Vertex algebras, Hopf algebras and lattices

Albert Zhang

Supervisor: Daniel Murfet

School of Mathematics and Statistics
University of Melbourne

A thesis submitted in partial fulfilment of the requirements of the degree of
Master of Science (Mathematics and Statistics)

October 2018

Abstract

The purpose of this thesis is to give an accompanying exposition of Borcherds's construction of vertex algebras as commutative algebras in a category with singular multilinear maps. The theory of vertex algebras is first introduced in the usual formal power series framework and an alternative definition and uniqueness theorem is provided. The key example will be the vertex algebra associated to an even integral lattice, whose underlying vector space will be the universal measuring algebra, in the sense of Sweedler, associated to the Hopf algebra of the one-dimensional formal additive group and the group ring of the lattice.

Acknowledgements

I owe my deepest gratitude to my supervisor Daniel Murfet for his tireless commitment and support throughout my research. I have greatly benefited from his mathematical insights and have been inspired to pursue mathematics with the same vigour.

Many thanks to my friends and family for their patience and unconditional love. I would also like to thank the School of Mathematics and Statistics for providing the educational opportunity and fostering a community of talented minds.

Lastly, I give a special thanks to the Potter Espresso Bar and Castro's Kiosk for their numerous caffeine beverages.

Table of contents

Chapter 1

Introduction

The purpose of this thesis is to provide a complementary exposition of Borcherds's construction in [4] of lattice vertex operator algebras as singular commutative algebras in a multilinear category. One focus of this thesis is then to setup the required machinery to comprehend Borcherds's construction. This will involve introducing the theory of vertex algebras through formal distributions, and then providing an exposition on Hopf algebras and algebraic groups. Given a coalgebra C and an algebra A, Sweedler has defined a universally associated algebra $\beta(C,A)$ whose spectrum is in bijection with scheme maps $Spec(C^*) \to Spec(A)$. If C is the dual numbers these scheme maps are in bijection with tangent vectors and for $C = (k[t]/t^{n+1})^*$ this constructs n-jets. The direct limit of all these coalgebras of infinitesimals is a Hopf algebra H, which can be viewed as the coordinate ring of the affine algebraic group of translations of $\mathbb{A}^1_k$. If $\mathbb{C}[L]$ is the group ring of a lattice L then the algebra $V_L = \beta(H, \mathbb{C}[L])$ is the coordinate ring of the infinite jet space of $X = Spec(\mathbb{C}[L])$, and is the underlying vector space of the lattice vertex operator algebra. By a universal construction, the pairing on the lattice can be extended to a twisted multiplication on V_L, which defines a singular multilinear map making V_L into a vertex operator algebra. In the end, we fell short of completely understanding Borcherds's construction of the lattice vertex operator algebra with some details still eluding us. We will discuss the gaps in our knowledge at the end of Chapter 4, but we hope this thesis will be a valuable resource for readers who want to understand Borcherds's paper [4].

1.1 What is a vertex algebra?

One of the most powerful applications of mathematics is to provide deep insights into physical phenomena. Thus, our story of vertex algebras begins in mathematical physics. Consider the Hilbert space of a quantum field theory, in which fields are modelled mathematically by distributions valued in (densely-defined) operators on the Hilbert space [1]. The multiplication of two fields may not necessarily be well-defined for all coordinates. In quantum field theory, this interaction is described by the operator product expansion, which separates the parts of the multiplication into the "regular" parts, which are well-defined, and the "singular" parts. This description should be the guiding physical intuition of the formal concepts we introduce. The operator product expansion in conformal field theories is axiomitised by vertex algebras which consists of a vector space, with a distinguished vacuum vector, an infinitesimal translation operator, and a set of fields satisfying various compatibility axioms motivated by Wightman's quantum field theory axioms [13].

Historically, vertex algebras were formally defined by Richard Borcherds in 1986 [3]. Prior to this, the first vertex operators were written down in string theory for the one-dimensional even integral lattice. One can theorise that the name "vertex" arose from assigning an operator to each vertex of the lattice. More surprisingly was the appearance of vertex algebras in various other mathematical theories such as finite group theory, Lie theory, modular functions, and string theory. The Monster vertex algebra or "moonshine module" was constructed by Frenkel, Lepowsky, and Meurman in 1988 [8] in studying the representation theory of the Monster group. This allowed Borcherds to prove the Monstrous moonshine conjecture [3] stated by Conway and Norton in 1979.

In general, the vertex algebra definition and the vertex operators are quite laborious to write down and are even more difficult to generalise to higher-dimensional quantum field theories. Lepowsky and Li state in their introductory textbook, "in vertex operator algebra theory, there are essentially no examples, [...] that are easy to construct and for which the axioms can be easily proved" [9, Section 1.6].

1.2 Thesis outline

In Chapter 2, the theory of vertex algebras is introduced from a formal calculus framework following [16] and [10]. The key results will be stated, with this chapter providing an accompanying exposition to the theory. The definition of a vertex algebra will not be explicitly stated until the end of the chapter. The examples mentioned in this thesis will be the Heisenberg vertex algebra, associated to a rank 1 free boson, and the vertex algebra associated to the even integral lattice. A modification of the uniqueness theorem stated in Tuite's paper [16] will be presented.

In Chapter 3, the theory of coalgebras and Hopf algebras is introduced following [15] to motivate the construction of the universal measuring algebra. Coalgebras are the natural algebraic structures of operator-valued distributions, and Hopf algebras generalise the group theory of space-time symmetries. The Hopf algebras associated to the complex additive and multiplicative algebraic groups will be explicitly calculated. The chapter ends with proving the existence of the universal measuring algebra, $\beta(C,A)$, which will be the underlying vector space of the lattice vertex algebra.

In Chapter 4, Borcherds's (A,H,S) vertex algebras will be defined as a commutative algebra in a multilinear category of functors with singular multilinear maps. This definition then generalises the ordinary vertex algebra to higher dimensions. An example of a singular multiplication is constructed by the twisting of the multiplication on an algebra by bicharacters. In his paper, Borcherds then explains how a universal procedure lifts the pairing on the lattice to a singular multiplication on $\beta(H,\mathbb{C}[L])$, making it a vertex algebra. The final example will provide some details for the vertex operator algebra associated to the even integral lattice as defined in Chapter 2. However, work still remains to completely understand the construction.

Chapter 2

Vertex algebras

For new readers to the area of vertex algebras a lot of machinery is required as a prerequisite before even being able to provide the formal definition. The aim of this chapter is to introduce the algebraic properties of a vertex algebra as stated in most introductory texts. A more complete treatment of vertex algebra theory can be found in [10] or [7].

To capture the Wightman axioms of a quantum field theory, we require that these fields be "mutually local", "creative" and that there exists a compatible infinitesimal translation operator T. The usual definitions of vertex algebras axiomise these properties by defining a vector space, a vacuum vector, a linear map $Y : V \to End(V)[[z^{\pm 1}]]$, called the "state-field correspondence", associating each state $v \in V$ to a field $Y(v,z)$, and a translation operator T, satisfying certain properties. We will discuss why the data of the translation operator actually arises from the compatibility condition for a choice of fields by modifying Theorem 33 in [16], and then suggest an alternative definition for a vertex algebra.

2.1 Formal distributions

The advantage of defining vertex algebras via formal distributions is that it allows us to carry over the theory of power series in a convenient way. Hence, like complex functions, we can take formal operations such as derivatives and residues. Fix k to be a field of characteristic zero and A an associative unital k-algebra.

Definition 2.1. The space of ***(A-valued) formal polynomials***, ***formal Taylor series*** and ***formal distributions*** are defined:

$$A[z] := \{\sum_{n=0}^{N} a_n z^n \mid a_n \in A, \quad N \in \mathbb{Z}_{\geq 0}\}$$

$$A[[z]] := \{\sum_{n\in\mathbb{Z}_{\geq 0}} a_n z^n \mid a_n \in A\}$$

$$A[[z,z^{-1}]] := \{\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in A\}$$

respectively, where z is a formal variable. The space $A[[z,z^{-1}]]$ is synonymous with $A[[z^{\pm 1}]]$, and we denote:

$$a(z) := \sum_n a_n z^n$$

for the respective formal power series in each of the spaces above.

To guide the formalism, A will be set as $End(V)$, the k-algebra of endomorphisms for some Hilbert space V. Hence, throughout this chapter, the formal distributions can be thought of as operator-valued distributions, which are the fields studied in quantum field theory. Alternatively, the space $A[z]$ can also be viewed as the set of functions:

$$A[z] = \{f : \mathbb{Z}_{\geq 0} \to A \mid f \text{ is finitely supported }\}$$

Similarly, the space $A[[z]]$ can be viewed as the set of functions $\{f : \mathbb{Z}_{\geq 0} \to A\}$ and $A[[z,z^{-1}]]$ as the set of functions $\{f : \mathbb{Z} \to A\}$. The spaces $A[z]$ and $A[[z]]$ form k-algebras with the usual addition and multiplication induced by A. However, we need to be cautious with formal distributions as they do not in general share this algebraic property. If $a(z), b(z) \in A[[z^{\pm 1}]]$ are two formal distributions, then the usual multiplication leads to:

$$\begin{aligned} a(z)\cdot b(z) &= \left(\sum_{n\in\mathbb{Z}} a_n z^n\right)\cdot\left(\sum_{m\in\mathbb{Z}} b_m z^m\right) \\ &= \sum_{n,m\in\mathbb{Z}} a_n b_m z^{n+m} \\ &= \sum_{k\in\mathbb{Z}}\left(\sum_{n+m=k} a_n b_m\right) z^k \end{aligned}$$

For a fixed $k \in \mathbb{Z}$, the equation $n+m=k$ admits infinite solutions of pairs $(n,m) \in \mathbb{Z}^2$. Hence, the coefficient for z^k is an infinite sum of elements in A. In general, there is no way to guarantee convergence in A. Thus, multiplication in the space of formal distributions is not well-defined.

Example 2.2. An amusing paradox, presented by Lepowsky and Li [9, §2.1.17], illustrates why we need to be careful of products of infinite series. We have the identity:

$$(\sum_{n\in\mathbb{Z}_{\geq 0}} x^n)(1-x) = \sum_{n\in\mathbb{Z}_{\geq 0}} x^n - \sum_{n\in\mathbb{Z}_{\geq 0}} x^{n+1} = 1$$

and consider the formal delta series defined by $\delta(x) := \sum_{n\in\mathbb{Z}} x^n$. Then the following calculations produces an absurdity:

$$\delta(x) = 1\cdot\delta(x) = \left((\sum_{n\in\mathbb{Z}_{\geq 0}} x^n)(1-x)\right)\delta(x) = (\sum_{n\in\mathbb{Z}_{\geq 0}} x^n)\left((1-x)\delta(x)\right) = (\sum_{n\in\mathbb{Z}_{\geq 0}} x^n)\cdot 0 = 0$$

The calculation above is invalid because the triple product does not exist and the associative law only applies where it is defined.

Remark. Define the expression $A[z,z^{-1}]$, by identifying this with the algebra $A[z,u]/(zu-1)$, constructed by quotienting the algebra $A[z,u]$ by the two-sided ideal generated by $(zu-1)$. This algebra is defined to be the space of ***formal Laurent polynomials*** in one variable.

Lemma 2.3. The space of formal distributions in one variable, $A[[z^{\pm 1}]]$, is an $A[z,z^{-1}]$-module.

Proof. Suppose $a(z) := \sum_{n\in\mathbb{Z}} a_n z^n$ is a formal distribution and $\varphi(z) = \sum_{m=M'}^{M} \varphi_m z^m$ is a formal Laurent polynomial, where $M, M' \in \mathbb{Z}$ and $M' \leq M$. To show that $A[[z^{\pm 1}]]$ is an $A[z,z^{-1}]$-module, it suffices to define the action of $\varphi(z)$ on $a(z)$. This can be defined by the usual multiplication of polynomials:

$$\begin{aligned}\varphi(z)\cdot a(z) &:= \left(\sum_{m=M'}^{M} \varphi_m z^m\right)\left(\sum_{n\in\mathbb{Z}} a_n z^n\right)\\ &= \sum_{m=M'}^{M}\sum_{n\in\mathbb{Z}} \varphi_m a_n z^{m+n}\\ &= \sum_{k\in\mathbb{Z}}\left(\sum_{\substack{n+m=k\\ M'\leq m\leq M}} \varphi_m a_n\right) z^k\end{aligned}$$

for all $\varphi(z) \in A[z,z^{-1}]$ and $a(z) \in A[[z^{\pm 1}]]$. Note that for a fixed $k \in \mathbb{Z}$, there are only finitely many pairs $(n,m) \in \mathbb{Z}^2$ such that they satisfy the equation $n+m=k$ and the condition

$M' \leq m \leq M$. Therefore, the coefficients of z^k are well-defined and $\varphi(z) \cdot a(z)$ is a well-defined element of $A[[z^{\pm 1}]]$. □

Additionally, the canonical algebra inclusion $A[z] \hookrightarrow A[z, z^{-1}]$, defines a natural $A[z]$-module structure on $A[[z^{\pm 1}]]$ as well. Likewise, we can define actions of the algebra A and the field k on $A[[z^{\pm 1}]]$.

Definition 2.4. Define $A^{\mathbb{Z}}$ to be the set of doubly infinite sequences:

$$A^{\mathbb{Z}} = \{\alpha := \{\alpha_n\}_{n \in \mathbb{Z}} \mid \alpha_n \in A\}$$

For each $\alpha \in A^{\mathbb{Z}}$, define its ***formal generating series***, $\alpha(z)$, to be the formal distribution:

$$\alpha(z) := \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1} \in A[[z, z^{-1}]]$$

where $\alpha_n \in A$. Alternatively, for any formal distribution of the form

$$\alpha(z) = \sum_{m \in \mathbb{Z}} \alpha_m z^m \in A[[z^{\pm 1}]]$$

the ***Fourier expansion*** of the formal distribution $\alpha(z)$ is defined to be the formal generating series:

$$\alpha(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

for the doubly infinite sequence $\{\alpha_n\}_{n \in \mathbb{Z}}$, where each α_n is defined by:

$$\alpha_n := \alpha_{-m-1} \in A.$$

The coefficients α_n for all $n \in \mathbb{Z}$ are called the ***Fourier modes*** of $\alpha(z)$.

Example 2.5. The formal Dirac's distribution, $\delta(z, w)$, is defined by:

$$\delta(z, w) := \sum_{n \in \mathbb{Z}} z^{-1} \left(\frac{w}{z}\right)^n = \sum_{n \in \mathbb{Z}} w^{-1} \left(\frac{z}{w}\right)^n \in A[[z^{\pm 1}, w^{\pm 1}]].$$

It is an easy verification to show that the Fourier expansion of the formal Dirac distribution is the formal generating series:

$$\delta(z, w) = \sum_{n, m \in \mathbb{Z}} \delta_{-n-1, m} z^{-n-1} w^{-m-1}$$

where the Fourier modes are doubly indexed sequences defined by the usual delta function $\delta_{n,m}$.

Definition 2.6. If $\alpha(z)$ is a formal generating series for $\alpha \in A^{\mathbb{Z}}$, then define its ***formal derivative*** $\partial\alpha(z)$ to be the formal distribution:

$$\partial\alpha(z) := \sum_{n\in\mathbb{Z}} \alpha_n(-n-1)z^{-n-2} = \sum_{m\in\mathbb{Z}} (-m\alpha_m)z^{-m-1}$$

and define the expression $\partial^{(i)}\alpha(z) := \frac{1}{i!}\partial^i\alpha(z)$, where $i \in \mathbb{Z}_{\geq 0}$ and we assume the rational numbers act on A.

If $\alpha(z)$ is a formal generating series for $\alpha \in A^{\mathbb{Z}}$, then define the ***formal residue***, as the map:

$$Res_z : A[[z^{\pm 1}]] \to A, \quad Res_z\alpha(z) := \alpha_0$$

The fourier modes can now be recovered from the formal generating series by the formal residue:

$$\alpha_n = Res_z\big(z^n\alpha(z)\big)$$

Remark. Suppose A is a $\mathbb{C}$-algebra. For any formal distribution $\alpha(z)$ and a formal laurent polynomial $\varphi(z) \in \mathbb{C}[z,z^{-1}]$, then we can define a non-degenerate pairing:

$$\begin{aligned} \langle \mid \rangle : A[[z^{\pm 1}]] \otimes \mathbb{C}[z,z^{-1}] \to A \\ \alpha(z) \otimes \varphi(z) \mapsto \langle \alpha \mid \varphi \rangle := Res_z\varphi(z)\cdot\alpha(z) \end{aligned}$$

where $\varphi(z)\cdot\alpha(z)$ is the usual polynomial multiplication. Thus, the formal Laurent polynomials with coefficients in $\mathbb{C}$ are a space of test functions for the formal distributions. In fact all continuous A-linear maps on the space of Laurent polynomials arise in this manner [11].

2.2 Locality

The first physical property to establish for the formal distributions is that they are mutually local. This has the interpretation that measurements at space-like separated points are independent. See [10, Chapter 1].

As per the discussion above, the space $A[[z^{\pm 1}]]$ does not carry a well-defined multiplication. However, we can define a notion of multiplication between spaces of different formal variables. This roughly represents the interaction of two fields at different coordinates. In algebraic terms, we define A-linear maps from $A[[z^{\pm 1}]] \otimes_A A[[w^{\pm 1}]]$ to $A[[z^{\pm 1}, w^{\pm 1}]]$.

Definition 2.7. Suppose $\alpha(z) \in A[[z^{\pm 1}]]$ and $\beta(w) \in A[[w^{\pm 1}]]$ are two formal generating series. The formal multiplication and commutator for $\alpha(z)$ and $\beta(w)$ are defined to be:

$$\alpha(z) \cdot \beta(w) := \sum_{m,n \in \mathbb{Z}} \alpha_m \beta_n z^{-m-1} w^{-n-1} \in A[[z^{\pm 1}, w^{\pm 1}]]$$
$$[\alpha(z), \beta(w)] := \sum_{m,n \in \mathbb{Z}} [\alpha_m, \beta_n] z^{-m-1} w^{-n-1} \in A[[z^{\pm 1}, w^{\pm 1}]]$$

respectively, where $[\alpha_m, \beta_n] = \alpha_m \beta_n - \beta_n \alpha_m$ is the usual commutator. Note that $\alpha(z) \cdot \beta(w)$ and $[\alpha(z), \beta(w)]$ are the formal generating series for the doubly index seqeucnes $\{\alpha_m \beta_n\}_{m,n \in \mathbb{Z}}$ and $\{[\alpha_m, \beta_n]\}_{m,n \in \mathbb{Z}}$ respectively.

Another way to introduce a multiplication between two formal distributions is to borrow from physics the notion of normal ordering. In this case, we consider the modes as annihilation and creation operators. The normal ordering then places creation operators to the left of annihilation operators.

Definition 2.8. Let $\alpha(z) \in A[[z^{\pm 1}]]$ and $\beta(z) \in A[[w^{\pm 1}]]$ be two formal generating series. The ***normal ordered product*** of $\alpha(z)$ and $\beta(w)$ is given by:

$$: \alpha(z)\beta(w) := \sum_{n \in \mathbb{Z}} (\sum_{m<0} \alpha_m \beta_n z^{-m-1} + \sum_{m \geq 0} \beta_n \alpha_m z^{-m-1}) w^{-n-1}$$

This term can be expressed as positive and negative modes of $\alpha(z)$ by:

$$\alpha_-(z) := \sum_{n \geq 0} \alpha_n z^{-n-1}, \quad \alpha_+(z) := \sum_{n<0} \alpha_n z^{-n-1}$$

such that, the normal ordered product can be expressed by:

$$: \alpha(z)\beta(w) := \alpha_+(z)\beta(w) + \beta(w)\alpha_-(z)$$

Before defining mutual locality, the following section clarifies some of the notation prevalent in literature. Define $0(z) \in A[[z^{\pm 1}]]$ to be the *zero formal distribution* where each Fourier mode is the additive identity $\mathbf{0}$ of A:

$$0(z) := \sum_{n \in \mathbb{Z}} \mathbf{0}\, z^{-n-1} = ... + \mathbf{0}\, z^0 + \mathbf{0}\, z^1 + ...$$

For example, if $A = End(V)$, then $\mathbf{0}$ will be the zero endomorphism that maps all vectors to 0. In most literature, the zero formal distribution is usually just written as 0. Here $0(z)$ plays the role of the additive identity in $A[[z^{\pm 1}]]$.

Likewise, define a multiplicative identity $I(z) \in A[[z^{\pm 1}]]$ to be the *identity formal distribution*, where if $\mathbf{1}$ is the multiplicative identity of A, then:

$$I(z) := \mathbf{1}\, z^0$$

In the case of $A = End(V)$ as above, then $\mathbf{1} = id_V$ is the linear identity map of V.

Definition 2.9. Suppose $\alpha(z) \in A[[z^{\pm 1}]]$ and $\beta(z) \in A[[w^{\pm 1}]]$ are two formal generating series. They are said to be ***mutually local***, if there exists some integer $n \in \mathbb{Z}_{\geq 0}$, such that the bivariate series $C^n(\alpha(z), \beta(w))$ defined by:

$$C^n\big(\alpha(z), \beta(w)\big) := (z-w)^n [\alpha(z), \beta(w)] \in A[[z^{\pm 1}, w^{\pm 1}]]$$

satisfies:

$$C^n\big(\alpha(z), \beta(w)\big) = 0(z).$$

The smallest $N \in \mathbb{Z}_{\geq 0}$ such that $C^N\big(\alpha(z), \beta(w)\big) = 0(z)$ is called the *order of locality* and it is denoted by $\alpha(z) \overset{N}{\sim} \beta(z)$. Any two formal distributions $\alpha(z)$, $\beta(w)$ are said to be mutually local if their Fourier expansions are mutually local. A formal distributions may not necessarily be mutually local to itself.

