st} m_{r+1+t}(\text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t}) = 0. \quad (2.3)$$

It is instructive to unpack the A_∞ -relations for small n :

- $n = 1$: We have $m_1 m_1 = 0$, so (A, m_1) is the data of a cochain complex.
- $n = 2$: We have

$$m_1 m_2 = m_2(m_1 \otimes \text{id} + \text{id} \otimes m_1)$$

as maps $A^{\otimes 2} \rightarrow A$, where id here denotes the identity map on A . This says that m_1 is a (graded) derivation with respect to the multiplication m_2 .

- $n = 3$: We have

$$m_2(\text{id} \otimes m_2 + m_2 \otimes \text{id}) = m_1 m_3 + m_3(m_1 \otimes \text{id} \otimes \text{id} + \text{id} \otimes m_1 \otimes \text{id} \otimes \text{id} \otimes m_1)$$

as maps $A^{\otimes 3} \rightarrow A$. The left hand side is the associator for m_2 and the right hand side is the differential of m_3 in the complex $\text{Hom}_{\mathbb{k}}(A^{\otimes 3}, A)$. This says that m_2 is associative up to homotopy m_3 .

We will refer to the maps m_n as *higher products* or A_∞ -products interchangeably.

Example 2.3.1. When $m_n = 0$ for $n \geq 3$, the A_∞ relations make m_2 into an associative multiplication with m_1 a compatible derivation, so such an A_∞ -algebra is precisely a DG-algebra.

Example 2.3.2. If $A^p = 0$ for all $p \neq 0$ then an A_∞ -structure on A is the same as an associative algebra structure.

Lemma 2.3.1. *Let V be a graded R module and write $\overline{TV} = \bigoplus_{n \geq 1} V^{\otimes n}$ for the reduced tensor coalgebra. If we are given a family of maps $m_n : V^{\otimes n} \rightarrow V$, $n \geq 0$ then it can be lifted uniquely to a coderivation $M : \overline{TV} \rightarrow \overline{TV}$ such that the following diagram commutes*

$$\begin{array}{ccc} & & \overline{TV} \\ & \nearrow M & \downarrow \text{proj} \\ \overline{TV} & \xrightarrow{m} & V \end{array}$$

where $M_{n,m} : V^{\otimes n} \rightarrow V^{\otimes m}$ is given by

$$\sum_{r+s+t=n} \text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t}. \quad (2.4)$$

Moreover, any coderivation $M' : \overline{TV} \rightarrow \overline{TV}$ restricts to a family of maps $m'_n : V^{\otimes n} \rightarrow V$, $n \geq 1$ given by

$$m'_n : V^{\otimes n} \subset \overline{TV} \xrightarrow{M'} \overline{TV} \rightarrow V, \quad (2.5)$$

where the last map is the natural projection.

To avoid signs it is convenient to work with maps $b_n : A[1]^{\otimes n} \rightarrow A[1]$ of degree $+1$. Write $S : A \rightarrow A[1]$ for the suspension map which is the identity on elements, and shifts the degree by $+1$. There is a bijection between degree 1 maps $b_n : A[1]^{\otimes n} \rightarrow A[1]$ and degree $2 - n$ maps $m_n : A^{\otimes n} \rightarrow A$, $n \geq 1$ given by the square

$$\begin{array}{ccc} A[1]^{\otimes n} & \xrightarrow{b_n} & A[1] \\ \uparrow S^{\otimes n} & & \uparrow S \\ A^{\otimes n} & \xrightarrow{m_n} & A \end{array}$$

where $S^{\otimes n}$ has Koszul signs

$$S^{\otimes n}(a_1 \otimes \cdots \otimes a_n) = (-1)^{(n-1)|a_n| + (n-2)|a_{n-1}| + \cdots + |a_2|} a_1 \otimes \cdots \otimes a_n. \quad (2.6)$$

We call the maps $\{b_n\}$ the *suspended products*. This grading shift allows us to write the definition of an A_∞ -algebra without signs. Going forward, it is our convention to use m to always refer to the higher products as above, and b for the suspended higher products.

Lemma 2.3.2. (Keller [29]) *Let $A = \bigoplus A^p$ be a graded \mathbb{k} -vector space, and let $m_n : A^{\otimes n} \rightarrow A$, $n \geq 1$, be a family of linear maps. The following are equivalent:*

1. *The maps $\{m_n\}_{n \geq 1}$ define an A_∞ -algebra structure on A .*
2. *The coderivation $B : \overline{TA}[1] \rightarrow \overline{TA}[1]$ associated to the family of maps $\{b_n\}$ satisfies $B^2 = 0$.*
3. *For each $n \geq 1$,*

$$\sum_{r+s+t=n} b_{r+1+t} \circ (\text{id}^{\otimes r} \otimes b_s \otimes \text{id}^{\otimes t}) = 0. \quad (2.7)$$

Proof. Write $\mathbf{v} = v_1 \otimes \cdots \otimes v_n$. To show that the first and third items are equivalent we just have to account for signs. First, note that $m_n = s^{-1} \circ b_n \circ s^{\otimes n}$, so

$$m_s(v_{r+1} \otimes \cdots \otimes v_{r+s}) = (-1)^{(s-1)|v_{r+1}| + \cdots + |v_{r+s-1}|} b_s(v_{r+1} \otimes \cdots \otimes v_{r+s}) \quad (2.8)$$

It follows, after applying the Koszul sign convention, that

$$\begin{aligned} & (\text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t})(\mathbf{v}) \\ &= (-1)^{(2-s+1)(|v_1| + \cdots + |v_r| + (s-1)|v_{r+1}| + \cdots + |v_{r+s-1}|)} (\text{id}^{\otimes r} \otimes b_s \otimes \text{id}^{\otimes t})(\mathbf{v}) \end{aligned}$$

and finally, since $|m_s(v_{r+1} \otimes \cdots \otimes v_{r+s})| = |v_{r+1}| + \cdots + |v_{r+s}| - 2 + s$, we have

$$\begin{aligned} & (-1)^{r+st} m_{r+1+s} (\text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t})(\mathbf{v}) \\ &= (-1)^{r+st} (-1)^{(r+t)|v_1| + \cdots + t(|v_{r+1}| + \cdots + |v_{r+s}| - 2 + s) + \cdots + |v_{n-1}|} \\ & \quad (-1)^{(2-s+1)(|v_1| + \cdots + |v_r|) + (s-1)|v_{r+1}| + \cdots + |v_{r+s-1}|} b_{r+1+t} (\text{id}^{\otimes r} \otimes b_s \otimes \text{id}^{\otimes t})(\mathbf{v}) \end{aligned}$$

To determine the sign on the right hand side we will write down the coefficients of each $|v_i|$ in the exponent.

- $i \leq r$: $r + t + 2 - s - i \equiv r + t - s - i \equiv r + t - s + 2s - i \equiv n - i$.
- $r + 1 \leq t \leq r + s - 1$: $t + s - i + r \equiv n - i$.

- $i = r - s$: $t \equiv n - i$.
- $r + s + 1 \leq i \leq n - 2$: $r + 1 + t - i + s - 1 \equiv n - i$.

Hence, after including the extra term $t(2 - s) + r + r + st \equiv 0$, we have

$$\begin{aligned} \sum_{r+s+t} (-1)^{r+st} m_{r+1+t} (\text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t}) (\mathbf{v}) \\ = (-1)^{\sum_{i=1}^{n-1} (n-i)|v_i|} \sum_{r+s+t} b_{r+1+t} (\text{id}^{\otimes r} \otimes b_s \otimes \text{id}^{\otimes t}) (\mathbf{v}) \end{aligned}$$

so the family $\{m_n\}$ defines an A_∞ -structure if and only if the sum on the right vanishes.

To show the second and third statements are equivalent, observe that B^2 is a coderivation. Thus, $B^2 = 0$ if and only if the composition

$$A[1]^{\otimes n} \subset \overline{TA}[1] \xrightarrow{B^2} \overline{TA}[1] \rightarrow A[1]$$

vanishes for all $n \geq 1$. But this is simply the A_∞ -relation

$$\sum_{r+s+t=n} b_{r+1+t} \circ (\text{id}^{\otimes r} \otimes b_s \otimes \text{id}^{\otimes t}) = 0.$$

□

Example 2.3.3. Let $A = \mathbb{k}[\varepsilon]/\varepsilon^2 \cong \mathbb{k} \oplus \mathbb{k}\varepsilon$, with $|\varepsilon| = 1$, and consider the products

$$\begin{aligned} b_2(x_1 \otimes x_2) &= (-1)^{|x_1|+1} x_1 x_2, \\ b_d(\varepsilon \otimes \cdots \otimes \varepsilon) &= 1. \end{aligned}$$

We will show that (A, b_2, b_d) has the structure of an A_∞ -algebra. We need to check that the A_∞ -relations

$$\sum_{r+s+t=n} b_{r+1+t} \circ (\text{id}^{\otimes r} \otimes b_s \otimes \text{id}^{\otimes t}) = 0$$

hold for all $n > 0$. Since there are only two higher products, the cases to check are reduced to $n \in \{3, d+1, 2d-1\}$.

- $n = 3$: We need to check that

$$b_2 \circ (b_2(x_1 \otimes x_2) \otimes x_3) + (-1)^{|x_1|+1} b_2 \circ (x_1 \otimes b_2(x_2 \otimes x_3)) = 0$$

If any two x_i are equal to ε then the relation above is immediate, since $\varepsilon^2 = 0$.

Suppose that only $x_i = \varepsilon$. Then

$$\begin{aligned} & b_2 \circ (b_2(x_1 \otimes x_2) \otimes x_3) + (-1)^{|x_1|+1} b_2 \circ (x_1 \otimes b_2(x_2 \otimes x_3)) \\ &= ((-1)^{|x_1|+1+|x_1x_2|+1} + (-1)^{|x_2|+1}) x_1x_2x_3 \\ &= ((-1)^{|x_2|} - (-1)^{|x_2|}) x_1x_2x_3 = 0. \end{aligned}$$

- $n = d + 1$: In this case, the A_∞ -relations reduce to

$$b_2 \circ (b_d \otimes \text{id}) + b_2 \circ (\text{id} \otimes b_d) + \sum_{r+s=d-1} b_d (\text{id}^{\otimes r} \otimes b_2 \otimes \text{id}^{\otimes s}) = 0.$$

Because all terms contain b_d , any input which doesn't vanish trivially must have at least d inputs equal to ε . First we check the inputs which preserve one of the first two terms. For $\varepsilon^{\otimes d} \otimes x_{d+1}$

$$\begin{aligned} LHS &= b_2(1 \otimes x_{d+1}) + 0 + b_d(\varepsilon^{\otimes(d-1)} \otimes (-1)\varepsilon x_{d+1}) \\ &= -x_{d+1} + x_{d+1} = 0. \end{aligned}$$

Likewise, for $x_1 \otimes \varepsilon^{\otimes d}$

$$\begin{aligned} LHS &= 0 - b_2(x_1 \otimes 1) + b_d(x_1\varepsilon \otimes \varepsilon^{\otimes(d-1)}) \\ &= x_1 - x_1 = 0. \end{aligned}$$

Next consider the input $\varepsilon^{\otimes(d-i)} \otimes x_{i+1} \otimes \varepsilon^{\otimes i}$ with $0 < i < d$. The first two terms will vanish and the only two terms surviving from the sum will have opposite signs, causing them to vanish.

- $n = 2d - 1$: The A_∞ -relations become

$$\sum_{r+s=d-1} b_d (\text{id}^{\otimes r} \otimes b_d \otimes \text{id}^{\otimes s}) = 0.$$

This is the simplest case. For each summand, if the inner b_d vanishes then the whole summand vanishes, and if the inner b_d doesn't vanish then the outer b_d does vanish because its input is not ε^d . Hence the sum will vanish for all inputs.

We have shown above that for all $n \in \{3, d + 1, 2d - 1\}$ the A_∞ -relations are satisfied, and hence (A, b_2, b_d) is an A_∞ -algebra.

There are two different types of morphisms of A_∞ -algebras that are used in the literature: strict and weak morphisms, the latter being more useful for our purposes. We will omit the word weak when referring to morphisms.

Definition 2.3.2. Given A_∞ -algebras (A, b^A) and (B, b^B) , a morphism of A_∞ -algebras $f : A \rightarrow B$ is a collection of homogeneous \mathbb{k} -linear maps

$$f_n : A^{\otimes n} \rightarrow B, \quad n \geq 1,$$

of degree zero which satisfy the following identity

$$\sum_{r+s+t=n} f_{r+1+t}(\text{id}^{\otimes r} \otimes b_s^A \otimes \text{id}^{\otimes t}) = \sum_{i_1+\dots+i_p=n} b_p^B(f_{i_1} \otimes \dots \otimes f_{i_p}). \quad (2.9)$$

Remark 2.3.1. One can show that there is a bijection between morphisms of A_∞ -algebras $f : A \rightarrow B$ and morphisms of differential coalgebras $F : \overline{TA}[1] \rightarrow \overline{TB}[1]$.

When $n = 1$ the A_∞ -relations dictate that $f_1 \circ b_1^A = b_1^B \circ f_1$, so $f_1 : (A, b_1^A) \rightarrow (B, b_1^B)$ is a morphism of cochain complexes. When $n = 2$ the A_∞ -relations are

$$f_1 \circ b_2^A + f_2 \circ (b_1^A \otimes \text{id}) + f_2 \circ (\text{id} \otimes b_1^A) = b_1^B \circ f_2 + b_2^B \circ (f_1 \otimes f_1)$$

so f_1 is compatible with b_2^A up to a homotopy given by f_2 .

Definition 2.3.3. A *quasi-isomorphism* of A_∞ -algebras is an A_∞ -algebra morphism $f : A \rightarrow B$ such that f_1 induces an isomorphism $H^*(A) \rightarrow H^*(B)$, with cohomology taken with respect to the differential b_1 .

Definition 2.3.4 (Keller [29]). Two A_∞ -algebra morphisms $f, g : A \rightarrow B$ are *homotopic* if there is a homogeneous map $H : \overline{TA}[1] \rightarrow \overline{TB}[1]$ of degree -1 such that

$$\Delta \circ H = F \otimes H + H \otimes G \quad \text{and} \quad F - G = m^B \circ H + H \circ m^A. \quad (2.10)$$

the homotopy category of A_∞ -algebras has objects A_∞ -algebras and morphisms homotopy classes of weak A_∞ -morphisms.

Theorem 2.3.1 (Prouté [55], Corollary 4.23). *Let A and B be A_∞ -algebras.*

1. *Homotopy is an equivalence relation on the set of A_∞ -algebra morphisms $A \rightarrow B$.*
2. *A morphism of A_∞ -algebras $A \rightarrow B$ is a quasi-isomorphism if and only if it is a homotopy equivalence.*

Definition 2.3.5. An A_∞ -algebra is (A, b) called *minimal* if $m_1 = 0$.

Theorem 2.3.2 (Minimal Model Theorem, Kadeishvili [24]). *Let (A, m) be an A_∞ -algebra. The cohomology H^*A has a minimal A_∞ -structure b^{H^*A} such that $b_2^{H^*A}$ is induced by b_2^A and there is a quasi-isomorphism of A_∞ -algebras $H^*A \rightarrow A$ which extends the identity morphism on H^*A . Moreover, this structure is unique up to (non-unique) isomorphism of A_∞ -algebras.*

Proof. We will delay the proof until we give the generalised version of The Minimal Model Theorem for DG-categories in the next chapter. □

2.4 A_∞ -modules and A_∞ -bimodules

It is a ubiquitous theme in algebra that to understand an object like a group or a ring, it is fruitful to understand its actions on sets or abelian groups. The appropriate notion for A_∞ -modules is the following:

Definition 2.4.1 (Keller [29]). Let (A, b) be an A_∞ -algebra. A left A_∞ -module over A is a graded \mathbb{k} -vector space equipped with homogeneous \mathbb{k} -linear maps

$$\nu_n : A[1]^{\otimes(n-1)} \otimes M[1] \rightarrow M[1], \quad n \geq 1,$$

of degree $+1$ satisfying

$$\sum_{r+s+t=n} \nu_{r+1+t} \circ (\text{id}^{\otimes r} \otimes \nu_s \otimes \text{id}^{\otimes t}) = 0. \quad (2.11)$$

Here we need to interpret the inner ν_s as b_s when $t > 0$. Right A_∞ -modules are defined similarly.

For example, an A_∞ -algebra A can be viewed as a module over itself, with all higher products simply being given by the A_∞ -algebra products. We call this the free A -module of rank one.

Example 2.4.1. Recall the A_∞ -algebra (A, b_2, b_d) of Example 2.2.3, and consider the $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $M = \mathbb{k} \oplus \mathbb{k}\eta$, $|\eta| = 1$, and the higher products $\nu_n : A[1]^{\otimes(n-1)} \otimes M[1] \rightarrow M[1]$ with $\nu_n = 0$ for $m \notin \{2, i+1, d-i+1\}$, $2 < i < d-1$, given by

$$\begin{aligned} \nu_2(1, m) &= -x, \\ \nu_{i+1}(\varepsilon \otimes \cdots \otimes \varepsilon \otimes x) &= \eta^* \lrcorner x, \\ \nu_{d-i+1}(\varepsilon \otimes \cdots \otimes \varepsilon \otimes x) &= \eta \wedge x, \end{aligned}$$

where $\eta^* \lrcorner (-)$ is the contraction operator on the exterior algebra.

We will show that this defines an A_∞ -module over A . To do so, we need to check that the relation (2.11) holds when $n \in \{3, i+2, d-i+2, 2i+1, d+1, 2d-2i+1\}$. Throughout we will let $x \in M$ be an arbitrary element.

- $n = 3$: We need to check that

$$\nu_2(\text{id} \otimes \nu_2) + \nu_2(b_2 \otimes \text{id}) = 0.$$

If either of the first two inputs are ε then this vanishes instantly since $\nu_2(\varepsilon, -) = 0$ and $b_2(\varepsilon, \varepsilon) = 0$, so consider $1 \otimes 1 \otimes x$. Then

$$LHS = -\nu_2(1 \otimes \nu_2(1 \otimes x)) + \nu_2((-1) \otimes x) = 0.$$

- $n = i+2$: In this case, the A_∞ -relation reduces to

$$\sum_{j+k=i} \nu_{i+1} \circ (\text{id}^{\otimes j} \otimes \nu_2 \otimes \text{id}^{\otimes k}) + \nu_2 \circ (\text{id} \otimes \nu_{i+1}) = 0.$$

The only input which doesn't immediately vanish is $1 \otimes \varepsilon^{\otimes i} \otimes x$, in which case

$$\begin{aligned} LHS &= \nu_{i+1}(\nu_2(1 \otimes \varepsilon) \otimes \varepsilon \otimes \cdots \otimes \varepsilon \otimes x) + \nu_2^M(1 \otimes (\eta^* \lrcorner x)) \\ &= \eta^* \lrcorner x - \eta^* \lrcorner x = 0. \end{aligned}$$

- $n = d-i+2$: We must check that

$$\sum_{j+k=d-i} \nu_{d-i+1} \circ (\text{id}^{\otimes j} \otimes b_2 \otimes \text{id}^{\otimes k}) + \nu_2 \circ (\text{id} \otimes \nu_{d-i+1}) = 0.$$

The inputs which don't trivially vanish are those of the form $\varepsilon^{\otimes p} \otimes 1 \otimes \varepsilon^{\otimes q} \otimes x$ with $p+q=i$. When $p \neq 0$ the last term vanishes, and the sum reduces to

$$\nu_{d-i+1} \circ (\varepsilon^{\otimes p-1} \otimes b_2(\varepsilon \otimes 1) \otimes \varepsilon^{\otimes q} \otimes x) + \nu_{d-i+1} \circ (\varepsilon^{\otimes p} \otimes b_2(1 \otimes \varepsilon) \otimes \varepsilon^{\otimes q-1} \otimes x),$$

and this vanishes because $b_2(1 \otimes \varepsilon) = -b_2(\varepsilon \otimes 1)$. When $p=0$ we have

$$\begin{aligned} LHS &= \nu_{d-i+1}(-\varepsilon^{\otimes(d-i)} \otimes x) + \nu_2(1 \otimes \eta \wedge x) \\ &= \eta \wedge x - \eta \wedge x = 0. \end{aligned}$$

- $n = 2i+1$: The relations reduce to

$$\nu_{i+1} \circ (\text{id}^{\otimes i} \otimes \nu_{i+1}) = 0,$$

which is immediately satisfied because $\eta \wedge \eta = 0$.

- $n = d + 1$: The relations reduce to

$$\nu_{i+1} \circ (\text{id}^{\otimes i} \otimes \nu_{d-i+1}) + \nu_{d-i+1} \circ (\text{id}^{\otimes(d-i)} \otimes \nu_{i+1}).$$

The only input which doesn't cause both terms to vanish is $\varepsilon^{\otimes d} \otimes x$, in which case

$$\begin{aligned} LHS &= \nu_{i+1}(\varepsilon^{\otimes i} \otimes \eta \wedge x) + \nu_{d-i+1}(\varepsilon^{\otimes(d-i)} \otimes \eta^* \lrcorner x) \\ &= \eta^* \lrcorner (\eta \wedge a) + \eta \wedge (\eta^* \lrcorner a) \\ &= a - a = 0. \end{aligned}$$

- $n = 2d - 2i + 1$: We must check that

$$\nu_{d-i+1} \circ (\text{id}^{\otimes(d-i)} \otimes \nu_{d-i+1}) = 0,$$

which is immediately satisfied because $\eta^* \lrcorner (\eta^* \lrcorner x) = (\eta \wedge \eta)^* \lrcorner x = 0$.

The above calculations show that the A_∞ -relations hold for $(M, \nu_2, \nu_{i+1}, \nu_{d-i+1})$ over (A, b_2, b_d) , and M is an A_∞ -module as claimed.

Definition 2.4.2. Let (A, b) be an A_∞ -algebra and let (M, ν^M) and (N, ν^N) be A_∞ -modules over (A, b) . A morphism of A_∞ -modules $f : M \rightarrow N$ consists of a family of graded maps

$$f_n : A[1]^{\otimes n-1} \otimes M[1] \rightarrow N[1], \quad n \geq 1,$$

of degree +1 satisfying

$$\sum_{\substack{n=r+s+t \\ s \geq 1, r, t \geq 0}} f_{r+1+t} \circ (\text{id}^{\otimes r} \otimes \nu_s \otimes \text{id}^{\otimes t}) = \sum_{\substack{n=r+s \\ r \geq 1, s \geq 0}} \nu_{1+s}^M \circ (f_r \otimes \text{id}^s), \quad (2.12)$$

where ν_s in the left-hand sum is interpreted as either ν_s^M or b_s where appropriate.

A morphism of A_∞ -modules is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism. The identity morphism is given by $f_1 = \text{id}_M$ and $f_i = 0$ for all $i > 1$.

Definition 2.4.3. A morphism of $f : M \rightarrow N$ of A_∞ -modules over A is *nullhomotopic* if there is a family of graded maps

$$h_n : A^{\otimes(n-1)} \otimes M \rightarrow N, \quad (2.13)$$

homogeneous of degree 0 such that

$$f_n = \sum_{\substack{n=r+s+t \\ s \geq 1, r, t \geq 0}} h_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes \nu_s \otimes \mathrm{id}^{\otimes t}) = \sum_{\substack{n=r+s \\ r \geq 1, s \geq 0}} \nu_{1+s}^M \circ (h_r \otimes \mathrm{id}^s). \quad (2.14)$$

Two morphisms f, g of A_∞ -modules are called *homotopy equivalent* if $f - g$ is nullhomotopic. The *homotopy category* $H_\infty(A)$ of A_∞ -modules has objects all A_∞ -modules over A with morphisms given by morphisms of A_∞ -modules modulo nullhomotopic morphisms.

Definition 2.4.4 (Keller). Let A be an A_∞ -algebra. The *derived category* $D_\infty A$ is defined as the localisation of the category of *right* A_∞ -modules over A (with degree 0 morphisms) with respect to the class of quasi-isomorphisms.

More concretely, the derived category has objects all right A_∞ -modules over A and morphisms are obtained from morphisms of A_∞ -modules by formally inverting all quasi-isomorphism (for instance via Verdier localisation [50]). It turns out that $D_\infty A$ is naturally a triangulated category.

Theorem 2.4.1 (Keller [31]). *Every quasi-isomorphism of A_∞ -modules is a homotopy equivalence.*

Corollary 2.4.1. *The derived category $D_\infty(A)$ is equivalent to the homotopy category $H_\infty(A)$.*

Definition 2.4.5. Let A be an A_∞ -algebra. The category of *perfect* A_∞ -modules $\mathrm{Perf}_\infty(A)$ is the triangulated subcategory of $D_\infty A$ generated by the free A -module of rank one.

Following Hovey [23] we will call a triangulated category \mathcal{T} *algebraic* if it is the homotopy category of a stable \mathbb{k} -linear Quillen model category. The class of such categories is closed under passage to triangulated subcategories and Verdier quotients, so it contains in particular the derived category of modules over a ring, and the homotopy category of matrix factorisations over a hypersurface singularity.

Theorem 2.4.2 (Lefevre-Hasegawa, [39], Section 7.6). *Let \mathcal{T} be a \mathbb{k} -linear algebraic triangulated category with split idempotents and a generator G . Then there exists the*

structure of a minimal A_∞ -algebra on

$$A = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(G, G[n])$$

such that b_2 is given by composition and the functor

$$\mathcal{T} \rightarrow \text{GrMod}(A, b_2), \quad X \mapsto \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(G, X[n]),$$

lifts to a triangulated equivalence

$$\mathcal{T} \rightarrow \text{Perf}_\infty(A).$$

Remark 2.4.1. This theorem says that we can encode all of the information in \mathcal{T} in a lossless way in an A_∞ -algebra A .

In ring theory, the Eillenberg-Watt Theorem [13, 64] states that there is a bijection between the set of colimit preserving functors $\text{Mod}A \rightarrow \text{Mod}B$ and the set of A - B -bimodules. The correspondence sends an A - B -bimodule N to the functor

$$- \otimes_A N : \text{Mod}A \rightarrow \text{Mod}B.$$

With this example as motivation, we will define A_∞ -bimodules and show that they are a natural source of functors between categories of A_∞ -modules. In light of Theorem 2.4.2, this gives a way to study the functor category $\text{Fun}(\mathcal{T}, \mathcal{S})$ between well-behaved triangulated categories.

Definition 2.4.6 (Tradler, Keller [29, 61]). Let (A, b^A) and (B, b^B) be A_∞ -algebras. An A_∞ - A - B -bimodule is a graded \mathbb{k} -vector space M equipped with homogeneous linear maps

$$\nu_{m,n} : A[1]^{n-1} \otimes M[1] \otimes B[1]^{m-1} \rightarrow M[1], \quad i, j \geq 1,$$

of degree $+1$ satisfying

$$\sum_{r+a+b+t=n+m-1} \nu_{r+1,t+1} \circ (\text{id}^{\otimes r} \otimes \nu_{a,b} \otimes \text{id}^{\otimes t}) = 0. \quad (2.15)$$

Here we should interpret $\nu_{a,0}$ as ν_a^A if $r+a \leq n-1$ and $\nu_{0,b}$ as ν_b^B if $r > n-1$.

If A and B are A_∞ -algebras and N is an A_∞ - A - B -bimodule, the *derived tensor product* as a triangulated functor [29]

$$N \widehat{\otimes}_B (-) : D_\infty B \rightarrow D_\infty A, \quad M \mapsto \bigoplus_{i \geq 0} N \otimes B[1]^{\otimes i} \otimes M. \quad (2.16)$$

It follows that A_∞ -bimodules are a canonical source of functors of A_∞ -modules.

Chapter 3

A_∞ -Categories and the Minimal Model Theorem

3.1 Background

The algebraic objects of interest to us in connection to the geometry of hypersurface singularities are DG-categories and A_∞ -categories, which are categorifications of DG-algebras and A_∞ -algebras. There are many excellent surveys on DG-categories in the literature, including [30, 42, 60, 62], although we will only use them in a strictly elementary way in this thesis. For A_∞ -categories we will follow the exposition of Lazarioiu [38].

Definition 3.1.1. A *differential graded category*, or DG-category, \mathcal{D} , consists of a set of objects $\text{ob}\mathcal{D}$ and, for each pair of objects $a, b \in \text{ob}\mathcal{D}$, a DG-module $\text{Hom}_{\mathcal{D}}(a, b)$.

By choosing a grading on the Hom-modules we can write $\text{Hom}_{\mathcal{D}}(a, b) = \bigoplus \text{Hom}_{\mathcal{D}}^n(a, b)$ with a degree +1 linear map $d : \text{Hom}_{\mathcal{D}}^n(a, b) \rightarrow \text{Hom}_{\mathcal{D}}^{n+1}(a, b)$ satisfying $d \circ d = 0$. Hence, $\text{Hom}_{\mathcal{D}}^n(a, b)$ is required to be a cochain complex. Furthermore, the composition of morphisms $\text{Hom}_{\mathcal{D}}(b, c) \otimes \text{Hom}_{\mathcal{D}}(a, b) \rightarrow \text{Hom}_{\mathcal{D}}(a, c)$ is required to be a map of cochain complexes.

Example 3.1.1. The category Ch_R of cochain complexes of modules over a ring R is the prototypical example of a DG-category. Given two cochain complexes (A, d^A) and (B, d^B) , the mapping complex $\text{Hom}_{\mathcal{D}}(A, B)$ has the differential

$$d(f) = f \circ d^A + (-1)^{|f|} f \circ d^B.$$

Example 3.1.2. A DG-category with one object is the same as a DG-algebra.

Definition 3.1.2. An A_∞ -category \mathcal{A} consists of a class of objects $\text{ob}\mathcal{A}$ together with graded modules $\text{Hom}_{\mathcal{A}}(a, b)$ for all $a, b \in \text{ob}\mathcal{A}$, such that, for every finite collection a_0, \dots, a_n , there is a linear map

$$\mu_{a_n, \dots, a_0} : \text{Hom}_{\mathcal{A}}(a_{n-1}, a_n) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}}(a_0, a_1) \rightarrow \text{Hom}_{\mathcal{A}}(a_0, a_n)$$

of degree $2 - n$ which satisfies the A_∞ -relations

$$\sum_{r+s+t=n} (-1)^{r+st} \mu_{a_0, \dots, a_r, a_{r+s}, \dots, a_n} (\text{id}^{\otimes r} \otimes \mu_{a_r, \dots, a_{r+s}} \otimes \text{id}^{\otimes t}) = 0, \quad n \geq 1. \quad (3.1)$$

It is immediate from the definition that an A_∞ -category with one object is an A_∞ -algebra.

Definition 3.1.3. For any pair of objects $a, b \in \text{ob}\mathcal{A}$ in an A_∞ -category there is a linear map

$$S_{ab} : \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{A}}(a, b)[1], \quad (3.2)$$

called the *shift operator*, which is the identity on objects and shifts the degree by $+1$.

We write $\tilde{x} := |x| - 1$ for the degree of elements $x \in \text{Hom}_{\mathcal{A}}(a, b)[1]$. This shifted grading is referred to as the *tilde grading*. We also write $S_{ab}^n := S_{ab} \circ \dots \circ S_{ab}$ with n factors. As with A_∞ -algebras, the main reason we work with the shift operator is to simplify the notoriously troublesome signs in the A_∞ -relations. To this end, we introduce the *forward composition*

$$m_{a_0, \dots, a_n}(x_1 \otimes \cdots \otimes x_n) = (-1)^{\sum_{1 \leq i \leq j \leq n} |x_i||x_j|} \mu_{a_n, \dots, a_0}(x_n \otimes \cdots \otimes x_1), \quad (3.3)$$

and the *forward suspended composition*

$$r_{a_0, \dots, a_n} = s_{a_0 a_n} \circ m_{a_0, \dots, a_n} \circ (s_{a_0 a_1}^{-1} \circ \cdots \circ s_{a_{n-1} a_n}^{-1}). \quad (3.4)$$

The latter is a map $\text{Hom}_{\mathcal{A}}(a_{n-1}, a_n)[1] \otimes \cdots \otimes \text{Hom}_{\mathcal{A}}(a_0, a_1)[1] \rightarrow \text{Hom}_{\mathcal{A}}(a_0, a_n)[1]$.

Lemma 3.1.1. *The A_∞ relations can be rewritten in terms of the forward suspended compositions as*

$$\sum_{r+s+t=n} r_{a_0, \dots, a_r, a_{r+s}, \dots, a_n} (\text{id}^{\otimes r} \otimes r_{a_r, \dots, a_{r+s}} \otimes \text{id}^{\otimes t}) = 0 \quad (3.5)$$

with no signs.

Proof. This follows by the same reasoning as the proof of Lemma 2.3.2 .

□

Definition 3.1.4. Let \mathcal{A} be an A_∞ -category. The *cohomology category* $H(\mathcal{A})$ is the (possibly non-unital) associative graded category with the same objects as \mathcal{A} , morphisms

$$\mathrm{Hom}_{H(\mathcal{A})}(a, b) = H_{\mu_{ab}}^*(\mathrm{Hom}_{\mathcal{A}}(a, b)), \quad (3.6)$$

and morphism compositions

$$\mathrm{Hom}_{H(\mathcal{A})}(b, c) \otimes \mathrm{Hom}_{H(\mathcal{A})}(a, b) \rightarrow \mathrm{Hom}_{H(\mathcal{A})}(a, c)$$

given by $[x] \circ [y] = [\mu_{abc}(x \otimes y)]$. We denote by $H^0(\mathcal{A})$ the full subcategory with only degree zero morphisms.

Definition 3.1.5. Given A_∞ -categories \mathcal{A} and \mathcal{B} , an A_∞ -functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a map $F : \mathrm{ob}(\mathcal{A}) \rightarrow \mathrm{ob}(\mathcal{B})$ together with linear maps

$$F_{a_0 \dots a_n} : \mathrm{Hom}_{\mathcal{A}}(a_0, a_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(a_{n-1}, a_n) \rightarrow \mathrm{Hom}_{\mathcal{B}}(F(a_0), F(a_n)),$$

of degree $1 - n$, such that the forward suspended maps

$$F_{a_0 \dots a_n}^S = s_{F(a_0)F(a_n)} \circ F_{a_0 \dots a_n} \circ (s_{a_0 a_1}^{-1} \otimes \cdots \otimes s_{a_{n-1} a_n}^{-1}) \quad (3.7)$$

satisfy, for all $n \geq 1$, the following condition:

$$\begin{aligned} & \sum_{p=1}^n \sum_{0 < i_1 < \cdots < i_{p-1} < n} r_{F(a_0) \dots F(a_n)}^{\mathcal{B}} \circ \left(F_{a_0 \dots a_{i_1}}^S \otimes \cdots \otimes F_{a_{i_{p-1}} \dots a_n}^S \right) \\ &= \sum_{0 \leq i < j \leq n} F_{a_0 \dots a_i a_j \dots a_n}^S \circ \left(\mathrm{id}_{a_0 a_1} \otimes \cdots \otimes \mathrm{id}_{a_{i-1} a_i} \otimes r_{a_i \dots a_j}^{\mathcal{A}} \otimes \mathrm{id}_{a_j a_{j+1}} \otimes \cdots \otimes \mathrm{id}_{a_{n-1} a_n} \right). \end{aligned}$$

By using the map F_{ab} on morphisms, the map on objects induces a functor $H(F) : H(\mathcal{A}) \rightarrow H(\mathcal{B})$ of associative graded categories. We say that F is a *quasi-isomorphism* if $H(F)$ is an isomorphism. An A_∞ -functor F is called *strict* if $F_{a_0 \dots a_n} = 0$ for all $n > 1$. In this case the condition above reduces to

$$r_{F(a_0) \dots F(a_n)}^{\mathcal{B}} \circ \left(F_{a_0 a_1}^S \otimes \cdots \otimes F_{a_{n-1} a_n}^S \right) = F_{a_0 a_n}^S \circ r_{a_0 \dots a_n}^{\mathcal{A}}. \quad (3.8)$$

3.2 Sector decomposition

When performing calculations on A_∞ -categories it is often convenient to repackage the data using the following trick [37]. Let \mathcal{A} be an A_∞ -category, let $R_{\mathcal{A}}$ be the commuta-

tive associative \mathbb{k} -algebra generated by $\{\varepsilon_a\}_{a \in \text{ob } \mathcal{A}}$ with relations $\varepsilon_a \varepsilon_b = \delta_{ab} \varepsilon_a$.

Remark 3.2.1. Note that $R_{\mathcal{A}}$ is not necessarily unital. In fact, $R_{\mathcal{A}}$ is unital if and only if \mathcal{A} has a finite number of objects. Indeed, if $r = \sum_{k=1}^n r_k \varepsilon_{a_k} \in R_{\mathcal{A}}$, and $\iota = \sum_{k=1}^m i_k \varepsilon_{b_k}$, then

$$\iota r = r \iota = \sum b_j r_j \varepsilon_{a_j}.$$

Therefore ι is a unit if and only if $b_j = 1$ for all j , and $\{a_1, \dots, a_n\} \subseteq \{b_1, \dots, b_m\}$. The only option is to take $\iota = \sum_{a \in \text{ob } \mathcal{A}} \varepsilon_a$, which is an element of $R_{\mathcal{A}}$ only when \mathcal{A} has a finite number of objects.

Definition 3.2.1. (Lazaroiu, [38]) Consider the graded \mathbb{k} -module

$$\mathcal{H}_{\mathcal{A}} = \bigoplus_{a, b \in \text{ob } \mathcal{A}} \text{Hom}_{\mathcal{A}}(a, b)$$

with grading taken by degree. We call $\mathcal{H}_{\mathcal{A}}$ the *sector decomposition* of \mathcal{A} . Write $p_{ab} : \mathcal{H}_{\mathcal{A}} \rightarrow \text{Hom}_{\mathcal{A}}(a, b)$ for the projections. There is an $R_{\mathcal{A}}$ -bimodule structure on $\mathcal{H}_{\mathcal{A}}$. Namely, ε_a acts on the left by the projection of $\mathcal{H}_{\mathcal{A}}$ onto

$$\varepsilon_a \cdot \mathcal{H} := \bigoplus_{b \in \text{ob } \mathcal{A}} \text{Hom}_{\mathcal{A}}(a, b),$$

and ε_b acts on the right by the projection of $\mathcal{H}_{\mathcal{A}}$ onto

$$\mathcal{H} \cdot \varepsilon_b := \bigoplus_{a \in \text{ob } \mathcal{A}} \text{Hom}_{\mathcal{A}}(a, b).$$

In the following we will suppress \mathcal{A} from the notation and write $R = R_{\mathcal{A}}$ and $\mathcal{H} = \mathcal{H}_{\mathcal{A}}$.

Lemma 3.2.1. *The \mathbb{k} -module $\mathcal{H}^{\otimes R^n} = \mathcal{H} \otimes_R \cdots \otimes_R \mathcal{H}$ is given by*

$$\mathcal{H}^{\otimes R^n} \cong \bigoplus_{a_0, \dots, a_n \in \text{ob } \mathcal{A}} \text{Hom}_{\mathcal{A}}(a_0, a_1) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}}(a_{n-1}, a_n), \quad (3.9)$$

with the obvious R -bimodule structure.

Proof. It suffices to show the case where $n = 2$. The relation imposed by the tensor product is

$$\mathcal{H} \otimes_R \mathcal{H} = (\mathcal{H} \otimes_{\mathbb{k}} \mathcal{H}) / (x \varepsilon_b \otimes_{\mathbb{k}} y - x \otimes_{\mathbb{k}} \varepsilon_b y) \quad (3.10)$$

where x and y are any homogeneous elements. From this fact the claim is clear. \square

Definition 3.2.2. The total A_∞ -products $r_n : \mathcal{H}[1]^{\otimes R^n} \rightarrow \mathcal{H}[1]$ are given by

$$r_n(x^{(1)} \otimes \cdots \otimes x^{(n)}) := \bigoplus_{a_0, a_n \in \text{ob} \mathcal{A}} \sum_{a_1, \dots, a_{n-1} \in \text{ob} \mathcal{A}} r_{a_0, \dots, a_n} (x_{a_0 a_1}^{(1)} \otimes \cdots \otimes x_{a_{n-1} a_n}^{(n)}) \quad (3.11)$$

where $x^{(j)} = \bigoplus_{a, b \in \text{ob} \mathcal{A}} x_{ab}^{(j)}$ and $x_{ab}^{(j)} \in \text{Hom}_{\mathcal{A}}(a, b)[1]$. Observe that these maps are R -bilinear.

Lemma 3.2.2. *The maps $\{r_n\}_{n>0}$ satisfy the A_∞ -relations*

$$\sum_{q+s+t=n} r_{q+1+t} \left(\text{id}_{\mathcal{H}[1]}^{\otimes q} \otimes r_s \otimes \text{id}_{\mathcal{H}[1]}^t \right) = 0. \quad (3.12)$$

Proof. These are the same relations satisfied by the maps $r_{a_0 \dots a_n}$ so the claim follows from bilinearity of the maps r_n . □

This lemma implies that we can recover the categorical A_∞ -products $r_{a_0 \dots a_n}$ from the r_n , using R -linearity. By pre-composing with the quotient $\mathcal{H}[1]^{\otimes n} \rightarrow \mathcal{H}[1]^{\otimes R^n}$ we can define the r_n on $\mathcal{H}[1]^{\otimes n}$, and $(\mathcal{H}[1], \{r_n\}_{n>0})$ is therefore an A_∞ -algebra over \mathbb{k} .

Remark 3.2.2. This is not an A_∞ -algebra over R in general, because \mathcal{H} is an R bi-module with left and right actions which do not necessarily agree.

3.3 Minimal Models

Let \mathcal{A} be an A_∞ -category, and R, \mathcal{H} as in Section 3.4. For simplicity of notation, we will view r_n as defined on \mathcal{H} by applying the tilde grading.

Definition 3.3.1. An A_∞ -category is called *minimal* if all unary compositions r_{ab} vanish. A *minimal model* of an A_∞ -category \mathcal{A} is a minimal A_∞ -category \mathcal{B} which is quasi-isomorphic to A_∞ .

Definition 3.3.2. A *strict homotopy retraction* of \mathcal{A} is a strict homotopy retraction of (\mathcal{H}, r_1) . That is, the data of maps

$$\begin{aligned} P_{ab} &: \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{A}}(a, b), \\ G_{ab} &: \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{A}}(a, b)[-1], \end{aligned}$$

of degree zero for all $a, b \in \text{ob}\mathcal{A}$, satisfying the relations

$$P_{ab}^2 = P_{ab}, \quad \text{id} - P_{ab} = [(r_1)_{ab}, G_{ab}], \quad (3.13)$$

where $[-, -]$ is the graded commutator.

The submodule

$$B = \text{im}P \subseteq \mathcal{H}$$

is given by $\bigoplus_{a,b \in \text{ob}\mathcal{A}} \text{im}P_{ab}$. It follows from the relations (3.13) that B is a sub-complex of \mathcal{H} . We write $\iota : B \rightarrow \mathcal{H}$ for the inclusion and $p : \mathcal{H} \rightarrow B$ for the map induced by P , so that $\iota \circ p = P$.

Remark 3.3.1. For terminology on rooted trees, refer to Appendix A.

Given a valid plane tree T let $E(T)$ denote the set of all edges, $E_i(T)$ all internal edges, $E_e(T)$ all external edges, and $V_i(T)$ all internal vertices. We write $e_i(T)$ and $v_i(T)$ for the number of internal edges and internal vertices respectively.

Definition 3.3.3. Given a valid plane tree T the *augmentation* $A(T)$ of T is the plane tree obtained from T by inserting a new vertex of valency 2 on each internal edge of T .

Definition 3.3.4. Given homotopy retract data (P, G) and a tree $T \in \mathcal{T}_n$, a *decoration* D_T of $A(T)$ by (P, G) is defined as follows:

1. Place the inclusion $\iota : B \rightarrow \mathcal{H}$ on every non-root leaf of $A(T)$.
2. Place the surjection $p : \mathcal{H} \rightarrow B$ on the root of $A(T)$.
3. Place r_k on each internal vertex of valency $k + 1 > 2$.
4. Place G on each vertex of valency 2, i.e. on each internal edge.

We then obtain a morphism of graded R -bimodules $\rho_T \in \text{Hom}(B^{\otimes n}, B)$ as

$$\rho_T = (-1)^{e_i(T)} \langle D_T \rangle, \quad (3.14)$$

where $\langle D_T \rangle$ is the *denotation* of the decorated tree.

Remark 3.3.2. There are two natural ways to compose the maps on a tree: either we can proceed branch by branch, obtaining a nested composition, or we can work level by level. These are called the branch and height denotations respectively. Due to Koszul

signs, with $|r_n| = +1$ and $|G| = -1$, these two conventions do not agree in general. Indeed, in the example shown in Figure 3.3, the branch denotation gives

$$p \circ r_2 \circ (G \circ r_2 \circ \iota^{\otimes 2} \otimes G \circ r_2 \circ \iota^{\otimes 2}),$$

whereas the height denotation gives

$$p \circ r_2 \circ (G \otimes G) \circ (r_2 \otimes r_2) \circ (\iota \otimes \iota \otimes \iota \otimes \iota) = -p \circ r_2 \circ (G \circ r_2 \circ \iota^{\otimes 2} \otimes G \circ r_2 \circ \iota^{\otimes 2}).$$

In the context of proving the minimal model theorem, the branch denotation is much simpler, so that is what we will use in the remainder of the thesis.

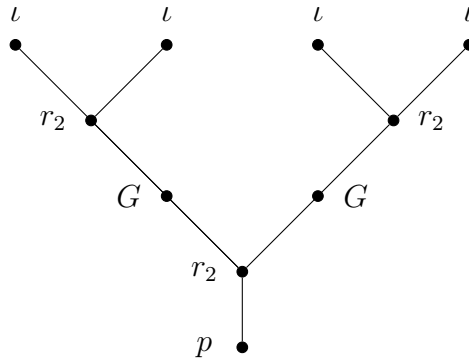


Figure 3.1: An example of an augmented tree decorated by homotopy retract data.

Lemma 3.3.1. *For any $T \in \mathcal{T}_n$, the associated map ρ_T is homogeneous of degree $2 - n$, and hence has degree $+1$ as a map $B[1]^{\otimes n} \rightarrow B[1]$.*

Proof. Since G has degree -1 , and r_k has degree $+1$, the degree of ρ_T is

$$(-1) \cdot e_i(T) + \sum_{v \in V_i(T)} (3 - \text{valency}(v)).$$

For every internal vertex of valency $k + 1$ there are k edges, so

$$\sum_{v \in V_i(T)} (\text{valency}(v) - 1) = e_i(T) + v_i(T) + n,$$

and the expression for the degree of ρ_T reduces to

$$2(v_i(T) - e_i(T)) - n.$$

But the injection which sends every internal edge to its source only misses the vertex adjacent to the root, so $v_i(T) - e_i(T) = 1$, and the degree of ρ_T is therefore $2 - n$.

□

Definition 3.3.5. For $n \geq 2$ define the degree +1 map

$$\rho_n := \sum_{T \in \mathcal{T}_n} \rho_T,$$

and set $\rho_1 := p \circ r_1 \circ \iota = r_1|_B$.

The Minimal Model Theorem was first stated by Kadeishvili [25] in 1980 where it was used to compute A_∞ -infinity structures on the homology and cohomology of fibre spaces. The original proof, which was published in Russian and translated by Merkulov in [44] with applicaitons to Kähler manifolds, constructed the higher products recursively and was extended to A_∞ -categories by Fukaya [17] and Lefevre-Hasegawa [39].

The proof that we present below, using decorated plane rooted trees, was first sketched for A_∞ -algebras by Kontsevich-Soibelman [34]. In [38] Lazaroiu presented the same proof in more detail for A_∞ -categories, using the sector decomposition approach which we follow closely here. However, to the author's knowledge, the literature is lacking a proof which carefully keeps tracks of all Koszul signs, especially those which arise from the choice of denotation (see Appendix A). We aim to remedy this in the remainder of this section.

Theorem 3.3.1 (Minimal Model Theorem). *The maps $\{\rho_n\}_{n \geq 1}$ satisfy the forward suspended A_∞ -relations*

$$\sum_{1 \leq i+j \leq n} \rho_{n-j+1} \circ \left(\text{id}_{B[1]}^{\otimes i} \otimes \rho_j \otimes \text{id}_{B[1]}^{\otimes (n-i-j)} \right) = 0, \quad n \geq 1. \quad (3.15)$$

Before we can give the proof we must establish a few technical results. First, observe that for $n = 1$ the A_∞ -relation is immediate because $r_1^2 = 0$. Next, consider the following R -bilinear maps $\mathcal{H}[1]^{\otimes n} \rightarrow \mathcal{H}[1]$,

$$(r)_1^n = r_1 \circ r_n + \sum_{i=0}^{n-1} r_n \circ \left(\text{id}_{\mathcal{H}[1]}^{\otimes i} \otimes r_1 \otimes \text{id}_{\mathcal{H}[1]}^{\otimes (n-i-1)} \right), \quad (3.16)$$

and

$$(\rho)_1^n = \rho_1 \circ \rho_n + \sum_{i=0}^{n-1} \rho_n \circ \left(\text{id}_{B[1]}^{\otimes i} \otimes \rho_1 \otimes \text{id}_{B[1]}^{\otimes (n-i-1)} \right). \quad (3.17)$$

We can rewrite the A_∞ -relations on the maps $\{r_n\}$ as $r_1^2 = 0$ and

$$(r)_1^n = - \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} r_{k-j+1} \circ \left(\text{id}_{\mathcal{H}[1]}^{\otimes i} \otimes r_j \otimes \text{id}_{\mathcal{H}[1]}^{\otimes (k-j-i)} \right), \quad n \geq 2. \quad (3.18)$$

Since $\rho_1^2 = 0$, it suffices to show that

$$(\rho)_1^n = - \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} \rho_{k-j+1} \circ \left(\text{id}_{B[1]}^{\otimes i} \otimes \rho_j \otimes \text{id}_{B[1]}^{\otimes (k-j-i)} \right), \quad n \geq 2, \quad (3.19)$$

from which the A_∞ -relations will follow. The idea of the proof is to perturb the ρ_T maps by adding r_1 to an internal edge and then exploit the relation $[G, r_1] = \text{id} - P$ to compute $(\rho)_1^n$ in two different ways to give the desired relation. In what follows we will suppress the subscripts from the id morphisms.

Given any $f \in \text{End}_{R\text{Mod}_R}(\mathcal{H})$ of degree zero, and any $T \in \mathcal{T}_n$ with an internal edge e , let $D_{f,e}$ be the decoration of $A(T)$ which puts f rather than G at the vertex corresponding to e . We write

$$\rho_{T,e}^f := (-1)^{e_i(T)} \langle D_{f,e} \rangle, \quad (3.20)$$

and set

$$\rho_n^f := \sum_{T \in \mathcal{T}_n} \sum_{e \in E_i(T)} \rho_{T,e}^f. \quad (3.21)$$

Similarly, if e is an external edge adjacent to the i^{th} leaf or the root (which we denote by 0), then we define a new decoration $D_{e,i}$ by replacing ι (resp. p) with $\iota \circ \rho_1$ (resp. $p \circ r_1$). We now define *perturbed* bilinear maps $\widehat{\rho}_{T,e} \in \text{Hom}(B^{\otimes n}, B)$ as

$$\widehat{\rho}_{T,e} = \begin{cases} \rho_{T,e}^{r_1 G + G r_1}, & e \in E_i(T), \\ (-1)^{e_i(T)} \langle D_{e,i} \rangle, & e \in E_e(T) \text{ adjacent to } i, \end{cases} \quad (3.22)$$

and finally, by summing over all edges and all trees,

$$\widehat{\rho}_n := \sum_{T \in \mathcal{T}_n} \sum_{e \in E(T)} \widehat{\rho}_{T,e}. \quad (3.23)$$

Lemma 3.3.2. For $0 \leq i \leq n - 1$,

$$\rho_n \circ \left(\text{id}^{\otimes(i-1)} \otimes \rho_1 \otimes \text{id}^{\otimes(n-1)} \right) = \sum_{T \in \mathcal{T}_n} \widehat{\rho}_{T,e}, \quad (3.24)$$

where e is the edge in T which is adjacent to the i^{th} leaf.