Example 2.10. Given some power series R, it may have different power series expansions $i_{z,w}R$ and $i_{w,z}R$ by expanding in the domains $|z| > |w|$ and $|w| > |z|$ respectively. For example, the power series $\frac{1}{(z-w)^{j+1}}$ has the two converging expansions:

$$i_{z,w}\frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty} \binom{m}{j} z^{-m-1} w^{m-j}, \quad |z| > |w|$$
$$i_{w,z}\frac{1}{(z-w)^{j+1}} = -\sum_{m=-1}^{-\infty} \binom{m}{j} z^{-m-1} w^{m-j}, \quad |w| > |z|$$

such that:

$$i_{z,w}\frac{1}{(z-w)^{j+1}} - i_{w,z}\frac{1}{(z-w)^{j+1}} = \sum_{m\in\mathbb{Z}} \binom{m}{j} z^{-m-1} w^{m-j} = \partial_w^{(j)}\delta(z,w)$$

Note that the bivariate series $C^n\big(\alpha(z), \beta(w)\big)$ can be extended to all $n \in \mathbb{Z}$ by defining:

$$C^n\big(\alpha(z), \beta(w)\big) = (z-w)^n \alpha(z)\beta(w) - (-w+z)^n \beta(w)\alpha(z),$$

which agrees with the original definition for $n \geq 0$ and for $n \leq 0$, the terms $(z-w)^n, (-w+z)^n$ denote the different expansions in the domains $|z| > |w|$ and $|w| > |z|$ respectively. The data of locality can be encoded by taking the residue of the bivariate series $C^n\big(\alpha(z), \beta(w)\big)$. This induces another multiplication called the residue product.

Definition 2.11. Suppose $\alpha(z), \beta(z) \in A[[z^{\pm 1}]]$ are two formal distributions. If $n \in \mathbb{Z}_{\geq 0}$, then the ***n-th residue product***, $*_n$, is defined by:

$$\alpha(z) *_n \beta(z) := Res_w C^n\big(\alpha(w), \beta(z)\big) \in A[[z^{\pm 1}]]$$

We identify $(\alpha *_n \beta)(z) := \alpha(z) *_n \beta(z)$ and note that if $\alpha(z) \overset{N}{\sim} \beta(z)$, then for all $n \geq N$:

$$(\alpha *_n \beta)(z) = 0(z).$$

Note that the residue product can also be expressed as the expansion:

$$(\alpha *_n \beta)(z) = \sum_{k=0}^{n} \binom{n}{k} (-z)^k [\alpha_{n-k}, \beta(z)]$$

Additionally, the residue product relates the usual product and the normal ordered product.

Lemma 2.12. Suppose $\alpha(z), \beta(z) \in A[[z^{\pm 1}]]$ are two formal distributions such that there exits $N \in \mathbb{Z}_{\geq 0}$ with $\alpha(z) \overset{N}{\sim} \beta(z)$, then:

$$\alpha(z)\beta(w) =: \alpha(z)\beta(w) : + \sum_{j=0}^{N-1} \frac{(\alpha *_j \beta)(w)}{(z-w)^{j+1}}$$

Proof. See [16, Corollary 17] □

The above expression is related to the operator product expansion in chiral conformal field theory, where the normal ordering product $: \alpha(z)\beta(z) :$ represents the well-behaved interactions between two fields and the right-hand expression represents the "singular" parts of the product. We refer to Tuite's paper [16] for detailed methods of how to calculate the operator product expansion from the commutation relations.

2.3 Creative fields

Now consider $A = End(V)$, where $End(V)$ is the associative, unital k-algebra of k-linear endomorphisms of a k-vector space V, for some a field of characteristic zero k. Refer to the elements of V as *states*. A *vacuum vector* or *vacuum state* is a distinguished vector identified as $\mathbb{1} \in V$. Usually, this will be a non-zero state such as the highest weight in a Verma module.

Our formal distributions above can be naturally considered "fields", which create some state when acting on the distinguished vacuum state. To get a well-defined collection of fields for a vector space, we want to associate precisely one field to each state, which is also compatible with locality and an infinitesimal translation operator.

Definition 2.13. Suppose $\alpha(z) \in End(V)[[z^{\pm 1}]]$ is a formal generating series. $\alpha(z)$ is called a ***field***, if for any $v \in V$, then there exists an integer $N \in \mathbb{Z}_{\geq 0}$ such that for all $n \geq N$:

$$\alpha_n(v) = 0.$$

Remark. Given a collection of fields, we can define an action of the fields on the states as follows: if $\alpha(z) \in End(V)[[z^{\pm 1}]]$ is a formal generating series expressed as:

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1},$$

and $v \in V$ is a state, then define the action:

$$\alpha(z)(v) := \sum_{n \in \mathbb{Z}} \alpha_n(v) z^{-n-1} \in V[[z^{\pm 1}]].$$

Give two mutually local fields, then the n-th residue product preserves the pairwise locality of the resulting field. This was proved by Dong and is stated in the following lemma:

Lemma 2.14 (Dong's Lemma)**.** Let $\alpha(z), \beta(z), \gamma(z)$ be pairwise mutually local fields. Then $(\alpha *_n \beta)(z)$ and $\gamma(z)$ are mutually local fields for all $n \in \mathbb{Z}$.

Proof. See [16, Lemma 27]. □

When calculating with mutually local fields, we use the following identity due to Borcherds-Frenkel-Lepowsky-Meurmann, which relates the residue product of two fields and their associated bivariate series. It is sometimes refered to as the foundational axiom due to its importance.

Theorem 2.15 (Borcherds-Frenkel-Lepowsky-Meurmann identity)**.** Suppose $\alpha(z), \beta(z) \in End(V)[[z^{\pm 1}]]$ are mutually local fields. Then for all $l, m, n \in \mathbb{Z}$, we have:

$$\sum_{i \geq 0} \binom{l}{i} (\alpha *_{n+i} \beta)_{l+m-i} = \sum_{i \geq 0} (-1)^i \binom{n}{i} (\alpha_{l+n-i} \beta_{m+i} - (-1)^n \beta_{m_n - i} \alpha_{l+i})$$

Proof. A detailed proof can be found in [16, Theorem 24]. □

Definition 2.16. Let $\mathbb{1} \in V$ be a distinguished vacuum vector. A field defined by:

$$C_v(z) = \sum_{n\in\mathbb{Z}} (C_v)_n z^{-n-1} \in End(V)[[z^{\pm 1}]]$$

with Fourier modes $(C_v)_n \in End(V)$, is called a ***creative field*** for $v \in V$, if it satisfies the following conditions:

(a) $(C_v)_{-1}(\mathbb{1}) = v$

(b) $(C_v)_n(\mathbb{1}) = 0$ for all $n \geq 0$.

If $i \in \mathbb{Z}$, and $z^i \in End(V)[[z^{\pm 1}]]$ is a formal distribution, then there always exists a formal distribution $z^{-i} \in End(V)[[z^{\pm 1}]]$, such that the multiplication $z^i \cdot z^{-i} = z^{-i} \cdot z^i = z^0$. Suppose $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$ is a formal distribution in $End(V)[[z^{\pm 1}]]$. Then $f(z)$ satisfies the expression $f(z) = O(z^i)$ if and only if the formal distribution defined by:

$$\frac{f(z)}{z^i} := \sum_{n\in\mathbb{Z}} f_n z^n \cdot z^{-i} = \sum_{n\in\mathbb{Z}} f_n z^{n-i}$$

is a formal power series in $End(V)[[z]]$. Note that $O(z^i)$ has the algebraic properties $O(z^i) + O(z^j) = O(z^{min\{i,j\}})$, and $O(z^i) \cdot O(z^j) = O(z^{i+j})$. It is then convenient to write the axioms of a creative field as the single condition:

$$C_v(z)(\mathbb{1}) = vz^0 + O(z) \in V[[z]]$$

where it is also common to write $C_v(z)(\mathbb{1}) = v + O(z)$ omitting the z^0 term.

The above notation for creative fields differs from conventional literature because we want to emphasis that we have not yet established a one-to-one correspondence between fields $C_v(z)$ and the states $v \in V$. In other words, we want to distinguish between possibly different creative fields for v by writing expressions such as $C_v(z)$ and $C'_v(z)$. Hence, the following section will retain the awkward notation. When we write $(C_v)_n$ we refer to the n-th mode of the field $C_v(z)$. Recall, this is well-defined and can be recovered by setting:

$$(C_v)_n = Res_z\big(z^n C_v(z)\big)$$

Proposition 2.17. If $v, w \in V$ are states such that $C_v(z)$ is a creative field for v and $C_w(z)$ is a creative field for w, then

(a) for all $n \in \mathbb{Z}$, $(C_v *_n C_w)(z)$ is a creative field for $(C_v)_n(w) \in V$,

(b) if $\lambda, \mu \in k$ then $\lambda C_v(z) + \mu C_w(z)$ is a creative field for $\lambda v + \mu w \in V$,

(c) $C_v(z) + zC_w(z)$ is a creative field for v.

Proof. (a) Fix $n \in \mathbb{Z}$. By the residue product expansion and creativity of $C_w(z)$, we have:

$$\begin{aligned}(C_v *_n C_w)(z)(\mathbb{1}) &= \sum_{i \in \mathbb{Z}_{\geq 0}} \binom{n}{i} \Big((-z)^i (C_v)_{n-i} C_w(z) - (-z)^{n-i} C_w(z)(C_v)_i \Big)(\mathbb{1}) \\ &= \sum_{n \in \mathbb{Z}_{\geq 0}} \binom{n}{i} (-z)^i (C_v)_{n-i}(w + O(z)) \\ &= (C_v)_n(w) + O(z).\end{aligned}$$

Therefore, $(C_v *_n C_w)(z)$ is a creative field for $(C_v)_n(w) \in V$.
(b) If $\lambda, \mu \in k$, then,

$$\begin{aligned}\big(\lambda C_v(z) + \mu C_w(z)\big)(\mathbb{1}) &= \lambda C_v(z)(\mathbb{1}) + \mu C_w(z)(\mathbb{1}) \\ &= \lambda\big(v + O(z)\big) + \mu\big(w + O(z)\big) \\ &= \lambda v + \mu w + O(z).\end{aligned}$$

Therefore, $\lambda C_v(z) + \mu C_w(z)$ is a creative field for $\lambda v + \mu w \in V$.
(c) Since $C_w(z)$ is a creative field for $w \in V$, then $(C_w)_1(w) = 0$, and:

$$\begin{aligned}zC_w(z)(\mathbb{1}) &= z \sum_{n \in \mathbb{Z}} (C_w)_n(w) z^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} (C_w)_n(w) z^{-n} \\ &= \ldots + (C_v)_1(w) z^0 + (C_v)_0(w) z^1 + (C_v)_{-1}(w) z^2 + \ldots \\ &= 0 + O(z).\end{aligned}$$

Therefore, $zC_w(z)$ is a creative field for $0 \in V$. Since $C_v(z)$ is a creative field for v, then:

$$\begin{aligned}(C_v(z) + zC_w(z))(\mathbb{1}) &= C_v(z)(\mathbb{1}) + zC_w(z)(\mathbb{1}) \\ &= v + 0 + O(z) \\ &= v + O(z).\end{aligned}$$

Thus, $C_v(z) + zC_w(z)$ is a creative field for v. □

Note that if both $C_v(z), C_w(z)$ are mutually local to some field $\gamma(z)$, then $C_v(z) + zC_w(z)$ is also mutually local to $\gamma(z)$. In general, $C_v \neq C_v(z) + zC_w(z)$, hence for each state there may be many mutually local creative fields associated to it. Thus, we can define many bijective collections of fields for a vector space. The introduction of the translation operator in the next section establishes a criteria of uniqueness for such a collection.

2.4 Translation covariance and uniqueness

Suppose V is a k-vector space with a distinguished vacuum vector $\mathbb{1} \in V$. Let

$$\mathscr{F} := \{C_v(z) \mid v \in V\}$$

be a set of fields containing precisely one creative field $C_v(z) \in \mathscr{F}$ for each $v \in V$ such that it satisfies, if $C_w(z) \in \mathscr{F}$ is another field creative for $w \neq v$ in V, then there exists some $N \in \mathbb{Z}_{\geq 0}$ such that $C_v(z) \overset{N}{\sim} C_w(z)$. We call $\mathscr{F}$ is a *set of mutually local creative fields* for $(V, \mathbb{1})$.

The set $\mathscr{F}$ is not necessary unique since by Proposition 2.17(c) if $C_v(z) \in \mathscr{F}$ is a creative field for $v \in V$ and $C_w(z) \in \mathscr{F}$ is a creative field for $w \in V$, then we can define another set of mutually local creative fields for $(V, \mathbb{1})$ by:

$$\mathscr{F}' := \mathscr{F} \setminus \{C_v(z)\} \cup \{C_v(z) + zC_w(z)\}$$

Lemma 2.18. Suppose $\phi(z)$ is a creative field for the zero state $0 \in V$. If $0(z)$ is the zero formal distribution, then:

$$\phi(z)(\mathbb{1}) = 0 \iff \phi(z) = 0(z)$$

Proof. If $\phi(z) = 0(z)$, then by definition $\phi(z)(\mathbb{1}) = 0(z)(\mathbb{1}) = 0$.

Conversely, suppose $\phi(z)(\mathbb{1}) = 0$. Let $\alpha(z) \in \mathscr{F}$ be another creative field for $v \in V$. By the property of being mutually local, there exists some $N \in \mathbb{Z}_{\geq 0}$ such that, $\alpha(z) \overset{N}{\sim} \phi(z)$. In other words, $\phi(z)$ and $\alpha(w)$ satisfy:

$$(z-w)^N[\phi(z), \alpha(w)] = 0(z)$$

Applying these distributions to the vacuum vector gives the following calculation:

$$0 = z^{-N}0(z)(\mathbb{1}) = z^{-N}(z-w)^N[\phi(z), \alpha(w)](\mathbb{1}) = z^{-N}(z-w)^N\phi(z)\alpha(w)(\mathbb{1}) = \phi(z)(v) + O(w)$$

Therefore, $\phi(z)(v) = 0$ for all $v \in V$ and, $\phi(z) = 0(z)$. □

Lemma 2.19. If $C_v(z), C'_v(z)$ are creative fields for $v \in V$, then:

$$C_v(z)(\mathbb{1}) = C'_v(z)(\mathbb{1}) \iff C_v(z) = C'_v(z)$$

Proof. Set $\phi = C_v(z) - C'_v(z)$ and apply Lemma 2.18. □

Thus, creative fields are determined for each state by how they act on $\mathbb{1}$. If a creative field exists, then pairwise mutual locality and compatibility with a translation operator are sufficient requirements to define a unique set of creative fields.

Definition 2.20. Suppose $(V, \mathbb{1})$ and a set of mutually local creative fields $\mathscr{F} := \{C_v(z) \mid v \in V\}$ for $(V, \mathbb{1})$. A $\mathscr{F}$*−**translation operator***, $T \in End(V)$, is an endomorphism of V, such that:

(a) $T(\mathbb{1}) = 0$

(b) $[T, C_v(z)] = \partial C_v(z)$ for all $C_v(z) \in \mathscr{F}$.

Condition (b) is equivalent to saying that all Fourier modes of $C_v(z)$ satisfy:

$$[T, (C_v)_n] = -n(C_v)_{n-1}$$

Definition 2.21. A set $\mathscr{F}$ of mutually local creative fields and $T \in End(V)$ such that $T(\mathbb{1}) = 0$ are ***compatible*** if condition (b) is satisfied. The pair $(\mathscr{F}, T)$ is then said to be *translation covariant*.

The following theorem then modifies Tuite's uniqueness theorem [16, Tuite, Theorem 33] and we have added further details to clarify what the theorem is actually trying to state.

Theorem 2.22. Suppose $(V, \mathbb{1})$ is a vector space with a distinguished vacuum vector.

(a) Given $T \in End(V)$ a $\mathscr{F}$-translation operator, then there exists at most one set of mutually local fields $\mathscr{F}$ for $(V, \mathbb{1})$ compatible with T.

(b) Given a set $\mathscr{F}$ of mutually local creative fields for $(V, \mathbb{1})$, then there exists at most one $\mathscr{F}$-translation operator $T \in End(V)$ compatible with $\mathscr{F}$,

(c) Given a set $\mathscr{F}$ of mutually local cretive fields $\{C_v(z) \mid v \in V\}$ satisfying:

(i) $C_{\lambda v + \mu w}(z) = \lambda C_v(z) + \mu C_w(z)$,

(ii) $C_{\mathbb{1}}(z) = I(z)$,

(iii) $C_{(C_v)_n(w)}(z) = (C_v * C_w)(z)$.

then a compatible $\mathscr{F}$-translation operator T exists for $\mathscr{F}$.

Proof. (a) Given T a $\mathscr{F}$-translation operator. Suppose there exists another compatible set of mutually local creative fields $\mathscr{F}'$. Let $C_v(z)$ be a creative field for $v \in V$ in $\mathscr{F}$ and for any $k \in \mathbb{Z}$ we have that:

$$(T \cdot (C_v)_k - (C_v)_k \cdot T)(\mathbb{1}) = -k(C_v)_{k-1}(\mathbb{1})$$

By the first property of the translation operator, $T(\mathbb{1}) = 0$, and exchanging indices, $k \mapsto -k$, the above expression becomes:

$$T(C_v)_{-k}(\mathbb{1}) = k(C_v)_{-k-1}(\mathbb{1})$$

For $k = 1$ and by creativity of $C_v(z)$ we have, $T(v) = T(C_v)_{-1}(\mathbb{1}) \in V[[z^{\pm 1}]]$. Applying the operator T iteratively, we have the following expression:

$$T^n(v) = n!(C_v)_{n-1}(\mathbb{1})$$

for all $n \in \mathbb{Z}$. Define the formal distribution e^{zT} by:

$$e^{zT} := \sum_{n=0}^{\infty} \frac{1}{n!} T^n z^n,$$

then the following calculation:

$$\begin{aligned} e^{zT}(v) &= \sum_{n=0}^{\infty} \frac{1}{n!} T^n(v) z^n \\ &= \sum_{n=0}^{\infty} (C_v)_{n-1}(\mathbb{1}) z^n \\ &= \sum_{m=-1}^{-\infty} (C_v)_m(\mathbb{1}) z^{-m-1} \\ &= C_v(z)(\mathbb{1}) \end{aligned}$$

determines the field $C_v(z)(\mathbb{1})$. Thus, if $C'_v(z)$ is another creative field for $v \in V$ in $\mathscr{F}'$, then it must be the case that:

$$C'_v(z)(\mathbb{1}) = e^{zT}(v) = C_v(z)(\mathbb{1})$$

By Lemma 2.19, we conclude that $C'_v(z) = C_v(z)$. Since this is true for arbitrary $v \in V$ we must have $\mathscr{F} = \mathscr{F}'$.

Note that if $\phi(z) \in \mathscr{F}$ is a creative field for $\mathbb{1}$, then since $T(\mathbb{1}) = 0$ and $T^0 := id_v$, the following calculation with Lemma 2.19:

$$\begin{aligned}\phi(z)(\mathbb{1}) &:= e^{zT}(\mathbb{1})\\ &= \sum_{n=0}^{\infty} T^n(\mathbb{1})z^n\\ &= \mathbb{1}z^0\\ &= I(z)(\mathbb{1})\end{aligned}$$

shows that $I(z)$ is the creative field for $\mathbb{1}$ and determines the creative field $C_{\mathbb{1}}(z)$ in the collection $\mathscr{F}$.

(b) This is immediate from part (a) since any translation operator T must satisfy the commutation relation:

$$T(C_v)_{-k}(\mathbb{1}) = k(C_v)_{-k-1}(\mathbb{1}).$$

for all $k \in \mathbb{Z}$. Thus, evaluating at $k = 1$, then determines $T \in End(V)$ by:

$$T(v) = T(C_v)_{-1}(\mathbb{1}) = (C_v)_{-2}(\mathbb{1}).$$

(c) Given a set $\mathscr{F}$ of mutually local creative fields for $(V, \mathbb{1})$, then for each $v \in V$, let $C_v(z) = \sum_{n\in\mathbb{Z}} (C_v)_n z^{-n-1}$ be the creative field for $v \in V$ in $\mathscr{F}$. Define a map by:

$$T(v) = (C_v)_{-2}(\mathbb{1})$$

We claim that T is a linear endomorphism and that it is a compatible translation operator with $\mathscr{F}$. If $v, w \in V$ are states with creative fields $C_v(z) = \sum_{n\in\mathbb{Z}} (C_v)_n z^{-n-1}$ and $C_w(z) = \sum_{n\in\mathbb{Z}} (C_v)_n z^{-n-1}$ respectively in $\mathscr{F}$ and $\lambda, \mu \in k$, then since $C_{\lambda v+\mu w}(z) = (\lambda C_v + \mu C_w)(z)$, the following calculation:

$$\begin{aligned}T(\lambda v + \mu w) &= (C_{\lambda v+\mu w})_{-2}(\mathbb{1})\\ &= (\lambda C_v + \mu C_w)_{-2}(\mathbb{1})\\ &= \lambda(C_v)_{-2}(\mathbb{1}) + \mu(C_w)_{-2}(\mathbb{1})\\ &= \lambda T(v) + \mu T(w)\end{aligned}$$

shows that T is linear.