Proof. Fix $0 \leq i \leq n - 1$ and $T \in \mathcal{T}_n$. Then

$$\langle D_{e,i} \rangle = (-1)^s \langle D_T \rangle \circ \left(\text{id}^{\otimes(i-1)} \otimes \rho_1 \otimes \text{id}^{\otimes(n-1)} \right), \quad (3.25)$$

where s is the sum of the degrees $|\phi_p|$ of the maps on $A(T)$ at vertices q such that $q > i$ under the ordering on T defined in section A.1. Observe that, since internal edges are paired exactly with internal vertices which are not adjacent to the root, those vertices contributing to S come in pairs of opposite degree. Hence we conclude that $s = 0$. \square

Lemma 3.3.3. Using the notation above, we have

$$\widehat{\rho}_n = (\rho)_1^n + \rho_n^{\text{id}_{\mathcal{H}}} - \rho_n^P. \quad (3.26)$$

Proof. It is obvious that $\rho_1 \circ \rho_n = \sum_{T \in \mathcal{T}_n} \widehat{\rho}_{T,e}$ if e is adjacent to the root. Hence, using the fact that $\text{id} - P = [r_1, G]$ we have

$$\begin{aligned} (\rho)_1^n &= \sum_{T \in \mathcal{T}_n} \sum_{e \in E_e(T)} \widehat{\rho}_{T,e} \\ &= \widehat{\rho}_n - \sum_{T \in \mathcal{T}_n} \sum_{e \in E_e(T)} \widehat{\rho}_{T,e} \\ &= \widehat{\rho}_n - \sum_{T \in \mathcal{T}_n} \sum_{e \in E_e(T)} \rho_{T,e}^{[r_1, G]} \\ &= \widehat{\rho}_n - \sum_{T \in \mathcal{T}_n} \sum_{e \in E_e(T)} (\rho_{T,e}^{\text{id}} - \rho_{T,e}^P) \\ &= \widehat{\rho}_n - \rho_{T,e}^{\text{id}_{\mathcal{H}}} + \rho_{T,e}^P. \end{aligned}$$

\square

Now we will compute $\widehat{\rho}_n$ in a different way. Set

$$\widehat{\rho}_T := \sum_{e \in E(T)} \widehat{\rho}_{T,e},$$

so that $\widehat{\rho}_n = \sum_{T \in \mathcal{T}_n} \widehat{\rho}_T$. Let v be a vertex of valency greater than 2 on a tree $T \in \mathcal{T}_n$

with $n \geq 2$. Set

$$\widehat{\rho}_{T,v} = \sum_{\substack{e \in E_i(T) \\ e \text{ ends at } v}} \rho_{T,e}^{r_i G} + \sum_{\substack{e \in E_i(T) \\ e \text{ begins at } v}} \rho_{T,e}^{Gr_i} + \sum_{\substack{e \in E_i(T) \\ e \text{ incident at } v}} \widehat{\rho}_{T,e}. \quad (3.27)$$

Then we can rewrite $\widehat{\rho}_T$ as

$$\widehat{\rho}_T = \sum_{v \in V_i(T)} \widehat{\rho}_{T,v}.$$

Recall now that $\rho_{T,e}^{r_1 G}$, $\rho_{T,e}^{Gr_1}$, and $\widehat{\rho}_{T,e}$ are the same - up to signs - as $\langle D_{r_1 G, e} \rangle$, $\langle D_{Gr_1, e} \rangle$, and $\langle D_{e, i} \rangle$ respectively. Suppose that the vertex v has k incoming edges. Then the contribution of v to $\langle D_T \rangle$ is the operator

$$r_k \circ (\langle D_{T_1} \rangle \otimes \cdots \otimes \langle D_{T_k} \rangle),$$

where the T_i is the i^{th} sub-tree of $A(T)$ above v . Let us now introduce a convenient piece of notation. Since we are only looking at changes to D_T on the edges either side of v , we will write $\langle \langle - \rangle \rangle_{T,v}$ for the denotation which has $(-)$ contributed around the vertex v . This is where using the branch denotation makes life much easier, since describing local deviations from D_T will not hide unexpected signs. For example,

$$\langle \langle r_k \circ (\langle D_{T_1} \rangle \otimes \cdots \otimes \langle D_{T_k} \rangle) \rangle \rangle_{T,v} = \langle D_T \rangle.$$

Now by the same reasoning as in the proof of Lemma 4.3, $|T_i| = 0$ since degree 1 and degree -1 maps are inserted in pairs on every sub-tree of the standard decoration. Hence, writing e_i for the edge adjacent to the i^{th} leaf,

$$\begin{aligned} \langle D_{r_1 G, e_i} \rangle &= \langle \langle r_k \circ (\langle D_{T_1} \rangle \otimes \cdots \otimes r_1 \langle D_{T_i} \rangle \otimes \cdots \otimes \langle D_{T_k} \rangle) \rangle \rangle_{T,v} \\ &= \langle \langle r_k \circ (\text{id}^{\otimes(i-1)} \otimes r_1 \otimes \text{id}^{\otimes(k-i)}) \circ (\langle D_{T_1} \rangle \otimes \cdots \otimes \langle D_{T_k} \rangle) \rangle \rangle_{T,v}. \end{aligned}$$

It follows then that since $r_1^2 = 0$ we can write

$$\begin{aligned} \widehat{\rho}_{T,v} &= (-1)^{e_i(T)} \langle \langle (r_1^k) \rangle \rangle_{T,v} \\ &= (-1)^{e_i(T)} \left\langle \left\langle - \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} r_{k-j+1} \circ \left(\text{id}^{\otimes i} \otimes r_j \otimes \text{id}^{\otimes(k-j-i)} \right) \right\rangle \right\rangle_{T,v} \\ &= (-1)^{e_i(T)+1} \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} \left\langle \left\langle r_{k-j+1} \circ \left(\text{id}^{\otimes i} \otimes r_j \otimes \text{id}^{\otimes(k-j-i)} \right) \right\rangle \right\rangle_{T,v}. \end{aligned}$$

Lemma 3.3.4. *Given an valid plane tree $T \in \mathcal{T}_n$, an internal vertex v of valency $k+1$, and integers $i \geq 0, j \geq 2$ with $i+j \leq k, j \leq k-1$, let $T' = \text{ins}(T, v, i, j)$ be the tree defined in Section A.3, and let e' be the created edge. Write $D'_{\text{id}, e'}$ for the decoration of $A(T')$ obtained from the standard one by inserting $\text{id}_{\mathcal{H}[1]}$ rather than G at e' . Then*

$$\left\langle \left\langle r_{k-j+1} \circ \left(\text{id}^{\otimes i} \otimes r_j \otimes \text{id}^{\otimes (k-j-i)} \right) \right\rangle \right\rangle_{T,v} = \langle D'_{\text{id}, e'} \rangle. \quad (3.28)$$

Proof. Because operators of opposite sign come in pairs in sub-decorations, we have

$$\begin{aligned} \left\langle \left\langle r_{k-j+1} \circ \left(\text{id}^{\otimes i} \otimes r_j \otimes \text{id}^{\otimes (k-j-i)} \right) \right\rangle \right\rangle_{T,v} &= \langle \langle r_{k-j+1} \circ (\text{id} \otimes r_j \otimes \text{id}) \circ (\langle D_{T_1} \rangle \otimes \cdots \otimes \langle D_{T_k} \rangle) \rangle \rangle_{T,v} \\ &= \langle \langle r_{k-j+1} \circ (\langle D_{T_1} \rangle \otimes \cdots \otimes r_j \langle D_{T_j} \rangle \otimes \cdots \otimes \langle D_{T_k} \rangle) \rangle \rangle_{T,v} \\ &= \langle D'_{\text{id}, e'} \rangle, \end{aligned}$$

as claimed. \square

Using Lemma 3.3.4 we can write

$$\widehat{\rho}_{T,v} = (-1)^{e_i(T)+1} \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} \langle D'_{\text{id}, e'} \rangle. \quad (3.29)$$

We can partition \mathcal{T}_n by the number of internal edges, so we write

$$\mathcal{T}_n = \coprod_{d \geq 0} \mathcal{T}_n^{(d)},$$

where $\mathcal{T}_n^{(d)}$ denotes the subset of trees which have d internal edges. By Lemma A.3.1 of section A.3, there is a bijection when $n > 2$

$$(\mathcal{T}_n^{(d+1)})^+ \leftrightarrow \{(Q, v, i, j) \mid Q \in \mathcal{T}_n^{(d)}, v \in V_i(T), i \geq 0, i+j \leq |v|-1, 2 \leq j \leq |v|-2\},$$

where $(\mathcal{T}_n^{(d+1)})^+$ is the set of pairs (T, e) with $T \in \mathcal{T}_n^{d+1}$ and $e \in E_i(T)$. Writing

$$\mathcal{T}_n^+ := \coprod_{d \geq 0} (\mathcal{T}_n^{(d+1)})^+,$$

this shows that

$$\begin{aligned}
\widehat{\rho}_n &= \sum_{T \in \mathcal{T}_n} \sum_{v \in V_i(T)} \widehat{\rho}_{T,v} \\
&= \sum_{t \in \mathcal{T}_n} \sum_{v \in V_i(T)} \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} (-1)^{e_i(T)+1} \langle D'_{\text{id}, e'} \rangle \\
&= \sum_{T' \in \mathcal{T}_n} \sum_{e' \in E_i(T')} (-1)^{e_i(T')+1} \langle D'_{\text{id}, e'} \rangle \\
&= \rho_n^{\text{id}\mathcal{H}}.
\end{aligned}$$

Corollary 3.3.1. *For $n > 2$, $(\rho)_1^n = \rho_n^p$.*

Proof. In Lemma 3.3.3 we showed that $\widehat{\rho}_n = (\rho)_1^n + \rho_n^{\text{id}\mathcal{H}} - \rho_n^p$. From the result above that $\widehat{\rho}_n = \rho_n^{\text{id}\mathcal{H}}$, it follows that, for $n > 2$, $(\rho)_1^n = \rho_n^p$. \square

Proof of theorem. Recall that $P = i \circ p$. Since $\rho_1^2 = 0$ it suffices to show that for $n \geq 2$,

$$\rho_n^p = - \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} \rho_{n-j+1} \circ (\text{id}^{\otimes i} \otimes \rho_j \otimes \text{id}^{\otimes n-j-i}).$$

This, together with the above results will imply that

$$\sum_{1 \leq i+j \leq n} \rho_{n-j+1} \circ (\text{id}^{\otimes i} \otimes \rho_j \otimes \text{id}^{\otimes (n-i-j)}) = 0. \quad (3.30)$$

Recall that by definition

$$\rho_n^p = \sum_{T \in \mathcal{T}_n} \sum_{e \in E_i(T)} (-1)^{e_i(T)} \langle D_{p,e} \rangle.$$

For a fixed n there is a bijection

$$\gamma: \prod_{2 \leq j \leq n-1} \mathcal{T}_{n-j+1} \times \mathcal{T}_j \times \{0, \dots, n-j\} \longrightarrow \mathcal{T}_n^+,$$

which is defined by taking (T, T', i) to the tree formed by attaching T' to T at the $(i+1)^{\text{st}}$ leaf, creating a new marked internal edge e' . This results in a tree \widehat{T} with

$e_i(\widehat{T}) = e_i(T) + e_i(T') + 1$. We can now apply the previous lemmas to compute

$$\begin{aligned}
\rho_n^p &= \sum_{\substack{2 \leq j \leq n-1 \\ 0 \leq i \leq n-j}} \sum_{\substack{T \in \mathcal{T}_{n-j+1} \\ Q' \in \mathcal{T}_j}} (-1)^{e_i(\widehat{T})} \langle D_T \rangle \circ \left(\text{id}^{\otimes i} \otimes \langle D_{T'} \rangle \otimes \text{id}^{\otimes (n-j-i)} \right) \\
&= - \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} \left(\sum_{T \in \mathcal{T}_{n-j+1}} (-1)^{e_i(T)} \right) \circ \left(\text{id}^{\otimes i} \otimes \left(\sum_{T' \in \mathcal{T}_j} \langle D_{T'} \rangle \right) \otimes \text{id}^{\otimes (n-j-i)} \right) \\
&= - \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k, j \leq k-1}} \rho_{n-j+i} \circ \left(\text{id}^{\otimes i} \otimes \rho_j \otimes \text{id}^{\otimes (n-j-i)} \right)
\end{aligned}$$

as required. □

Chapter 4

Matrix factorisations

4.1 Preliminaries

In the 1980's Eisenbud studied the homological properties of hypersurface singularities and complete intersections using a deceptively simple, yet highly effective, piece of technology: matrix factorisations. A matrix factorisation is exactly what it sounds like: a factorisation of one linear operator into two other linear operators. These objects date back to Dirac's famous factorisation of the four-dimensional Laplacian, Δ , as

$$\left(\sum_{\mu=1}^4 \gamma_{\mu} \frac{\partial}{\partial x_{\mu}} \right)^2 = \Delta \cdot \text{id}, \quad (4.1)$$

where the γ_{μ} are the *Dirac matrices*. However, it was Eisenbud's application of the idea to the theory of singular rings which built the idea into the powerful theory we have today.

Let (R, \mathfrak{m}) be a regular local ring and let M be a finitely generated R -module. A sequence $x_1, \dots, x_n \in R$ is called an *M -regular sequence* if x_i is a nonzerodivisor in $M/(x_1, \dots, x_{i-1})M$ for all $1 \leq i \leq r$. The *depth* of M is the length of a maximal M -sequence. The *projective dimension*, $\text{pd}_R(M)$, of M is the length of a minimal free resolution of M . The following theorem is a cornerstone of homological commutative algebra.

Theorem 4.1.1 (Auslander-Buchsbaum [1]). *Let M be a finitely generated module over a regular local ring R . The projective dimension and the depth of M are related to the Krull dimension of R by the following formula:*

$$\text{pr}_R(M) = \dim(R) - \text{depth}(M). \quad (4.2)$$

Let $W \in R$. Although this formula does not hold when $S = R/(W)$ is singular, we can still use it to deduce information about certain classes of modules over S .

Definition 4.1.1. We say that a finitely generated S -module M is *Cohen-Macaulay* if $\text{depth}(M) = \dim(S)$.

The following construction is due to Eisenbud [14]. Let M be a Cohen-Macaulay module over $S = R/(W)$. This is equivalent to considering M as an R -module such that $\text{depth}(M) = \dim(R) - 1$ and W annihilates M . The Auslander-Buchsbaum formula then says that

$$\text{proj dim}_R(M) = \dim(R) - \text{depth}(M) = 1, \quad (4.3)$$

so M admits an R -free resolution of length 1. In other words, we can find an exact sequence of R -modules

$$0 \rightarrow X_0 \xrightarrow{\phi} X_1 \rightarrow M \rightarrow 0$$

such that X_0 and X_1 are free. Since M is annihilated by W , we can construct a splitting ψ of ϕ such that $\psi\phi = W \cdot \text{id}_R$. It follows that $\psi\phi\psi = W \cdot \psi$. Since ψ is a monomorphism this implies that $\phi\psi = W \cdot \text{id}_R$. This implies in turn that X_0 and X_1 have the same rank, and therefore ψ and ϕ are in fact square matrices. This motivates the following definition.

Definition 4.1.2 (Eisenbud). Let (R, \mathfrak{m}) be a regular local ring and let $W \in \mathfrak{m}$ be some non-zero element. A *matrix factorisation* of W is a free $\mathbb{Z}/2\mathbb{Z}$ -graded R -module X of finite rank equipped with an R -linear map d of odd degree such that $d^2 = W \cdot \text{id}_X$.

This definition is equivalent to writing $X = X_0 \oplus X_1$ together with degree +1 maps

$$X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_0,$$

such that $d_0 \circ d_1 = d_1 \circ d_0 = W \cdot \text{id}_X$ and

$$d = \begin{bmatrix} 0 & d_1 \\ d_0 & 0 \end{bmatrix}.$$

We will refer to d as the *twisted differential* of the matrix factorisation.

A morphism of matrix factorisations of W is a $\mathbb{Z}/2\mathbb{Z}$ -graded linear map $f : X \rightarrow Y$ such that $f \circ d^X = d^Y \circ f$. Equivalently, it is a pair R -module homomorphisms $f_0 : X_0 \rightarrow Y_0$ and $f_1 : X_1 \rightarrow Y_1$ such that the following diagram commutes:

$$\begin{array}{ccccc}
X_0 & \xrightarrow{d_0^X} & X_1 & \xrightarrow{d_1^X} & X_0 \\
f_0 \downarrow & & \downarrow f_1 & & \downarrow f_0 \\
Y_0 & \xrightarrow{d_0^Y} & Y_1 & \xrightarrow{d_1^Y} & Y_0
\end{array}$$

Definition 4.1.3. The *DG-category of matrix factorisations* $\mathrm{mf}(R, W)$ of $W \in R$ is the $\mathbb{Z}/2\mathbb{Z}$ -graded DG-category consisting of the following data:

- The objects of $\mathrm{mf}(R, W)$ are matrix factorisations (X, d) of W .
- The mapping complexes $\mathrm{mf}(R, W)(X, Y)$ are given by the R -module of $\mathbb{Z}/2\mathbb{Z}$ -graded R -linear maps $f : (X, d^X) \rightarrow (Y, d^Y)$ with the differential given by

$$d(f) = d_Y \circ f - (-1)^{|f|} f \circ d_X.$$

The *homotopy category of matrix factorisations* $\mathrm{hmf}(R, W)$ of $W \in R$ is obtained by taking the 0^{th} cohomology of the mapping complexes $\mathrm{mf}(R, W)(X, Y)$.

Remark 4.1.1. We will usually drop the ring R from the notation, writing simply $\mathrm{mf}(W)$ and $\mathrm{hmf}(W)$.

The following is a simple but important example:

Example 4.1.1. Consider the polynomial $W = x^n \in \mathbb{C}[x]$. The family of matrices

$$A_i = \begin{bmatrix} 0 & x^{n-i} \\ x^i & 0 \end{bmatrix}, \quad i \in \{1, \dots, n\}, \quad (4.4)$$

satisfy $A_i^2 = x^n \cdot \mathrm{id}$. Therefore $\{(\mathbb{C}[x]^2, A_i)\}$ is a family of matrix factorisations of W . It was shown in [21] that all matrix factorisations of $W = x^n$ are given by direct sums of these factorisations.

Example 4.1.2. Consider the polynomial $W = x^d - y^d \in \mathbb{C}[x, y]$, and write $\eta = e^{2\pi i/d}$ for the d^{th} root of unity. We can factor W as

$$W = \prod_{i=0}^{d-1} (x - \eta^i y). \quad (4.5)$$

As in the previous example we can now write down a family of matrix factorisations

$$X_i = \begin{bmatrix} 0 & \prod_{i \notin I} (x - \eta^i y) \\ \prod_{i \in I} (x - \eta^i y) & 0 \end{bmatrix}, \quad I \subseteq \{1, \dots, d\}. \quad (4.6)$$

Following the physics literature [4, 15], we will refer to this family of matrix factorisations as *permutation defects*.

We will return to these examples in the context of A_∞ -algebras and A_∞ -modules in Sections 5.4 and 6.1.

Definition 4.1.4. Given a matrix factorisation (X, d) of $W \in R$, the dual matrix factorisation is $X^\vee := \text{Hom}_R(X, R)$, with twisted differential $d^\vee(f) = (-1)^{|f|+1} f \circ d$. This is a matrix factorisation of $-W$.

Definition 4.1.5. Let X and Y be matrix factorisations of W and V respectively. The tensor product, $X \otimes_R Y$, has a natural $\mathbb{Z}/2\mathbb{Z}$ -grading, with twisted differential $d_{X \otimes Y} = d_X \otimes \text{id} + \text{id} \otimes d_Y$. It is a matrix factorisation of $W + V$.

After choosing a basis, we can write $X \otimes_R Y$ as

$$(X_0 \otimes Y_0) \oplus (X_1 \otimes Y_1) \xrightarrow{d_0^{X \otimes Y}} (X_1 \otimes Y_0) \oplus (X_0 \otimes Y_1) \xrightarrow{d_1^{X \otimes Y}} (X_0 \otimes Y_0) \oplus (X_1 \otimes Y_1),$$

where $d_i^{X \otimes Y} = d_i^X \otimes \text{id} + \text{id} \otimes d_{i+1}^Y$ with the necessary Koszul signs.

Definition 4.1.6. The *shift functor* $(-)[1] : \text{hmf}(W) \rightarrow \text{hmf}(W)$ is given on matrix factorisations (X, d) by

$$\left(X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_0 \right) [1] = X_1 \xrightarrow{-d_1} X_0 \xrightarrow{-d_0} X_1. \quad (4.7)$$

We write $X[n]$ for the n -fold application of the shift functor.

Remark 4.1.2. Observe that $X[2] = X$, so we can define shifts of negative degree by $X[-1] = X[1]$ and more generally $X[-n] = X[n]$.

4.2 Koszul factorisations

We will be particularly interested in a certain class of matrix factorisations, which we call Koszul factorisations.

Definition 4.2.1. Let R be a commutative ring and let $W \in R$. Suppose we have sequences a_1, \dots, a_n and b_1, \dots, b_n in R such that

$$W = \sum_{i=1}^n a_i b_i.$$

Consider the $\mathbb{Z}/2\mathbb{Z}$ -graded R -module $F_\psi = R\psi_1 \oplus \dots \oplus R\psi_n$ with $|\psi_i| = 1$, and define operators

$$\delta_+ = \sum_{i=1}^n a_i \psi_i^* \lrcorner (-), \quad \delta_- = \sum_{i=1}^n b_i \psi_i \wedge (-). \quad (4.8)$$

Here $\psi_i^* \lrcorner (-)$ is the contraction operator defined by the formula

$$\psi_i^* \lrcorner (\psi_{i_1} \wedge \dots \wedge \psi_{i_p}) = \sum_{k=1}^p (-1)^{k+1} \delta_{i, i_k} \psi_{i_1} \wedge \dots \wedge \widehat{\psi_{i_k}} \wedge \dots \wedge \psi_{i_p}, \quad (4.9)$$

where $\widehat{\psi_{i_k}}$ means that ψ_{i_k} is omitted. We will write ψ_i^* when we mean $\psi_i^* \lrcorner (-)$, and ψ_i when we mean $\psi_i \wedge (-)$. The *Koszul matrix factorisation* of W with respect to the chosen decomposition is

$$\{\underline{a}, \underline{b}\} := \left(\bigwedge F_\psi, \delta_+ + \delta_- \right),$$

Example 4.2.1. If we take $R = \mathbb{k}[x]$, the Koszul factorisation $\{x^i, x^{n-i}\}$ of $W = x^n$ is simply the matrix factorisation

$$\mathbb{k}[x] \xrightarrow{x^i} \mathbb{k}[x] \xrightarrow{x^{n-i}} \mathbb{k}[x]$$

of Example 4.1.1.

Example 4.2.2. Suppose $\mathbb{k} = R/(\underline{x} - P)$ is the residue field at a singular point $P = (p_1, \dots, p_n)$ of $Z(W)$, the zero locus of W . Then for some $W_1, \dots, W_n \in R$, write,

$$W = \sum_{i=1}^n (x_i - p_i) W_i.$$

The associated Koszul factorisation is denoted

$$\mathbb{k}_W^{stab}(P) := \{\underline{x} - P, (W_1, \dots, W_n)\}. \quad (4.10)$$

Lemma 4.2.1. *Let X and Y be matrix factorisations of W and V respectively.*

1. *There is a natural isomorphism of matrix factorisations of $-W - V$*

$$\begin{aligned}\Psi : X^\vee \otimes Y^\vee &\rightarrow (X \otimes Y)^\vee, \\ \Psi(f \otimes g)(x \otimes y) &= (-1)^{|x||y|} f(x)g(y).\end{aligned}$$

2. *There is a canonical isomorphism of matrix factorisations of $-W$*

$$\begin{aligned}\Psi : \{\underline{b}, -\underline{a}\} &\rightarrow \{\underline{a}, \underline{b}\}^\vee \\ \psi_{i_1} \wedge \cdots \wedge \psi_{i_p} &\mapsto (-1)^{\binom{p}{2}} (\psi_{i_1} \wedge \cdots \wedge \psi_{i_p})^*, \quad (i_1 < \cdots < i_p).\end{aligned}$$

3. *There is a canonical isomorphism of matrix factorisation of W*

$$\{\underline{b}, -\underline{a}\} \cong \{\underline{a}, -\underline{b}\}[m]. \quad (4.11)$$

Proof. See [8], Appendix B. □

Corollary 4.2.1. *Let X be a matrix factorisation of $W = \sum_{i=1}^n x_i W_i \in R$. Then there is an R -linear isomorphism of complexes*

$$\mathrm{Hom}_R(\mathbb{k}_W^{stab}(p), X) \cong (\{\underline{x} - \underline{p}, -\underline{w}\} \otimes_R X) [n], \quad (4.12)$$

where $\underline{w} = (W_1, \dots, W_n)$.

Proof. We compute

$$\begin{aligned}\mathrm{Hom}_R(\mathbb{k}_W^{stab}(p), X) &= \mathrm{Hom}_R(\{\underline{x} - \underline{p}, \underline{w}\}, X) \\ &\cong \{\underline{x} - \underline{p}, \underline{w}\}^\vee \otimes_R X \\ &\cong \{\underline{w}, -(\underline{x} - \underline{p})\} \otimes_R X && \text{Lemma 4.2.1} \\ &\cong (\{\underline{x} - \underline{p}, -\underline{w}\} \otimes_R X) [n], && \text{Lemma 4.2.1.}\end{aligned}$$

□

4.3 Compact generator of hmf(W)

In this section we will follow [48] to show that $\mathbb{k}_W^{stab}(p)$ is a compact generator of $\text{hmf}(W)$.

Definition 4.3.1 (Neeman, [50]). Let \mathcal{T} be a triangulated category. A triangulated subcategory \mathcal{T}' is a *thick* subcategory if it is closed under taking direct summands of its objects. An object X of \mathcal{T} is a *split generator* of \mathcal{T} if the smallest thick subcategory of \mathcal{T} containing X is \mathcal{T} itself. An object X of \mathcal{T} is *compact* if the functor $\text{Hom}_{\mathcal{T}}(X, -)$ commutes with arbitrary coproducts.

Remark 4.3.1. This definition of compactness is equivalent to the property that every morphism $X \rightarrow \bigoplus_i Y_i$ factors through a finite coproduct.

Lemma 4.3.1 (Dyckerhoff, [10]). *There is an isomorphism for any (possibly infinite rank) matrix factorisation X of W*

$$H^* \text{Hom}_R(\mathbb{k}_W^{stab}(p), X) \cong H^*(X \otimes_R R/(\underline{x} - p))[n] \quad (4.13)$$

as $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces.

Proof. This was originally proved by Dyckerhoff [10] using different methods. Without loss of generality take $\underline{p} = 0$. By Section 5.3.1 there is a strong deformation retract

$$H \hookrightarrow (\bigwedge (\bigoplus_{i=1}^n \mathbb{k}\theta_i) \otimes_{\mathbb{k}} R, d_K = \sum_{i=1}^n x_i \theta_i^*) \xleftarrow[\sigma]{\pi \otimes \text{id}} (\mathbb{k}, 0).$$

Where $H = [\Delta, d_K]\Delta$ and $\Delta = \sum_{i=1}^n \frac{\partial}{\partial x_i} \theta_i$. By taking the tensor product with X we obtain another strong deformations retract

$$\text{id}_X \otimes H \hookrightarrow (X \otimes_R \bigwedge (\bigoplus_{i=1}^n \mathbb{k}\theta_i) \otimes_{\mathbb{k}} R, d_K) \xleftarrow[\text{id} \otimes \sigma]{\text{id} \otimes \pi \otimes \text{id}} (X \otimes_R \mathbb{k}, 0).$$

We can perturb the differential on the left hand side to $d_K + \delta$, where

$$\delta = \sum_{i=1}^n (-W_i) \theta_i + d_X. \quad (4.14)$$

Since $(H\delta)^m = 0$ for $m \gg 0$, we can apply the Homological Perturbation Lemma (5.2.1) to obtain a strong deformation retract

$$H_\infty \hookrightarrow (X \otimes_R \bigwedge (\bigoplus_{i=1}^n \mathbb{k}\theta_i) \otimes_{\mathbb{k}} R, d_K + \delta) \xleftarrow[\sigma_\infty]{\text{id} \otimes \pi \otimes \text{id}} (X \otimes_R \mathbb{k}, d_X \otimes 1).$$

By Corollary 4.2.1, the left hand side is equal to $(\{\underline{x} - \underline{p}, -\underline{w}\} \otimes_R X) \cong \text{Hom}_R(\mathbb{k}_W^{stab}(p), X)[n]$. The right hand side is isomorphic to $X \otimes_R R/(\underline{x})$, so the strong deformation retract gives the result. \square

Lemma 4.3.2 (Infinite dimensional Gaussian reduction). *Let X be a (possibly infinite rank) matrix factorisation of $W \in \mathbb{k}[[\mathbf{x}]]$. Then X decomposes as $X \cong X_{\min} \oplus Y$ with Y contractible and X_{\min} a matrix factorisation such that $d_{X_{\min}}(X_{\min}) \subseteq \mathfrak{m} \cdot X_{\min}$.*

Proof. Let C be the contractible matrix factorisation

$$C = \left(R \cdot c_0 \oplus R \cdot c_1, d_C = \begin{bmatrix} 0 & 1 \\ W & 0 \end{bmatrix} \right). \quad (4.15)$$

Then given any matrix factorisation X of W , a morphism $f : C \rightarrow X$ is determined completely by the image of c_1 . Indeed $d_C(c_1) = c_0$, so $f(c_0) = f(d_C(c_1)) = d_X(f(c_1))$. Therefore, given $x \in X$ we can define a morphism $f_x : C[[x]] \rightarrow X$ by $f_x(c_1) = x$.

Any $\mathbb{Z}/2\mathbb{Z}$ -graded submodule of X closed under d_X which is isomorphic to a direct sum of copies of C will be a contractible submodule, so the idea of the proof is to use Zorn's Lemma to show that the set of such submodules has a maximal element and a complement in X . This maximal element will be Y and its complement will be X_{\min} .

Let \mathcal{P} denote the set of triples (Z, U, P) where $Z, P \subseteq X$ are $\mathbb{Z}/2\mathbb{Z}$ -graded submodules with complement P , i.e. $Z \oplus P = X$, such that $d_X(Z) \subseteq Z$ and $U \subset Z$ is a subset of homogeneous elements. Consider the subset

$$\mathcal{Q} = \left\{ (Z, U, P) \mid \sum_{x \in U} f_x : \bigoplus_{x \in U} C[[x]] \rightarrow Z \text{ is an isomorphism} \right\} \subseteq \mathcal{P}. \quad (4.16)$$

This has the structure of a poset with $(Z, U, P) \leq (Z', U', P')$ if $Z \subseteq Z'$, $P' \subseteq P$, and $U = Z \cap U'$. Let $\{(Z_i, U_i, P_i) \mid i \in I\}$ be a chain in \mathcal{Q} and set

$$(Z, U, P) = \left(\bigcup_{i \in I} Z_i, \bigcup_{i \in I} U_i, \bigcap_{i \in I} P_i \right). \quad (4.17)$$

Then $Z_i \subseteq Z$, $U_i = Z_i \cap U$, and $P \subseteq P_i$ for all i , and we claim that $(Z, U, P) \in \mathcal{Q}$.

First observe that the following square commutes

$$\begin{array}{ccc}
\bigoplus_{x \in U_i} C[[x]] & \xrightarrow{\cong} & Z_i \\
\downarrow & & \downarrow \\
\bigoplus_{x \in U_j} C[[x]] & \xrightarrow{\cong} & Z_j
\end{array}$$

It follows that we can form a commuting ladder of inclusions and isomorphisms. When we take the limit of this entire diagram we obtain an isomorphism

$$\lim_{\rightarrow} Z_i \cong \lim_{\rightarrow} \bigoplus_{x \in U_i} C[[x]].$$

These limits are isomorphic to Z and $\bigoplus_{x \in U} C[[x]]$ respectively, so we see that

$$\sum f_x : \bigoplus_{x \in U} C[[x]] \rightarrow Z$$

is an isomorphism. Next, it is certainly true that $X = Z + P$, so we just need to show that $Z \cap P = \{0\}$. If $0 \neq z \in Z$ then $z \in Z_i$ for some i , so $z \notin P_i$ because $Z_i \cap P_i = \{0\}$. But then $z \notin P$, so $X = Z \oplus P$.

This shows that every chain has an upper bound, so by Zorn's Lemma \mathcal{Q} has a maximal element which we denote by (Y, U, X_{\min}) . If X_{\min} were to have a unit in the matrix representation of $d_{X_{\min}}$ we could use classical Gaussian elimination to split off a single copy of C . But this would contradict the maximality of Y . Hence it must be the case that $d_{X_{\min}}(X_{\min}) \subseteq \mathfrak{m} \cdot X_{\min}$. \square

The following corollary was first proved by Schoutens [56], in the context of the stable derived category of Cohen-Macaulay modules, and independently by Dyckerhoff [10]. Seidel [57, 11.1] also observed that a slightly weaker form of the result can be deduced from results of Orlov [52]. The proof presented here is due to Murfet [48].

Corollary 4.3.1. *If \mathbb{k} is a field and $W \in \mathbb{k}[[\mathbf{x}]]$ has an isolated singularity at 0, then $\mathbb{k}_W^{\mathrm{stab}}$ is a split generator of $\mathrm{hmf}(W)$.*

Proof. Let $R = \mathbb{k}[[\mathbf{x}]]$. Consider $\mathrm{hmf}(W)$ as a subcategory of $\mathrm{HMF}(W)$, the homotopy category of infinite rank matrix factorisations of W . This is a triangulated category with infinite coproducts, and it is easy to see that $\mathrm{hmf}(W) \subseteq \mathrm{HMF}(W)^c$, the subcategory of compact objects. Indeed, if X is a finitely generated matrix factorisation with chosen basis $\{e_j\}_{j=1, \dots, n}$, then any morphism $f : X \rightarrow \bigoplus_{i \in I} Y_i$ factors as

$$\begin{array}{ccc}
X & \xrightarrow{f} & \bigoplus_{i \in I} Y_i \\
& \searrow & \nearrow \\
& & \bigoplus_{j=1}^n Y_j
\end{array}$$

where $f(e_i) \in Y_j$. In particular \mathbb{k}_W^{stab} is a compact object in $\text{HMF}(W)$.

We will show that \mathbb{k}_W^{stab} is a compact generator of $\text{HMF}(W)$. From this it will follow, using the general theory of triangulated categories (see [50]), that the smallest thick triangulated subcategory of $\text{HMF}(W)$ containing \mathbb{k}_W^{stab} , written $\langle \mathbb{k}_W^{stab} \rangle$, is $\text{HMF}(W)^c$. But $\text{hmf}(W)$ is a thick subcategory containing \mathbb{k}_W^{stab} . Therefore we have inclusions of subcategories

$$\langle \mathbb{k}_W^{stab} \rangle \subseteq \text{hmf}(W) \subseteq \text{HMF}(W) = \langle \mathbb{k}_W^{stab} \rangle, \quad (4.18)$$

from which it follows that $\langle \mathbb{k}_W^{stab} \rangle = \text{hmf}(W)$.

Suppose $\text{Hom}_{\mathcal{T}}(\mathbb{k}_W^{stab}, X[i]) = 0$ for $i \in \mathbb{Z}/2\mathbb{Z}$ and some $X \in \text{HMF}(W)$. Then, by Lemma 4.3.1,

$$0 = H^* \text{Hom}_R(\mathbb{k}_W^{stab}, X) \cong H^*(X \otimes \mathbb{k})[n]. \quad (4.19)$$

But using Lemma 4.3.2 we see that X is homotopy equivalent over R to a matrix factorisation X_{min} whose differential, in matrix form, contains no units. It follows that the differential on $X_{min} \otimes_R \mathbb{k}$ vanishes, so

$$0 = H^*(X \otimes_R \mathbb{k}) \cong H^*(X_{min} \otimes_R \mathbb{k}) \cong X_{min} \otimes_R \mathbb{k}. \quad (4.20)$$

This implies that $X \cong 0$ in \mathcal{T} . It then follows that \mathbb{k}_W^{stab} compactly generates $\text{HMF}(W)$. \square

As a consequence of Theorem 2.4.2 and Corollary 4.3.1, there is a quasi-isomorphism of DG-categories

$$\text{mf}(W) \xrightarrow{\cong} \text{Perf}(\text{End}_R(\mathbb{k}_W^{stab})), \quad X \mapsto \text{Hom}_R(\mathbb{k}_W^{stab}, X), \quad (4.21)$$

In particular the DG-algebra $\text{End}_R(\mathbb{k}_W^{stab})$ contains *all* of the information of $\text{hmf}(W)$.

4.4 Orlov's theorem

Matrix factorisations package the homological data of hypersurface singularities into an elementary and computable package. It turns out, surprisingly, that the category $\text{hmf}(W)$ contains *all* of the homological information (up to quasi-isomorphism) of the singularity in question. For example, Dyckerhoff showed in [10] that we one can recover the Jacobi algebra $\text{Jac}(W)$ as the Hochschild cohomology of the DG-category $\text{mf}(W)$.

Let R be a Noetherian regular local ring and suppose $W \in R$ has a singular point at 0. Let $S = R/(W)$. We denote by $D^b(S)$ the *bounded derived category* of S -modules. The objects are bounded complexes of finitely generated S -modules with morphisms given up to homotopy and with all quasi-isomorphisms formally inverted.

Recall that the *global dimension* of a ring is defined to be the supremum of the lengths of projective resolutions of S -modules. Recall the following homological criterion for regularity.

Theorem 4.4.1 (Serre [58]). *A commutative local ring S is regular if and only if it has finite global dimension.*

Definition 4.4.1. A complex of S -modules is *perfect* if it is quasi-isomorphic to a bounded complex of projective S -modules.

Perfect S modules form a triangulated subcategory $\text{Perf}(S)$ of $D^b(S)$. Using this terminology, Serre's theorem implies that S is regular if and only if $D^b(S)$ is equivalent to $\text{Perf}(S)$. It follows that the failure of this equivalence when S is singular ought to provide insight into the nature of the singularities of $Y = \text{Spec}(S)$.

Definition 4.4.2 (Orlov [51]). Let $S = R/W$. The derived category of singularities is given by the Verdier quotient

$$D_{sg}(S) := D^b(S)/\text{Perf}(S).$$

Proposition 4.4.1. *Every object A in $D_{sg}(S)$ is isomorphic to $N[k]$ for some finitely generated S -module N .*

The derived category of singularities has one major pitfall: its morphisms are extremely difficult to work with. Unlike the bounded derived category $D^b(S)$, whose morphisms can be computed as Ext groups, the morphisms in $D^{sg}(S)$ depend in a very non-trivial way on how $\text{Perf}(S)$ embeds in $D^b(S)$ as a subcategory. This is where matrix factorisations enter the picture, in the form of the following remarkable theorem of Orlov:

Theorem 4.4.2 (Orlov). *Let $S = R/W$. There is a triangulated equivalence of categories,*

$$\text{coker} : \text{hmf}(W) \rightarrow D^{sg}(S). \quad (4.22)$$

$$(4.23)$$

Proof. See [51], Theorem 3.9. □

In one direction, this functor sends a matrix factorisation

$$X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_0 \quad (4.24)$$

to the R -module $N = \text{coker}(d_1)$. Since N is annihilated by W , it can be thought of as an S -module. We can then identify N with the object of $D^b(S)$ which has N in degree 0 and 0 in all other degrees.

We will now describe a partial inverse to coker following [10]. Proposition 4.4.1 tells us that every object in $D^{sg}(S)$ is isomorphic to an S -module N . Let us restrict to those S -modules which are isomorphic to $R/(f_1, \dots, f_n)$, where $f_1, \dots, f_n \in R$ is a regular sequence such that $W \in (f_1, \dots, f_n)$. Then $W = \sum_{i=1}^n w_i f_i$, for some $W_i \in R$, and we have constructed the Koszul matrix factorisation $\{\underline{w}, \underline{f}\}$ in $\text{hmf}(W)$.

In the general case we use a theorem of Eisenbud [14], which implies that, given any object N of $D^{sg}(S)$, the minimal free resolution of N is eventually 2-periodic, and this gives the desired matrix factorisation.

This theorem was originally due to Buchweitz [6] but was not published at the time. It was rediscovered by Orlov in the context of mathematical physics, following the suggestion of Kontsevich that the homotopy category of matrix factorisations ought to be a good model for the category of D -branes in Landau Ginzberg models.

Chapter 5

Minimal model construction

5.1 Introduction

Motivated by Theorem 4.4.2 and Corollary 4.3.1, we wish to compute the A_∞ -minimal model of the DG-algebra $\text{End}_R(\mathbb{k}_W^{stab})$. We begin this chapter with an account of the Homological Perturbation Lemma. Using this, we will construct a strong deformation retract on $\text{End}_R(\mathbb{k}_W^{stab})$ in order to apply the Minimal Model Theorem of Section 3.3. We then introduce configurations and Feynman interactions as a tool to compute higher products explicitly, following Murfet [47]. This chapter concludes with two examples of A_∞ -minimal models.

5.2 Homological perturbation lemma

Suppose we are given some complex of R modules (M, d) on which we wish to perform concrete computations. In a best case scenario, M will be finite dimensional and calculations can be implemented using any number of computer packages. Unfortunately, this not the case in the majority of problems.

In many cases of interest our chain complex M will have finite dimensional cohomology. This often means that we are able to find some finite dimensional complex (N, d') which is quasi-isomorphic to M , in such a way that any information contained in the cohomology of M can be computed using N . In this case we call N a *finite model* of M .

Example 5.2.1. Consider the Koszul complex of the element $x \in \mathbb{k}[x]$:

$$K(x) : \quad 0 \rightarrow \mathbb{k}[x] \xrightarrow{x} \mathbb{k}[x]. \quad (5.1)$$

As a graded \mathbb{k} -vector space, $K(x)$ is infinite dimensional. It is easy to see however, that $H^*K(x) \cong \mathbb{k}$. In particular, we can construct a quasi-isomorphism $K(X) \rightarrow \mathbb{k}$ as follows. Consider the morphism of complexes

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{k}[x] & \xrightarrow{x} & \mathbb{k}[x] \\
\downarrow & & \downarrow 0 & & \downarrow \pi \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{k}
\end{array}$$

with π given by

$$\pi : \mathbb{k}[x] \rightarrow \mathbb{k}, \quad \sum_{i=1}^n a_i x^i \mapsto a_0. \quad (5.2)$$

Let $\iota : \mathbb{k} \rightarrow \mathbb{k}[x]$ be the inclusion of constants, and define $H : \mathbb{k}[x] \rightarrow \mathbb{k}[x]$ by extending linearly $H(x^n) = x_{n-1}$, $H(1) = 0$. Then $\pi \circ \iota = \text{id}$ and

$$xH = \text{id} - \iota \circ \pi.$$

It follows that H defines a contracting homotopy, and thus π is a quasi-isomorphism.

In most cases, the construction of a contracting homotopy isn't this simple, and finding a finite model can be somewhat of an art-form. The Homological Perturbation Lemma provides an algorithm to compute a finite model in the case where we can exhibit the differential d on M as a perturbation of a simpler differential.

More explicitly, suppose we have a cochain complex (M, d) such that the differential can be decomposed as $d = \tau + \gamma$ and (M, τ) is a chain complex with a known contracting homotopy h . The Homological Perturbation Lemma allows us, under certain 'convergence' conditions, to deform h via γ to a contracting homotopy of (M, d) .

Definition 5.2.1. A *deformation retract* of chain complexes (M, d_M) and (N, d_N) consists of a triple of morphisms (π, σ, h)

$$M \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\sigma} \end{array} N \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{h} \end{array} N$$

where $h : N \rightarrow N$ is a degree -1 operator satisfying

$$\begin{aligned} \pi\sigma &= \text{id} \\ \text{id} - \sigma\pi &= [d_N, h]. \end{aligned}$$

A deformation retract is called *strong* if in addition it satisfies the following conditions

$$h^2 = 0, \quad h\sigma = 0, \quad \pi h = 0.$$

Remark 5.2.1. Any deformation retract can be made into a strong deformation retract by appropriately adjusting H .

Theorem 5.2.1 (Homological Perturbation Lemma). *Suppose*

$$(M, d_M) \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\sigma} \end{array} (N, d_N) \begin{array}{c} \hookrightarrow \\ \hookleftarrow \end{array} h$$

is a strong deformation retract. If $d_N + \tau$ has finite order, i.e. $(h\tau)^n = 0$ for some n , then

$$(M, d_\infty) \begin{array}{c} \xrightarrow{\pi_\infty} \\ \xleftarrow{\sigma_\infty} \end{array} (N, d_N + \tau) \begin{array}{c} \hookrightarrow \\ \hookleftarrow \end{array} h_\infty$$

is a strong deformation retract, where $A = \sum_{m \geq 0} (-1)^m \tau (h\tau)^m$ and

$$\begin{aligned} \sigma_\infty &= \sigma - hA\sigma \\ \pi_\infty &= \pi - \pi Ah \\ d_\infty &= d - \pi A\sigma \\ h_\infty &= \sum_{m \geq 0} (-1)^m (h\tau)^m. \end{aligned}$$

Proof. See [9]. □

5.3 The construction

In this section we will construct a strong deformation retract of $\text{End}_R(\mathbb{k}_W^{stab})$ onto a finite dimensional sub-complex in order to apply the Minimal Model Theorem. Similar techniques have been used by Seidel [57] and Efimov [12] in the context of mirror symmetry. We begin with a description of the DG-algebra $\text{End}_R(\mathbb{k}_W^{stab})$.

Definition 5.3.1. Let $W \in \mathbb{k}[x_1, \dots, x_n]$, and write $\partial_{x_i} = \frac{\partial}{\partial x_i}$ for the partial derivatives. We say that W is a *potential* if it satisfies the following conditions:

1. With $I = (\partial_{x_1} W, \dots, \partial_{x_n} W)$, R/I is a finitely generated free \mathbb{k} -module;
2. The sequence $\partial_{x_1} W, \dots, \partial_{x_n} W$ is quasi-regular;
3. The Koszul complex of $\partial_{x_1} W, \dots, \partial_{x_n} W$ is exact except in degree zero.

In order to construct a strong deformation retract which generalises from $\text{End}(\mathbb{k}_W^{stab})$ to the whole DG-category $\text{MF}(W)$, we embed $\text{End}(\mathbb{k}_W^{stab})$ into a larger chain complex \mathcal{H} , and then construct a strong deformation retract on \mathcal{H} using the Homological Perturbation Lemma. This allows us to recover a deformation retract of $\text{End}(\mathbb{k}_W^{stab})$ onto a finite dimensional vector space by constructing a split idempotent from the natural Clifford algebra structure on $\text{End}(\bigwedge(\bigoplus_{i=1}^n \mathbb{k}\theta_i))$ [49].

Let us fix some notation for later chapters. Let \mathbb{k} be a \mathbb{Q} -algebra and let $W \in \mathfrak{m}^3 \subset R = \mathbb{k}[x_1, \dots, x_n]$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Choose a decomposition $W = \sum x_i W_i$ with $W_i \in \mathfrak{m}^2$. We use the following $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{k} -vector spaces

$$\begin{aligned} F_\psi &= \mathbb{k}\psi_1 \oplus \dots \oplus \mathbb{k}\psi_n, \\ F_{\psi^*} &= \mathbb{k}\psi_1^* \oplus \dots \oplus \mathbb{k}\psi_n^*, \\ F_\theta &= \mathbb{k}\theta_1 \oplus \dots \oplus \mathbb{k}\theta_n. \end{aligned}$$

Here, $|\psi_i| = |\psi_i^*| = |\theta_i| = 1$ for all i , ψ_i^* and ψ_i are the contraction and wedge product operators of Section 4.2, and the θ_i are formal variables. We can then write \mathbb{k}_W^{stab} as

$$\mathbb{k}_W^{stab} = \left(R \otimes_{\mathbb{k}} \bigwedge F_\psi, d_W = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W_i \psi_i \right). \quad (5.3)$$

Using this presentation, the endomorphism DG-algebra is given by

$$\text{End}_R(\mathbb{k}_W^{stab}) = \left(R \otimes_{\mathbb{k}} \text{End}_{\mathbb{k}} \left(\bigwedge F_\psi \right), d_{\text{End}} = \sum_{i=1}^n x_i [\psi_i^*, -] + \sum_{i=1}^n W_i [\psi_i, -] \right). \quad (5.4)$$

An important role in the A_∞ -minimal model will be played by the following operators:

Definition 5.3.2. The *Atiyah classes* of $\text{End}_R(\mathbb{k}_W^{stab})$ are the R -linear operators

$$\text{At}_i^W := [d_{\text{End}}, \partial_{x_i}] : \text{End}_R(\mathbb{k}_W^{stab}) \rightarrow \text{End}_R(\mathbb{k}_W^{stab}). \quad (5.5)$$

Lemma 5.3.1. $\text{At}_i = -[\psi_i^*, -] - \sum_{q=1}^n \partial_{x_i} W_q [\psi_q, -]$.

Proof. By direct computation we have

$$\begin{aligned}
\text{At}_i &= [[d_{\mathbb{k}_W^{stab}}, -], \partial_{x_i}] \\
&= \sum_q [x_q[\psi_q^*, -], \partial_{x_i}] + \sum_q [W_q[\psi_q, -], \partial_{x_i}] \\
&= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W_q)[\psi_q, -] \\
&= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W_q)[\psi_q, -].
\end{aligned}$$

□

It is proven in [49] there is, up to homotopy, an induced module structure over the Clifford algebra on $\text{End}_R(\mathbb{k}_W^{stab})$ generated by

$$\gamma_i = \text{At}_i, \quad \gamma_i^\dagger = -\psi_i, \quad (5.6)$$

with relations $[\gamma_i \gamma_j^\dagger] \delta_{ij}$.

Lemma 5.3.2. *The map $e = \gamma_1 \dots \gamma_n \gamma_n^\dagger \dots \gamma_1^\dagger$ is, up to homotopy, an idempotent which is split by a projection $H^* \text{End}_R(\mathbb{k}_W^{stab}) \rightarrow \bigwedge F_{\psi^*}$.*

Proof. See [49], Section 2.4. □

5.3.1 The deformation retract

Consider the Koszul complex of $(x_1, \dots, x_n) \in R$ given by

$$K = \left(R \otimes_{\mathbb{k}} \bigwedge F_\theta, d_K = \sum_{i=1}^n x_i \theta_i^* \right) \quad (5.7)$$

again writing $\theta_i^* = \theta_i^* \lrcorner (-)$ in d_K . By combining K and $\text{End}_R(\mathbb{k}_W^{stab})$ we obtain

$$\mathcal{H} := \left(R \otimes_{\mathbb{k}} \text{End}_{\mathbb{k}} \left(\bigwedge F_\psi \right) \otimes_{\mathbb{k}} \bigwedge F_\theta, d_{\text{End}} + d_K \right). \quad (5.8)$$

Recall that a contracting homotopy on a chain complex is a special case of a deformation retract onto a complex concentrated in degree zero. Thus, if we can find a contracting homotopy on K , then The Homological Perturbation Lemma will give us a deformation retract of \mathcal{H} .