By assumption, $I(z) = id_V z^0$ is the creative field for $\mathbb{1} \in V$. Hence, all the Fourier modes of $I(z)$ except for $n = -1$ are the zero endomorphism. Thus, property (a) of a translation operator is satisfied since:

$$T(\mathbb{1}) = I_{-2}(\mathbb{1}) = \mathbf{0}(\mathbb{1}) = 0$$

Consider the vector $(C_v)_n(w) \in V$, where $(C_v)_n \in End(V)$ and $w \in V$. If $C_v(z), C_w(z)$ are creative fields for $v, w \in V$ respectively, then $(C_v *_n C_w)(z) \in F$ is a creative field for $(C_v)_n(w)$ for all $n \in \mathbb{Z}$. By expanding the residue product expression:

$$\begin{aligned} T\big((C_v)_n(w)\big) &= (C_v *_n C_w)_{-2}(\mathbb{1}) \\ &= \sum_{i \geq 0} (-1)^i \binom{n}{i} \big((C_v)_{n-i}(C_w)_{i-2} - (-1)^n (C_w)_{n-i-2}(C_v)_i\big)(\mathbb{1}) \end{aligned}$$

Note that for all ≥ 0, $(C_v)_i(\mathbb{1}) = 0$. Thus, the second term in the summand above vanishes. Likewise, only the $(C_w)_{-2}$ and $(C_w)_{-1}$ terms do not annihilate the vacuum vector, hence, the expression reduced down to:

$$\begin{aligned} T\big((C_v)_n(w)\big) &= (C_v)_n(C_w)_{-2}(\mathbb{1}) - n(C_v)_{n-1}(C_w)_{-1}(\mathbb{1}) \\ &= (C_v)_n(T(w)) - n(C_v)_{n-1}(w) \end{aligned}$$

Therefore, for all $w \in V$:

$$\begin{aligned} T\big((C_v)_n(w)\big) - (C_a)_n\big(T(w)\big) &= -n(C_v)_{n-1}(w) \\ [T, (C_v)_n](w) &= -n(C_v)_{n-1}(w) \end{aligned}$$

Hence, the second property is satisfied, and T is a $\mathscr{F}$-translation operator.

□

2.5 Vertex algebra definitions

The canonical definition of a vertex algebra is given in [16], [10], or [7] as follows:

Definition 2.23. A ***vertex algebra*** is the data $(V, Y, \mathbb{1}, T)$, where V is a vector space, $\mathbb{1} \in V$ is a distinguished vacuum vector, $T \in End(V)$ a linear endomorphism of V, and a linear map

Y, called the *state-field correspondence*, defined by:

$$Y : V \to End(V)[[z, z^{-1}]]$$
$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

where $a_n \in End(V)$, subjected to the following axioms:

(a) **locality:** $Y(a,z) \sim Y(b,z)$, for all $a, b \in V$

(b) **creativity:** $Y(a,z)(\mathbb{1}) = a + O(z)$

(c) **translation covariance:** $[T, Y(a,z)] = \partial Y(a,z)$, and $T(\mathbb{1}) = 0$.

In general, the fields are defined first for the vertex algebra then the correspondence between the states and fields is given. Additionally, in this definition we are not given the modes $\{a_n \in End(V) \mid n \in \mathbb{Z}\}$ a priori, thus it is not clear how the fields are defined with respect for each state. We provide an alternative definition for a vertex algebra, which provides a field-theoretic definition.

Definition 2.24. A ***vertex algebra*** is the data $(V, \mathbb{1}, \mathscr{F})$, where V is a vector space, $\mathbb{1} \in V$ is a distinguished vacuum vector, and $\mathscr{F}$ is a set of fields, satisfying the following axioms:

(a) **creativity:** for all $v \in V$, there exists precisely one $C_v(z) \in \mathscr{F}$ such that $C_v(z)(\mathbb{1}) = v + O(z)$, and $I(z)$ is the creative field for the vacuum vector $\mathbb{1}$,

(b) **locality:** if $C_v(z), C_w(z) \in \mathscr{F}$ then there exists $N \in \mathbb{Z}_{\geq 0}$ such that $C_v(z) \overset{N}{\sim} C_w(z)$,

By Theorem 2.22, there exists at most one $T \in End(V)$ compatible with $\mathscr{F}$. Given a vertex algebra $(V, \mathbb{1}, \mathscr{F})$, where $\mathscr{F} = \{C_a(z) \mid a \in V\}$, then the state-field correspondence Y, is defined by:

$$Y : V \to End(V)[[z^{\pm 1}]]$$
$$a \mapsto Y(a,z) := C_a(z)$$

Lemma 2.25. This map is an injective linear homomorphism.

Proof. This is a consequence of Proposition 2.17 and Theorem 2.22. □

Hence, for each state $a \in V$ there is precisely one mutually local creative field $C_a(z)$. We can now make the bijective correspondence and identify:

$$a(z) := Y(a,z) = C_a(z)$$

Proposition 2.26. The two definitions of vertex algebras are equivalent.

Proof. By the discussion above, given a $(V, \mathbb{1}, \mathscr{F})$ vertex algebra there exists an endomorphism T and we can define a state-field correspondence Y to define a $(V, \mathbb{1}, Y, T)$ vertex algebra. Conversely, given $(V, \mathbb{1}, Y, T)$ we can define a set of fields $\mathscr{F} := \{Y(a,z) \mid a \in V\}$ with T a compatibile $\mathscr{F}$-translation operator. Hence, this defines a $(V, \mathbb{1}, \mathscr{F})$ vertex algebra. □

In most cases, vertex algebras are constructed by defining the set of creative fields acting on the vaccum state. The following theorem allows us to explicitly define a unique vertex algebra structure on a vector space V generated by the modes of some fields. This extends fields defined on the basis of a vector space to fields for all states.

Theorem 2.27 (Reconstruction/Generating Theorem)**.** Suppose V is a vector space with a vacuum vector $\mathbb{1} \in V$. If $\mathscr{F}' = \{a^i(z) \mid i \in I\}$ is a collection of mutually local creative fields, where I is some indexing set, satisfying:

$$V = span\{a^{i_1}_{n_1}...a^{i_k}_{n_k}(\mathbb{1}) \mid n_1,...n_k \in \mathbb{Z}, i_1,...,i_k \in I\}$$

By the term $a^{i_j}_{n_j} \in End(V)$, we mean the n_j-th mode of the field $a^{i_j}(z)$. Then,

$$Y(a^{i_1}_{n_1}(z)...a^{i_k}_{n_k}(\mathbb{1}), z) = a^{i_1} *_{n_1} (a^{i_2} *_{n_2} (...(a^{i_k} *_{n_k} I)))(z)$$

defines a creative field for $a^{i_1}_{n_1}(z)...a^{i_k}_{n_k}(\mathbb{1})$, such that $Y(a^i, z) = a^i(z)$. Defining

$$\mathscr{F} = \{Y(a^{i_1}_{n_1}(z)...a^{i_k}_{n_k}(\mathbb{1}), z) \mid n_1,...n_k \in \mathbb{Z}, i_1,...,i_k \in I\}$$

then makes $(V, \mathbb{1}, \mathscr{F})$ a vertex algebra.

Proof. See [16, Theorem 40]. □

Example 2.28. The canonical example of a vertex algebra is the Heisenberg vertex algebra associated to the rank one free boson, as defined in [16] or [7]. Consider $\mathbb{C}[t, t^{-1}]$ as the abelian Lie algebra of Laurent polynomials in one variable. Define the Heisenberg Lie algebra, $\mathscr{H}$, as the central extension:

$$0 \to \mathbb{C}\mathbb{1} \to \mathscr{H} \to \mathbb{C}[t, t^{-1}] \to 0$$

with the cocycle,

$$c(f, g) = -Res_{t=0} f dg$$

This Lie algebra has a basis $h_n := t^n$ for $n \in \mathbb{Z}$ and a central element $\mathbb{1}$. The cocycle evaluated on basis elements give:

$$\begin{aligned} c(h_n, h_m) &= -Res_{t=0} t^n dt^m \\ &= -mRes_{t=0} t^{n+m-1} dt \\ &= -m\delta_{n,-m} = n\delta_{n,-m} \end{aligned}$$

Therefore, the basis elements satisfy the following Lie brackets:

$$[h_n, h_m] = n\delta_{n,-m}\mathbb{1}, \quad [\mathbb{1}, h_n] = 0$$

Recall, that the universal enveloping algebra, denoted $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, as defined in [5], is obtained as a quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n\geq 0} g^{\otimes n}$ by the two-sided ideal generated by the elements of the form $x \otimes y - y \otimes x - [x,y]$ for all $x, y \in \mathfrak{g}$. Then $U(\mathfrak{g})$ is an associative algebra with unit, and any representation of $\mathfrak{g}$ is immediately an $U(\mathfrak{g})$-module.

By the Poincare-Birkhoff-Witt theorem [5, Chapter 14.3], the set of all monomials:

$$h_{n_1} h_{n_2} ... h_{n_{k-1}} h_{n_k}$$

with $n_1 \leq n_2 \leq ... \leq n_k$, $k \geq 0$, forms a basis for $U(\mathscr{H})$, satisfying the relations:

$$h_n h_m - h_m h_n = n\delta_{n,-m}\mathbb{1}, \quad h_n\mathbb{1} - \mathbb{1}h_n = 0.$$

Construct M_0 as the Verma module of the Heisenberg Lie algebra by the quotient,

$$M_0 := U(\mathscr{H})/U(\mathscr{H}_+)$$

where $U(\mathscr{H}_+)$ is the two-sided ideal generated by the positive modes h_n for all $n > 0$. As a vector space, the Verma module can be considered as the span of the Poincare-Birkhoff-Witt basis with negative indices:

$$M_0 = span\{h_{-n_1} ... h_{-n_k}(\mathbb{1}) \mid n_1, ..., n_k \geq 1\}$$

where $h_n(v) = 0$ for all $n \geq 0$ and $\mathbb{1}(v) = v$ for all $v \in M_0$. Define a map:

$$T = \sum_{n\in\mathbb{Z}} h_{-n-1} h_n$$

and a field encoding the basis elements as the Fourier modes,

$$h(z) := \sum_{n\in\mathbb{Z}_{\geq 0}} h_n z^{-n-1}$$

Thus, $h(z)$ is the creative field for $h := h_{-1}(\mathbb{1})$ since,

$$h(z)(\mathbb{1}) = \sum_{n\in\mathbb{Z}} h_n(\mathbb{1})z^{-n-1} = h_{-1}(\mathbb{1}) + O(z)$$

This field is also compatible with T since:

$$[T, h(z)] = Th(z) - h(z)T = \partial h(z)$$

and $T(\mathbb{1}) = \sum_{n\in\mathbb{Z}} h_{-n-1}h_n(\mathbb{1}) = 0$. Therefore, by Theorem 2.27 and Proposition 2.17(b) the field $h(z)$ uniquely generates the Heisenberg vertex algebra $(M_0, \mathbb{1}, \mathscr{F})$ with fields defined by:

$$Y(h_{-n_1}...h_{-n_k}(\mathbb{1})) := (h *_{n_1} (h *_{n_2} ...(h *_{n_k} I)))(z)$$

for $n_1, ..., n_k \geq 1$.

Example 2.29. Let $N = 2k$ for $k \in \mathbb{Z}_{>0}$, then the one-dimensional lattice $L := \sqrt{N}\mathbb{Z}$ induces a vertex algebra $V_{\sqrt{N}\mathbb{Z}}$ by taking infinite copies of the Verma module M_0 as defined above. Define $M_\lambda := M_0$ for all $\lambda \in L$, then the vertex algebra has an underlying vector space:

$$V_{\sqrt{N}\mathbb{Z}} := \bigoplus_{\lambda\in\sqrt{N}\mathbb{Z}} M_\lambda$$

By Theorem 2.27 above, it suffices to define fields $V_\lambda(z) := Y(|\lambda\rangle, z)$ for $\lambda \in \sqrt{N}\mathbb{Z}$ that satisfy the various conditions of locality, creativity and translation covariance. This is explicitly writen down in [7, §4.2.8] by:

$$V_\lambda(z) := S_\lambda z^{\lambda h_0} exp(-\lambda \sum_{n<0} \frac{h_n}{n} z^n) exp(-\lambda \sum_{n>0} \frac{h_n}{n} z^n) \tag{2.1}$$

where S_λ is the shift operator $M_\lambda \to M_{\lambda+\mu}$ defined by the conditions: $S_\lambda|\mu\rangle = c_{\lambda,\mu}|\lambda+\mu\rangle$, for a certain choice of constants $c_{\lambda,\mu}$. Note that for $\lambda = m\sqrt{N}$, then $z^{\lambda h_0} = z^{mkN}$. The fields $V_\lambda(z)$ are called bosonic vertex operators and were originally introduced in string theory [10, Remark 4.2.7]. In Chapter 4, we will reconstruct this vertex algebra by defining a singular multiplication of two states extending the bicharacter of the lattice.

Remark. We conclude this chapter by roughly illustrating the connection between vertex algebras and coalgebras. Firslty, recall that there does not exist a natural multiplication on the space of distributions but we can define a natural comultiplication and counit, hence a coalgebra structure on the space of distributions. This can also be realised by the fact that the Sweedler dual is a functor from the category of algebras to the category of coalgebras by associating an algebra to a maximal subspace of its dual space. If our algebra is the algebra of operators on some space then the dual space will be the space of linear functionals or operator-valued distributions. Secondly, the coordinate ring of the affine algebraic group of translation of $\mathbb{A}^1_k$ is a Hopf algebra. Then since the underlying symmetries of the one-dimensional lattice are the translations of $\mathbb{A}^1_k$, it is not surprising that Hopf algebras appear in the theory of lattice vertex operator algebras. We explore this natural connection between vertex algebras, coalgebras, and Hopf algebras in the next chapter.

Chapter 3

Hopf algebras

The aim of this chapter is to introduce the theory of Hopf algebras and prove the existence of the universal measuring algebra, which will be the underlying object of the lattice vertex algebra. We motivate the reader with a discussion about associating Hopf algebras to algebraic groups. Hence, the properties of an algebraic group can be determined by the algebraic properties of its corresponding Hopf algebra.

The following definitions and results largely follow Sweedler's textbook "Hopf algebras" [15]. Fix a ground field k of characteristic zero, which can be assumed to be $\mathbb{C}$. All vector spaces are to be considered vector spaces over k. Additionally, "map" always means a k-linear homomorphism of vector spaces, and tensor products $V \otimes W$ are to be understood to be $V \otimes_k W$.

Recall that the definition of an associative unital *algebra* over k is a triple (A, ∇, μ) with A a vector space, a map $\nabla : A \otimes A \to A$ called *multiplication* and a map $\eta : k \to A$ called the *unit*, satisfying the associative and unit axioms, which can be expressed by the commutative diagrams:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes \nabla} & A \otimes A \\ {\scriptstyle \nabla \otimes id}\downarrow & & \downarrow{\scriptstyle \nabla} \\ A \otimes A & \xrightarrow{\nabla} & A \end{array} \qquad \begin{array}{ccccc} & \overset{\eta \otimes id}{\nearrow} & A \otimes A & \overset{id \otimes \eta}{\nwarrow} & \\ & & \downarrow{\scriptstyle \nabla} & & \\ k \otimes A & \xrightarrow{\cong} & A & \xleftarrow{\cong} & A \otimes k \end{array}$$

The dimension of an associative unital algebra is the dimension of the underlying vector space. Hence, an associative unital algebra is finite dimensional if it is a finite dimensional vector space. An *algebra morphism* between (A, ∇_A, η_A) and (B, ∇_B, η_B) is a map $f : A \to B$

that makes the following diagrams commute:

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\nabla_A} & A \\
{\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\
B \otimes B & \xrightarrow{\nabla_B} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
{\scriptstyle \eta_A}\nwarrow & & \nearrow{\scriptstyle \eta_B} \\
 & k &
\end{array}
$$

Denote $\mathbf{Alg}_k(A,B)$ the set of all algebra morphisms from an algebra A to an algebra B.

3.1 Coalgebras

The diagrammatic definition of an algebra and its properties can be dualised by "reversing all the arrows" to give the corresponding definition of a coalgebra.

Definition 3.1. A ***coassociative counital coalgebra*** is the triple (C,Δ,ε) with C a vector space, a map $\Delta : C \to C\otimes C$ called *comultiplication*, and a map $\varepsilon : C \to k$ called the *counit*, making the following diagrams commute:

$$
\begin{array}{ccc}
C\otimes C\otimes C & \xleftarrow{id\otimes\Delta} & C\otimes C \\
{\scriptstyle \Delta\otimes id}\uparrow & & \uparrow{\scriptstyle \Delta} \\
C\otimes C & \xleftarrow{\quad\Delta\quad} & C
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & C\otimes C & & \\
 & {\scriptstyle \varepsilon\otimes id}\swarrow & \uparrow{\scriptstyle \Delta} & \searrow{\scriptstyle id\otimes\varepsilon} & \\
k\otimes C & \xleftarrow{\cong} & C & \xrightarrow{\cong} & C\otimes k
\end{array}
$$

A coalgebra is additionally called ***cocommutative*** if the following diagram commutes,

$$
\begin{array}{ccc}
 & C & \\
{\scriptstyle \Delta}\swarrow & & \searrow{\scriptstyle \Delta} \\
C\otimes C & \xrightarrow{\quad\sigma\quad} & C\otimes C
\end{array}
$$

where $\sigma(c_1\otimes c_2) = c_2\otimes c_1$ for all $c_1,c_2\in C$. Given two coalgebras, $(C_1,\Delta_1,\varepsilon_1)$ and $(C_2,\Delta_2,\varepsilon_2)$, a ***coalgebra morphism*** is a map $f : C_1 \to C_2$ such that the following diagrams commute:

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\Delta_1} & C_1\otimes C_1 \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes f} \\
C_2 & \xrightarrow{\Delta_2} & C_2\otimes C_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
C_1 & \xrightarrow{\quad f \quad} & C_2 \\
{\scriptstyle \varepsilon_1}\searrow & & \swarrow{\scriptstyle \varepsilon_2} \\
 & k &
\end{array}
$$

Denote $\mathbf{CoAlg}_k(C,D)$ the set of all coalgebra morphisms from C to D.

Example 3.2. Suppose S is a set. Denote kS the vector space with S as a basis. Defining the comultiplication and counit by:

$$\Delta(s) = s \otimes s, \quad \varepsilon(s) = 1$$

for all $s \in S$, makes $(kS, \Delta, \varepsilon)$ into a cocommutative coalgebra.

Remark. Suppose (C, Δ, ε) is a coalgebra. Given an element $c \in C$, we will adopt the sigma notation to assist in calculations:

$$\Delta(c) := \sum c_{(1)} \otimes c_{(2)}$$

where $c_{(1)}, c_{(2)} \in C$. Similarly, set the notation:

$$(\Delta \circ \Delta)(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$$

and in general,

$$\Delta^{n-1}(c) = \sum c_{(1)} \otimes \ldots \otimes c_{(n)}.$$

Given a vector space V (not necessarily finite dimensional), define its linear dual space $V^* = Hom_k(V, k)$ to be the vector space of maps from V to the ground field k. In general, we can define an injective linear homomorphism:

$$\rho_{V,W} : V^* \otimes W^* \to (V \otimes W)^*,$$

by $\rho_{V,W}(f \otimes g)(v \otimes w) = f(v)g(w)$ for all $f \in V^*$, $g \in W^*$, $v \in V$, and $w \in W$, which is natural in both variables. If both V and W are finite dimensional, then $\rho_{V,W}$ determines an isomorphism $(V \otimes W)^* \cong V^* \otimes W^*$. This is no longer true when either V or W is infinite dimensional.

If $f : V \to W$ is a map between vector spaces, then there is an induced map on the linear duals $f^* : W^* \to V^*$, called the *transpose*, defined by the precomposition $f^*(g) = g \circ f$ for all $g \in W^*$. Consider a coalgebra (C, Δ, ε) and it's dual space C^*. The comultiplication $\Delta : C \to C \otimes C$ induces a multiplication $\nabla : C^* \otimes C^* \xrightarrow{\rho_{C,C}} (C \otimes C)^* \xrightarrow{\Delta^*} C^*$ where Δ^* is the transpose of Δ. Likewise, the counit induces a unit $\eta : k \cong k^* \xrightarrow{\varepsilon^*} C^*$ where ε^* is the transpose of ε. These maps also satisfy the associative and unit axioms. Hence, the triple (C^*, ∇, η), defined as above, is an algebra.

Proposition 3.3. There is a contravariant functor $(\ )^* : \mathbf{CoAlg}_k \to \mathbf{Alg}_k$ between the category of coassociative counital coalgebras to the category of associative unital algebras, where each coalgebra C is mapped to its dual space C^* and every coalgebra morphism, $f : C \to D$, is mapped to an algebra morphism $f^* : D^* \to C^*$ defined by $f^*(\psi) = \psi \circ f$ for every $\psi \in D^*$.