To construct a contracting homotopy on the Koszul complex we require some machinery. First, let us identify $R \otimes_{\mathbb{k}} \bigwedge F_{\theta}$ with $\Omega_{R|\mathbb{k}}$, the algebra of *Kähler differential forms* on R .

Definition 5.3.3 (Loday [40], Section 8.1). Let M be an R -module. A *connection* on M is a \mathbb{k} -linear map

$$\Delta^0 : M \rightarrow M \otimes_R \Omega_{R|\mathbb{k}},$$

such that

$$\Delta^0(x\omega) = \Delta(x)\omega + (-1)^{|\omega|} x d_K(\omega), \quad (5.9)$$

for $x \in M \times \Omega_{R|\mathbb{k}}$ and $\omega \in \Omega_{R|\mathbb{k}}$.

Such a map always extends to a linear map $\Delta : M \otimes_R \Omega_{R|\mathbb{k}} \rightarrow M \otimes_R \Omega_{R|\mathbb{k}}$. A connection Δ is called *flat* if $\Delta^2 = 0$.

Consider the connection $\Delta : R \otimes \bigwedge F_{\theta} \rightarrow R \otimes \bigwedge F_{\theta}$ defined on $r \in R$ and $\omega \in \bigwedge F_{\theta}$ by

$$\Delta(r \otimes \omega) = \sum_{i=1}^n \partial_{x_i}(r) \otimes (\theta_i \wedge \omega). \quad (5.10)$$

In the remainder of this chapter we will abuse notation and write $r \cdot \omega := r \otimes \omega$.

Remark 5.3.1. Note that this connection is a natural generalisation of the contracting homotopy used in example 5.2. Observe also that, given $r \in R$ and $\omega \in \bigwedge F_{\theta}$,

$$\begin{aligned} \Delta^2(r \cdot \omega) &= \sum_{i=1}^n \partial_{x_i} \theta_i^* \sum_{j=1}^n \partial_{x_j} \theta_j^*(r \cdot \omega) \\ &= \sum_{i \neq j} \partial_{x_i} \partial_{x_j}(r) \theta_i^* \theta_j^*(\omega) \\ &= 0 \end{aligned}$$

since $\partial_{x_i} \partial_{x_j}(r) \theta_i^* \theta_j^*(\omega) = -\partial_{x_j} \partial_{x_i}(r) \theta_j^* \theta_i^*(\omega)$, so Δ is flat.

Observe that R and $\bigwedge F_{\theta}$ have natural \mathbb{Z} -gradings, with degrees $|x^n| = n$ and $|\theta_i \theta_j| = 2$.

Lemma 5.3.3. *If $r \in R$ and $\omega \in \bigwedge F_{\theta}$ are homogeneous then*

$$[d_K, \Delta](f \cdot \omega) = (|r|_{\mathbb{Z}} + |\omega|_{\mathbb{Z}}) r \cdot \omega. \quad (5.11)$$

Proof. We proceed by direct calculation,

$$\begin{aligned}
d_K \Delta(r \cdot \omega) &= \sum_{i=1}^n x_i \theta_i^* \sum_{j=1}^n \partial_{x_j} \theta_j(r \cdot \omega) \\
&= \sum_{i \notin \omega} x_j \partial_{x_j}(r) \omega - \sum_{i \neq j} x_i \partial_{x_j}(f) \theta_j \theta_i^* \lrcorner(\omega). \\
\Delta d_K(r \cdot \omega) &= \sum_{j=1}^n \partial_{x_j} \theta_j \sum_{i=1}^n x_i \theta_i^*(r \cdot \omega) \\
&= \sum_{i=1}^n \partial_{x_i}(x_i r) \theta_i \theta_i^* \lrcorner(\omega) + \sum_{i \neq j} x_i \partial_{x_j}(r) \theta_j \theta_i^* \lrcorner(\omega) \\
&= |\omega|_{\mathbb{Z}} r \cdot \omega + \sum_{i \in |\omega|} x_i \partial_{x_i}(r) \omega + \sum_{i \neq j} x_i \partial_{x_j}(r) \theta_j \theta_i^* \lrcorner(\omega).
\end{aligned}$$

here we write $i \in \omega$ to mean those indices i such that θ_i is a component of ω . Combining these terms gives the desired form. \square

It follows that $[d_K, \Delta]$ is invertible. Explicitly, if $r = \sum_{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}}$ for some multi-index $\mathbf{i} \in \mathbb{N}^n$, we have

$$[d_K, \Delta]^{-1}(r \theta_{i_1} \dots \theta_{i_p}) = \sum_{\mathbf{i}} \frac{r(\mathbf{i})}{p + |\mathbf{i}|} x^{\mathbf{i}} \theta_{i_1} \dots \theta_{i_p}, \quad (5.12)$$

where $r(\mathbf{i})$ is the coefficient of $x^{\mathbf{i}}$ in r and $|\mathbf{i}| = i_1 + \dots + i_n$.

Definition 5.3.4. Let H be the degree -1 \mathbb{k} -linear map on K defined by

$$H = [d_K, \Delta]^{-1} \Delta. \quad (5.13)$$

Lemma 5.3.4. $H^2 = 0$ on K .

Proof. It suffices to show that $\Delta[d_K, \Delta]^{-1} \Delta = 0$. Now $\Delta[d_K, \Delta] = [d_K, \Delta] \Delta$, so

$$[d_K, \Delta] \Delta [d_K, \Delta]^{-1} \Delta = \Delta [d_K, \Delta] [d_K, \Delta]^{-1} \Delta = \Delta^2 = 0.$$

The result follows, since $[d_K, \Delta]$ is multiplication by a constant. \square

Write $\pi : R \rightarrow R/\mathfrak{x}R \cong \mathbb{k}$ for the natural projection map, and $\sigma : \mathbb{k} \rightarrow R$ for the inclusion of constants. Note that $\pi\sigma = \text{id}$, and, away from degree zero, $\sigma\pi = \text{id}$.

Lemma 5.3.5. *The map H gives a homotopy between $\sigma\pi$ and id on K .*

Proof. We need to show that $d_K H + \sigma\pi = \text{id}$ on the degree zero part of $R \otimes_{\mathbb{k}} \bigwedge F_\theta$. Obviously on any constant term $d_K H$ will vanish, so it suffices to check for $\mathbf{x}R$. For any $r \in \mathbf{x}R$

$$\begin{aligned} (1 - d_K H - \sigma\pi)(r) &= r - d_K[d_K, \Delta]^{-1} \Delta(r) \\ &= r - \sum_{i=1}^n \sum_{j=1}^n x_j \frac{1}{|r|_{\mathbb{Z}} - 1 + |\theta_i|_{\mathbb{Z}}} \partial_{x_i}(r) \theta_j^* \theta_i \\ &= r - \frac{1}{|r|_{\mathbb{Z}}} \sum_{i=1}^n x_i \partial_{x_i}(r) \\ &= r - r = 0. \end{aligned}$$

□

It follows that there is a homotopy equivalence of cochain complexes between K and the complex with \mathbb{k} in degree zero and zero in all other degrees. This is called the *de Rham contraction*. This data gives us a strong deformation retraction

$$H \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} (\mathcal{H}, d_K) \begin{array}{c} \xrightarrow{\pi \otimes \text{id}} \\ \xleftarrow{\sigma} \end{array} (\text{End}(\bigwedge F_\psi) \otimes \mathbb{k}, 0)$$

where $\sigma(\Psi) = 1 \otimes 1 \otimes \Psi$ and π the surjective composition

$$K \rightarrow R \rightarrow R/\mathfrak{m}.$$

We can now apply the Homological Perturbation Lemma to the above deformation retract to obtain a strong deformation retract

$$H_\infty \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} (\mathcal{H}, d_{\text{End}} + d_K) \begin{array}{c} \xrightarrow{\pi \otimes \text{id}} \\ \xleftarrow{\sigma_\infty} \end{array} (\text{End}(\bigwedge F_\psi) \otimes \mathbb{k}, 0)$$

where H_∞ and σ_∞ are as in the statement of the Homotopy Perturbation Lemma.

Lemma 5.3.6 (Murfet). *There is an isomorphism of DG-algebras*

$$(\mathcal{H}, d_{\text{End}}) \begin{array}{c} \xrightarrow{\exp(-\delta)} \\ \xleftarrow{\exp(\delta)} \end{array} (\mathcal{H}, d_{\text{End}} + d_K),$$

where $\delta = \sum_{i=1}^n (\psi_i \wedge (-)) \theta_i^*$.

Proof. First we will prove the useful identity

$$[d_{\text{End}}, \delta^n] = n\delta^{n-1}d_K, \quad n \geq 1. \quad (5.14)$$

When $n = 1$ this is easy because $\delta \circ d_{\text{End}} = 0$ and

$$d_{\text{End}} \circ \delta = \left(\sum_{i=1}^n x_i[\psi_i^*, -] + \sum_{i=1}^n W_i[\psi_i, -] \right) \sum_{j=1}^n (\psi_j \wedge (-))\theta_j^* = d_K.$$

Now for $n > 1$ we have

$$[d_{\text{End}}, \delta^n] = d_{\text{End}}\delta^n - \delta^n d_{\text{End}} = \sum_{i=0}^{n-1} \delta^i [d_{\text{End}}, \delta] \delta^{n-i-1} = \sum_{i=0}^{n-1} \delta^i d_K \delta^{n-i-1} = n\delta^{n-1}d_K.$$

To prove the lemma we need to show that $\exp(-\delta)$ and $\exp(\delta)$ intertwine the differentials. Using the identity above we have

$$\begin{aligned} (d_{\text{End}} + d_K) \exp(-\delta) - \exp(-\delta) &= [d_{\text{End}}, \exp(-\delta)] + d_K \exp(-\delta) \\ &= \sum_{m \geq 1} \frac{(-1)^m}{m!} [d, \delta^m] + d_K \exp(-\delta) \\ &= \sum_{m \geq 1} \frac{(-1)^m}{m!} m\delta^{m-1}d_K + d_K \exp(-\delta) \\ &= d_K \frac{d}{d\delta} \exp(-\delta) + d_K \exp(-\delta) \\ &= -d_K \exp(-\delta) + d_K \exp(-\delta) \\ &= 0. \end{aligned}$$

The same calculation shows that $(d_{\text{End}} + d_K) \exp(\delta) - \exp(\delta) = 0$, completing the proof. \square

Collecting all of the result of this section so far we have the following theorem:

Theorem 5.3.1 (Murfet, [47]). *Using the following even homogeneous operators on \mathcal{H} ,*

$$\begin{aligned} \Delta &= \sum_{i=1}^n \partial_{x_i} \theta_i, & H &= [d_K, \Delta]^{-1} \Delta, \\ \sigma_\infty &= \sum_{m=0}^{\infty} (-1)^m (H d_{\text{End}})^m \sigma & H_\infty &= \sum_{m=0}^{\infty} (-1)^m (H d_{\text{End}})^m H \end{aligned}$$

there is a strong deformation retract

$$h_\infty \circlearrowleft (\mathcal{H}, d_{\text{End}}) \xleftarrow[\exp(\delta)]{\exp(-\delta)} (\mathcal{H}, d_{\text{End}} + d_K) \xleftarrow[\sigma_\infty]{\pi} (\text{End}_{\mathbb{k}}(\bigwedge F_\psi), 0)$$

where $h_\infty = \exp(\delta) \circ H_\infty \circ \exp(-\delta)$.

This theorem provides a deformation retract of \mathcal{H} onto a finite dimensional vector space, and by the standard minimal model algorithm we can compute the induced A_∞ -structure b on $\text{End}_{\mathbb{k}}(\bigwedge F_\psi)$. We find that the contraction operators ψ_i^* span an A_∞ -subalgebra

$$\bigwedge F_{\psi^*} \subseteq \text{End}_{\mathbb{k}}(\bigwedge F_\psi).$$

The strong deformation retract of Theorem 5.3.1 implies that $\bigwedge F_{\psi^*}$ is quasi-isomorphic to $H^*\text{End}_R(\mathbb{k}_W^{\text{stab}}) \subset H^*\mathcal{H}$, and therefore the A_∞ -structure b on $\bigwedge F_{\psi^*}$ gives the minimal model of $\text{End}_R(\mathbb{k}_W^{\text{stab}})$. In what follows we set

$$\mathcal{A}_W = \left(\bigwedge F_{\psi^*}, b \right). \quad (5.15)$$

Putting all of this together, the Minimal Model Theorem allows us to write A_∞ -higher products as

$$b_d = \sum_{T \in \mathcal{T}_d} (-1)^{e_i(T)} \langle D_T \rangle : \mathcal{A}_W[1]^{\otimes d} \rightarrow \mathcal{A}_W[1]. \quad (5.16)$$

Here, $e_i(T)$ is the number of internal edges in T , and $\langle D_T \rangle$ is the decoration of $A(T)$ by assignment of modules:

- \mathcal{A}_W to each leaf of T , including the root;
- \mathcal{H} to each internal edge;

and the assignment of maps:

- $\exp(\delta)\sigma_\infty$ to each non-root leaf;
- h_∞ to each internal edge;
- r_2 , the forward suspended multiplication, to each internal vertex of valency three;
- $\pi \exp(-\delta)$ to the root.

In fact, the following proposition simplifies things even more.

Proposition 5.3.1. *The higher A_∞ -products are given by*

$$b_d(\Lambda_1 \otimes \cdots \otimes \Lambda_d) = \sum_{T \in \mathcal{T}_d} (-1)^{Q(T, \Lambda_1 \otimes \cdots \otimes \Lambda_d)} \langle D'_T \rangle (\Lambda_1 \otimes \cdots \otimes \Lambda_d), \quad (5.17)$$

where

$$Q(T, \Lambda_1 \otimes \cdots \otimes \Lambda_d) = e_i(T) + d + 1 + \sum_{1 \leq i \leq j \leq d} \tilde{\Lambda}_i \tilde{\Lambda}_j + \sum_{i=1}^d \tilde{\Lambda}_i C_i. \quad (5.18)$$

Here, $\tilde{\Lambda}_i = |\Lambda_i| - 1$, and C_i is the number of times a path from the i^{th} input enters an internal vertex of T from the left. The decoration D'_T is given by the same assignment of modules as D_T but with maps:

- σ_∞ for each input;
- H_∞ for each internal edge;
- $m_2 \exp(-\Xi)$ for each vertex of valency three, where $\Xi = \sum_{i=1}^n [\psi_i, -] \otimes \theta_i^*$;
- π for the root.

Proof. See [47], Proposition 4.18. □

Remark 5.3.2. Observe that

$$\exp(-\Xi) = \sum_{m=0}^{\infty} (-1)^m \sum_{1 \leq j_1 < \cdots < j_m \leq n} \prod_{i=1}^m [\psi_{j_i}, -] \otimes \theta_{j_i}^*. \quad (5.19)$$

It may seem at first like this construction has made the picture a whole lot more complicated: we started with a very concrete object, a matrix factorisation, and we've produced higher A_∞ -products which appear to be a whole lot harder to work with. However, hidden beneath the mass of complicated expressions is one major advantage: we have a finite dimensional vector space and a discrete data structure, so we can perform concrete computations. In this subsection we will explain a method to break down the higher A_∞ -products into sums of *configurations* resembling Feynman diagrams on trees.

Proposition 5.3.2. *The maps σ_∞ and H_∞ can be reduced to the maps*

$$\sigma_\infty = \sum_{m=0}^{\infty} (-1)^m \sum_{j, z, \gamma, \nu} \frac{1}{|\gamma_1| \cdots |\gamma_m|} \left[\prod_{i=1}^m W_{j_i}(\gamma_i) \partial_{x_{z_\nu(i)}}(\mathbf{x}^{\gamma_i}) \theta_{z_\nu(i)}[\psi_{j_i}, -] \right] \circ \sigma, \quad (5.20)$$

$$H_\infty = \sum_{m=0}^{\infty} (-1)^m \sum_{\underline{j}, \underline{z}, \underline{\gamma}, \underline{\nu}} \sum_{t=1}^n \frac{1}{a} \frac{\zeta_a(|\gamma_1|, \dots, |\gamma_m|)}{|\gamma_1| \dots |\gamma_m|} \left[\prod_{i=1}^m W_{j_i}(\gamma_i) \partial_{x_{z_{\nu(i)}}}(\mathbf{x}^{\gamma_i}) \theta_{z_{\nu(i)}}[\psi_{j_i}, -] \right] \circ \partial_{x_t} \theta_t. \quad (5.21)$$

Here we write $\mathbf{x}^{\gamma_i} = x_1^{\gamma_{i1}} \dots x_m^{\gamma_{im}}$, $|\gamma_i| = \sum_{j=1}^m \gamma_{ij}$, $\underline{j} = \{1 \leq j_1 < \dots < j_m \leq n\}$, $\underline{z} = \{1 \leq z_1 < \dots < z_m \leq n\}$, $\underline{\gamma}$ ranges over sequences of length m in $\mathbb{N}^n \setminus \{0\}$, and $\underline{\nu}$ ranges over elements of the symmetric group \mathcal{S}_n . Here, ζ_a is the function

$$\zeta_a(\lambda_1, \dots, \lambda_m) = \sum_{\sigma \in \mathcal{S}_m} \frac{\lambda_1 \dots \lambda_m}{(a + \lambda_{\sigma(1)}) \dots (a + \lambda_{\sigma(1)} + \dots + \lambda_{\sigma(m)}),} \quad (5.22)$$

and $W_{j_i}(\gamma_i)$ is equal to the coefficient of x^{γ_i} in W_{j_i} .

Proof. See [47], Proposition 4.6. □

5.4 Configurations and Feynman calculus

We can see now, looking at the forms of σ_∞ , H_∞ , and $\exp(-\Xi)$, that the maps in the decoration $D_{T'}$ are really built from simple operators:

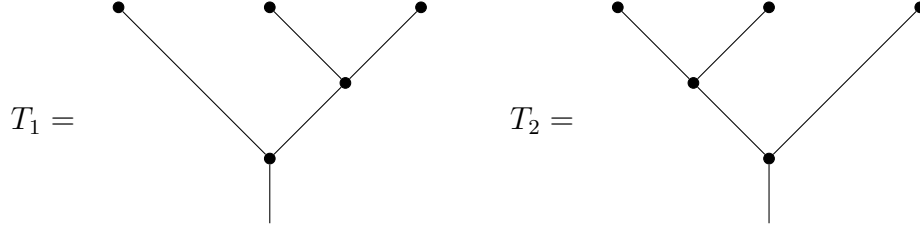
$$\sigma, \quad \partial_{x_i} \theta_i, \quad m_2 \circ ([\psi_{j_i}, -] \otimes \theta_{j_i}^*), \quad W^{j_i}(\gamma_i) \partial_{x_{z_{\nu(i)}}} (x^{\gamma_i}) \theta_{z_{\nu(i)}}[\psi_{j_i}, -], \quad (5.23)$$

such that

$$[\theta_i^*, \theta_i] = 1, \quad [\psi_i, \psi_i^*] = 1, \quad \theta_i \psi_i^* = -\psi_i^* \theta_i. \quad (5.24)$$

This motivates a change of perspective: rather than obtaining the higher products b_n by summing over trees $T \in \mathcal{T}_d$ decorated by the unwieldy maps of Proposition, we can pull all of the summations out of $\langle D_{T'} \rangle$ and instead sum over the *configurations* of the aforementioned simple operators on the trees. We will make this idea precise below, but first we will present a simple yet illustrative example.

Example 5.4.1. Let $W = x^3$ with the canonical decomposition $x \cdot x^2$. Let's look at the structure of the first of the higher A_∞ -products b_3 . First, observe that the underlying $\mathbb{Z}/2\mathbb{Z}$ -graded module of \mathcal{A}_W is isomorphic to $\mathbb{k} \oplus \mathbb{k}\psi$, with $|\psi| = 1$. There are two trees to consider:



The potential W has only one variable, giving a dramatic reduction in complexity:

$$\begin{aligned}\sigma_\infty &= \sigma - \partial_x(x^2)\theta[\psi, -] \circ \sigma, \\ H_\infty &= \partial_x\theta - \partial_x(x^2)\theta[\psi, -] \circ \partial_x\theta, \\ \exp(-\Xi) &= [\psi, -] \otimes \theta.\end{aligned}$$

There are 16 terms once we expand the summations in $\langle D_T \rangle$. The product b_3 is then

$$\begin{aligned}b_3 &= \pi \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes (\partial_x \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes \sigma))) \\ &\quad - \pi \circ m_2([\psi, -] \otimes \theta) \circ ((x\theta[\psi, -] \circ \sigma) \otimes (\partial_x \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes \sigma))) \\ &\quad + \pi \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes (\partial_x \circ m_2([\psi, -] \otimes \theta) \circ ((x\theta[\psi, -] \circ \sigma) \otimes \sigma))) \\ &\quad - \pi \circ m_2([\psi, -] \otimes \theta) \circ ((x\theta[\psi, -] \circ \sigma) \otimes (\partial_x \circ m_2([\psi, -] \otimes \theta) \circ ((x\theta[\psi, -] \circ \sigma) \otimes \sigma))) \\ &\quad \vdots \\ &\quad + \pi \circ m_2([\psi, -] \otimes \theta) \circ ((x\theta[\psi, -] \circ \sigma) \otimes ((x\theta[\psi, -] \circ \partial_x) \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes \sigma))).\end{aligned}$$

It is left to the reader to check that there is only one term which doesn't vanish, leaving

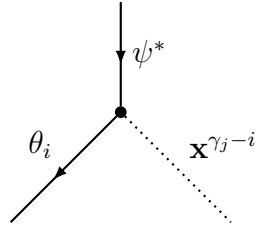
$$b_3 = -\pi \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes (\partial_x \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes (x\theta[\psi, -] \circ \sigma)))).$$

We can then check that this gives

$$b_3(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3) = \begin{cases} 1, & \Lambda_1 = \Lambda_2 = \Lambda_3 = \psi \\ 0, & \text{otherwise.} \end{cases}$$

Definition 5.4.1. Let $W \in \mathfrak{m}^3$ be a potential in $\mathbb{k}[x_1, \dots, x_n]$. Make a choice of decomposition $W = \sum_{i=1}^n x_i W_i$. Write $\mathbf{x}^\gamma = x_1^{\gamma_1} \dots x_n^{\gamma_n}$ for the monomials in W_i where γ ranges over a finite subset of \mathbb{N}^n .

- An *A-type interaction* is an operator



$$\theta_i \mathbf{x}^{\gamma-i} [\psi_j, -] \in \text{End}_{\mathbb{k}}(\mathcal{H}).$$

with $i \geq 1, j \leq n, \gamma \in \mathbb{N}^n$.

- A *B-type interaction* in an operator



$$\theta_t \partial_{x_t} \in \text{End}_{\mathbb{k}}(\mathcal{H}).$$

with $t \in \{1, \dots, n\}$.

- A *C-type interaction* is an operator



$$m_2([\psi_j, -] \otimes \theta_j^*) \in \text{Hom}_{\mathbb{k}}(\mathcal{H}^{\otimes 2}, \mathcal{H}).$$

with $j \in \{1, \dots, n\}$.

A *configuration* $D_{T,C}$ on a tree T is an assignment of *A*, *B*, and *C*-type interactions to the vertices of $A(T)$, as well as π to the root, subject to the following constraints:

- *A*-type interactions can only occur on non-root leaves and internal edges;
- there must be exactly one *B*-type interaction on each internal edge;

- C -type interactions can only occur at trivalent vertices;
- There may be up to n A -type or C -type interactions on a given vertex;
- The A -type interactions with indices i, j, γ are given the coefficient

$$-\frac{\gamma_i}{|\gamma|} W^j(\gamma),$$

and each B -type and C -type interaction appears with coefficient -1 .

We will write $\text{Con}(T)$ for the set of all configurations on T .

Remark 5.4.1. We will refer to the pictorial representations of the interactions as *Feynman interactions*. It must be stressed that we are not claiming any direct connection to Feynman diagrams in quantum field theory, and that the pictures are only intended as a tool to provide illustration. We think of the configurations as determining paths of ψ_i^* , x_i , and θ_i variables which propagate down the tree from the leaves to the root.

Lemma 5.4.1. *The denotation $\langle D_{T,C} \rangle$ of a decoration is a linear map $\mathcal{A}_W^{\otimes d} \rightarrow \mathcal{A}_W$.*

Proof. A priori the denotation $\langle D_{T,C} \rangle$ is a linear map $\mathcal{A}_W^{\otimes d} \rightarrow \mathcal{H}$. Recall that the map $\pi : \mathcal{H} \rightarrow \mathcal{H}$ acts as the identity on \mathcal{A}_W and sends each θ_i and x_i term to zero. This implies that the only non-zero denotations must consist of compositions of commutators $[\psi_i, -]$. Since the submodule $\mathcal{A}_W \subset \mathcal{H}$ is closed under such compositions, the image of $\langle D_{T,C} \rangle$ is in \mathcal{A}_W . □

Definition 5.4.2. Given $T \in \mathcal{T}_d$ and $C \in \text{Con}(T)$, we define the \mathbb{k} -linear operator

$$\mathcal{O}(T, C) : \mathcal{A}_W^{\otimes d} \rightarrow \mathcal{A}_W,$$

to be the denotation $\mathcal{O}(T, C) = \langle D_{T,C} \rangle$, defined without Koszul signs.

We will say that a configuration $C \in \text{Con}(T)$ *vanishes* if $\mathcal{O}(T, C)$ is the zero map.

Remark 5.4.2. For notational convenience, we represent a configuration C on a tree T by the following data at each non-root vertex of $A(T)$:

- An integer $m(v) \geq 0$;
- A subset $J(v) \subseteq \{1, \dots, n\}$ with $|J(v)| = m(v)$;

- If v is an input, or comes from an internal edge of T , a pair

$$(a_j(v), \gamma_j(v)) \in \{1, \dots, n\} \times \mathbb{N}^n$$

for each $j \in J(v)$, with $\gamma_j(v)_{a_j(v)} \geq 1$;

- If v comes from an internal edge, an integer $t(v) \in \{1, \dots, n\}$.

The integer $m(v)$ counts the number of A -type or C -type interactions which take place at v . The set $J(v)$ consists of all j indices appearing in interactions at v . On input vertices and vertices corresponding to internal edges the pair $(a_j(v), \gamma_j(v))$ are the indices which define the A -type interaction at v , and on internal edges the integer $t(v)$ determines the index the B -type interaction.

Example 5.4.2. In the previous example, the unique nonvanishing term was

$$-\pi \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes (\partial_x \circ m_2([\psi, -] \otimes \theta) \circ (\sigma \otimes (x\theta[\psi, -] \circ \sigma))))).$$

This corresponds to the configuration with $n = 1$,

$$\begin{aligned} m(i_1) &= m(i_2) = m(i_3) = 1, \\ J(i_3) &= J(v_1) = J(v_2) = \{1\}, \\ t(e) &= 1. \end{aligned}$$

In other words, the configuration with one A -type interaction at the third vertex, one C -type interaction on each internal vertex, and the mandatory B -type interaction on the one internal edge.

Definition 5.4.3. Let e be an internal edge of T , and define

$$N_C(e) = \sum_{v < e} \sum_{j \in J(v)} |\gamma_j(v)| - \sum_{z < e} m(z), \quad (5.25)$$

where v ranges over all inputs and internal edges of T which are strictly above e in T , and z ranges over all internal vertices of T which are strictly above e .

Remark 5.4.3. The integer $N_C(e)$ counts the number of x_i and θ_j terms which flow into e from above.

Definition 5.4.4. Given a tree T and a configuration $T \in \text{Con}(T)$, write

$$Z(T, C) = \prod_e \frac{\zeta_{N_C(e)}(|\gamma_{j_1}|, \dots, |\gamma_{j_m}|)}{N_C(e)} \in \mathbb{Q} \quad (5.26)$$

where e ranges over all internal edges in T with $N_C(e) \neq 0$, and $\{j_1, \dots, j_m\} = J(e)$. For an edge with $m(e) = 0$, so that $J(e) = \emptyset$, we set $\zeta_{N_C(e)} = 1$. If T has no internal edges we set $Z(T, C) = 1$.

Example 5.4.3. In the example we have used so far our tree has only one internal edge, and all A -type interactions are on the leaves. This means that $Z(T, C) = 1$.

If $m(e) = 1$ then $J(e) = \{j\}$ and we write $N_C(e) = N$, $\gamma_j(e) = \gamma$. Then

$$\zeta_N(|\gamma|) = \frac{|\gamma|}{N + |\gamma|}.$$

The reason for introducing $N_C(e)$ and $Z(T, C)$ is that they arise as the coefficients of $\mathcal{O}(T, C)$ in the A_∞ -higher products. From a combinatorial point of view, these two numbers are the most important pieces of information encoded in the higher products.

We are now in a position to write down a formula for the higher A_∞ -products in terms of configurations:

Theorem 5.4.1. (*Murfet, [47]*) *The A_∞ -products $b_d : \mathcal{A}_W[1]^{\otimes d} \rightarrow \mathcal{A}_W[1]$ can be written as*

$$b_d(\Lambda_1 \otimes \cdots \otimes \Lambda_d) = \sum_{T \in \mathcal{T}_d} \sum_{C \in \text{Con}(T)} (-1)^{Q(T, \Lambda_1 \otimes \cdots \otimes \Lambda_d)} Z(T, C) \cdot \mathcal{O}(T, C)(\Lambda_1 \otimes \cdots \otimes \Lambda_d), \quad (5.27)$$

for any $\Lambda_i \in \mathcal{A}_W[1]^{\otimes d}$, where, as in Proposition 5.3.1, the sign is given by

$$Q(T, \Lambda_1 \otimes \cdots \otimes \Lambda_d) = 1 + \sum_{1 \leq i < j \leq d} \tilde{\Lambda}_i \tilde{\Lambda}_j + \sum_{i=1}^d \tilde{\Lambda}_i C_i. \quad (5.28)$$

In conclusion, (\mathcal{A}_W, b) is the A_∞ -minimal model of $\text{End}_R(\mathbb{k}_W^{stab})$, and in particular there is an equivalence of categories

$$\text{hmf}(\mathbb{k}[[\mathbf{x}]], W) \cong \text{Perf}_\infty(\mathcal{A}_W, b), \quad (5.29)$$

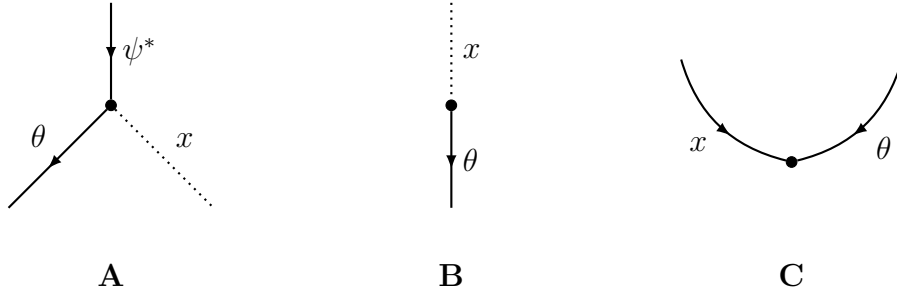
by Theorem 2.4.2.

Lemma 5.4.2. *Given any potential W , the associated A_∞ -algebra must have a product*

$$b_2(\Lambda_1 \otimes \Lambda_2) = (-1)^{\tilde{\Lambda}_1 + 1} \Lambda_1 \cdot \Lambda_2. \quad (5.30)$$

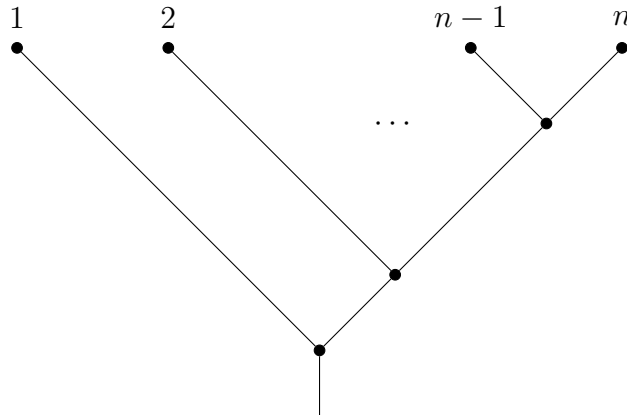
Proof. This is precisely the product obtained from the empty configuration on the tree with two leaves. □

Example 5.4.4. We will now compute the A_∞ -minimal model for the A_d hypersurface singularity $W = x^d = x \cdot x^{d-1}$. Since $n = 1$ the underlying graded vector space is $\mathbb{k}[\psi]/\psi^2 \cong \mathbb{k} \oplus \mathbb{k}\psi$. The Feynman interactions in this case are



Suppose $T \in \mathcal{T}_n$ and C is a nonvanishing configuration on T . Observe first that we cannot have any A -type interactions on the internal edges, because the θ term produced would annihilate with the θ produced by the B -type interaction. This forces all A -type interactions to the leaves of T . Next consider an arbitrary internal vertex v in T . Then the sub-tree T_v entering v from the left must consist of a single external edge. Otherwise, the edge connecting T_v to v must produce a θ term which annihilates at the subsequent edge.

The same argument then tells us that there can only be one A -type interaction, and that it must be the rightmost leaf of T . Thus, T must be a *comb-shaped* tree of the form:



This tree will have $n - 2$ internal edges. Since all x variables must be annihilated before reaching the root, we must have $n = d$. This is the only nonvanishing configuration. To feed the C -type interactions at each vertex and the A -type interaction on the

rightmost leaf, the input pattern must be $\psi^* \otimes \cdots \otimes \psi^*$ with d factors. The resulting higher products are therefore

$$\begin{aligned} b_2(\Lambda_1 \otimes \Lambda_2) &= (-1)^{\bar{\Lambda}_1+1} \Lambda_1 \cdot \Lambda_2 \\ b_d(\psi^* \otimes \cdots \otimes \psi^*) &= 1, \end{aligned}$$

for any $\Lambda_1, \Lambda_2 \in \mathcal{A}_W[1]$.

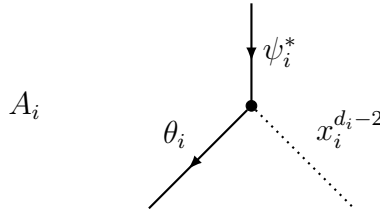
5.5 Bounds on the higher products

In this section we will address the computability of the construction in section 5.4.

Lemma 5.5.1. *Let T be a binary rooted tree with d leaves equipped with a nonvanishing configuration coming from a linear potential $W = \sum_{i=1}^n x_i^{d_i}$. Then*

$$d \leq \frac{2}{1 - n + \sum_{i=1}^n \frac{2}{d_i}}. \quad (5.31)$$

Proof. The potential W produces n A -type interactions:



Observe that each A_i -type interaction requires d_i ψ_i^* inputs: one for the A_i -interaction itself, one for the C -type interactions removing the θ_i produced by the A_i -interaction, and one for each of the θ_i 's produced by the B -types necessary to eliminate the x_i variables. Let ε_i be the number of A_i interactions. Then, since $\psi_i^* \psi_i^* = 0$, $d_i \varepsilon_i \leq d$. Moreover, since each A_i requires $d_i - 2$ internal edges, we have

$$\sum_{i=1}^n (d_i - 2) \varepsilon_i = d - 2. \quad (5.32)$$

It follows that

$$d - 2 = \sum_{i=1}^n (d_i - 2) \varepsilon_i \leq \sum_{i=1}^n \frac{d_i - 2}{d_i} d,$$

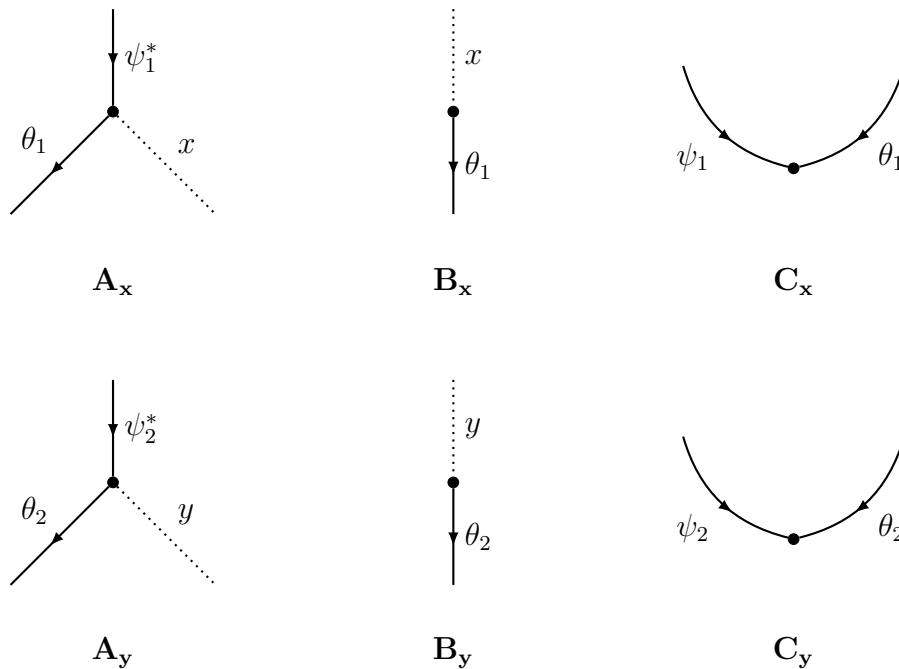
from which the bound follows. \square

Observe that if $n > 2$ or $\deg(W) > 3$ then the bound no longer provides any information. For instance, if $W = x^4 + y^4$ then Equation 5.27 becomes $0 \leq 2$, which is tautological. However, this is the only bound on the number of leaves available to us from the data of a configuration. This leads us to suspect that if either the degree of W is greater than 3, or there are more than three variables, then there will be an *infinite* number of higher products.

5.6 The A_∞ -algebra associated to $W = x^3 - y^3$

Let $W = x^3 - y^3 = x \cdot x^2 - y \cdot y^2$. In this section we will compute the A_∞ -minimal model associated to W , which is original in this thesis.

By Lemma 5.5.1, only trees with $q \leq 6$ leaves can contribute configurations to the higher products. Moreover, if $q = 5$ then there will be precisely three internal edges, requiring three A -type interactions to produce x and y variables. This requires either six ψ_1^* inputs or six ψ_2^* inputs, which is impossible. Thus we eliminate $q \in \{2, 3, 4, 6\}$. The Feynman interactions for W are :

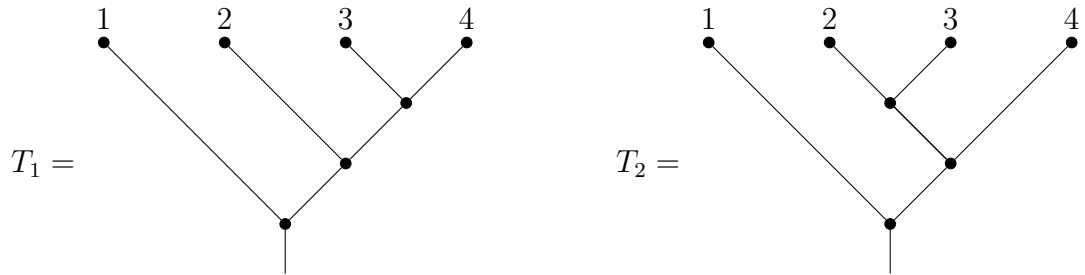


The $q = 2$ case will be the standard suspended multiplication b_2 , so all that remains is the $q \in \{3, 4, 6\}$ cases.

$q = 3$: There is only one tree with trivial leftmost branch, and, since there is only one internal edge, any non-zero configuration must have exactly one A -type interaction. Since this lies on the rightmost input, the only choice is whether the A -type interaction is A_x or A_y . The configuration is completely determined by this choice of A -type interaction, so we have

$$m_3(x, y, z) = [\psi_1^* + \psi_2^*, x] \otimes [\psi_1^* + \psi_2^*, y] \otimes [\psi_1^* + \psi_2^*, z].$$

$q = 4$: There are five binary rooted trees with four leaves, but only two have a trivial leftmost branch:



Numerical simulations reveal that T_1 has 16 nonvanishing configurations, and T_2 has 4 nonvanishing configurations. These trees have two internal edges, so there must be two A -type interactions. Moreover, we must have one A -type interaction of each type, since otherwise we would require six ψ_i^* inputs to feed the A and C type interactions.

We first consider T_2 . We cannot place A -type interactions on either of the edges, as otherwise there would be no chance to remove the x or y without causing θ_i terms to annihilate. Moreover, we must have at least one A -type interaction on 3, otherwise the θ_i would annihilate at the next edge. For the same reason we also cannot have A -type interactions on both 2 and 3.

Suppose we have two A -type interactions at 3. Then there must be two C -type interactions on the next vertex, otherwise we would have two θ_i terms entering the last edge, leading to annihilation. Next, we can choose to have either a B_x or B_y interaction on the first edge and then have the complementary B -type interaction on the second edge. Finally, we must have two C -type interactions at the bottom vertex. This gives

$$m_4(\psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes x) = 2x.$$

The next two configurations must have A -type interactions at 3 and 4. Since the Feynman rules are unchanged by swapping x and y , it makes no difference

whether we place A_x or A_y at 3. These configurations have no more choices, and therefore contribute

$$\begin{aligned} m_4(\psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \otimes \psi_2^*) &= 1, \\ m_4(\psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_2^* \otimes \psi_1^*) &= 1. \end{aligned}$$

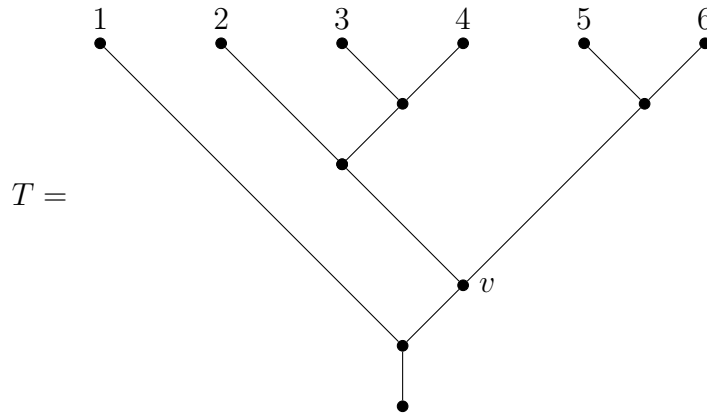
The configurations on T_1 are more numerous, so we will leave it to the reader to check that the configurations contribute

$$\begin{aligned} m_4(\psi_2^* \otimes \psi_1^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^*) &= 1, \\ m_4(\psi_1^* \otimes \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^*) &= 1, \\ m_4(\sigma(\psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \otimes \psi_1^* \psi_2^*)) &= 2, \end{aligned}$$

where $\sigma \in \mathcal{S}_4 \setminus \{(23), (14)\}$ acts by permuting the tensors:

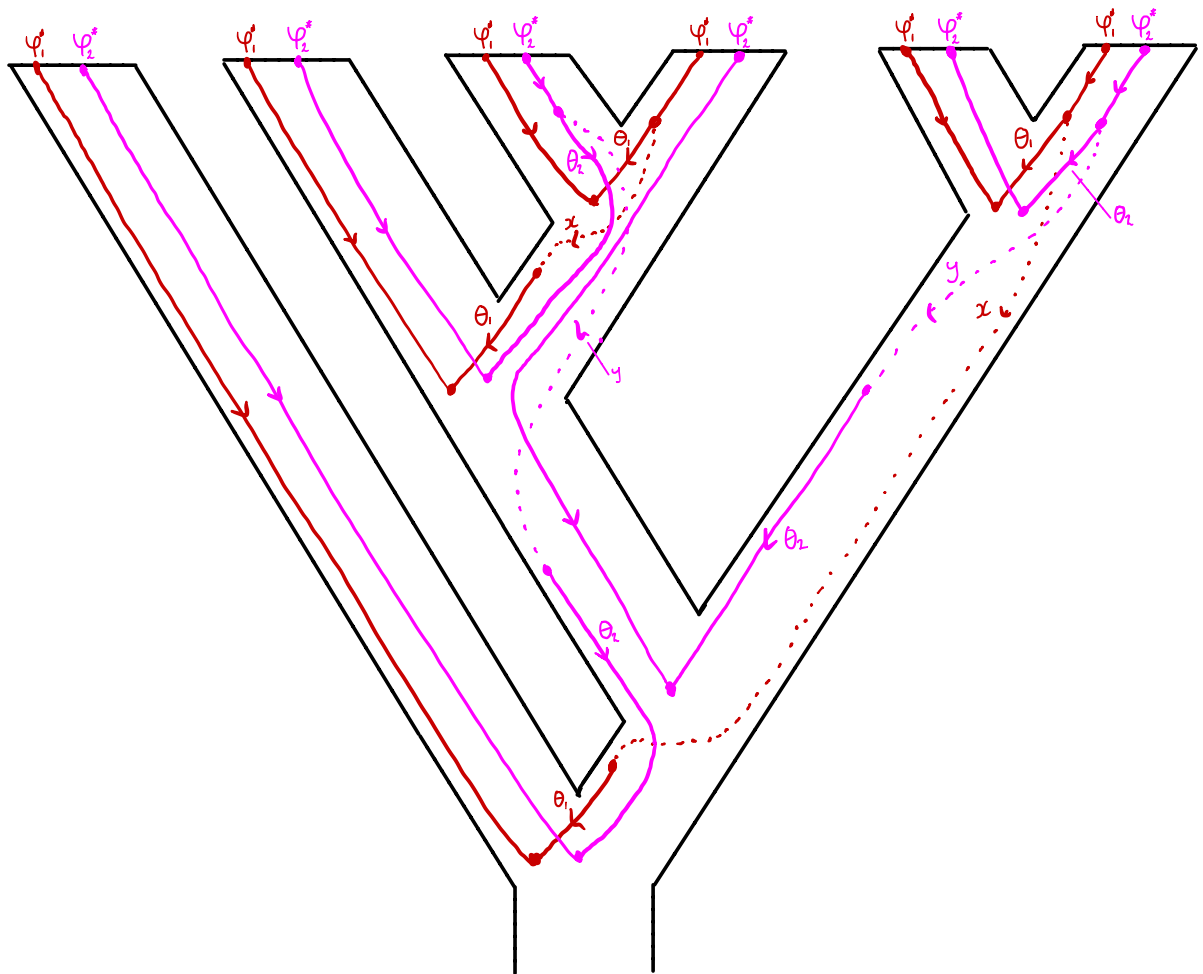
$$\sigma(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)} \otimes v_{\sigma^{-1}(4)}$$

$q = 6$: Numerical computations show that there is only one tree to check:



There are four nonvanishing configurations. As a starting point, observe that we must have two A -type interactions on the $(2, 3, 4)$ sub-tree and two on the $(5, 6)$ sub-tree. Next, note that we cannot have any A -type interactions on 2 or 5, as both would result in a $\theta_1 \theta_2$ term appearing at v and annihilating at the subsequent edge. Likewise we cannot have any A -type interactions on internal edges. We must therefore have two A -type interactions on 6, and we are free to choose which is used by the B -type interaction first.

On the $(3, 4)$ sub-tree we can either have one A -type on each leaf or have both on 4. Both cases have two choices, namely which variable is used in the first B -type

Figure 5.1: A propagation pattern on T .

Chapter 6

A_∞ -minimal models of modules and bimodules

In this chapter we will extend the methods of Chapter 5 to provide original examples of a family of A_∞ -modules and A_∞ -bimodules.

6.1 A_∞ -minimal models of modules

In this section we will extend the construction of the minimal model of the DG-algebra $\text{End}_{\mathbb{k}}(\mathbb{k}_W^{stab})$ to an explicit description of the functor

$$\Psi : \text{hmf}(W) \rightarrow \text{Perf}_\infty(\mathcal{A}_W) \quad (6.1)$$

of Theorem 2.4.2. Recall that this sends a matrix factorisation X of W to the minimal A_∞ -module, over \mathcal{A}_W ,

$$H^* \text{Hom}_R(\mathbb{k}_W^{stab}, X).$$

As in the case of the construction of the A_∞ -algebra minimal model, we will not actually compute the A_∞ -structure on cohomology, but rather on a homotopically equivalent model. Fix a matrix factorisation (X, d_X) of W and assume that $d_X(X) \subseteq (x_1, \dots, x_n)X$.