Proof. Given the construction above there is a well-defined algebra (C^*, ∇, η) corresponding to every coalgebra (C, Δ, ε). It suffices to check that for every coalgebra morphism $f : (C, \Delta_C, \varepsilon_C) \to (D, \Delta_D, \varepsilon_D)$, then $f^* : D^* \to C^*$ is an algebra morphism. Indeed, this follows since the following diagrams commute:

$$\begin{array}{ccccc} D^* \otimes D^* & \xrightarrow{\rho_{D,D}} & (D \otimes D)^* & \xrightarrow{\Delta_D^*} & D^* \\ \big\downarrow{\scriptstyle f^* \otimes f^*} & & \big\downarrow{\scriptstyle (f \otimes f)^*} & & \big\downarrow{\scriptstyle f^*} \\ C^* \otimes C^* & \xrightarrow{\rho_{C,C}} & (C \otimes C)^* & \xrightarrow{\Delta_C^*} & C^* \end{array} \qquad \begin{array}{ccc} & k \cong k^* & \\ {\scriptstyle \varepsilon_D^*} \swarrow & & \searrow {\scriptstyle \varepsilon_C^*} \\ D^* & \xrightarrow{\quad f^* \quad} & C^* \end{array}$$

where the first square commutes by the naturality of ρ, the second square and triangle commute since they are the dual of commuting diagrams in $\mathbf{CoAlg}_k$. □

Example 3.4. Consider the coalgebra kS defined in example 1.2. Note that kS has a universal property. There is a natural inclusion map of sets $i : S \to kS$, and for every map of sets $f : S \to k$, there is a k-linear map $\tilde{f} : kS \to k$ that makes the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{i} & kS \\ & {\scriptstyle f} \searrow & \big\downarrow{\scriptstyle \tilde{f}} \\ & & k \end{array}$$

Thus, there is a bijection of sets $(kS)^* := Hom_k(kS, k) \cong Map(S, k)$. We can then identify the dual linear space of kS as the vector space of set maps from S to k, where the algebraic structure of $(kS)^*$ is given by:

$$\begin{aligned} (f+g)(s) &= f(s) + h(s) \\ (fg)(g) &= f(s)g(s) \\ (\alpha f)(s) &= \alpha f(s) \end{aligned}$$

for all $f, g \in Map(S, k)$, $s \in S$ and $\alpha \in k$.

A natural question to consider is whether there is a dual functor $(\ )^*$ from the category of algebras to the category of coalgebras. Given a finite dimensional algebra, A, an analogous construction can be used to define a multiplication and unit map. However, for infinite dimensional algebras, these maps are no longer well-defined. Instead, we need to consider a maximal subspace of A^* as defined in [15, Chapter 6].

Definition 3.5. Suppose (A, ∇, η) is an algebra over k. Define the ***Sweedler dual space***:

$$A^\circ := \{f \in A^* \mid ker(f) \text{ contains a cofinite two-sided ideal } I\}$$

where a two-sided ideal I is *cofinite* if the quotient A/I is a finite dimensional algebra over k.

Remark. $f \in A^\circ$ if and only if $f : A \to k$ factors via $A \xrightarrow{q} A/I \xrightarrow{\tilde{f}} k$, where A/I is a finite dimensional algebra over k. If A is finite dimensional, then obviously $A^\circ = A^*$ by taking $I = 0$ for all $f \in A^*$.

Lemma 3.6. Suppose $f : A \to B$ is an algebra morphism between A and B. Then the Sweedler dual satisfies the following conditions:

(a) If $f^* : B^* \to A^*$ is the respective dual map, then $f^*(B^\circ) \subset A^\circ$

(b) $A^\circ \otimes B^\circ \cong (A \otimes B)^\circ$

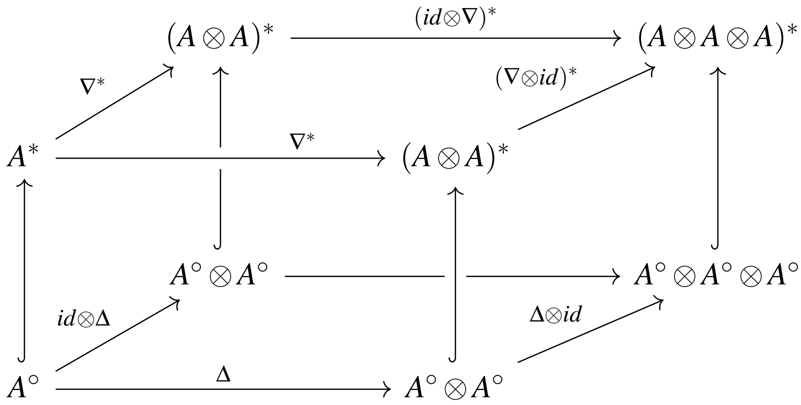
(c) If $\nabla : A \otimes A \to A$ is the product of A and $\nabla^* : A^* \to (A \otimes A)^*$ the induced dual map, then $\nabla^*(A^\circ) \subset A^\circ \otimes A^\circ$

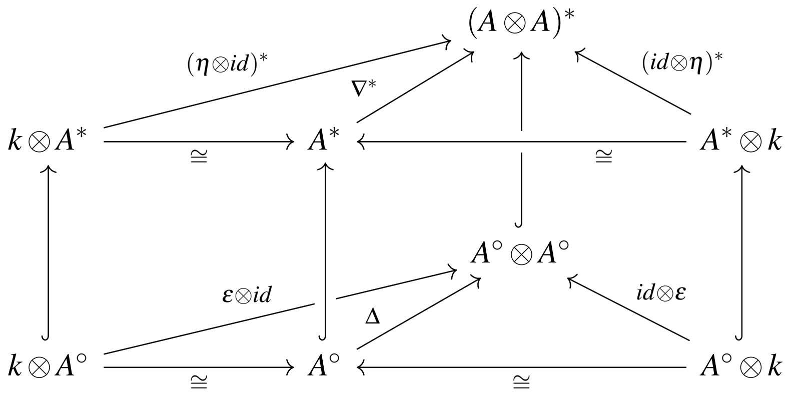
Proof. See [15, Chapter 6, Lemma 6.0.1]. □

Given an algebra (A, ∇, η), the multiplication $\nabla : A \otimes A \to A$ induces a comultiplication $\Delta : A^\circ \to A^\circ \otimes A^\circ$ by defining $\Delta := \nabla^* |_{A^\circ}$. Similarly, define a counit $\varepsilon : A^\circ \to k$ by setting $\varepsilon(f) = f(1)$ for all $f \in A^\circ$. We can then express the statement of Lemma 1.5(c) as a commutative diagram:

$$\begin{array}{ccc} A^\circ & \xrightarrow{\Delta} & A^\circ \otimes A^\circ \\ \downarrow & & \downarrow \\ A^* & \xrightarrow{\nabla^*} & A^* \otimes A^* \end{array} \tag{3.1}$$

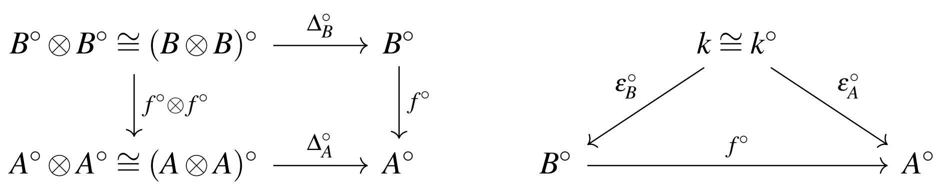
which is natural in A. Thus, $(A^\circ, \Delta, \varepsilon)$ as defined above is a coassociative counital coalgebra since the following diagrams commute:





Proposition 3.7. There is a contravariant functor $(\ )^o : \mathbf{Alg}_k \to \mathbf{CoAlg}_k$ between the category of associative unital algebras and the category of coassociative counital coalgebras.

Proof. It suffices to check that for every algebra morphism $f : (A, \nabla_A, \eta_A) \to (B, \nabla_B, \eta_B)$, then $f^\circ : B^\circ \to A^\circ$ is a coalgebra morphism. Indeed, this follows since the following diagrams:



are the dual of commutating diagrams in $\mathbf{Alg}_k$. $\square$

Theorem 3.8. The functors $(\ )^* : \mathbf{CoAlg}_k \to \mathbf{Alg}_k$ and $(\ )^\circ : \mathbf{Alg}_k \to \mathbf{CoAlg}_k$ are adjoint to each other. In other words, there is a bijection of sets, $\mathbf{Alg}_k(A, C^*) \cong \mathbf{CoAlg}_k(C, A^\circ)$ natural in both variables.

Proof. See [15, Chapter 6, Theorem 6.0.5]. □

Remark. To see the bijection, for all $f \in \mathbf{Alg}_k(A, C^*)$, define $\Psi(f) \in \mathbf{CoAlg}_k(C, A^\circ)$ to be the composite of coalgebra morphisms:

$$C \to C^{*\circ} \xrightarrow{f^\circ} A^\circ$$

where the map $C \to C^{*\circ}$ is in fact the natural map $C \to C^{**}$ proved in [15, Lemma 6.0.4] and f° is the Sweedler dual of f.

Since the injection map $i : A^\circ \to A^*$ induces a transpose map $i^* : A^{**} \to A^{\circ *}$. We define the map $\pi : A \to A^{\circ *}$ to be the composite:

$$\pi : A \to A^{**} \xrightarrow{i^*} A^{\circ *}$$

where $A \to A^{**}$ is the natural injection map. If $g \in \mathbf{CoAlg}_k(C, A^\circ)$, then define $\Phi(f) \in \mathbf{Alg}_k(A, C^*)$ to be the composite of algebra morphisms:

$$A \xrightarrow{\pi} A^{\circ *} \xrightarrow{g^*} C^*$$

where π is defined above and g^* is the transpose of g. The maps $\Psi : \mathbf{Alg}_k(A, C^*) \to \mathbf{CoAlg}_k(C, A^\circ)$, $\Phi : \mathbf{CoAlg}_k(C, A^\circ) \to \mathbf{Alg}_k(A, C^*)$, then give the required bijection.

Definition 3.9. Suppose (C, Δ, ε) is a coalgebra. An element $x \in C$ is called **grouplike** if $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$. Denote $G(C)$ the set of grouplike elements of C. Suppose $x \in G(C)$, an element $x' \in C$ is called **primitive** over x if $\Delta(x') = x \otimes x' + x' \otimes x$. Denote $P(C)$ the set of all primitive elements of C.

Lemma 3.10. If $f : C \to D$ is a coalgebra morphism, and $c \in G(C)$ then $f(c) \in G(D)$. Likewise, if $c \in P(C)$ then $f(c) \in P(D)$. In other words, coalgebra morphisms send grouplike elements to grouplike elements and primitive elements to primitive elements.

Proof. Suppose $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ are coalgebras. If $f : C \to D$ is a coalgebra morphism and $c \in G(C)$, then $\Delta_D f(c) = (f \otimes f)\Delta_C(c) = (f \otimes f)(c \otimes c) = f(c) \otimes f(c) \in G(D)$. If $c \in P(C)$ with $\Delta(c) = c \otimes x + x \otimes c$ for some $x \in G(C)$, then $\Delta_D f(c) = (f \otimes f)\Delta_C(c) = (f \otimes f)(c \otimes x + x \otimes c) = f(c) \otimes f(x) + f(x) \otimes f(c) \in P(D)$ since $f(x) \in G(D)$. □

Lemma 3.11. Suppose C is a coalgebra. There exists natural bijections of sets:

(a) $\mathbf{CoAlg}_k(k, C) \cong G(C)$,

(b) $\mathbf{CoAlg}_k((k[t]/(t^2))^*, C) \cong P(C)$.

Proof. (a) Each coalgebra morphism, $f : k \to C$, corresponds to an element $f(1)$ which is a grouplike element of C by the previous lemma. Conversely, if $x \in G(C)$, then there is a corresponding coalgebra morphism $f : k \to C$ defined by $f(\alpha) = \alpha x$ for all $\alpha \in k$.

(b) Set $J := (k[t]/(t^2))^*$, then J is a vector space with basis $\{1^*, t^*\}$. The coalgebra structure on J is defined by

$$\Delta(1^*) = 1^* \otimes 1^*, \quad \Delta(t^*) = t^* \otimes 1^* + 1^* \otimes t^*$$
$$\varepsilon(1^*) = 1, \quad \varepsilon(t^*) = 0$$

Thus, 1^* is a grouplike element and t^* is a primitive element of J. For every coalgebra morphism, $f : J \to C$, $f(t^*)$, there is a corresponding primitive element $f(t^*)$ in C by the previous lemma. Conversely, for every primitive element y over $x \in G(C)$, there is a coalgebra morphism $f : J \to C$ defined by $f(1^*) = x$ and $f(t^*) = y$. □

Lemma 3.12. Suppose A is an algebra. There exists natural bijections of sets:

(a) $\mathbf{Alg}_k(A, k) \cong G(A^\circ)$,

(b) $\mathbf{Alg}_k(A, k[t]/(t^2)) \cong P(A^\circ)$.

Proof. This is a consequence of applying Theorem 3.8 and Lemma 3.11 by setting the coalgebra $C = A^\circ$:

$$\mathbf{Alg}_k(A,k) \overset{3.8}{\cong} \mathbf{CoAlg}_k(k, A^\circ) \overset{3.11(a)}{\cong} G(A^\circ)$$

$$\mathbf{Alg}_k(A, k[t]/(t^2)) \overset{3.8}{\cong} \mathbf{CoAlg}_k((k[t]/(t^2))^*, A^\circ) \overset{3.11(b)}{\cong} P(A^\circ).$$

□

Example 3.13. Consider the polynomial ring in two variables over k, $A = k[x, y]$, where k is an algebraically closed field of characteristic 0. Then as sets:

$$MaxSpec(A) = \{(x - \lambda, y - \mu) \mid \lambda, \mu \in k\} \cong \{(\lambda, \mu) \mid \lambda, \mu \in k\} = k^2$$

where $MaxSpec(A)$ is the set of maximal ideals of A. Note that an algebra morphism $\zeta : A \to k$ is completely determined by the images of the variables x and y. Thus, for each morphism, $\zeta \in \mathbf{Alg}_k(A, k)$, we associate a pair $(\lambda, \mu) \in k^2$ by defining $\lambda = \zeta(x)$ and $\mu = \zeta(y)$. Conversely, given a pair (λ, μ) we associate an algebra morphism $\zeta_{\lambda,\mu} : A \to k$ by evaluation $f \mapsto f(\lambda, \mu)$ for all $f \in A$.

These correspondences, $\zeta \mapsto (\lambda,\mu)$ and $(\lambda,\mu) \mapsto \zeta_{\lambda,\mu}$ are inverse equivalences, and with Lemma 3.12, we have the bijections of sets:

$$k^2 \cong \mathbf{Alg}_k(A,k) \cong G(A^\circ).$$

In general, for $A = k[x_1,...,x_n]$ the polynomial ring in n variables over k, the associated affine space is in bijection with the group-like elements of A°:

$$k^n \cong G(A^\circ).$$

Example 3.14. With $A = k[x,y]$ as above, set $B = k[t]/(t^2)$, the ring of dual numbers. By identifying B with the direct sum $k \oplus kt$, we can define an algebraic structure on B with multiplication defined by $(a,b)\cdot(c,d) := (ac, ad+bc)$ and a unit $(1,0)$.

Suppose $v := (\lambda,\mu) \in k^2$ and consider a tangent vector $\delta := (\alpha,\beta) \in T_v(k^2)$. For a pair (v,δ) we can associate a linear map $\zeta_{v,\delta} : A \to B$ by defining

$$\zeta_{v,\delta}(f) = \left(f(\lambda,\mu), \alpha \frac{\partial f(x,y)}{\partial x}|_{x=\lambda,y=\mu} + \beta \frac{\partial f(x,y)}{\partial y})|_{x=\lambda,y=\mu} \right)$$

for all $f \in A$. This is an algebra morphism since by the Leibniz rule we have:

$$\begin{aligned}
&\zeta_{v,\delta}(f)\cdot\zeta_{v,\delta}(g) \\
&= \left(f(\lambda,\mu), \alpha\frac{\partial f}{\partial x}|_{\lambda,\mu} + \beta\frac{\partial f}{\partial y}|_{\lambda,\mu} \right) \cdot \left(g(\lambda,\mu), \alpha\frac{\partial g}{\partial x}|_{\lambda,\mu} + \beta\frac{\partial g}{\partial y}|_{\lambda,\mu} \right) \\
&= \Big(f(\lambda,\mu)g(\lambda,\mu), \alpha f(\lambda,\mu)\frac{\partial g}{\partial x}|_{\lambda,\mu} + \beta f(\lambda,\mu)\frac{\partial g}{\partial y}|_{\lambda,\mu} \\
&\quad + \alpha g(\lambda,\mu)\frac{\partial f}{\partial x}|_{\lambda,\mu} + \beta g(\lambda,\mu)\frac{\partial f}{\partial y}|\lambda,\mu \Big) \\
&= \left(f(\lambda,\mu)g(\lambda,\mu), \alpha\frac{\partial(fg)}{\partial x}|_{\lambda,\mu} + \beta\frac{\partial(fg)}{\partial y}|_{\lambda,\mu} \right) \\
&= \zeta_{v,\delta}(fg)
\end{aligned}$$

for all $f,g \in A$ and $\zeta_{v,\delta}(1) = (1,0)$.

Conversely, for every algebra morphism, $\zeta : A \to k[t]/(t^2)$, we can associate a point and a tangent vector $(v,\delta) \in T(k^2)$. Define the element $\theta = t + (t^2)$ in the ring of dual numbers and we can identity $k[t]/(t^2)$ as the algebra:

$$k[\theta] = \{a + b\theta \mid a,b \in k, \theta^2 = 0\}$$

If $f \in A$, then write the image of f under ζ as:

$$\zeta(f) = a(f) + b(f)\theta$$

where $a(f), b(f) \in k$. Define the projection algebra maps

$$\pi_1 : k[\theta] \to k, \quad \pi_1(a + b\theta) = a$$

and,

$$\pi_2 : k[\theta] \to k, \quad \pi_2(a + b\theta) = b$$

for $a, b \in k$. The composition,

$$f \mapsto \zeta(f) = a(f) + b(f)\theta \mapsto a(f)$$

is then an algebra morphism with some maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{m}$ is the kernel of this composition. Every maximal ideal of A is of the form $(x - \lambda, y - \mu)$ for some $\lambda, \mu \in k$. Thus, for this maximal ideal $\mathfrak{m}$, set $v := (\lambda, \mu)$ the corresponding point.

Likewise, the composition of algebra maps, $\delta := \pi_2 \circ \zeta$:

$$f \mapsto \zeta(f) = a(f) + b(f)\theta \mapsto b(f)$$

defines a deviation on k^2, since:

$$\begin{aligned}
\delta(fg) &= \pi_2 \circ \zeta(fg) \\
&= \pi_2\big(\zeta(f) \cdot \zeta(g)\big) \\
&= \pi_2\big((a(f) + b(f)\theta) \cdot (a(g) + b(g)\theta)\big) \\
&= \pi_2\big(a(f)a(g) + (a(f)b(g) + a(g)b(f))\theta\big) \\
&= a(g)b(f) + a(f)b(g) \\
&= a(g)\delta(f) + a(g)\delta(g).
\end{aligned}$$

Therefore, for every algebra morphism, $\zeta : A \to k[t]/(t^2)$, we can associate a point and tangent vector, $(v, \delta) \in T(k^2)$. Applying Lemma 3.12, we then have bijections of sets:

$$T(k^2) \cong \mathbf{Alg}_k(A, k[t]/(t^2)) \cong P(A^\circ).$$

In other words, the tangent space of the affine plane is in bijection with the primitive elements of A°.

Remark. Can we see the Zariski topology on $G(A^\circ)$? Let A be a commutative algebra over a field of characteristic zero. The Zariski topology on the space $MaxSpec(A) \cong \mathbf{Alg}_k(A,k)$ is defined by closed sets, where a subset $Z \subset \mathbf{Alg}_k(A,k)$ is closed if it is the image of a map $\phi_I : \mathbf{Alg}_k(A/I,k) \to \mathbf{Alg}_k(A,k)$ for some ideal I of A. Consider the following diagram:

$$\begin{array}{ccccc} \mathbf{Alg}_k(A,k) & \xrightarrow{\cong} & \mathbf{CoAlg}_k(k,A^\circ) & \xrightarrow{\cong} & G(A^\circ) \\ {\scriptstyle \phi_I}\uparrow & & {\scriptstyle \phi_I \circ \_}\uparrow & & {\scriptstyle G(\phi_I)}\uparrow \\ \mathbf{Alg}_k(A/I,k) & \xrightarrow{\cong} & \mathbf{CoAlg}_k(k,(A/I)^\circ) & \xrightarrow{\cong} & G((A/I)^\circ) \end{array}$$

Thus, we define the Zariski topology on $G(A^\circ)$ by, a subset $Z \subset G(A^\circ)$ is closed if it is the image of $G(\phi_I) : G((A/I)^\circ) \to G(A^\circ)$ for some ideal I of A. This is equivalent to the Zariski topology defined on $\mathbf{Alg}_k(A,k)$. For finitely presented algebras we have an alternate description of this topology.

Choose a presentation of $A \cong k[x_1,...,x_n]/(f_1, f_2,..., f_m)$ giving a diagram in the category of associative unital algebras:

$$k[y_1,...,y_m] \underset{0}{\overset{y_i \mapsto f_i}{\rightrightarrows}} k[x_1,...,x_n] \longrightarrow A$$

such that A is the coequaliser of the two arrows in $\mathbf{Alg}_k$. Since $(\ )^\circ$ is a contravariant functor with a left adjoint A° is the equaliser of the following diagram in the category of coalgebras:

$$k[y_1,...,y_m]^\circ \leftleftarrows k[x_1,...,x_n]^\circ \longleftarrow A^\circ$$

Therefore, it suffices to define a topology on $G(k[x_1,...,x_n]^\circ)$, which will then induce the subspace topology on $G(A^\circ) \subset G(k[x_1,...,x_n]^\circ)$.

Lemma 3.15. Let A be a finitely generated commutative algebra and $G(A^\circ)$ the group-like elements of the maximal coalgebra A° in A^*, then $G(A^\circ)$ is an algebraic variety with the Zariski topology.