Let T be the DG-category with one object \bullet which has endomorphism DG-algebra $(\bigwedge F_\theta, 0)$, where $F_\theta = \bigoplus_{i=1}^n \mathbb{k}\theta_i$. Consider the DG-subcategory of $\text{mf}(W)$ given by

$$\mathcal{C} : \begin{array}{ccc} & \text{Hom}(\mathbb{k}_W^{stab}, X) & \\ & \xrightarrow{\quad} & \\ \mathbb{k}_W^{stab} & \xleftrightarrow{\quad} & X \\ & \xleftarrow{\quad} & \\ & \text{Hom}(X, \mathbb{k}_W^{stab}) & \\ \uparrow & & \uparrow \\ \text{End}(\mathbb{k}_W^{stab}) & & \text{End}(X) \end{array}$$

We form the tensor product

$$T \otimes_{\mathbb{k}} \mathcal{C} : \begin{array}{ccc} \mathbb{k}_W^{stab} & \begin{array}{c} \xleftarrow{\wedge F_\theta \otimes_{\mathbb{k}} \text{Hom}(\mathbb{k}_W^{stab}, X)} \\ \xrightarrow{\wedge F_\theta \otimes_{\mathbb{k}} \text{Hom}(X, \mathbb{k}_W^{stab})} \end{array} & X \\ \uparrow & & \uparrow \\ \wedge F_\theta \otimes_{\mathbb{k}} \text{End}(\mathbb{k}_W^{stab}) & & \wedge F_\theta \otimes_{\mathbb{k}} \text{End}(X) \end{array}$$

This is still a DG-category with the action

$$\begin{array}{c} (\wedge F_\theta \otimes_{\mathbb{k}} \text{End}(\mathbb{k}_W^{stab})) \otimes_{\mathbb{k}} (\wedge F_\theta \otimes_{\mathbb{k}} \text{Hom}(\mathbb{k}_W^{stab}, X)) \\ \downarrow \cong \\ (\wedge F_\theta \otimes_{\mathbb{k}} \wedge F_\theta) \otimes_{\mathbb{k}} (\text{End}(\mathbb{k}_W^{stab}) \otimes_{\mathbb{k}} \text{Hom}(\mathbb{k}_W^{stab}, X)) \\ \downarrow \\ \wedge F_\theta \otimes_{\mathbb{k}} \text{Hom}(\mathbb{k}_W^{stab}, X) \end{array}$$

After applying the construction of Section 5.3 we obtain strong deformations retracts

$$\begin{array}{c} h_\infty \hookrightarrow (\wedge F_\theta \otimes \text{End}(\mathbb{k}_W^{stab}), d_{\text{End}}) \xrightarrow{\quad} (\text{End}_{\mathbb{k}}(\wedge F_\psi), 0) \\ \hat{h}_\infty \hookrightarrow (\wedge F_\theta \otimes \text{Hom}(\mathbb{k}_W^{stab}, X), d_{\text{Hom}}) \xrightarrow{\quad} (\underline{\text{Hom}}(\mathbb{k}_W^{stab}, X), d_{\underline{\text{Hom}}}) \end{array}$$

where $\underline{\text{Hom}}(A, B) := \text{Hom}(A, B) \otimes_{\mathbb{k}[x]} \mathbb{k}$, and the tensor product is taken with all x_i acting as 0 on \mathbb{k} . All of the maps here are the same as Theorem 5.3.1, with the exception of the homotopy \hat{h}_∞ and the morphism $\hat{\sigma}_\infty$ which are given by

$$\hat{h}_\infty = \sum_{m=0}^{\infty} (-1)^m (H d_{\text{Hom}})^m H, \quad (6.2)$$

$$\hat{\sigma}_\infty = \sum_{m=0}^{\infty} (-1)^m (H d_{\text{Hom}})^m \sigma, \quad (6.3)$$

where d_{Hom} is the differential on $\text{Hom}(\mathbb{k}_W^{stab}, X)$ given by

$$d_{\text{Hom}}(\alpha) = d_X \circ \alpha - (-1)^{|\alpha|} \sum_{i=1}^n W_i \alpha \circ \psi_i - (-1)^{|\alpha|} \sum_{i=1}^n x_i \alpha \circ \psi_i^*. \quad (6.4)$$

Applying the Minimal Model Theorem of Section 3.5 yields a minimal A_∞ -category quasi-isomorphic to $T \otimes_{\mathbb{k}} \mathcal{C}$. We denote this A_∞ -category by

$$\begin{array}{ccc}
 \bullet & \xrightarrow{(\underline{\text{Hom}}(\mathbb{k}_W^{stab}, X), \{\nu_n\})} & \bullet \\
 \uparrow \cup & & \\
 (\underline{\text{End}}(\mathbb{k}_W^{stab}), \{b_n\}) & &
 \end{array}$$

where $\{\nu_n\}$ denote the induced A_∞ -module higher products

$$\nu_n : \underline{\text{End}}(\mathbb{k}_W^{stab})^{\otimes(n-1)} \otimes \underline{\text{Hom}}(\mathbb{k}_W^{stab}, X) \rightarrow \underline{\text{Hom}}(\mathbb{k}_W^{stab}, X), \quad (6.5)$$

and the $\{b_n\}$ are the A_∞ -algebra higher products of Theorem 5.4.1. The higher products $\{\nu_n\}$ are computed using trees in the usual way. The decoration D_T given by the same assignment of modules and maps as in Lemma 5.3.1, with the exception of the rightmost branch which is decorated by the following maps:

- $\hat{\sigma}_\infty$ for the input;
- \hat{h}_∞ for each internal edge,;
- $r_2 \exp(-\Xi)$ for each internal vertex, where $\Xi = \sum_{i=1}^n [\psi_i, -] \otimes \theta_i^*$, and r_2 is the action

$$\left(\bigwedge F_\theta \otimes \underline{\text{End}}(\mathbb{k}_W^{stab}) \right) \otimes \left(\bigwedge F_\theta \otimes \underline{\text{Hom}}(\mathbb{k}_W^{stab}, X) \right) \rightarrow \bigwedge F_\theta \otimes \underline{\text{Hom}}(\mathbb{k}_W^{stab}, X),$$

given above.

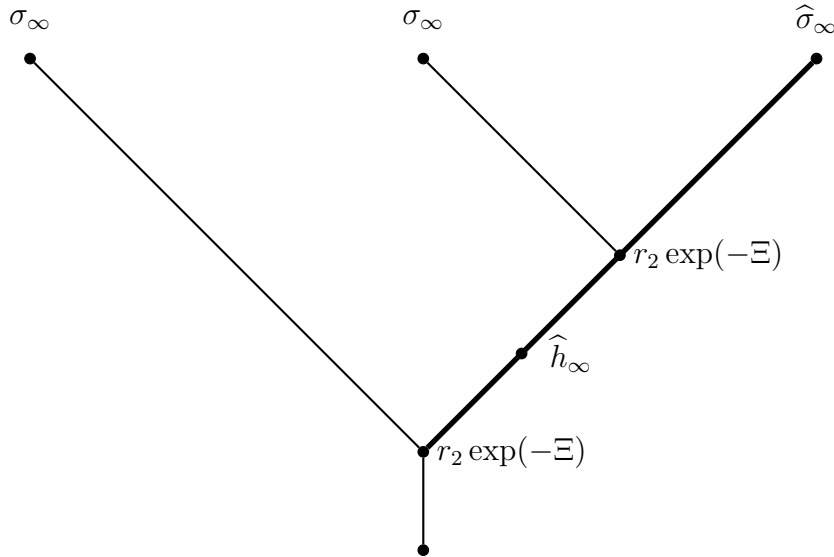


Figure 6.1: An example of a decorated tree

The configurations that we use to compute the higher products will also be the same as those in Section 5.4, with the exception of the rightmost channel. In addition to the usual B , and C type interactions will have the following *exotic* interactions arising from d_{Hom} :

1. $\theta_i \partial_{x_i}(d_{\text{Hom}}) \circ (-)$;
2. $\theta_i \partial_{x_i}(W_j) \circ (-) \circ \psi_j$;
3. $\theta_i \circ (-) \circ \psi_i^*$.

We can write the underlying graded module of $\text{Hom}(\mathbb{k}_W^{\text{stab}}, X)$ as

$$\begin{aligned} \text{Hom}(\mathbb{k}_W^{\text{stab}}, X) &\cong \left(\bigwedge F_\psi \right)^* \otimes_{\mathbb{k}} X \\ &\cong \bigwedge F_{\psi^*} \otimes_{\mathbb{k}} X. \end{aligned}$$

Using this presentation, a contraction $\psi_i^* \in \text{End}(\mathbb{k}_W^{\text{stab}})$ acts as

$$\psi_i^* \cdot \left(\psi_{i_1}^* \wedge \cdots \wedge \psi_{i_p}^* \otimes x \right) = (-1)^p \psi_i^* \wedge \psi_{i_1}^* \wedge \cdots \wedge \psi_{i_p}^* \otimes x, \quad (6.6)$$

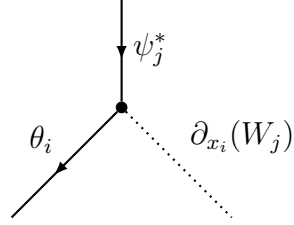
$$\psi_j \cdot \left(\psi_{i_1}^* \wedge \cdots \wedge \psi_{i_p}^* \otimes x \right) = (-1)^p \psi_j \lrcorner (\psi_{i_1}^* \wedge \cdots \wedge \psi_{i_p}^*) \otimes x. \quad (6.7)$$

Therefore pre-composing with ψ_i^* acts as a wedge product, and pre-composing with ψ_i acts as a contraction. We can therefore present the exotic interactions diagrammatically as follows:

- Exotic A -type interactions of the first kind which depends on the structure of d_X :

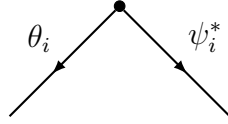


- The standard A -type interactions



$$\theta_i \partial_{x_i}(W_j) \psi_j \lrcorner (-).$$

- Exotic A -type interaction of the second kind which produces θ and ψ_i^* from nothing



$$\theta_i \psi_i^* \wedge (-).$$

Definition 6.1.1. Given a tree $T \in \mathcal{T}_n$, an *exotic configuration* on T is consists of a configuration on T (as in Definition 5.4.1) together with the assignment of exotic interactions to the leaf and edges of the rightmost branch.

It is clear from the deformation retracts that

$$\bigwedge F_\theta \otimes_{\mathbb{k}} H^* \text{Hom}_R(\mathbb{k}_W^{stab}, X) \cong \underline{\text{Hom}}(\mathbb{k}_W^{stab}, X), \quad (6.8)$$

so all that remains is to find the summand of $\underline{\text{Hom}}(\mathbb{k}_W^{stab}, X)$ which corresponds to $\mathbb{k} \cdot 1 \otimes H^* \text{Hom}_R(\mathbb{k}_W^{stab}, X) \subset \bigwedge F_\theta \otimes_{\mathbb{k}} H^* \text{Hom}_R(\mathbb{k}_W^{stab}, X)$. We use the Clifford action of $\text{End}(\bigwedge F_\theta)$ on $\underline{\text{Hom}}(\mathbb{k}_W^{stab}, X)$ with

$$\gamma_i = \text{At}_i = -\partial_{x_i}(d_{\text{Hom}}), \quad \gamma_i^\dagger(\alpha) = -(-1)^{|\alpha|} \alpha \circ \psi_i. \quad (6.9)$$

If we assume that $W \in \mathfrak{m}^3$ then this simplifies to

$$\gamma_i(\alpha) = -\partial_{x_i}(d_X) + (-1)^{|\alpha|} \alpha \circ \psi_i^* \quad (6.10)$$

We can therefore identify the intersection of the $\ker(\gamma_i^\dagger)$ as i varies with $H^* \text{Hom}_R(\mathbb{k}_W^{stab}, X)$,

giving an isomorphism

$$\begin{aligned}
 H^* \mathrm{Hom}_R(\mathbb{k}_W^{stab}, X)[n] &\cong \bigcap_{i=1}^n \ker(\gamma_i^\dagger) \\
 &= \bigcap_{i=1}^n \ker(\psi_{i \lrcorner}(-)) \\
 &= \mathbb{k} \cdot 1 \otimes_{\mathbb{k}} \underline{X} \\
 &\cong \underline{X}.
 \end{aligned} \tag{6.11}$$

Combining this with the split idempotent of Section 5.3 we obtain higher products

$$\nu_n : \mathcal{A}_W[1]^{\otimes(n-1)} \otimes \underline{X}[1] \rightarrow \underline{X}[1]. \tag{6.12}$$

The outcome of this construction is the following theorem:

Theorem 6.1.1. *Let \mathcal{A}_W denote the A_∞ -minimal model of $\mathrm{End}(\mathbb{k}_W^{stab})$. The A_∞ -module corresponding a matrix factorisation $X[n]$ has underlying vector space*

$$\underline{X} \cong H^* \mathrm{Hom}_R(\mathbb{k}_W^{stab}, X[n]),$$

and higher products

$$\nu_n : \mathcal{A}_W[1]^{\otimes(n-1)} \otimes \underline{X}[1] \rightarrow \underline{X}[1], \tag{6.13}$$

given by summing over exotic configurations on trees $T \in \mathcal{T}_n$.

Example 6.1.1. Let $W = x^d$ and let X be the Koszul matrix factorisation

$$\left(\bigwedge \mathbb{k}\xi \otimes_{\mathbb{k}} R, d_X = x^i \xi^* + x^{d-i} \xi \right).$$

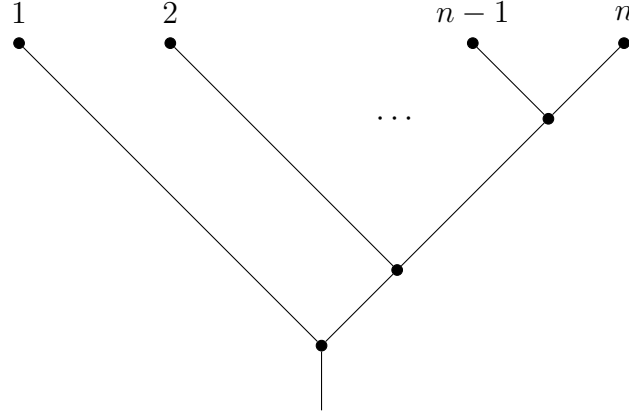
Then $\partial_x(d_X) = ix^{i-1}\xi^* + (d-i)x^{d-i-1}\xi$, and the exotic A -type interaction of the first becomes two interactions,



The exotic interaction of the second kind does not contribute, because it does not produce an x variable for the subsequent B -type interaction. If an x variable were to

enter the rightmost branch from an incoming sub-tree the it would carry with it a θ term, thus anniilating at the subsequent internal edge.

The same arguments of Example 5.4.4 show that nonvanishing configurations can only appear on trees with the shape



We can only have one exotic A -type interaction on this tree, and it must be at the rightmost vertex. It follows that we have two configurations to consider, one for each of the exotic A -type interactions, and we can read off the higher products as

$$\begin{aligned} m_2(x \otimes y) &= (-1)^{x+1} x \cdot y, \\ m_i(\psi^* \otimes \cdots \otimes \psi^* \otimes x) &= \xi \lrcorner x, \\ m_{d-i}(\psi^* \otimes \cdots \otimes \psi^* \otimes x) &= \xi \wedge x. \end{aligned}$$

6.2 A_∞ -minimal models of bimodules

As a result of Theorem 2.4.2 and Corollary 4.3.1, we can completely understand the homotopy category of matrix factorisations $\text{hmf}(W)$ by computing A_∞ -modules over the minimal model of $\text{End}(\mathbb{k}_W^{\text{stab}})$. This result has one major drawback: the computational difficulty rises steeply as we make W more complicated. As a stark example, we have only been successful in computing by hand the minimal model of one potential with two variables, namely $W = x^3 - y^3$ (see Section 5.6). Surprisingly it is often possible to explicitly compute A_∞ -module and A_∞ -bimodule structures arising from matrix factorisations, even when we know little about the higher products of the A_∞ -algebras involved. In this section will will show how one can use the construction of the previous section to produce an A_∞ -bimodule structure on a matrix factorisation of $V - W$.

Lemma 6.2.1. *Let $V, W \in \mathbb{k}[x_1, \dots, x_n]$ be potentials. Then,*

1. $(\mathbb{k}_W^{\text{stab}})^\vee \cong \mathbb{k}_{-W}^{\text{stab}}[n]$ as matrix factorisations of $-W$.

2. $\mathbb{k}_W^{stab} \otimes \mathbb{k}_V^{stab} \cong \mathbb{k}_{W+V}^{stab}$ as matrix factorisations of $V + W$.

Proof. The first statement is a direct consequence of Lemma 4.2.1, and the second follows by direct computation. \square

Let X be a matrix factorisation of $W - V$. Using the above lemma we have

$$\begin{aligned}
\mathrm{Hom}(\mathbb{k}_{W-V}^{stab}, X) &\cong \mathrm{Hom}(\mathbb{k}_W^{stab} \otimes \mathbb{k}_{-V}^{stab}, X) \\
&\cong \mathrm{Hom}(\mathbb{k}_W^{stab} \otimes (\mathbb{k}_V^{stab}[n])^\vee, X) \\
&\cong \mathrm{Hom}(\mathbb{k}_W^{stab}, \mathrm{Hom}((\mathbb{k}_V^{stab}[n])^\vee, X)) \\
&\cong \mathrm{Hom}(\mathbb{k}_W^{stab}, \mathbb{k}_V^{stab}[n] \otimes X) \\
&\cong (\mathbb{k}_W^{stab})^\vee \otimes \mathbb{k}_V^{stab}[n] \otimes X.
\end{aligned} \tag{6.14}$$

As a consequence, we have a $\mathrm{End}(\mathbb{k}_V^{stab})$ - $\mathrm{End}(\mathbb{k}_W^{stab})$ -DG-bimodule structure given by

$$((\psi_{i_1} \dots \psi_{i_p})^* \otimes \phi_{i_1} \dots \phi_{i_q} \otimes x) \cdot \psi_i^* = (-1)^{q+|x|} (\psi_{i_1} \dots \psi_{i_p} \psi_i)^* \otimes \phi_{i_1} \dots \phi_{i_q} \otimes x \tag{6.15}$$

$$((\psi_{i_1} \dots \psi_{i_p})^* \otimes \phi_{i_1} \dots \phi_{i_q} \otimes x) \cdot \psi_j = (-1)^{q+|x|} (\psi_j \lrcorner (\psi_{i_1} \dots \psi_{i_p}))^* \otimes \phi_{i_1} \dots \phi_{i_q} \otimes x \tag{6.16}$$

$$\phi_j \cdot ((\psi_{i_1} \dots \psi_{i_p})^* \otimes \phi_{i_1} \dots \phi_{i_q} \otimes x) = (-1)^p (\psi_{i_1} \dots \psi_{i_p})^* \otimes \phi_i \phi_{i_1} \dots \phi_{i_q} \otimes x \tag{6.17}$$

$$\phi_i^* \cdot ((\psi_{i_1} \dots \psi_{i_p})^* \otimes \phi_{i_1} \dots \phi_{i_q} \otimes x) = (-1)^p (\psi_{i_1} \dots \psi_{i_p})^* \otimes \phi_i^* \lrcorner (\phi_{i_1} \dots \phi_{i_q}) \otimes x \tag{6.18}$$

for any $\psi_i^*, \psi_j \in \mathrm{End}(\mathbb{k}_W^{stab})$ and any $\phi_i^*, \phi_j \in \mathrm{End}(\mathbb{k}_V^{stab})$.

We let T be the category with one object and endomorphism DG-algebra $\bigwedge F_\theta$ and let \mathcal{D} be the category

$$\mathcal{D} : \quad \begin{array}{ccc} & \bullet & \xrightarrow{\mathrm{Hom}(\mathbb{k}_{W-V}^{stab}, X)} \bullet \\ & \uparrow \cup & \uparrow \cup \\ & \mathrm{End}(\mathbb{k}_V^{stab}) & \mathrm{End}(\mathbb{k}_V^{stab}) \end{array}$$

We take the tensor product with T to obtain

$$T \otimes_{\mathbb{k}} \mathcal{D} : \quad \begin{array}{ccc} & \bullet & \xrightarrow{\bigwedge F_\theta \otimes_{\mathbb{k}} \mathrm{Hom}(\mathbb{k}_{W-V}^{stab}, X)} \bullet \\ & \uparrow \cup & \uparrow \cup \\ & \bigwedge F_\theta \otimes_{\mathbb{k}} \mathrm{End}(\mathbb{k}_V^{stab}) & \bigwedge F_\theta \otimes_{\mathbb{k}} \mathrm{End}(\mathbb{k}_V^{stab}) \end{array}$$

After applying the homological perturbation of Section 5.3 we obtain strong deformations retracts

$$\begin{aligned} h_\infty^W &\hookrightarrow (\bigwedge F_\theta \otimes \text{End}(\mathbb{k}_W^{stab}), d_{\text{End}}^W) \xleftarrow{\quad} (\text{End}_{\mathbb{k}}(\bigwedge F_\psi), 0), \\ h_\infty^V &\hookrightarrow (\bigwedge F_\theta \otimes \text{End}(\mathbb{k}_V^{stab}), d_{\text{End}}^V) \xleftarrow{\quad} (\text{End}_{\mathbb{k}}(\bigwedge F_\psi), 0), \\ \widehat{h}_\infty &\hookrightarrow (\bigwedge F_\theta \otimes \text{Hom}(\mathbb{k}_{W-V}^{stab}, X), d_{\text{Hom}}) \xleftarrow{\quad} (\underline{\text{Hom}}(\mathbb{k}_W^{stab}, X), d_{\underline{\text{Hom}}}). \end{aligned}$$

Combining the Minimal Model Theorem and the split idempotent of the previous section together with calculations 6.11 and 6.14 gives the A_∞ -category

$$\mathcal{A} : \begin{array}{ccc} \bullet & \xrightarrow{\quad \underline{X} \quad} & \bullet \\ \uparrow \cup & & \uparrow \cup \\ (\mathcal{A}_V, \{b_n^V\}) & & (\mathcal{A}_W, \{b_n^W\}) \end{array}$$

which encodes the A_∞ -bimodule higher products

$$\nu_{r,s} : \mathcal{A}_V[1]^{\otimes r} \otimes \underline{X}[1] \otimes \mathcal{A}_W[1]^{\otimes s} \rightarrow \underline{X}[1],$$

giving the \mathcal{A}_V - \mathcal{A}_W - A_∞ -bimodule structure on X .

Let $T \in \mathcal{T}_n$ and choose a non-root leaf v of T whose path to the root we call the *trunk* of T . The decoration D_T that we use to compute the A_∞ -products is given as follows:

- If T' is a sub-tree off the left of the trunk we place h_∞^V on internal edges, $m_2 \exp(-\Xi)$ on each internal vertex, and σ_∞^V on each leaf. At the vertex where T' meets the trunk we place the operator $r_2^V \exp(-\Xi)$, where r_2^V is the operator

$$\left(\bigwedge F_\theta \otimes \text{End}(\mathbb{k}_V^{stab}) \right) \otimes \left(\bigwedge F_\theta \otimes \text{Hom}(\mathbb{k}_{W-V}^{stab}, X) \right) \rightarrow \bigwedge F_\theta \otimes \text{Hom}(\mathbb{k}_{W-V}^{stab}, X) \quad (6.19)$$

given in equations (6.14), (6.15);

- If T' is a sub-tree off the left of the trunk we place h_∞^W on internal edges, $m_2 \exp(-\Xi)$ on each internal vertex, and σ_∞^W on each leaf. At the vertex where T' meets the trunk we place the operator $r_2^W \exp(-\Xi^s)$, where

$$\Xi^s = \sum_{i=1}^n \theta_i^* \otimes [\psi, -], \quad (6.20)$$

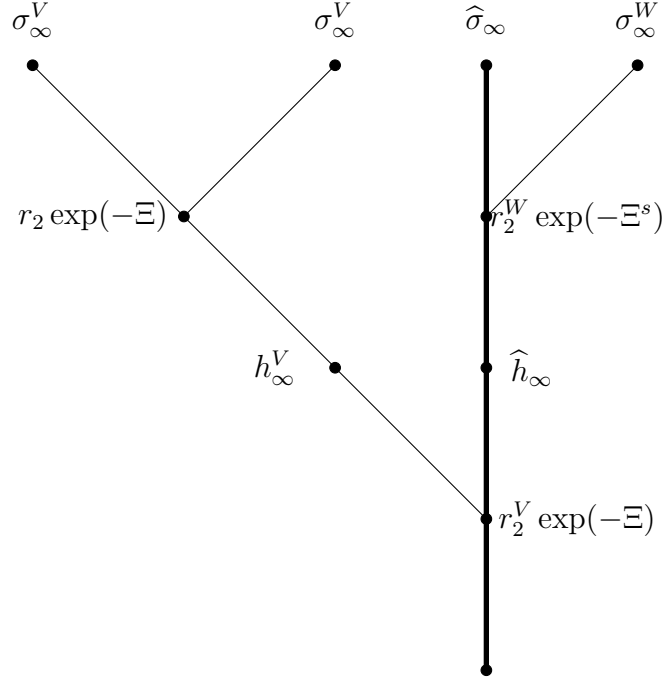


Figure 6.2: A decoration on a tree with the trunk in bold.

and r_2^W is the operator

$$\left(\bigwedge F_\theta \otimes \text{Hom}(\mathbb{k}_{W-V}^{stab}, X) \right) \otimes \left(\bigwedge F_\theta \otimes \text{End}(\mathbb{k}_V^{stab}) \right) \rightarrow \bigwedge F_\theta \otimes \text{Hom}(\mathbb{k}_{W-V}^{stab}, X) \quad (6.21)$$

given in equations (6.16), (6.17);

- We place \hat{h}_∞ on the internal edges of the trunk, and $\hat{\sigma}_\infty$ on v .

As a result, the configurations determining the higher products have standard A , B , and C type interactions of the trunk, exotic interactions on the trunk, and reversed C -type interactions on vertices of the trunk meeting branches which come from the right.

6.3 \mathcal{A}_{xd} - \mathcal{A}_{ye} -bimodules

In this section we will give an original example of a family of A_∞ -bimodule minimal models associated to permutation defects. Let $R = \mathbb{k}[x, y]$. Recall from Example 4.1.2 that we can factor

$$W = x^d - y^d = \prod_{i=0}^{d-1} (x - \eta^i y), \quad \eta = e^{2\pi i/d}, \quad d \geq 3. \quad (6.22)$$

Given a non-empty subset $I \subseteq \{0, \dots, d-1\}$, there is a matrix factorisation

$$P_I = \left(\bigwedge_{a \in I} \mathbb{k}\xi \otimes_{\mathbb{k}} R, \quad d_{P_I} = \prod_{i \in I} (x - \eta^i y)\xi^* + \prod_{i \in I^c} (x - \eta^i y)\xi \right) \quad (6.23)$$

called a permutation defect. Using the construction of Section 6.2 we will compute the A_∞ - \mathcal{A}_{xd} - \mathcal{A}_{ys} -bimodule structure on P_I .

To compute the higher products we need to understand the exotic Feynman interactions which arise from the permutation defects. These arise from the partial derivatives

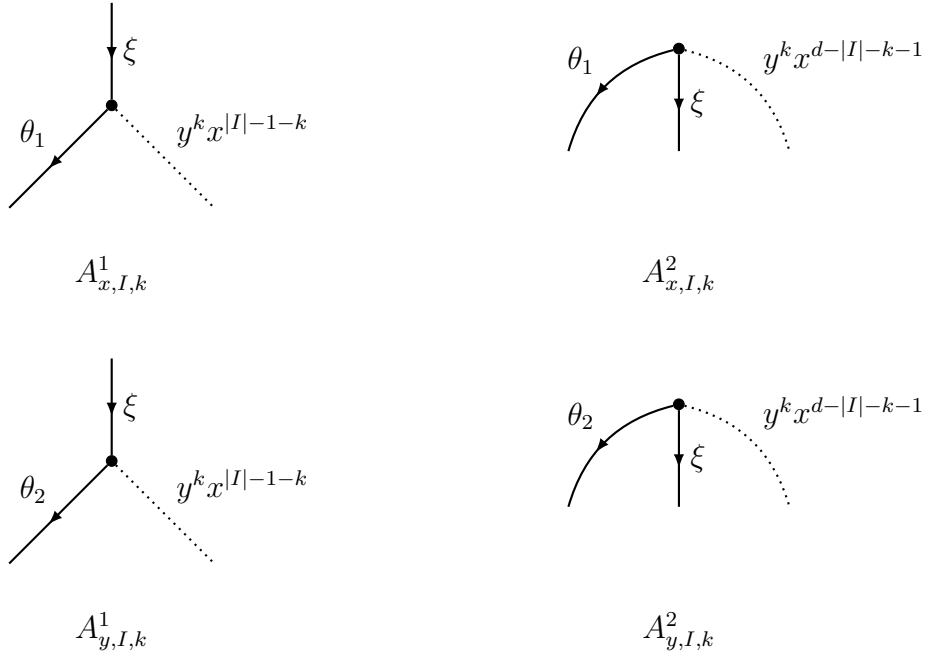
$$\partial_x(d_{P_I}) = \sum_{a \in I} \prod_{i \in I \setminus \{a\}} (x - \eta^i y)\xi^* + \sum_{a \in I^c} \prod_{i \in I^c \setminus \{a\}} (x - \eta^i y)\xi, \quad (6.24)$$

$$\partial_y(d_{P_I}) = - \sum_{a \in I} \eta^a \prod_{i \in I \setminus \{a\}} (x - \eta^i y)\xi^* - \sum_{a \in I^c} \eta^a \prod_{i \in I^c \setminus \{a\}} (x - \eta^i y)\xi. \quad (6.25)$$

Using a generalised binomial formula, we can expand the products as

$$\prod_{i \in I \setminus \{a\}} (x - \eta^i y) = \sum_{k=0}^{|I|-1} (-1)^k c_{a,k} y^k x^{|I|-k-1}, \quad c_{a,k} = \sum_{\substack{J \subseteq I \setminus \{a\} \\ |J|=k}} \prod_{j \in J} \eta^j.$$

Observe that there can not be any contribution from exotic A -type interactions of the second kind, because they do not produce an x or y variable for the subsequent B -type interaction. If an x or y variable were to come from a branch then it would bring with it a θ_i of the same type, causing the configuration to vanish at the next edge. Each monomial in these expansions will contribute one exotic A -type interaction, so the exotic Feynman interactions will be of the following four types (omitting coefficients and signs):



In what follows, suppose T is a tree supporting a nonvanishing configuration C .