Proof. Example 3.13 shows that there is a bijection:

$$G(k[x_1,...,x_n]^\circ) \cong k^n$$

Thus, the Zariski topology on k^n will induce the Zariski topology on $G(A^\circ)$. □

3.2 Hopf algebras

If (A,∇,η) is an algebra and (C,Δ,ε) is a coalgebra, then the multiplication of A and comultiplication of C induce an algebra structure on $Hom_k(C,A)$ with multiplication $*$, called *convolution*, to be the composition:

$$*: Hom_k(C,A)\otimes Hom_k(C,A) \hookrightarrow Hom_k(C\otimes C, A\otimes A) \xrightarrow{\nabla\circ\_\circ\Delta} Hom_k(C,A)$$

where the first map is the natural injection map and explicitly:

$$f*g := \nabla\circ(f\otimes g)\circ\Delta$$

for all $f,g\in Hom_k(C,A)$. Similarly, the unit $\eta_\circ$ is the map:

$$\eta_\circ : k \cong Hom_k(k,k) \to Hom_k(C,A)$$

explicitly defined by $\eta_\circ(x)(c) = \varepsilon(c)\eta(x)$ for all $c\in C$ and $x\in k$.

Remark. Note that $\eta_\circ(1)(c) = \varepsilon(c)\eta(1) = \eta\varepsilon(c)$. Thus, $\eta\varepsilon$ is the identity in $Hom_k(C,B)$.

Example 3.16. For the case where $A = k$ and $c\in G(C)$. Then $f,g\in Hom_k(C,k)$ satisfy $(f*g)(c) = f(c)g(c)$. In other words, the convolution multiplication is just pointwise multiplication on group-like elements.

Lemma 3.17. Suppose (H,∇,η) is an algebra and (H,Δ,ε) is a coalgebra, then the following are equivalent:

(a) Δ and η are coalgebra morphisms,

(b) ∇ and ε are algebra morphisms,

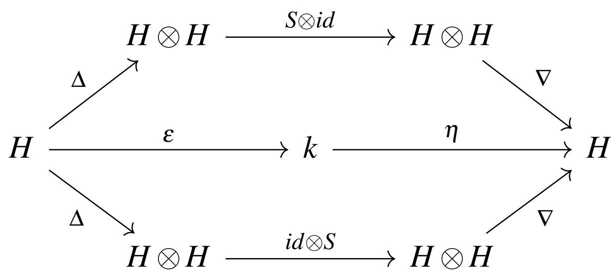
(c) $(H,\nabla,\eta,\Delta,\varepsilon)$ is a bialgebra.

Proof. See Proposition 3.1.1 in [15, Sweedler, Chapter III]. □

Definition 3.18. A ***bialgebra*** is the data $(H,\nabla,\eta,\Delta,\varepsilon)$, where H is a k-vector space, (H,Δ,ε) is a coassociative counital coalgebra over k and (H,∇,η) is an associative unital algebra over k such that $\nabla : H\otimes H\to H$ and $\eta : k\to H$ are coalgebra morphisms.

If H is a bialgebra, then denote $H^C := (H,\Delta,\varepsilon)$, the underlying coalgebra structure, and $H^A := (H,\nabla,\varepsilon)$, the underlying algebra structure. A map $S\in Hom_k(H^C,H^A)$ such that under the convolution multiplication $*$ is a two-sided inverse to the linear identity map

$id : H^C \to H^A$, is called an ***antipode***. In other words, $S * id = id * S = \eta\varepsilon$ or equivalently, $S : H \to H$ is a map that makes the following diagram commute:



Definition 3.19. A ***Hopf algebra*** is the data $(H, \nabla, \eta, \Delta, \varepsilon, S)$ where $(H, \nabla, \eta, \Delta, \varepsilon)$ is a bialgebra and $S : H \to H$ is an antipode.

Example 3.20. The first example of a Hopf algebra, given in [15, Chapter 4], is the group ring of a group. Let $(G, \cdot, e)$ be a group. Denote kG the vector space with G as a basis. The product and unit is given by the group operation and identity:

$$\nabla(g \otimes g') = g \cdot g', \quad \eta(1) = e$$

where $g, g' \in G$. The comultiplication, counit and antipode are defined by:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

for all $g \in G$. Thus defined, $(kG, \nabla, \eta, \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra and is commutative if G is commutative.

Example 3.21. The second example given in [15, Chapter 4] is the Hopf algebra associated to a Lie algebra $(\mathfrak{g}, [,])$. A Lie algebra is not an associative unital algebra as defined above, but we can consider the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ defined by the tensor algebra of $\mathfrak{g}$, $T(\mathfrak{g})$, and quotienting by the ideal generated by elements $X \otimes Y - Y \otimes X - [X, Y]$ for $X, Y \in \mathfrak{g}$. $\mathfrak{U}(\mathfrak{g})$ is an associative unital algebra by construction.

Defining the comultiplication, counit and antipode by:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(1) = 1 \otimes 1$$
$$\varepsilon(x) = 0, \quad \varepsilon(1) = 1,$$
$$S(1) = 1, \quad S(x) = -x.$$

for all $x \in \mathfrak{g}$, endows $\mathfrak{U}(\mathfrak{g})$ with a Hopf algebra structure.

Example 3.22. Similarly, the deformed quantum algebra $\mathfrak{U}_q(\mathfrak{sl}(2))$ is the algebra generated by four elements, E, F, K, K^{-1}, subjected to the following relations:

$$KK^{-1} = 1 = K^{-1}K$$
$$KEK = q^2E, \quad KFK = q^{-2}F$$
$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Define a comultiplication, counit, and anitpode by:

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1$$
$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}$$
$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF \quad S(K) = K^{-1}, \quad S(K^{-1}) = K,$$

which endows $\mathfrak{U}_q(\mathfrak{sl}(2))$ with a Hopf algebra structure. More details can be found in [12, Chapter VI, Definition VI.1.1].

3.3 Algebraic groups

The following section constructs an algebraic group associated to a Hopf algebra. The underlying set will be determine by the algebraic structure, where each maximal ideal corresponds to a point and the group operations will correspond to a comultiplication and counit map. These results can be found in [15, Sweedler, Chapter IV] with further details provided in this thesis. Suppose $(H, \Delta_H, \varepsilon_H, \nabla_H, \eta_H)$ is a Hopf algebra, and (A, ∇_A, η_A) is an algebra. The underlying coalgebra structure of H makes the triple $(Hom_k(H,A), *, \eta_A\varepsilon_H)$ into a monoid, with the convolution multiplicaiton $*$ and identity $\eta_A\varepsilon_H$. The convolution multiplication in this case will be defined by:

$$f * g = \nabla_A \circ (f \otimes g) \circ \Delta_H \in Hom_k(H,A)$$

for all $f, g \in Hom_k(H,A)$, which is associative since

$$((f * g) * h)(x) = \sum f(x_{(1)})g(x_{(2)})h(x_{(3)}) = (f * (g * h))(x)$$

for all $x \in H$, where we adopt sigma notation for $\Delta_H(x) = \sum x_{(1)} \otimes x_{(2)}$.

Lemma 3.23. If H is a Hopf algebra and A is commutative, then $\mathbf{Alg}_k(H,A)$ forms a multiplicative submonoid of $(Hom_k(H,A),*,\eta_A\varepsilon_H)$.

Proof. It suffices to show that the convolution of two algebras morphisms is again an algebra morphism. Let $f,g \in \mathbf{Alg}_k(H,A)$ be two algebra morphisms. Denote $f*g = \nabla_A \circ f \otimes g \circ \Delta_H$ the convolution product. Then since Δ_H is an algebra map:

$$(f*g)(xy) = (\nabla_A \circ (f\otimes g)\circ\Delta_H)(xy) = (\nabla_A \circ (f\otimes g))(\Delta_H(x)\Delta_H(y)) = \sum f(x_{(1)}y_{(1)})g(x_{(2)}y_{(2)})$$

where $\Delta_H(x) = \sum x_{(1)} \otimes x_{(2)}$ and $\Delta_H(y) = \sum y_{(1)} \otimes y_{(2)}$. Since f and g are algebra morphism and A is commutative we have:

$$\begin{aligned}(f*g)(xy) &= \sum f(x_{(1)}y_{(1)})g(x_{(2)}y_{(2)})\\ &= \sum f(x_{(1)})f(y_{(1)})g(x_{(2)})g(y_{(2)})\\ &= \sum f(x_{(1)})g(x_{(2)})f(y_{(1)})g(y_{(2)})\\ &= (f*g)(x)(f*g)(y)\end{aligned}$$

Therefore, $f*g$ is an algebra morphism. □

To make $\mathbf{Alg}_k(H,A)$ into group, we require an inverse map that defines an inverse element for all $R \in \mathbf{Alg}_k(H,A)$.

Lemma 3.24. Suppose A,B are algebras and C,D are coalgebras. If $R \in \mathbf{Alg}_k(A,B)$ and $S \in \mathbf{CoAlg}_k(C,D)$ then R,S induces a linear map:

$$\varphi : Hom_k(D,A) \to Hom_k(C,B)$$

via $f \mapsto R\circ f \circ S$. Additionally, φ is an algebra morphism.

Proof. See [15, Lemma 4.0.2]. □

Lemma 3.25. Suppose H is a Hopf algebra and A an algebra. If $R \in \mathbf{Alg}_k(H,A)$ then there exists an inverse R^{-1} under convolution $*$, defined by

$$R^{-1} := R\circ S \in Hom_k(H,A)$$

where S is the antipode of H.

Proof. By Lemma 3.24, R induces an algebra map:

$$\varphi : Hom_k(H,H) \to Hom_k(H,A)$$

via $\varphi(f) = R \circ f$. By definition, the antipode is the inverse to the linear identity map i.e. $S = I^{-1}$. Thus,

$$R \circ S = \varphi(S) = \varphi(I^{-1}) = \varphi(I)^{-1} = (R \circ I)^{-1} = R^{-1}.$$

□

Lemma 3.26. Suppose H is a Hopf algebra and A is a commutative algebra. Define the map $i : \mathbf{Alg}_k(H,A) \to \mathbf{Alg}_k(H,A)$ by $i(R) = R \circ S$. Then the data $(\mathbf{Alg}_k(H,A), *, \eta_A \varepsilon_H, i)$ forms a group.

Proof. By the previous proposition, there is an inverse element for every $R \in \mathbf{Alg}_k(H,A)$, and with Lemma 3.23, $\mathbf{Alg}_k(H,A)$ is then a group. □

Setting $A = k$, then by Lemma 3.12(a) we have the bijection of sets $G(H^\circ) \cong \mathbf{Alg}_k(H,k)$. Since k is trivially a commutative algebra over k, there is an induced group structure on $G(H^\circ)$ by identifying it as $\mathbf{Alg}_k(H,k)$ and applying the previous lemma. Therefore, for every Hopf algebra H, we can associate a group $G(H^\circ)$. In fact, by Lemma 3.15, $G(H^\circ)$ is also an algebraic variety. The convolution multiplication $*$, and the inversion map i, are morphisms of varieties by construction, therefore, $G(H^\circ)$ can be considered an algebraic group.

Proposition 3.27. There is a functor $G(\_^\circ)$ from the category of Hopf algebras to the category of algebraic groups:

$$G(\_^\circ) : \mathbf{Hopf}_k \to \mathbf{AlgGrp}.$$

Proof. The correspondence $H \mapsto G(H^\circ)$ associates a well-defined algebraic group for every Hopf algebra. Suppose H, L are Hopf algebras with a Hopf algebra morphism $f : H \to L$. Identifying $G(L^\circ) \cong \mathbf{Alg}_k(L,k)$ and $G(H^\circ) \cong \mathbf{Alg}_k(H,k)$ induces a map $G(f^\circ) : \mathbf{Alg}_k(L,k) \to \mathbf{Alg}_k(H,k)$ by precomposition with f. This map is then a group homomorphism, since:

$$G(f^\circ)(\varphi * \psi) = (\varphi * \psi) \circ f = (\varphi \circ f) * (\psi \circ f) = G(f^\circ)(\varphi) * G(f^\circ)(\psi)$$

for all $\varphi, \psi \in Alg_k(L,k)$. $G(f^\circ)$ is also a morphism of varieties. □

Definition 3.28. A Hopf algebra H is called ***affine*** if it is finitely generated as an algebra and commutative. Additionally, if H is an affine Hopf algebra, then $\mathbf{Alg}_k(H,k) \cong G(H^\circ)$ is called an ***affine algebraic group***.

Hence, for each Hopf algebra we can associate a well-defined algebraic group. The examples below provide a heuristic to recover the corresponding Hopf algebras associated to an algebraic group. The following lemma will be helpful with our calculations:

Lemma 3.29. Let A and B be algebras over a field k of characteristic 0. Then there is an bijection of sets:

$$MaxSpec(A) \times MaxSpec(B) \cong MaxSpec(A \otimes B)$$

Proof. Consider the algebra inclusion morphisms:

$$\pi_1 : A \hookrightarrow A \otimes B$$

defined by $\pi_1(a) = a \otimes 1$ for all $a \in A$, and

$$\pi_2 : B \hookrightarrow A \otimes B$$

defined by $\pi_2(b) = 1 \otimes b$ for all $b \in B$. Then the following correspondences:

$$\Phi : MaxSpec(A \otimes B) \to MaxSpec(A) \times MaxSpec(B)$$
$$\mathfrak{m} \mapsto (\pi_1^{-1}(\mathfrak{m}), \pi_2^{-1}(\mathfrak{m}))$$

$$\Psi : MaxSpec(A) \times MaxSpec(B) \to MaxSpec(A \otimes B)$$
$$(\mathfrak{n}_1, \mathfrak{n}_2) \mapsto \mathfrak{n}_1 \otimes 1 + 1 \otimes \mathfrak{n}_2$$

where $\mathfrak{n}_1$ is a maximal ideal of A and $\mathfrak{n}_2$ is a maximal ideal of B, give the bijection. □

Example 3.30. Consider the additive algebraic group $(\mathbb{C}, +, 0, i)$ with smooth maps, $m(\lambda, \mu) = \lambda + \mu$, $i(\theta) = -\theta$, for all $\lambda, \mu, \theta \in \mathbb{C}$ and identity $0 \in \mathbb{C}$. We will show that the associated Hopf algebra is the set $\mathbb{C}[x]$ with point-wise addition determining its algebra structure and recover the comultiplication and counit maps.

Let $A = \mathbb{C}[x]$ the polynomial algebra in one variable, then as sets there is a bijection:

$$MaxSpec(\mathbb{C}[x]) = \{(x - \lambda) \mid \lambda \in \mathbb{C}\} \cong \mathbb{C}$$

Hence, the data of the multiplication:

$$m : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \quad (\lambda, \mu) \mapsto \lambda + \mu$$

is equvialent to the data of a map:

$$m' : MaxSpec(A) \times MaxSpec(A) \to MaxSpec(A), \quad (\lambda, \mu) \mapsto \lambda + \mu$$

By Lemma 3.29 this is equivalent to a scheme map, $m'' : MaxSpec(A \otimes A) \to MaxSpec(A)$, which we expect to be induced by an algebra morphism:

$$\Delta : A \to A \otimes A.$$

Since we can identity:

$$\mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[x] \cong \mathbb{C}[x_1, x_2]$$

then, the maximal ideals of $A \otimes A$ are of the form $\{(x_1 - \lambda, x_2 - \mu) \mid \lambda, \mu \in \mathbb{C}\}$.

To determine the algebra map Δ, note that it induces m'' on $MaxSpec$ by:

$$\begin{aligned} m'' : MaxSpec(A \otimes A) &\to MaxSpec(A) \\ \mathfrak{m} &\mapsto \Delta^{-1}(\mathfrak{m}) \\ (x_1 - \lambda, x_2 - \mu) &\mapsto \Delta^{-1}(x_1 - \lambda, x_2 - \mu) \\ &= \{f \in \mathbb{C}[x] \mid \Delta(f) \in (x_1 - \lambda, x_2 - \mu)\} \\ &= \{f \mid \Delta(f)|_{x_1 = \lambda, x_2 = \mu} = 0 \in \mathbb{C}\} \\ &= \{f \mid f(\phi)|_{x_1 = \lambda, x_2 = \mu} = 0\} \\ &= \{f \mid f(\phi(\lambda, \mu)) = 0\} \\ &= \{f \mid f \in (x - \phi(\lambda, \mu))\} \\ &= (x - \phi(\lambda, \mu)) \end{aligned}$$

where $\phi := \Delta(x) \in \mathbb{C}[x_1, x_2]$. Hence, the algebra map $\Delta : A \to A \otimes A$ is determined by ϕ. The following calculation:

$$\begin{aligned} \phi(\lambda, \mu) &= m''(\lambda, \mu) \\ &= m(\lambda, \mu) \\ &= \lambda + \mu \end{aligned}$$

shows that we must define $\phi = x_1 + x_2$ or equivalently $\phi := \Delta(x) = x \otimes 1 + 1 \otimes x$. This is the required algebra morphism and comultiplication that induces m on $MaxSpec(A) \cong \mathbb{C}$. We can recover the counit map from the identity element map:

$$e : \{*\} \to \mathbb{C}$$

which is equivalent to a map:

$$e' : MaxSpec(\mathbb{C}) \to MaxSpec(A).$$

We expect this map to be induced by an algebra map $\varepsilon : A \to \mathbb{C}$ and in fact $\varepsilon = eval_0$. Therefore, if $A = \mathbb{C}[x]$ and defining the maps:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x$$

then A is a Hopf algebra over $\mathbb{C}$ such that $G(A^\circ) \cong \mathbf{Alg}_k(A,k) \cong MaxSpec(A) \cong \mathbb{C}$ is the additive algebraic group.

Example 3.31. If instead we consider the multiplicative algebraic group $(\mathbb{C}^\times, m, 1, i)$ with a smooth group operation $m(\lambda,\mu) = \lambda\mu$, a smooth inverse map $i(\theta) = \theta^{-1}$ for all $\lambda,\mu,\theta \in \mathbb{C}^\times$, and a unit $1 \in \mathbb{C}^\times$. We will show that the associated Hopf algebra, which induces $\mathbb{C}^\times$, is in fact $\mathbb{C}[x,x^{-1}] := \mathbb{C}[x,u]/(xu-1)$ and explicitly state the comultiplication and counit maps.

Setting $A = \mathbb{C}[x,u]/(xu-1)$, there is a natural algebra structure on A from the canonical multiplicaion and unit maps of $\mathbb{C}[x,u]$. Note that quotient algebra morphism $\varphi : \mathbb{C}[x,u] \to A$, defines an injective map on its maximal spectrum space:

$$MaxSpec(A) \hookrightarrow MaxSpec(\mathbb{C}[x,u]) \cong \mathbb{A}^2_{\mathbb{C}}$$

where $\mathfrak{m} \mapsto \varphi^{-1}(\mathfrak{m})$ for all $\mathfrak{m} \in MaxSpec(A)$. Since every element of $MaxSpec(\mathbb{C}[x,u])$ is a maximal ideal $(x-\lambda, u-\mu)$ for some $\lambda,\mu \in \mathbb{C}$, then the elements of $MaxSpec(A)$ are of the form $(x-\lambda, u-\mu)$ with some condition between λ and μ. In fact, the MaxSpec of A is explicitly defined:

$$MaxSpec(A) = \{(x-\lambda, u-\mu) \mid \lambda,\mu \in \mathbb{C}, \lambda\mu = 1\}$$

Therefore, there is a bijection of sets:

$$\begin{aligned} MaxSpec(A) &= \{(x-\lambda, u-\mu) \mid \lambda\mu = 1\} \\ &\cong \{(\lambda,\mu) \mid \lambda\mu = 1\} \\ &\cong \mathbb{C}^\times \end{aligned}$$

where the last bijection is given by the fact that for every element $\theta \in \mathbb{C}^\times$ there is a corresponding element (θ, θ^{-1}) such that $\theta\theta^{-1} = 1$, and for every pair (λ,μ) such that $\lambda\mu = 1$ there is a corresponding element $\lambda \in \mathbb{C}^\times$.

Under these bijection, and identifying $MaxSpec(A)$ as the pairs (θ, θ^{-1}), the group operation m is defined by:

$$m : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times, \quad (\theta_1, \theta_2) \mapsto \theta_1\theta_2$$

This is equivalent to the map on sets:

$$m' : MaxSpec(A) \times MaxSpec(A) \to MaxSpec(A)$$
$$\big((\theta_1, \theta_1^{-1}), (\theta_2, \theta_2^{-1})\big) \mapsto (\theta_1\theta_2, \theta_1^{-1}\theta_2^{-1})$$

where $\theta_1, \theta_2 \in \mathbb{C}^\times$. To construct an algebra morphism $\Delta : A \to A \otimes A$, such that Δ induces m' on $MaxSpec(A)$, and equivalently m we identify:

$$\mathbb{C}[x_1, u_1]/(x_1u_1 - 1) \otimes_{\mathbb{C}} \mathbb{C}[x_2, u_2]/(x_2u_2 - 1) \cong \mathbb{C}[x_1, u_1, x_2, u_2]/(x_1u_1 - 1, x_2u_2 - 1),$$

Then the following bijection of sets:

$$\begin{aligned} & MaxSpec(A \otimes A) \\ & \cong MaxSpec(\mathbb{C}[x_1, u_1, x_2, u_2]/(x_1u_1 - 1, x_2u_2 - 1)) \\ & = \{(x_1 - \lambda_1, u_1 - \mu_1, x_2 - \lambda_2, u_2 - \mu_2) \mid \lambda_1\mu_1 = \lambda_2\mu_2 = 1\} \\ & \cong \{(\lambda_1, \mu_1, \lambda_2, \mu_2) \mid \lambda_1\mu_1 = \lambda_2\mu_2 = 1\} \end{aligned}$$

and applying Lemma 3.29, shows that the following is a bijection of sets:

$$MaxSpec(A \otimes A) \xrightarrow{\cong} MaxSpec(A) \times MaxSpec(A)$$
$$(\lambda_1, \mu_1, \lambda_2, \mu_2) \mapsto ((\lambda_1, \mu_1), (\lambda_2, \mu_2)).$$

and the data of $m' : MaxSpec(A) \times MaxSpec(A) \to MaxSpec(A)$ is then equivalent to defining a map:

$$m'' : MaxSpec(A \otimes A) \to MaxSpec(A).$$

Therefore, it suffices to define an algebra morphism $\Delta : A \to A \otimes A$ which induces $m'' : MaxSpec(A \otimes A) \to MaxSpec(A)$ on $MaxSpec$. To define an algebra map:

$$\Delta : A \cong \mathbb{C}[x, u]/(xu - 1) \to A \otimes A \cong \mathbb{C}[x_1, u_1, x_2, u_2]/(x_1u_1 - 1, x_2u_2 - 1)$$

it suffices to define the images of x, u in $\mathbb{C}[x_1, u_1, x_2, u_2]$ i.e. explicitly define $\phi_1 := \Delta(x)$ and $\phi_2 := \Delta(u)$. The following calculation:

$$\begin{aligned} (\phi_1(\lambda_1, \mu_1, \lambda_2, \mu_2), \phi_2(\lambda_1, \mu_1, \lambda_2, \mu_2) &= m''(\lambda_1, \mu_1, \lambda_2, \mu_2) \\ &= m'((\lambda_1, \mu_1), (\lambda_2, \mu_2)) \\ &= (\lambda_1\lambda_2, \mu_1\mu_2), \end{aligned}$$

shows that if we set:

$$\phi_1 = x_1 x_2, \quad \phi_2 = u_1 u_2$$

i.e. $\Delta(x) = x \otimes x$ and $\Delta(u) = u \otimes u$, then this defines the required algebra morphism that induces m on $MaxSpec(A) \cong \mathbb{C}^\times$.

The counit map can be similar recovered from the identity element map:

$$e : \{*\} \to \mathbb{C}^\times$$

This is equivalent to a map:

$$e' : MaxSpec(\mathbb{C}) \to MaxSpec(A)$$

We expect that this map is induced by an algebra map $\varepsilon : A \to \mathbb{C}$ and in fact $\varepsilon = eval_{1,1}$. Therefore, if $A = \mathbb{C}[x,u]/(xu-1)$ and define:

$$\Delta(x) = x \otimes x, \quad \Delta(u) = u \otimes u, \quad S(x) = x^{-1}, \quad S(u) = u^{-1} \quad \varepsilon(x) = 1, \quad \varepsilon(u) = 1$$

then, A is the Hopf algebra over $\mathbb{C}$ such that $G(H^\circ) \cong \mathbf{Alg}_k(A,k) \cong MaxSpec(A) \cong \mathbb{C}^\times$ is the multiplicative algebraic group of the non-zero elements of $\mathbb{C}$.

3.4 The universal measuring algebra

Given an algebra A and a finite dimensional algebra T, there is an affine scheme, X, whose k-points are scheme morphisms:

$$Spec(T) \to Spec(A)$$

In other words, we have a bijection of sets:

$$Hom_{Sch_k}(Spec(T), Spec(A)) \cong Hom_{Sch_k}(Spec(k), X)$$

This scheme is obtained from the pair (C,A), where $C = T^*$, by a universal construction.

Suppose A, B, C are vector spaces, then there is a natural bijection of sets:

$$\Phi : Hom_k(A \otimes C, B) \xrightarrow{\cong} Hom_k(A, Hom_k(C,B))$$

which is defined by $\Phi(\psi)(a)(c) = \psi(a \otimes c)$ with inverse $\Psi(\phi)(a \otimes c) = \phi(a)(c)$ for all $a \in A$, $c \in C$.

Suppose B is an algebra and C is a coalgebra. Recall that there is an algebra structure on $Hom_k(C,B)$ defined by the convolution product $*$ and unit $\eta_\circ$. (See Section 1.2.) If A is also an algebra, then we can consider the subset $\mathbf{Alg}_k(A,Hom_k(C,B)) \subset Hom_k(A,Hom_k(C,B))$ of algebra morphisms from A to $Hom_k(C,B)$.

Proposition 3.32. Suppose $\psi \in Hom_k(A\otimes C,B)$ where A,B are algebras and C is a coalgebra. Then $\rho := \Phi(\psi) \in Hom_k(A,Hom_k(C,B))$ is an algebra morphism with respect to the convolution product if and only if ψ makes the following diagrams commute:

$$\begin{array}{ccccc} A\otimes A\otimes C & \xrightarrow{id_{A\otimes A}\otimes\Delta} & A\otimes A\otimes C\otimes C & \xrightarrow{\cong} & (A\otimes C)\otimes(A\otimes C) \\ \downarrow{\scriptstyle \nabla\otimes id} & & & & \downarrow{\scriptstyle \psi\otimes\psi} \\ A\otimes C & \xrightarrow{\psi} & B & \xleftarrow{\nabla} & B\otimes B \end{array} \tag{3.2}$$

$$\begin{array}{ccc} C\cong C\otimes k & \xrightarrow{id_C\otimes\eta} & C\otimes A \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \psi} \\ k & \xrightarrow{\eta} & B \end{array} \tag{3.3}$$

where ∇,η denotes the multiplication and unit map in A,B, and Δ,ε denote the comultiplication and counit maps in C.

Proof. See [15, Proposition 7.0.1] □

Definition 3.33. The pair (C,ψ), where C is a coalgebra and $\psi : A\otimes C \to B$ is a map satisfying (3.2) and (3.3), is said to ***measure*** A to B as defined in [15, Chapter VII]. Denote the set of all such measures $Meas(A,C;B)$. Where it is unambiguous, we write $\psi(a\otimes c) = c(a)$.

Example 3.34. If $g \in G(C)$ is group-like and $\psi \in Meas(A,C;B)$, then

$$\psi(g\otimes\_) : A \to B$$

is an algebra morphism. Since g is group-like this implies:

$$g(aa') = \sum_{(g)} g_{(1)}(a)g_{(2)}(a') = g(a)g(a'), \quad g(1) = \varepsilon(g)1 = 1$$

Remark. The convolution product defines a functor:

$$Hom_k(\_,\_) : \mathbf{CoAlg}_k \times \mathbf{Alg}_k^{op} \to \mathbf{Alg}_k$$

In other words, if $f : A \to A'$ is an algebra morphism, then it induces an algebra morphism:

$$Hom_k(C,A) \to Hom_k(C,A')$$

by post-composition with f, and if $g : C \to C'$ is a coalgebra morphism, then it induces the algebra morphism:

$$Hom(C',A) \to Hom(C,A)$$

by precomposition by g.

Remark. Given a measuring $\psi : A \otimes C \to B$ and an algebra morphism $\varphi : B \to B'$, by the previous remark

$$A \xrightarrow{\rho} Hom_k(C,B) \to Hom_k(C,B')$$

is an algebra map, where $\rho := \Psi(\psi)$. Thus, $\varphi \circ \psi$ is a measuring. Similarly, if $\gamma : A' \to A$ is an algebra morphism, then $\psi \circ (1 \otimes \gamma) : C \otimes A' \to B$ is a measuring, and if $\chi : C' \to C$ is a coalgebra morphism, then $\psi \circ (\chi \otimes 1) : C' \otimes A \to B$ is a measuring. Therefore, $Meas(\_,\_;\_)$ then defines a functor:

$$Meas : \mathbf{Alg}_k^{op} \times \mathbf{CoAlg}_k^{op} \times \mathbf{Alg}_k \to \mathbf{Sets}$$

There exists a pair of a universal commutative algebra and a morphism, representing these measures and called the *universal measuring algebra*. Proposition 7.0.3 of [15] states that given an algebra A over k and A° the maximal coalgebra in A^*, then A° with the evaluation $<,>: A^\circ \otimes A \to k$ measures A to k. Recall the Riesz–Markov–Kakutani representation theorem [11] which states, suppose X is a locally compact Hausdorff space. If $C^0(X)$ is the set of continuous functions on X, then for any continuous linear functional ψ on $C^0(X)$, there is a unique regular countably additive complex Borel measure μ on X such that:

$$\psi(f) = \int_X f(x) d\mu(x)$$

Thus, the elements of A° can be viewed as analogues of measures on A.

Theorem 3.35. Suppose A is a commutative algebra over k and C is a cocommutative coalgebra over k. Then there exists a commutative algebra $\beta(C,A)$ over k with a measuring:

$$\theta : A \otimes C \to \beta(C,A)$$

which is universal, in the sense that for any $\psi \in Meas(A,C;B)$ there is a unique algebra morphism $F : \beta(C,A) \to B$ that makes the following diagram commute:

$$\begin{array}{ccc} A \otimes C & \xrightarrow{\theta} & \beta(C,A) \\ & \searrow^{\psi} & \downarrow{\scriptstyle !F} \\ & & B \end{array}$$

This algebra with the morphism θ is defined to be the ***universal measuring algebra***.

Proof. We construct $(\beta(C,A),\theta)$ following [2]. Take the symmetric algebra $Sym(A \otimes C)$ of the underlying vector space of $A \otimes C$. Let $i : A \otimes C \to Sym(A \otimes C)$ be the inclusion morphism and denote m, η the multiplication and unit maps in $Sym(A \otimes C)$.

To construct $\beta(C,A)$, we want to define it as some quotient of $Sym(A \otimes C)$ such that it makes the diagrams (3.2) and (3.3) commute. If we let I be the ideal generated by the images of the maps (see the diagram below):

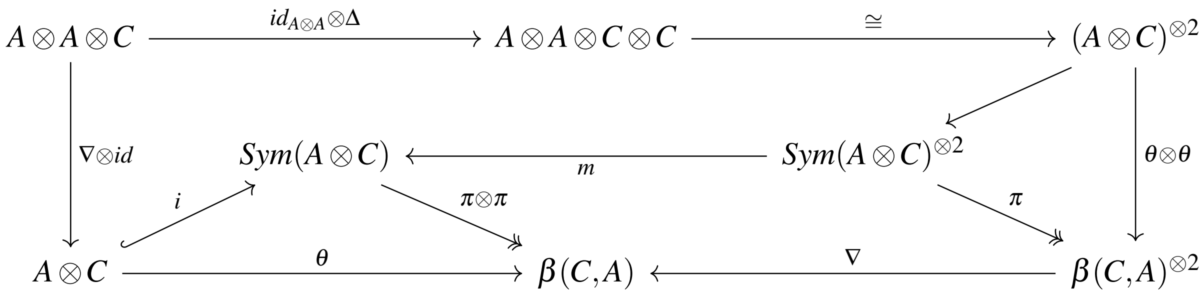
$$m \circ (i \circ i) \circ (id \otimes \Delta) - i \circ m \otimes id, \quad \eta \circ \varepsilon - i \circ (id \otimes \eta),$$

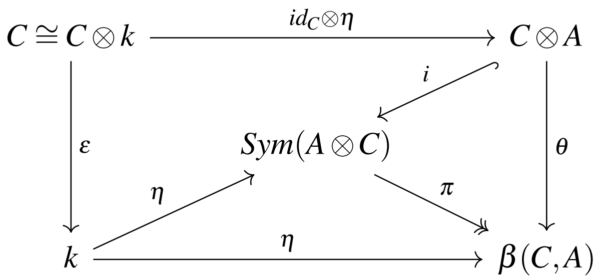
then the quotient $\pi : Sym(A \otimes C) \to Sym(A \otimes C)/I =: \beta(C,A)$, and $\theta := \pi \circ i : A \otimes C \to \beta(C,A)$ is our proposed universal commutative algebra. By the definition of the ideal, θ makes the following diagrams commute:

$$\begin{array}{ccccc} A \otimes A \otimes C & \xrightarrow{id_{A \otimes A} \otimes \Delta} & A \otimes A \otimes C \otimes C & \xrightarrow{\cong} & (A \otimes C)^{\otimes 2} \\ \downarrow{\scriptstyle \nabla \otimes id} & & & & \downarrow{\scriptstyle i \otimes i} \\ A \otimes C & \xhookrightarrow{i} & Sym(A \otimes C) & \xleftarrow{m} & Sym(A \otimes C)^{\otimes 2} \end{array}$$

$$\begin{array}{ccc} C \cong C \otimes k & \xrightarrow{id_C \otimes \eta} & C \otimes A \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle i} \\ k & \xrightarrow{\eta} & Sym(A \otimes C) \end{array}$$

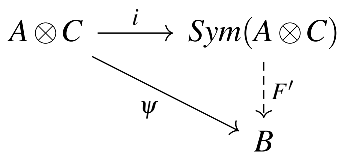
Therefore, θ is a measure since the boundaries of the following diagrams commute:



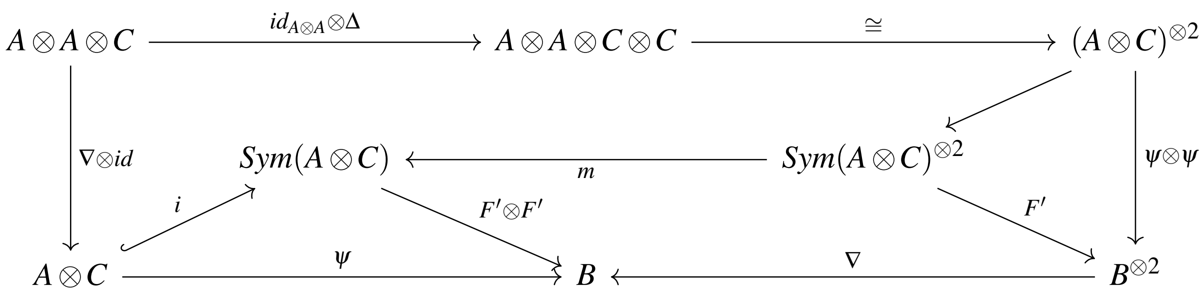


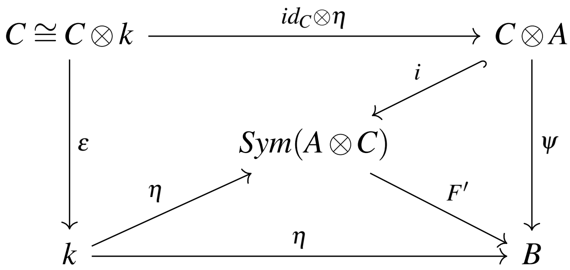
where the remaining squares and triangles commute since π, η are algebra morphisms and by the definition of $\beta(C,A)$.

It remains to show that $(\beta(C,A), \theta)$ is universal. Suppose there exists another $\psi \in Meas(A,C;B)$, since $Sym(A \otimes C)$ is a universal commutative unital associative algebra and ψ is linear homomorphism, there exists a unique algebra morphism $F' : Sym(A \otimes C) \to B$, which makes the following diagram commute:

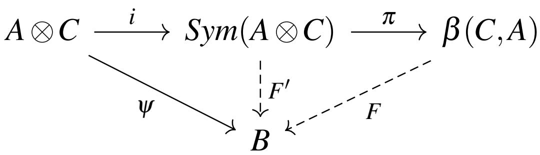


By definition, F' and the measure ψ makes the boundaries of the following diagrams commute:

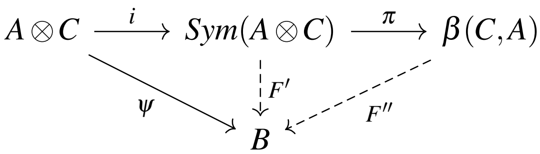




Since the ideal I is generated by the images of the maps making the inside diagrams commute, then F' must vanish on the ideal, and F' factors through the quotient $\pi : Sym(A \otimes C) \to \beta(C,A)$. This induces an algebra morphism $F : \beta(C,A) \to B$ satisfying $F \circ \pi = F'$ and making the outside of the following diagram commute:



If $F'' : \beta(C,A) \to B$ was another algebra morphism such that $F'' \circ \pi \circ i = \psi$ making the outside of the following diagram commute:



then we would have $F'' \circ \pi = F'$ by the universal property of i. Since π is surjective, then $F = F''$. Therefore, F is unique. □

Remark. The assignment $\beta(\_,\_)$ is functorial, that is, it defines a functor:

$$\beta : \mathbf{CoAlg}_k^{op} \times \mathbf{Alg}_k^{op} \to \mathbf{Alg}$$

and $Meas(A,C;B) \cong \mathbf{Alg}_k(\beta(C,A),B)$ is a bijection natural in all variables.

Example 3.36. Suppose C is a finite dimensional coalgebra and A is an algebra. Then the spectrum of the algebra $\beta(C,A)$ is in fact the proposed scheme at the start of this section.

Note that we have the following calculation:

$$\begin{aligned} Hom_{Sch_k}(Spec(C^*), Spec(A)) &\cong \mathbf{Alg}_k(A, C^*) \\ &\cong \mathbf{Alg}_k(A, Hom_k(C,k)) \quad \text{(Def. dual)} \\ &\cong Meas(C,A;k) \quad \text{(Def. measuring)} \\ &\cong \mathbf{Alg}_k(\beta(C,A),k) \quad \text{(Universal property)} \\ &\cong Hom_{Sch_k}(Spec(k), Spec(\beta(C,A)). \end{aligned}$$

Thus, the k-points of the scheme $Spec(\beta(C,A))$ are in bijection with the scheme morphisms $Spec(C^*) \to Spec(A)$.

Example 3.37. Consider the coalgebra $C = (k[t]/t^2)^*$. Let A be an algebra, and $X = Spec(A)$ the associated scheme. Then the above shows that the scheme $TX := Spec(\beta(C,A))$ has the k-points given by:

$$(TX)_k = Hom_{Sch_k}(Spec(k[t]/t^2), X)$$

thus, we can identify TX as the tangent bundle of X.

Definition 3.38. Suppose A is a commutative k-algebra. Define the ***n-jet algebra*** $J_nA := \beta(C,A)$ for $C = (k[t]/(t^{n+1}))^*$, and observe that the scheme $J_nX = Spec(J_nA)$ has the k-points:

$$(J_nX)_k := \mathbf{Alg}_k(A, (k[t]/(t^{n+1}))$$

The idea of jet spaces is that it generalise tangent spaces and we can consider the algebraic equivalence of higher-order partial derivatives. Further properties and definitions of jets can be found in [14, Saunders]. The $n = 1$ case obviously coincides with the tangent space.

Lemma 3.39. Suppose $H = k[x]$ is the symmetric coalgebra with $\Delta(x) = 1 \otimes x + x \otimes 1, \varepsilon(x) = 0$, and let H_n be the subcoalgebra spanned by $\{1, ..., x^n\}$ as a vector space. Then there is a coalgebra isomorphism:

$$H_n \cong (k[t]/(t^{n+1})^*$$

Proof. Set $C := k[t]/(t^{n+1})$ and let $\{t^i\}_{i \in \{0,\dots,n\}}$ be a k-basis of C. C^* forms a coalgebra since C is a finite dimensional vector and let $\{e^i\}_{i \in \{0,\dots,n\}}$ be the dual basis for C^*. Tthe comultiplication on C^* is then defined by $\Delta(e) = 1 \otimes e + e \otimes 1$. The vector space bijection $x^n \to e^n$, then defines a coalgebra isomorphism. This is clearly a bijection, and the following diagrams:

$$\begin{array}{ccc} H_n & \xrightarrow{\Delta_H} & H_n \otimes H_n \\ \phi \downarrow & & \downarrow \phi\otimes\phi \\ C^* & \xrightarrow{\Delta_{C^*}} & C^* \otimes C^* \end{array} \qquad \begin{array}{ccc} H_n & \xrightarrow{\phi} & C^* \\ & \searrow \quad \swarrow & \\ & k & \end{array}$$

commute by the definition of the comultiplications and counits. □

Example 3.40. Suppose L is a lattice and let $A := R[L]$ be the group ring over a commutative k-algebra R. Then the universal measuring algebra can be identified as

$$J_nA = \beta(C^*, A) \cong \beta(H_n, A)$$

The subcoalgebras $H_n \subset H$ form a direct system with colimit $H = \lim_{n\to\infty} H_n$. The inclusions induce algebra morphisms by the functorality of β

$$J_nA = \beta(H_n, A) \to \beta(H, A),$$

and associated morphisms of schemes

$$Spec(\beta(H, A)) \to Spec(\beta(H_n, A)) = J_nX$$

exhibit $Spec(\beta(H, A))$ as the inverse limit $J_\infty X := \varprojlim J_nX$ called the ***infinite jet scheme*** as in [7, §9.4.4]. Observe that by the universal properties of the universal measuring algebra, $J_\infty X = Spec(\beta(H, A))$ has the k-points:

$$(J_\infty X)_k = \mathbf{Alg}_k(A, H^*) \cong Hom_{Sch_k}(Spec(H^*), Spec(A)),$$

and the algebra $\beta(H, A)$ can be considered the coordinate ring of the infinite jet space of the scheme $Spec(A)$.