Lemma 6.3.1. *The configuration C must not allow there to be a $\theta_1\theta_2$ term on the central trunk.*

Proof. Because there are only ψ_1^* inputs on the left of the trunk and ψ_2^* inputs on the right, there cannot be more than one C -type interaction on each internal vertex. This means that each vertex of the trunk can only remove one θ_i term, so if a $\theta_1\theta_2$ term were to ever appear on the trunk, the configuration would not be able to remove it before the root, and the configuration would vanish. \square

Corollary 6.3.1. *The branches entering the trunk of T must have precisely one leaf with no A -type interactions.*

Proof. If there were a non-trivial sub-tree (or a leaf with an A -type interaction) entering the trunk then the connecting edge would generate a θ_i term. There is no way for a configuration to remove this θ_i before the next edge, where the B -type interaction would create another. By the lemma this would cause the configuration to vanish, so all inputs to the trunk must be leaves with no interactions. \square

Corollary 6.3.2. *There can only be one A -type interaction in C , and it must be an exotic A -type interaction occurring at the top of the trunk.*

All nonvanishing configurations will therefore consist of the following data:

- A choice of exotic A -type interaction, and
- a choice of ordering of the necessary B -type interactions.

It follows that the higher products come in one type for each of the four types of exotic A -type interaction, and are sums over all ways to choose orders of B -type interactions. Explicitly, the four types are

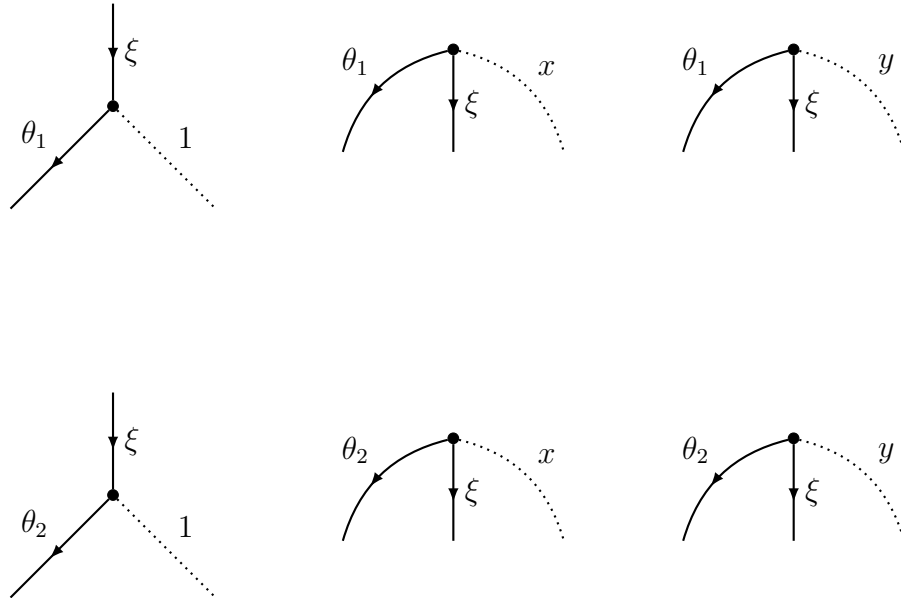
$$m_{|I|-k,k}((\psi_1^*)^{\otimes(|I|-k)} \otimes x \otimes (\psi_2^*)^{\otimes k}) = (-1)^k \binom{|I|-1}{k} \left(\sum_{a \in I} c_{a,k} \right) \xi^* \lrcorner x$$

$$m_{d-|I|-k,k}((\psi_1^*)^{\otimes(d-|I|-k)} \otimes x \otimes (\psi_2^*)^{\otimes k}) = (-1)^k \binom{|I|-1}{k} \left(\sum_{a \in I} c_{a,k} \right) \xi \wedge x$$

$$m_{|I|-1-k,k+1}((\psi_1^*)^{\otimes(|I|-1-k)} \otimes x \otimes (\psi_2^*)^{\otimes(k+1)}) = (-1)^{k+1} \binom{|I|-1}{k} \left(\sum_{a \in I} \eta^a c_{a,k} \right) \xi^* \lrcorner x$$

$$m_{d-|I|-1-k,k+1}((\psi_1^*)^{\otimes(d-|I|-1-k)} \otimes x \otimes (\psi_2^*)^{\otimes(k+1)}) = (-1)^{k+1} \binom{|I|-1}{k} \left(\sum_{a \in I} \eta^a c_{a,k} \right) \xi \wedge x$$

Example 6.3.1. We will consider the case when $d = 3$ and $I = \{1\}$. The exotic Feynman interactions are



We can immediately read off that the higher products are

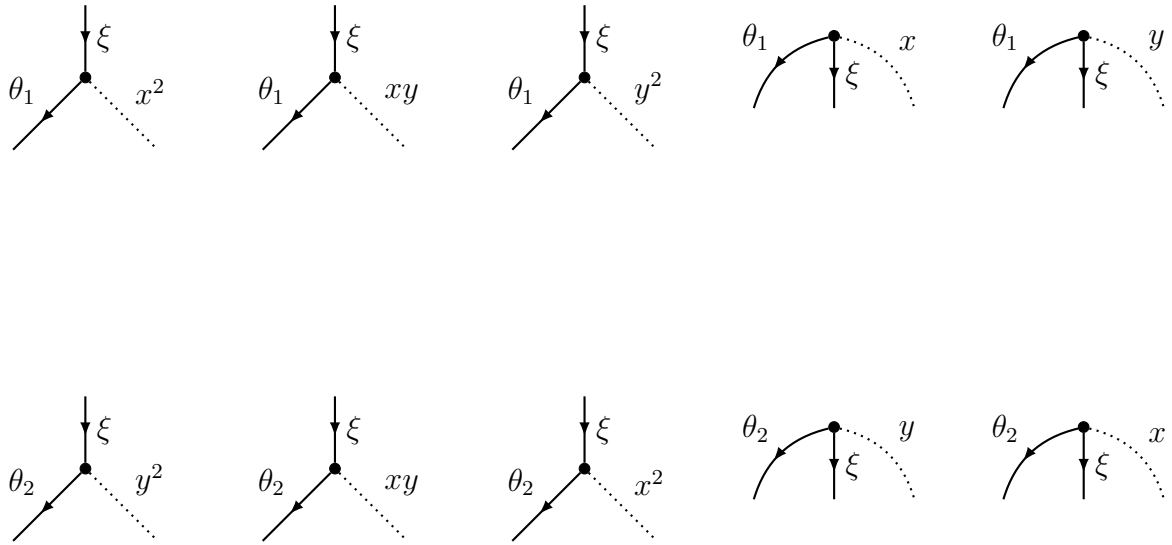
$$\begin{aligned} m_{2,0}(\psi_1^* \otimes \psi_1^* \otimes a) &= \xi \wedge a & m_{0,2}(b \otimes \psi_2^* \otimes \psi_2^*) &= \xi \wedge b \\ m_{1,0}(\psi_1^* \otimes a) &= \xi \lrcorner a & m_{0,1}(b \otimes \psi_2^*) &= \xi \lrcorner b \\ m_{1,0}(1, a) &= a & m_{0,1}(b, 1) &= b \\ m_{1,1}(\psi_1^* \otimes a \otimes \psi_2^*) &= 2\xi \wedge a. \end{aligned}$$

Example 6.3.2. When $d = 5$ and $I = \{1, 2, 3\}$ then we can expand

$$\prod_{i \in I} (x - \eta^i y) = x^3 - \eta(1 + \eta + \eta^2)x^2y + (1 + \eta^3 + \eta^4)xy^2 - \eta y^3 \quad (6.26)$$

$$\prod_{i \notin I} (x - \eta^i y) = x^2 - (\eta^4 + 1)xy + \eta^4 y^2 \quad (6.27)$$

From this we can read off that the exotic Feynman interactions are



All possible configurations are determined by placing one exotic Feynman interaction at the top of the trunk and then accounting for all possible choices in the ordering of the B -type interactions. For instance, if the monomial produced by the interaction contains only one variable then there are no choices: the B -type interactions are already determined. The picture is more interesting when we have a mixed monomial like xy . Then we can choose to either have the B_x or the B_y interaction highest in the tree, and this will subsequently determine the configuration.

Using this, we can write down the higher products as

$$\begin{aligned}
m_{3,0}(\psi_1^* \otimes \psi_1^* \otimes \psi_1^* \otimes a) &= 2\xi \wedge a \\
m_{0,3}(a \otimes \psi_2^* \otimes \psi_2^* \otimes \psi_2^*) &= -2\eta\xi \wedge a \\
m_{2,1}(\psi_1^* \otimes \psi_1^* \otimes a \otimes \psi_2^*) &= -2(\eta + \eta^3 + \eta^4)\xi \wedge a \\
m_{1,2}(\psi_1^* \otimes a \otimes \psi_2^* \otimes \psi_2^*) &= 2(1 + \eta^3 + \eta^4)\xi \wedge a \\
m_{2,0}(\psi_1^* \otimes \psi_1^* \otimes a) &= \xi \lrcorner a \\
m_{0,2}(a \otimes \psi_2^* \otimes \psi_2^*) &= \eta^4 \xi \lrcorner a \\
m_{1,1}(\psi_1^* \otimes a \otimes \psi_1^*) &= -2(\eta^4 + 1)\xi \lrcorner a \\
m_{1,0}(1, a) &= a \\
m_{0,1}(a, 1) &= a.
\end{aligned}$$

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Appendix A

Trees

In this appendix we will fix some notation and conventions for rooted trees as required for calculations of A_∞ -categories.

A.1 Basic definitions

Definition A.1.1. 1. A tree is a connected acyclic graph.

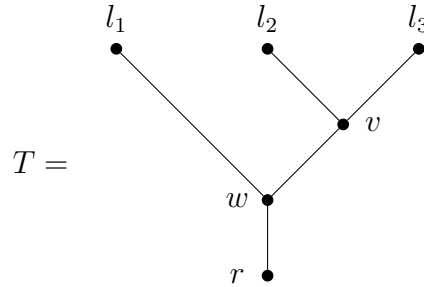
2. Any vertex with valency 1 is called a leaf vertex.
3. A rooted tree is a tree in which one leaf vertex is marked as the root. We give a rooted tree an orientation by orienting all edges in the direction of the root.
4. Given any vertex v in a rooted tree, there is a unique path to the root. We call the first vertex in this path the parent of v . A child of a vertex v is a vertex of which v is the parent.
5. A plane tree is a rooted tree with an ordering for the children of each vertex.
6. A plane tree is valid if it has $n + 1$ leaves (including the root) for some $n \geq 2$ and all non leaf vertices have valency at least three.

We will interchangeably refer to leaves as *external vertices*, and non-leaf vertices as *internal vertices*. We refer to the edges meeting leaves as *external edges*, and all other edges are called *internal edges*.

Definition A.1.2. A morphism of plane trees is a morphism of oriented graphs which preserves the root vertex and the ordering on children.

Definition A.1.3. Let v be a vertex of a rooted tree T . The depth of v is the length of the unique path from v to the root, and the height of v is the length of the longest path from v to a leaf. The height of T is the height of the root.

Example A.1.1. Consider the following plane tree:



The longest path from r to a leaf is that to either l_2 or l_3 which has length 3, so T has height 3. Likewise, w has height 2 and depth 1, and v has height 1 and depth 2. The vertex v has two children l_2 and l_3 .

Remark A.1.1. We can see from the example above that it is natural to assign clockwise order to a plane tree, so for instance the children of v are ordered (l_1, l_2) .

Definition A.1.4. Let T be a plane tree. We define a relation \prec on the vertices as follows: Let w and w' be vertices of T , and write P_w and $P_{w'}$ for the respective paths to the root. We say that $w \prec w'$ if there is a vertex v in $P_w \cap P_{w'}$ and children s, s' of v with $s < s'$ and $s \in P_w, s' \in P_{w'}$.

Remark A.1.2. This relation \prec subsumes the chosen order $<$ on the children of the vertices of T .

Lemma A.1.1. *If we restrict \prec to non-root leaves then \prec is a total order.*

The existence of this total order allow us to repackage the data of a valid plane tree in terms of purely combinatorial information.

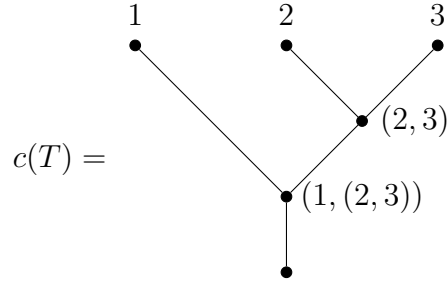
Definition A.1.5. Let T be a valid plane tree with $n + 1$ leaves. The combinatorial tree associated to T , denoted $c(T)$, is an ordered set defines as follows. Write v_1, \dots, v_n for the set of non-root leaves, written in clockwise order, and define $\alpha(v_i) = i$. We extend α recursively to internal vertices v with ordered children w_1, \dots, w_k by

$$\alpha(v) = (\alpha(w_1), \dots, \alpha(w_k)). \quad (\text{A.1})$$

We denote the set of combinatorial trees with $n + 1$ leaves by \mathcal{T}_n .

Because \prec is a total order on non-root leaves, there is an obvious bijection between the set of isomorphism classes of valid plane trees and combinatorial trees. We implicitly identify plane trees and combinatorial trees throughout the thesis.

Example A.1.2. The combinatorial tree associated to the tree in Example B.1.1 is given by



A.2 Decorations and denotations

We will now give details of how one can obtain a bilinear map from a plane tree T , via decorations and denotations. We will highlight the contrast between the height and branch denotations.

Definition A.2.1. Let R be a commutative ring and T a plane tree with at least one edge. A decoration of T , denoted by D_T , consists of the following data:

1. A graded R -module L_v for each leaf v ;
2. A graded R -module M_e for every edge e ;
3. For each internal vertex v with incoming edges (e_1, \dots, e_k) and outgoing edge e , an integer I_v (in either \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$) and a degree I_v R -linear map

$$\phi_v : M_1 \otimes \dots \otimes M_k \rightarrow M_e; \quad (\text{A.2})$$

4. For each non-root leaf l with incident edge e a degree zero R -linear map

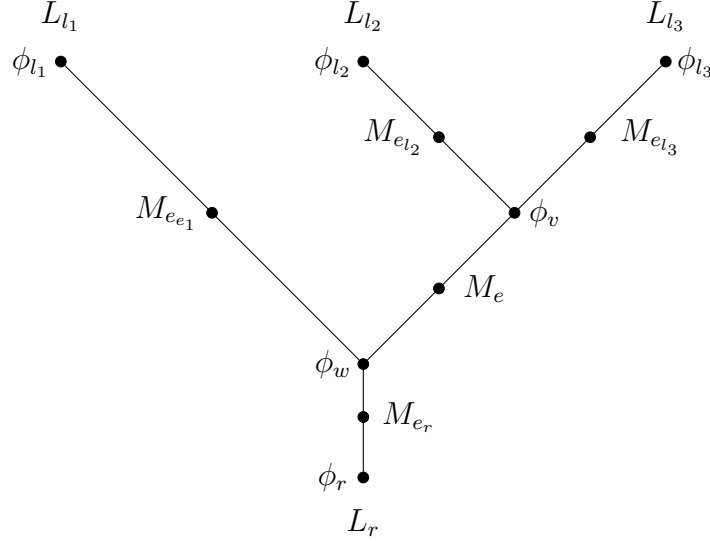
$$\phi_l : L_l \rightarrow M_e; \quad (\text{A.3})$$

5. A degree zero R -linear map

$$\phi_r : M_e \rightarrow L_r; \quad (\text{A.4})$$

where r is the root and e is the edge incident to r .

Example A.2.1. On the tree T of previous examples, a decoration is



Here, the maps are $\phi_{l_i} : L_{l_i} \rightarrow M_{e_{l_i}}$, $\phi_v : M_{e_{l_2}} \otimes M_{e_{l_3}} \rightarrow M_e$, and $\phi_w : M_{e_{l_1}} \otimes M_e \rightarrow M_{e_r}$.

Given a decoration D_T of a valid plane tree T , we can compose and take the tensor product of the maps to give a linear map $\langle D_T \rangle : L_{v_1} \otimes \cdots \otimes L_{v_n} \rightarrow L_r$. However, there are two obvious ways of doing this, either by height or by branch, and because of Koszul signs we will not always obtain the same linear map.

Definition A.2.2. Let T be a valid plane tree with a decoration D_T . The *height* denotation $\langle D_T \rangle_H$ is the homogeneous R -linear map $\langle D_T \rangle_H : L_{v_1} \otimes \cdots \otimes L_{v_n} \rightarrow L_r$ defined as follows:

1. Let h be the maximum depth of the leaves in T (or the height of T). For each leaf with v depth less than h we attach a bivalent vertex to the edge e_v coming from v until v had depth h , and to these new vertices we attach the identity operator on M_{e_e} .
2. For $1 \leq d \leq h$ let $V_d(T)$ denote the set of vertices of T (including the new vertices) of depth d . Make $V_d(T)$ totally ordered by \prec , say (v_1, \dots, v_k) , and for each v_i let $(e_{i,1}, \dots, e_{i,a_i})$ be the ordered set of edges terminating at v_i , and f_i the outgoing edge at v_i .

3. Define an outgoing R -linear map for each $1 \leq d < h$ by

$$\Phi_d := \phi_{v_1} \otimes \cdots \otimes \phi_{v_k} : \left(\bigotimes_j M_{e_{1,j}} \right) \otimes \cdots \otimes \left(\bigotimes_j M_{e_{k,j}} \right) \rightarrow M_{f_1} \otimes \cdots \otimes M_{f_k}, \quad (\text{A.5})$$

using the Koszul sign convention. Define $\Phi_h = \phi_{l_1} \otimes \cdots \otimes \phi_{l_n}$ and $\Phi_0 = \phi_r$.

4. Set $\langle D_T \rangle_H := \Phi_0 \otimes \cdots \otimes \Phi_h$.

Definition A.2.3. Let T be a valid plane tree with a decoration D_T . The *branch* denotation $\langle D_T \rangle_B$ is the homogeneous R -linear map $\langle D_T \rangle_B : L_{v_1} \otimes \cdots \otimes L_{v_n} \rightarrow L_r$ defined as follows:

1. If T has one edge from the leaf l_1 to the root r then $\langle D_T \rangle_B := \phi_r \circ \phi_{l_1}$.
2. If T has more than one edge then we recursively define

$$\langle D_T \rangle_B := \phi_r \circ \phi_v \circ (\langle D_{T_1} \rangle_B \otimes \cdots \otimes \langle D_{T_k} \rangle_B), \quad (\text{A.6})$$

where v is the vertex immediately before the root and T_1, \dots, T_k are the sub-trees of T whose root is v .

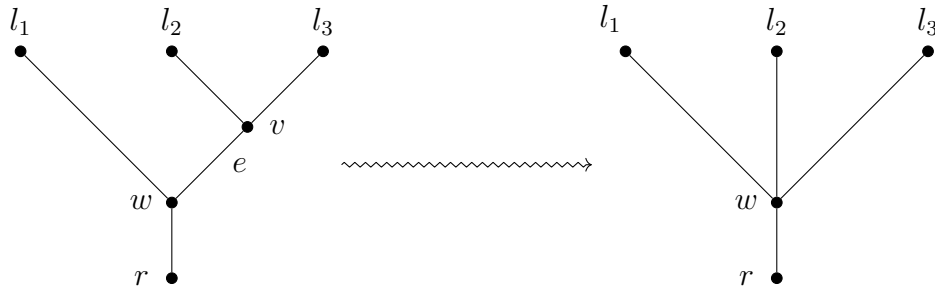
Example A.2.2. On the decoration of Example B.2.1 the height and branch denotation are

$$\begin{aligned} \langle D_T \rangle_H &= \phi_r \circ \phi_w \circ (\text{id} \otimes \phi_v) \circ (\phi_{l_1} \otimes \phi_{l_2} \otimes \phi_{l_3}) \\ \langle D_T \rangle_B &= \phi_r \circ \phi_w \circ (\phi_{l_1} \otimes (\phi_v \circ (\phi_{l_2} \otimes \phi_{l_3}))) \\ &= (-1)^{|\phi_v| + |\phi_{l_1}|} \phi_r \circ \phi_w \circ (\text{id} \otimes \phi_v) \circ (\phi_{l_1} \otimes \phi_{l_2} \otimes \phi_{l_3}) \\ &= (-1)^{|\phi_v|} \langle D_T \rangle_H. \end{aligned}$$

A.3 Edge contraction and insertion

Definition A.3.1. Let T be a plane tree and e an edge of T connecting vertices v and w . Suppose w is closer to the root than v , and that w has children w_1, \dots, w_k with $w_j = v$ and v has children v_1, \dots, v_m . The tree $T - e$ is the plane tree obtained by deleting v and attaching the children of v to w so that the ordered set of the children of w is $w_1, \dots, w_{j-1}, v_1, \dots, v_m, w_{j+1}, \dots, w_k$.

Example A.3.1. If T is the tree used in previous examples then the contraction of T at e is



Definition A.3.2. Let T be a plane tree and v an internal vertex with children (v_1, \dots, v_k) . Given integers $i \geq 0$ and $j \geq 1$ with $i+j \leq k$ the edge insertion $\text{ins}(T, v, i, j)$ is obtained from T by detaching the edges joining v_{i+1}, \dots, v_{i+j} to v , inserting a new vertex w to which v_{i+1}, \dots, v_{i+j} join, and inserting a new edge e joining w to v . The parent/child relations are unchanged by this relation with the exception that v_{i+1}, \dots, v_{i+j} now have parent w and the parent of w is v .

Example A.3.2. Let T' be the tree obtained in the previous example by contracting the edge e of T . We can recover T as the edge inserting $\text{ins}(T', w, 1, 1)$.

Remark A.3.1. It is obvious that if we start with a valid plane tree T and perform an edge contraction then the resulting tree will be valid. Likewise, if we begin with a valid plane tree and insert an edge over a vertex v of valency $k > 3$ and if $2 \leq j \leq k - 1$ then $\text{ins}(T, v, i, j)$ is also a valid plane tree.

Definition A.3.3. We denote by \mathcal{T}_n^+ the set of pairs (T, e) where $T \in \mathcal{T}_n$ and e is an edge of T .

Remark A.3.2. We can partition the set $\mathcal{T}_n = \coprod_{c \geq 0} \mathcal{T}_n^{(c)}$ where $\mathcal{T}_n^{(c)}$ is the subset of trees with c internal edges.

Lemma A.3.1. *For $n > 2$ and $c \geq 0$ there is a bijection*

$$(\mathcal{T}_n^{(c+1)})^+ \leftrightarrow \{(T, v, i, j) \mid T \in \mathcal{T}_n^{(c)}, v \in V_i(T), i \geq 0, i + j \leq |v| - 1, 2 \leq j \leq |v| - 2\},$$

where $|v|$ denotes the valency of v .

Proof. For any edge e in a tree, denote by $t(e)$ the vertex at which e terminates and $s(e)$ the vertex where e begins. Write v_1, \dots, v_k for the children of $t(e)$, where $s(e) = v_m$. We define a map from left to right by

$$(T, e) \rightarrow (T \neg e, t(e), m - 1, |s(e)| - 1). \quad (\text{A.7})$$

Clearly $T \neg e \in \mathcal{T}_n^{(c)}$, $t(e) \in V_i(T)$, and $m - 1 \geq 0$. It is also clear that $2 \leq |s(e)| - 1 \leq |t(e)| - 2$ because T is a valid plane tree. Moreover,

$$|t(e)| - 1 = k + |s(e)| - 1 - 1 \geq m + |s(e)| - 2 = m - 1 + |s(e)| - 1, \quad (\text{A.8})$$

so the map defined above really does land in the right hand side. To show that this is a bijection we will show that the map from the right to the left given by

$$(T, v, i, j) \mapsto \text{ins}(T, v, i, j), \quad (\text{A.9})$$

is a two-sided inverse. Again, this map clearly lands in $(\mathcal{T}_n^{(c+1)})^+$. It is also clear that

$$\text{ins}(T \neg e, t(e), m - 1, |s(e)| - 1) = T.$$

and that the composition in the other direction is the identity. Therefore the map B.7 is a bijection. □