Lemma 3.41. The universal measuring algebra $\beta(H, \mathbb{C}[L])$ is the commutative $\mathbb{C}$-algebra generated by the symbols:

$$\{|j, \alpha\rangle\}_{j\geq 0, \alpha\in L}$$

subject to the relations:

(a) $|j, \alpha + \beta\rangle = \sum_{0\leq i\leq j} |j, \alpha\rangle |j - i, \beta\rangle$

(b) $|j, 0\rangle = \delta_{j,0}1$

Proof. By the construction in Theorem 3.35 $\beta(H, \mathbb{C}[L])$ is a quotient of $Sym(H \otimes \mathbb{C}[L])$ so it is generated by the tensors $\varepsilon^j \otimes e^\alpha$, where $\varepsilon^j = (t^j)^*$. We denote the class of this tensor by $|j, \alpha\rangle$. The relation on $Sym(H_n \otimes C)$ given by:

$$m \circ (i \circ i) \circ (id \otimes \Delta) - i \circ m \otimes id,$$

implies that $e^\alpha \otimes e^\beta \otimes \varepsilon^j$ under the map $m \circ (i \circ i) \circ (id \otimes \Delta)$:

$$\begin{aligned} e^\alpha \otimes e^\beta \otimes \varepsilon^j &\mapsto \sum_{0 \le i \le j} e^\alpha \otimes e^\beta \otimes \varepsilon^i \otimes \varepsilon^{j-i} \\ &\mapsto \sum_{0 \le i \le j} (e^\alpha \otimes \varepsilon^i) \otimes (e^\beta \otimes \varepsilon^{j-i}) \\ &\mapsto \sum_{0 \le i \le j} |j, \alpha\rangle |j-i, \beta\rangle \end{aligned}$$

is equal to $(i \circ m \otimes id)(e^\alpha \otimes e^\beta \otimes \varepsilon^j) = |j, \alpha + \beta\rangle$. Likewise, $\eta \circ \varepsilon = i \circ (id \otimes \eta)$ implies the second condition. □

Let us consider the infinite jet algebra $J_\infty \mathbb{C}[L]$. We can directly calculate this algebra from the definition, as in [7, §9.4.4]. Observe that if we introduce the variables x_α for $\alpha \in L$, then

$$\mathbb{C}[L] = \mathbb{C}[\{x_\alpha\}_{\alpha \in L}]/(\{x_\alpha x_\beta - x_{\alpha+\beta}\}_{\alpha,\beta \in L}, x_0 - 1)$$

and therefore, writing the relations $P_{\alpha,\beta} = x_\alpha x_\beta - x_{\alpha+\beta}$ and $Q = x_0 - 1$, then:

$$J_\infty \mathbb{C}[L] = \mathbb{C}[\{x_\alpha\}_{\alpha \in L}]/(\{P_{\alpha,\beta,n}\}_{n \le 0}, \{Q_n\}_{n \le 0}),$$

where $P_{\alpha,\beta,n}$ is the coefficient of t^n in

$$P_{\alpha,\beta}(\{x_\gamma(t)\}_{\gamma \in L}) = x_\alpha(t) x_\beta(t) - x_{\alpha+\beta}(t),$$

where $x_\gamma(t) = \sum_{n \le 0} x_{\gamma,n} t^n$. Thus, for $j \ge 0$,

$$P_{\alpha,\beta,-j} = \sum_{0 \le i \le j} x_{\alpha,-i} x_{\beta,-(j-i)} - x_{\alpha+\beta,-j}$$

and for $j \ge 0$, $Q_{-j} = x_{0,-j} - \delta_{j,0} 1$. This implies that there is an algebra isomorphism

$$\beta(H, \mathbb{C}[L]) \xrightarrow{\cong} J_\infty \mathbb{C}[L], \quad \varepsilon^j \otimes e^\alpha \mapsto x_{\alpha,-j}.$$

Remark. With H and L as above, there exists a function

$$L \to \mathbb{C}[L] \xrightarrow{u \otimes 1} H \otimes_{\mathbb{C}} \mathbb{C}[L] \hookrightarrow Sym(H \otimes_{\mathbb{C}} \mathbb{C}[L]) \twoheadrightarrow \beta(H, \mathbb{C}[L])$$

which sends $\alpha \mapsto |0,\alpha\rangle := 1 \otimes e^{\alpha}$. Relation (a) in Lemma 3.41 implies that:

$$|0,\alpha\rangle|0,\beta\rangle = |0,\alpha+\beta\rangle,$$

thus, the function above defines a morphism of groups and induces a morphism of algebras:

$$\mathbb{C}[L] \to \beta(H,\mathbb{C}[L]).$$

With the scheme $X = Spec(\mathbb{C}[L])$, the corresponding scheme map is then the projection

$$X \twoheadleftarrow J_{\infty}X.$$

Chapter 4

Singular multilinear categories

The aim of this chapter is to define the categorical construction of a vertex algebra as presented by Borcherds in [4]. This definition extends vertex algebras to higher dimensions and generalises a well-known method for constructing twisted group rings from a bicharacter of a group to produce quantum vertex algebras. Borcherds then provides a theorem that relates his (A,H,S) vertex algebras with the classical definition stated in Chapter 2. We add further details to Borcherds's example for the vertex algebra associated to the even integral lattice, however, our exposition will not be complete. The end of this chapter will then discuss what details have not been understood and are omitted in this thesis.

4.1 Monoidal categories

The motivation behind a categorical construction is that it produces many examples of vertex algebras by describing the essential aspects of the structure i.e. the operations of addition, multiplication, differentiation and multiplication by functions on space-time. The definitions of a monoidal category and algebras/coalgebras/bialgebras in a category can be found in Appendix A.

Example 4.1. Define **Fin** to be the category whose objects are all finite sets, with morphisms given by functions. Define $\mathbf{Fin}^{\not\equiv}$ to be the category whose objects are all finite sets with an equivalence relation $\equiv$, and whose morphisms are the functions f preserving inequivalence i.e. if $f(a) \equiv f(b)$ then $a \equiv b$. The finite sets of the form $\{1,2,...,n\}$ will be the main interest of the construction since these sets represent isomorphism classes of **Fin**.

Example 4.2. There exists a forgetful functor $U : \mathbf{Fin}^{\not\equiv} \to \mathbf{Fin}$ taking a finite set I to the same underlying set I with the equivalence relation that all elements are equivalent. Likewise, morphisms preserving inequivalence are mapped to functions.

Denote $Fun(\mathscr{A},\mathscr{B})$ as the category of functors from a small category $\mathscr{A}$ to a category $\mathscr{B}$, with natural transformations between functors. The category $\mathscr{A}$ will be either **Fin** or **Fin**$^{\neq}$, and $\mathscr{C}$ will usually be viewed as an additive category such as the category of vector spaces. Therefore, for each object in $x \in \mathbf{Fin}$, a functor in $Fun(\mathbf{Fin},\mathscr{A})$ will assign a corresponding vector space. This roughly describes the notion of assigning a Hilbert space to finite coordinates in space-time.

Definition 4.3. Suppose T is a cocommutative bialgebra in $Fun(\mathscr{C}^{op},\mathscr{A})$. Define a T-*module* in $Fun(\mathscr{C},\mathscr{A})$ as a functor $V \in Fun(\mathscr{C},\mathscr{A})$, such that for each $I \in \mathscr{C}$, $V(I)$ is a module over $T(I)$ in $\mathscr{A}$, and it satisfies the condition:

$$f_*(f^*(g)(v)) = g(f_*(v))$$

for all $v \in V(I)$, $g \in T(J)$, $f : I \to J, f_* : V(I) \to V(J), f^* : V(J) \to V(I)$ for all $I,J \in \mathscr{C}$. Define $Fun(\mathscr{C},\mathscr{A},T)$ to be the category of T-modules in $Fun(\mathscr{C},\mathscr{A})$. Additionally, suppose S is a commutative algebra in $Fun(\mathscr{C},\mathscr{A},T)$, then $Fun(C,A,T,S)$ denotes the additive symmetric monoidal category of S-modules in $Fun(\mathscr{C},\mathscr{A},T)$.

We emphasize the fact that T is an object in $Fun(C^{op},\mathscr{A})$ where S is an object in $Fun(\mathscr{C},\mathscr{A})$. The bialgebra T can be viewed as the functor that takes any finite set I and associates $|I|$ commuting copies of Hopf algebras associated to some group of space-time automorphisms acting on $|I|$ space-time variables. This is formalised in the following:

Definition 4.4. Suppose M is a commutative algebra in $\mathscr{A}$. Define $T_*(M) \in Fun(\mathscr{C},\mathscr{A})$ by:

$$T_*(M)(I) = \otimes_{i \in I} M$$

for all $I \in \mathscr{A}$ and for each morphism $f : I \to J$ there is a corresponding morphism $T_*(M)(f)$ induced by the multiplication and unit in M. This is illustrated by the following example.

Example 4.5. Suppose $\mathscr{C} = \mathbf{Fin}$. Set $I = \{1,2\}$ and given a function $f : I \to I$ defined by $f(1) = f(2) = 2$. Then $T_*(M)(f) : T_*(M)(I) \to T_*(M)(I)$ maps $x_1 \otimes x_2 \mapsto 1 \otimes x_1x_2$. In general, if given some morphism $f : I \to J$ in the category of finite sets, and if $f(i) = f(j)$, then $T_*(M)(f)$ will multiply the elements x_i and x_j via the multiplication of M at position $f(i)$. If k is in the cokernel of f, then there is a unit at position k.

Definition 4.6. Similarly, if H is a cocommutative coalgebra in $\mathscr{A}$, then define $T^*(H) \in Fun(\mathscr{C}^{op},\mathscr{A})$ by:

$$T^*(H)(I) = \otimes_{i \in I} H$$

for all $I \in \mathscr{A}$ and morphism induced by the comultiplication and counit in H.

Example 4.7. Take $\mathscr{C} = \mathbf{Fin}$ and $f : I \to I$ as above. Then $T^*(H)(f) : T^*(H)(I) \to T^*(H)(I)$ maps $x_1 \otimes x_2 \mapsto \varepsilon(x_1) \otimes \Delta(x_2)$, where Δ denotes the comultiplication and ε denotes the counit in H.

The category $Fun(\mathbf{Fin}, \mathscr{A}, T^*(H), S)$ then forms an additive symmetric tensor category with the natural tensor product induced from $\mathscr{A}$ (See Appendix A). Commutative algebras in $Fun(\mathbf{Fin}, \mathscr{A}, T^*(H), S)$, then capture most of the formal properties of a classical field theory. However, to describe quantum field theories we require singular multilinear maps in $Fun(\mathbf{Fin}, \mathscr{A}, T^*(H), S)$.

To define a new product on our category, which allows for singular multilinear maps, we define a universal construction. The tensor product is the universal object in a category that represents multilinear maps, hence, we wish to describe an object that represents the universal property of singular multilinear maps. This section follows [4, Definition 3.10].

Definition 4.8. Consider the module category $Fun(\mathscr{C}, \mathscr{A}, T, S)$ where T is cocommutative bialgebra in $Fun(\mathscr{C}^{op}, \mathscr{A})$ and S some commutative algebra in $Fun(\mathscr{C}, \mathscr{A}, T)$. Let $U_1, U_2, ...$ and V be objects of $Fun(\mathscr{C}, \mathscr{A}, T, S)$. Define a ***singular multilinear map*** from $U_1, U_2, ...$ to V, as the set of maps:

$$U_1(I_1) \otimes_{\mathscr{A}} U_2(I_2) \otimes_{\mathscr{A}} ... \to V(I_1 \cup I_2 \cup ...)$$

for all $I_1, I_2, ... \in \mathscr{C}$ satisfying the following conditions:

(a) the maps commute with the action of T,

(b) the maps commute with the action of $S(I_1), S(I_2), ...$

(c) if we are given morphisms from $I_1 \to I_1', I_2 \to I_2', ...$ then the following diagram commutes:

$$\begin{array}{ccc} U_1(I_1) \otimes U_2(I_2) \otimes ... & \longrightarrow & V(I_1 \cup I_2 \cup ...) \\ \downarrow & & \downarrow \\ U_1(I_1') \otimes U_2(I_2') \otimes ... & \longrightarrow & V(I_1' \cup I_2' \cup ...) \end{array}$$

Definition 4.9. Define $U_1 \odot U_2$ to be the universal singular tensor product representing these singular multilinear maps, then there exists an explicit formula given by the direct limit:

$$(U_1 \odot U_2)(I) := \lim_{I_1 \cup I_2 \to I} (U_1(I_1) \otimes_{\mathscr{A}} U_2(I_2)) \otimes_{S(I_1) \otimes_{\mathscr{A}} S(I_2)} S(I)$$

The universal property can be explicitly stated for the bilinear case as follows. If given a set of maps $U_1(I_1) \otimes_{\mathscr{A}} U_2(I_2) \to V(I_1 \cup I_2)$ for all $I_1, I_2 \in \mathscr{C}$ satisfing the conditions (a), (b) and

(c) defined above, then there exists a set of maps $(U_1 \odot U_2)(I) \to V(I_1 \cup I_2)$, which make the following diagrams:

$$\begin{array}{ccc} U_1(I_1) \otimes_{\mathscr{A}} U_2(I_2) & \xrightarrow{\;i\;} & (U_1 \odot U_2)(I) \\ & \searrow & \downarrow \\ & & V(I_1 \cup I_2) \end{array}$$

commute for all $I_1, I_2 \in \mathscr{C}$, and where i is the limit inclusion map.

Definition 4.10. Define a ***singular algebra*** in $Fun(\mathscr{C}, \mathscr{A}, T, S)$ as an algebra in $Fun(\mathscr{C}, \mathscr{A}, T, S)$ with the singular tensor product $\odot$ interchanging with the tensor product $\otimes$, and satisfying the respective associative and unit axioms.

4.2 Bicharacters

To explicitly construct the singular multiplication for the vertex algebra associated to an even integral lattice, we generalise the construction of a twisted group ring from a bicharacter of a group. This universal procedure lifts the pairing on the lattice to a singular multiplication on $\beta(H, R[L])$, making it a vertex algebra. Recall that a bicharacter is defined for R a ring and L a group. Let $R^\times$ be the group of units in R. A $R^\times$***-valued bicharacter*** of L is a map $r : L \times L \to R^\times$ such that it satisfies the following for all $a, b, c \in L$:

(a) $r(1, a) = r(a, 1) = 1$

(b) $r(ab, c) = r(a, c)r(b, c)$

(c) $r(a, bc) = r(a, b)r(a, c)$

Suppose $M := R[L]$ is the group ring. We can extend a bicharacter $r : L \times L \to R^\times$ to an R-linear map $r : M \otimes M \to R$. This identifies the bicharacters of the group L with the bicharacters of its group ring $M = R[L]$.

Definition 4.11. Suppose M and N are bialgebras over R, and S is a commutative R-algebra. A ***bimultiplcative map*** from $M \otimes N \to S$ is a R-linear map $r : M \otimes N \to S$, such that the following diagrams commute:

$$\begin{array}{ccc} M \cong M\otimes R & \xrightarrow{1\otimes u} & M\otimes N \\ \varepsilon\downarrow & & \downarrow r \\ R & \xrightarrow{u} & S \end{array} \qquad \begin{array}{ccc} N \cong R\otimes N & \xrightarrow{1\otimes u} & M\otimes N \\ \varepsilon\downarrow & & \downarrow r \\ R & \xrightarrow{u} & S \end{array}$$

$$\begin{array}{ccc} M\otimes M\otimes N & \xrightarrow{\nabla\otimes 1} & M\otimes N \\ 1\otimes 1\otimes\Delta\downarrow & & \downarrow r \\ M\otimes M\otimes N\otimes N & & S \\ \cong\downarrow & & \uparrow\Delta \\ (M\otimes N)^{\otimes 2} & \xrightarrow{r\otimes r} & S\otimes S \end{array} \qquad \begin{array}{ccc} M\otimes N\otimes N & \xrightarrow{1\otimes\nabla} & M\otimes N \\ \Delta\otimes 1\otimes 1\downarrow & & \downarrow r \\ M\otimes M\otimes N\otimes N & & S \\ \cong\downarrow & & \uparrow\Delta \\ (M\otimes N)^{\otimes 2} & \xrightarrow{r\otimes r} & S\otimes S \end{array}$$

where u denotes the unit map, ∇ denotes the multiplication, Δ denotes comultiplication, and ε is the counit of M or N. These diagrams are equivalent to the following conditions on elements:

(a) $r(1\otimes a) = r(a\otimes 1) = \varepsilon(1)$

(b) $r(ab\otimes c) = \sum r(a,c')r(b,c'')$

(c) $r(a\otimes bc) = \sum r(a',b)r(a'',c)$

where $\Delta(a) = \sum a'\otimes a''$ and $\Delta(c) = \sum c'\otimes c''$.

Definition 4.12. Let S be an R-algebra. A ***S-valued bicharacter*** of an algebra M is a bimultiplicative map $r : M\otimes M \to S$. The bicharacter r is symmetric if $r(a\otimes b) = r(b\otimes a)$ for all $a,b\in M$.

Lemma 4.13. If G is a group and S is an R-algebra, any $S^\times$-valued bicharacter induces a S-valued bicharacter of $R[G]$, and this gives a bijection:

$$\{S^\times\text{- valued bicharacters of G }\} \cong \{S\text{- valued bicharacters of } R[G]\}$$

Proof. A bicharacter $r : G\times G \to S^\times$ lifts to a linear map $r : R[G]\otimes R[G] \to S$ and defining $r(a\otimes b) = r(a,b)$ and this is clearly bimultiplicative. Conversely, let $r : R[G]\otimes R[G] \to S$ be a given bicharacter on the algebra $R[G]$. Then r is determined by its values $r(g,g')$ and

$$r(g,g')r(g^{-1},g') = r(1,g') = 1$$

Thus, $r(g,g')\in S^\times$ for all $g,g'\in G$. □

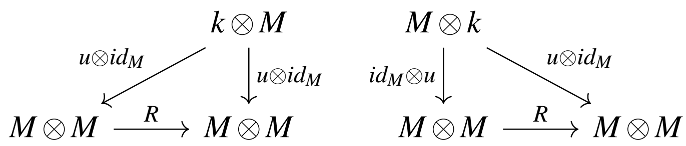
Example 4.14. Suppose $L = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$ is a free abelian group of rank n with basis $\{\alpha_1, ..., \alpha_n\}$, and suppose we are given $S^\times$-valued bicharacter $r(\alpha_i, \alpha_j) \in S^\times$ for some commutative R-algebra S. Then there exists an extension of r to a unique $S^\times$-valued bicharacter of the group ring $M = R[L]$ by defining:

$$\begin{aligned} r(\sum_{i=1}^{n} m_i\alpha_i, \sum_{j=1}^{n} n_i\alpha_j) &:= \prod_{i=1}^{n} r(\alpha_i, \sum_{j=1}^{n} n_j\alpha_j))^{m_i} \\ &= \prod_{i=1}^{n} \big[\prod_{j=1}^{n} r(\alpha_i, \alpha_j)^{n_j}\big]^{m_i} \\ &= \prod_{i,j=1}^{n} r(\alpha_i, \alpha_j)^{n_j m_i} \end{aligned}$$

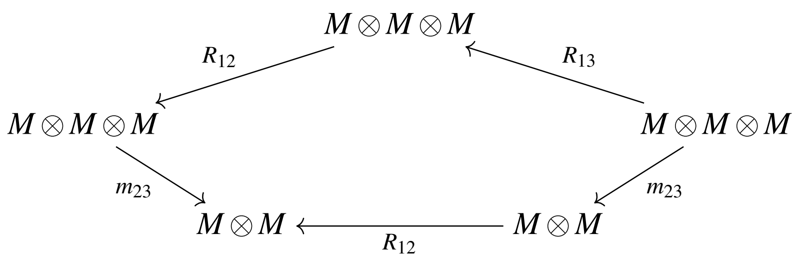
and hence extends to a S-valued bicharacter $r : R[L] \otimes R[L] \to S$, which can be used to determine a twisted multiplication on $R[L]$.

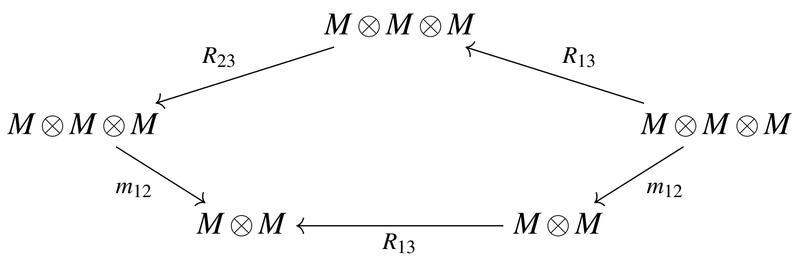
Definition 4.15. A R-**matrix** for a ring M with multiplication $m : M \otimes M \to M$ consists of a linear map $R : M \otimes M \to M \otimes M$ making the following diagrams commute:

(a) Compatibility with identity:

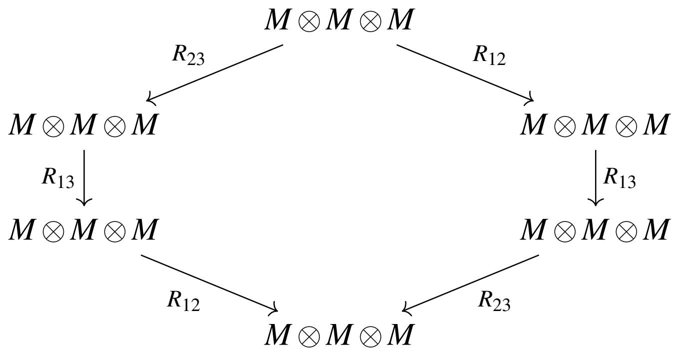


(b) Compatibility with multiplication





(c) Yang-Baxter equation



where m_{13} is the multiplication m applied to the first and third factors and R_{13} is the R-matrix applied to the first and third factors, and so on. These diagrams are equivalent to the following conditions for all $a \in M$:

(a) $R(1 \otimes a) = 1 \otimes a, \quad R(a \otimes 1) = a \otimes 1,$

(b) $m_{23}R_{12}R_{13} = R_{12}m_{23}, \quad m_{12}R_{23}R_{13} = R_{13}m_{12},$

(c) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

Lemma 4.16. Let M be a commutative and cocommutative bialgebra, and $r : M \otimes M \to R$ a bicharacter. The composition:

$$M \otimes M \xrightarrow{\Delta\otimes\Delta} M \otimes M \otimes M \otimes M \xrightarrow{id_M \otimes \sigma \otimes id_M} M \otimes M \otimes M \otimes M \xrightarrow{id_{M\otimes M} \otimes r} M \otimes M \otimes R \cong M \otimes M$$

defines a R-matrix for M, where $\sigma : M \otimes M \to M \otimes M$ is the map $\sigma(a \otimes b) := b \otimes a$ for all $a, b \in M$, and Δ is the comultiplication of M . In other words, we define the linear map for all $a, b \in M$:

$$R(a \otimes b) = \sum a_{(1)} \otimes b_{(1)} r(a_{(2)} \otimes b_{(2)}).$$

By defining $\circ : M \otimes M \to M$ the composite:

$$\circ : M \otimes M \xrightarrow{R} M \otimes M \xrightarrow{m} M$$

where m is the original multiplication of M, makes M an R-algebra with the new multiplication $\circ$ called the *twisting of M* by r, and is commutative if r is symmetric. This is equivalent to the linear map defined for all $a, b \in M$:

$$a \circ b := \sum a_{(1)} b_{(1)} r(a_{(2)} \otimes b_{(2)}).$$

Proof. The conditions for $R : M \otimes M \to M \otimes M$ to be an R-matrix can be easily checked. The element $1 \in M$ is an identity for the twisting of M since R is compatible with the identity. The twisting is an associative algebra because R satisfies the Yang-Baxter equation and is compatible with multiplication. The twisting is commutative if r is symmetric and M is commutative. □

Let H be a cocommutative coalgebra, M a commutative algebra, and

$$h : H \otimes M \to \beta(H,M)$$

the universal measuring. If H is in addition a bialgebra, then $Hom_k(H, \beta(H,M))$ is an algebra with the convolution product, and the map

$$\tau : H \otimes M \to Hom_k(H, \beta(H,M)),$$

is adjoint to the linear map,

$$H \otimes H \otimes M \xrightarrow{m \otimes 1} H \otimes M \xrightarrow{h} \beta(H,M),$$

which is also a measuring, since it is adjoint to the algebra map

$$M \xrightarrow{\widetilde{h}} Hom_k(H, \beta(H,M)) \xrightarrow{(\_) \circ m} Hom_k(H \otimes H, \beta(H,M)).$$

Note that we have used the facts that $m : H \otimes H \to H$ is a coalgebra map, and that the convolution algebra $Hom_k(H,A)$ is functorial in its two arguments. Since τ is a measuring, the universal property of $\beta(H,M)$ gives us an algebra map T making the following diagram commute:

$$\begin{array}{ccc} H \otimes M & \xrightarrow{\quad h \quad} & \beta(H,M) \\ & \searrow^{\tau} & \downarrow{T} \\ & & Hom_k(H, \beta(H,M)) \end{array}$$

Note that $\tau(h_1 \otimes m)(h_2) = h(h_1 h_2 \otimes m)$ and so T is defined by extending this to $\beta(H,M) = Sym(H \otimes M)/I$. Finally, T is adjoint to a linear map

$$H \otimes \beta(H,M) \to \beta(H,M)$$
$$h_0 \otimes ([h_1 \otimes m_1] \ldots [h_r \otimes m_r]) \mapsto [\tau(h_1 \otimes m_1) * \ldots * \tau(h_r \otimes m_r)](h_0)$$

which is by definition a measuring with $*$ the convolution multiplication. This map defines an action of H on $\beta(H,M)$.

4.3 Examples of Borcherds's vertex algebra

Borcherds's categorical definition of a vertex algebra is then given in [4, Definition 3.8] as follows:

Definition 4.17. Suppose $\mathscr{A}$ is an additive symmetric tensor category, H is a cocommutative bilagebra in $\mathscr{A}$, and S is a commutative algebra in $Fun(\mathrm{Fin}^{\not\equiv}, \mathscr{A}, T^*(H))$. An $(\mathscr{A}, H, S)$ ***vertex algebra*** is a singular commutative algebra in $Fun(\mathrm{Fin}^{\not\equiv}, \mathscr{A}, T^*(H), S)$.

Remark. Suppose H is the symmetric algebra $\mathbb{C}[x]$ viewed as a Hopf algebra with comltiplication $\Delta(x) := 1 \otimes x + x \otimes 1$, counit $\varepsilon(x) = 0$, and antipode $S(x) = -x$. This is the Hopf algebra associated to the additive algebraic group $(\mathbb{C}, +, 0, i)$ in Example 3.30. Borcherds defines H' as the formal group ring of the one dimensional additive formal group, viewed as the commutative cocommutative Hopf algebra over $\mathbb{C}$ with basis $D^{(i)}$ for $i \geq 0$, where the multiplication and comultiplication are defined:

$$D^{(i)}D^{(j)} = \binom{i+j}{i} D^{(i+j)}, \quad \Delta(D^{(i)}) = \sum_j D^{(j)} \otimes D^{(i-j)}$$

There is an isomorphism of Hopf algebras, $H \cong H'$ given by the bijection $\frac{1}{i!}x^i \mapsto D^{(i)}$. Note that it defines an algebra morphism since:

$$D^{(i)}D^{(j)} = \frac{x^i}{i!}\frac{x^j}{j!} = \frac{x^{i+j}}{i!j!} = \frac{(i+j)!}{i!j!}\frac{x^{i+J}}{(i+j)!} = \binom{i+j}{i} D^{(i+j)}$$

and, it defines a coalgebra morphism since:

$$\Delta(D^{(i)}) = \Delta(\frac{x^i}{i!}) = \frac{1}{i!}\Delta(x)^i = \frac{1}{i!}\sum_{j=0}\binom{i}{j} x^j \otimes x^{i-j} = \sum_{j=0}\frac{x^j}{j!} \otimes \frac{x^{i-j}}{(i-j)!} = \sum_j D^{(j)} \otimes D^{(i-j)}.$$

The following theorem then shows that the vertex algebra defined in Chapter 2 and Borcherds's definition are equivalent.

Theorem 4.18. Suppose k is an algebraically closed field and H is the symmetric algebra $k[x]$ viewed as a Hopf algebra with comltiplication $\Delta(x) := 1 \otimes x + x \otimes 1$, counit $\varepsilon(x) = 0$, and antipode $S(x) = -x$. Define S a commutative ring in $Fun(\mathbf{Fin}^{\not\equiv}, \mathbf{Vec}_k, T^*(H))$ by $S(I) =$ the k-algebra generated by elements $(x_i - x_j)^{\pm 1}$ for i and j not equivalent (Note that $S = k$

if all elements of I are equivalent) for $I \in \mathbf{Fin}^{\not\equiv}$. If V is a $(\mathbf{Vec}_k, H, S)$ vertex algebra, and 1 denotes the finite set $\{1\}$, then $V(1)$ is an ordinary vertex algebra (as defined in Chapter 2).

Proof. We follow Theorem 4.3 in [4]. By definition, $V(1)$ is a vector space over k. Thus, we just need to define a creative field for each state and show that these fields satisfy the required axioms. In other words, for each $u_1 \in V(1)$, we need to construct a vertex operator $Y(u_1, x_1) : V(1) \to V(1)[[x_1]][x_1^{-1}]$. If $u_2 \in V(2)$, then the singular product $u_1 \odot u_2 \in V(1:2) = V(1,2) \otimes S(1:2) = V(1,2)[(x_1 - x_2)^{\pm 1}]$ defines an element in $V(1,2)[(x_1 - x_2)^{\pm 1}]$. There exists a map by taking a vector w to its Taylor series expansion:

$$\begin{aligned} V(1,2) &\to V(1)[[x_1, x_2]] \\ w &\mapsto \sum_{i,j} f_{1,2\to 1}(D_1^{(i)} D_2^{(j)} w) x_1^i x_2^j \end{aligned}$$

where $f_{1,2\to 1} : V(1,2) \to V(1)$ is the map induced from the morphism of sets $f : \{1,2\} \to \{1\}$, and D_1 and D_2 indicate the two different actions of the Hopf algebra H on $V(1,2)$ (See previous remark). Let w^* be the image of w under this map. This induces another map:

$$\begin{aligned} V(1,2)[(x_1 - x_2)^{\pm 1}] &\to V(1)[[x_1, x_2]][(x_1 - x_2)^{\pm 1}] \\ \sum_n w_n (x_1 - x_2)^n &\mapsto \sum_n w^* (x_1 - x_2)^n \\ u_1 \odot u_2 &\mapsto u_1(x_1) u_2(x_2) \end{aligned}$$

where the coefficients of the $(x_1 - x_2)^n$ terms in $V(1,2)[(x_1 - x_2)^{\pm 1}]$ are mapped to coefficients of the $(x_1 - x_2)^n$ terms in $V(1)[[x_1, x_2]][(x_1 - x_2)^{-1}]$. We denote $u_1(x_1)u_2(x_2)$ the image of $u_1 \odot u_2$ under this map. Therefore, we can define $Y(u_1, x_1)$ by

$$Y(u_1, x)(u_2) := \lim_{x_2 \to 0} u_1(x_1) u_2(x_2) = u_1(x_1) u_2(0) \in V(1)[[x_1]][x_1^{-1}]$$

for all u_2, which defines our vertex operators. To show that these fields are mutually local, we can similarly define the terms:

$$Y(u_1, x_1) \ldots Y(u_n, x_n)(u_{n+1}) = u_1(x_1) \ldots u_n(x_n) u_{n+1}(0) \in V(1)[[x_1, \ldots, x_n]][\prod_{i \le j}^{n} (x_i - x_j)^{-1}]$$

Since V is a commutative singular algebra, we have that $u_1 \odot u_2 = u_2 \odot u_1$, which implies that:

$$u_1(x_1) u_2(x_2) u_3(0) = u_2(x_2) u_1(x_1) u_3(0) \in V(1)[[x_1, x_2]][(x_1 - x_2)^{-1}]$$

for all $u_1 \in V(1), u_2 \in V(2), u_3 \in V(3)$. i.e

$$Y(u_1,x_1)Y(u_2,x_2)(u_3) = Y(u_2,x_2)Y(u_1,x_1)(u_3) \in V(1)[[x_1,x_2]][(x_1-x_2)^{-1}]$$

There are two possible expansions of $(x_1-x_2)^{-1}$, which may not be equivalent in the space $V(1)[[x_1,x_2]][x_1^{-1},x_2^{-1}]$. Hence, in general $Y(u_1,x_1)Y(u_2,x_2)(u_3) \in V(1)[[x_1,x_2]][x_1^{-1}][x_2^{-1}]$ and $Y(u_2,x_2)Y(u_1,x_1)(u_3) \in V(1)[[x_1,x_2]][x_2^{-1}][x_1^{-1}]$ are not equal. However, there exists some sufficiently large positive integer N such that:

$$(x_1-x_2)^N Y(u_1,x_1)Y(u_2,x_2)(u_3) = (x_1-x_2)^N Y(u_2,x_2)Y(u_1,x_1)(u_3) \in V(1)[[x_1,x_2]]$$

which are then equivalent in the space $V(1)[[x_1,x_2]]$ implying the vertex operators are muturally local in the ordinary sense:

$$(x_1-x_2)^N \big(Y(u_1,x_1)Y(u_2,x_2) - Y(u_2,x_2)Y(u_1,x_1)\big)(u_3) = 0$$

for all $u_3 \in V(3)$. Note that there is a subtle difference in the way that the state-field operator is defined in this case. In the ordinary definition, the state-field correspondence is usually defined as such:

$$Y : V \to End(V)[[x]][x^{-1}]$$

However, Borcherds defines a vertex operator $\tilde{Y}$ by:

$$\tilde{Y} : V \otimes V \to V[[x]][x^{-1}]$$

These are equivalent definitions given by the assignment:

$$\tilde{Y}(v,w) = Y(v)(w)$$

□

Example 4.19. Let H be the Hopf algebra over $\mathbb{C}$ of the affine algebraic group $(C,+,0,i)$ as in Example 3.30 and $M = \mathbb{C}[L]$ the group algebra of an even integral lattice L with a pairing $(\_,\_) : L \times L \to \mathbb{Z}$. Following Example 4.4 of Borcherds we define (with an ordered basis $\alpha_1,...,\alpha_n$ of L):

$$c(\alpha_i,\alpha_j) = \begin{cases} 1 & \text{if } i \geq j \\ (-1)^{(\alpha_i,\alpha_j)} & \text{if } i < j \end{cases}$$

such that $c(\alpha,\beta) = (-1)^{(\alpha,\beta)}c(\beta,\alpha)$ for all $\alpha,\beta \in L$, where

$$c : L \times L \to \mathbb{C}^{\times}$$

defines a bicharacter on L. Let S be the $\mathbb{C}$-algebra $\mathbb{C}[(x_1 - x_2)^{\pm 1}]$. The bicharacter c can be extended to a symmetric S-valued bicharacter r of L by defining:

$$r : L \times L \to S^{\times}, \quad r(\alpha,\beta) = c(\alpha,\beta)(x_1 - x_2)^{(\alpha,\beta)}$$

This then corresponds to a symmetric S-valued bicharacter on $\mathbb{C}[L]$:

$$r' : \mathbb{C}[L] \otimes \mathbb{C}[L] \to S$$

by Lemma 4.13. Let $h : H \otimes \mathbb{C}[L] \to \beta(H,\mathbb{Z}[L])$ be the universal measuring. By Lemma 2.14 of Borcherds, there is a $H \otimes H$-invariant action on S and by Lemma 2.15, r' lifts uniquely to a $H \otimes H$-invariant bicharacter:

$$r'' : \beta(H,\mathbb{Z}[L]) \otimes \beta(H,\mathbb{Z}[L]) \to S$$

This then extends to a singular bicharacter of $T_*(\beta(H,\mathbb{Z}[L]))$ by Lemma 4.1 of Borcherds, and V_L is obtained by twisiting the multiplication by this bicharacter. This then characterises the singular multilinear map in the category $Fun(\mathrm{Fin}^{\neq}, \mathbf{Vec}_{\mathbb{C}}, T^*(H), S)$:

$$r''' : T_*(\beta(H,\mathbb{Z}[L])) \odot T_*(\beta(H,\mathbb{Z}[L])) \to S$$

where S is given in the statement of Theorem 4.3 of Borcherds, and the formula for r''' is given in Lemma 4.1 of Borcherds. The twisted multiplication associated to r''' on $V_L := T_*(\beta(H,\mathbb{Z}[L]))$ is then defined by:

$$V_L \odot V_L \xrightarrow{\Delta\odot\Delta} (V_L \otimes V_L) \odot (V_L \otimes V_L) \to (V_L \odot V_L) \otimes (V_L \odot V_L) \xrightarrow{1\otimes r'''} (V_L \odot V_L) \otimes S \to V_L \odot V_L \to V_L$$

where $(U \otimes V) \odot (W \otimes X) \to (U \odot W) \otimes (V \odot X)$ for all $U,V,W,X \in Fun(\mathrm{Fin}^{\neq}, \mathbf{Vec}_{\mathbb{C}}, T^*(H), S)$ is the interchange map and note that V_L is a S-module. This defines a $(\mathbf{Vec}_k, H, S)$ vertex algebra and by Theorem 4.18, $V_L(1)$ is the even integral lattice vertex operator algebra.

Remark. In the end, we fell short of completely understanding Borcherds's construction of the lattice vertex operator algebra. The following lists the gaps in our knowledge and the questions yet to be answered:

(a) How does the bialgebra structure of H induce a bialgebra structure on the universal measuring algebra $\beta(H,M)$?

(b) A detailed proof that shows the new singular multiplication defined by twisting forms an algebraic structure on A.

(c) How do functors in the category $Fun(\mathbf{Fin}, \mathbf{Vec}_k, T^*(H))$ promote to functors in the category $Fun(\mathbf{Fin}^{\not\equiv}, \mathbf{Vec}_k, T^*(H))$ by tensoring with S?

(d) Do the vertex operators defined by Borcherds's (A,H,S) vertex algebras coincide with the vertex operators defined in Chapter 2?

References

[1] Arthur Wightman, L. G. (1965). Fields as operator-valued distributions in relativistic quantum theory.

[2] Barr, M. (1974). Coalgebras over a commutative ring. *Jounral of Algebra, 600-610*.

[3] Borcherds, R. (1986). Vertex algebras, kac-moody algebras, and the monster. *arXiv preprint*.

[4] Borcherds, R. E. (1999). Quantum vertex algebras. *arXiv preprint*.

[5] Christoph Schweigert, J. F. (2003). *Symmetries, Lie Algebras and Representations*. Cambridge Monographs on Mathematical Physics.

[6] Dmitri Nikshych, Pavel Etingof, S. G. and Ostrik, V. (2010). *Tensor Categories*. American Mathematical Society.

[7] Edward Frenkel, D. B.-Z. (2004). *Vertex Algebras and Algebraic Curves*, volume 88. Mathematical Surveys and Monographs.

[8] Igor Frenkel, James Lepowsky, A. M. (1988). *Vertex Operator Algebras and the Monster*. Academic Press.

[9] James Lepowsky, H. (2004). *Introduction to Vertex Operator Algebras and Their Representations*. Springer Science+Business Media.

[10] Kac, V. (1998). *Vertex Algebras for Beginners*, volume 10. Unversity Lecture Series.

[11] Kakutani, S. (1941). Concrete representation of abstract (m)-spaces (a characterization of the space of continuous functions). *Annals of Mathematics*.

[12] Kassel, C. (1995). *Quantum Groups*. Springer-Science+Business Media, LLC.

[13] Raymond F. Streater, A. S. W. (1964). *PCT, Spin and Statistics, and All That*. Princeton University Press.

[14] Saunders, D. (1989). *The Geometry of Jet Bundles*. Cambridge University Press.

[15] Sweedler, M. E. (1969). *Hopf Algebras*. W. A. Benjamin , Inc.

[16] Tuite, M. (2017). Vertex algebras according to isaac newton. *arXiv preprint*.

Appendix A

Monoidal categories

The following is based on the definitions presented in [6, Etingof].

Definition A.1. A ***monoidal category*** or ***tensor category*** is a sextuple $(\mathscr{M}, \otimes, a, \mathbf{1}, l, r)$ where $\mathscr{M}$ is a category, $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is a bifunctor called the tensor product bifunctor, $a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ is a natural isomorphism:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \quad \text{for all } X, Y, Z \in \mathscr{M}$$

called the associativity constraint, $\mathbf{1}$ is an object of $\mathscr{M}$, and l, r are natural isomorphisms

$$l_X : \mathbf{1} \otimes X \to X, \quad r_X : X \otimes \mathbf{1} \to X, \quad \text{for all } X \in \mathscr{M}$$

called the left and right unit constraints or unit isomorphisms respectively, subject to the following axioms:

(a) **The pentagon axiom:** The diagram

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{a_{W,X,Y} \otimes id_Z} & & \searrow^{a_{W \otimes X,Y,Z}} & \\ (W \otimes (X \otimes Y)) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow{\scriptstyle a_{W,X \otimes Y,Z}} & & & & \downarrow{\scriptstyle a_{W,X,Y \otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{id_W \otimes a_{X,Y,Z}} & & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

is commutative for all objects $W, X, Y, Z \in \mathscr{M}$.

(b) **The triangle axiom:** The diagram

$$\begin{array}{ccc} (X\otimes 1)\otimes Y & \xrightarrow{a_{X,1,Y}} & X\otimes(1\otimes Y) \\ & {\scriptstyle r_X\otimes id_Y}\searrow \quad \swarrow{\scriptstyle id_X\otimes l_Y} & \\ & X\otimes Y & \end{array}$$

is commutative for all $X,Y\in\mathscr{M}$

Example A.2. Let k be any field. Then the category $\mathbf{Vec}_k$ of all k-vector spaces with k-linear homomorphisms is an example of a monoidal category, where we define the bifunctor $\otimes = \otimes_{\mathbf{k}}$ by the usual tensor product of vector spaces, and unit is the underlying field k. The natural isomorphisms are then induced by the usual isomorphisms.

Definition A.3. A tensor category $(\mathscr{M},\otimes,a,\mathbf{1},l,r)$ is called *symmetric* if there exists a natural isomorphism $s_{AB}: A\otimes B\to B\otimes A$ for all $A,B\in\mathscr{M}$, such that the following diagrams commute:

(a) The unit coherence:

$$\begin{array}{ccc} A\otimes\mathbf{1} & \xrightarrow{s_{A\mathbf{1}}} & 1\otimes A \\ & {\scriptstyle r_A}\searrow \quad \swarrow{\scriptstyle l_A} & \\ & A & \end{array}$$

(b) The associativity coherence

$$\begin{array}{ccc} (A\otimes B)\otimes C & \xrightarrow{s_{AB}\otimes id_C} & (B\otimes A)\otimes C \\ \downarrow & & \downarrow \\ A\otimes(B\otimes C) & & B\otimes(A\otimes C) \\ \downarrow & & \downarrow \\ (B\otimes C)\otimes A & \longrightarrow & B\otimes(C\otimes A) \end{array}$$

(c) The inverse law

$$\begin{array}{ccc} & B\otimes A & \\ {\scriptstyle s_{AB}}\nearrow & & \searrow{\scriptstyle s_{BA}} \\ A\otimes B & \xrightarrow{id_{A\otimes B}} & A\otimes B \end{array}$$

for all $A,B,C\in\mathscr{M}$.

Lemma A.4. Let $\mathscr{C}$ be a small category, $\mathscr{A}$ a monoidal category. Then the category $Fun(\mathscr{C},\mathscr{A})$ is a monoidal category.

Proof. The tensor product of $\mathscr{A}$ induces a tensor product on $Fun(\mathscr{C},\mathscr{A})$ by defining the bifunctor $(F\otimes G)(X) = F(X)\otimes_{\mathscr{A}} G(X)$ for all $F,G\in Fun(\mathscr{C},\mathscr{A})$ and $X\in ob(\mathscr{C})$. Likewise, our unit object $\mathbf{1}$ of $Fun(\mathscr{C},\mathscr{A})$ is induced by the unit object in $\mathscr{A}$, $\mathbf{1}(X) = \mathbf{1}_{\mathscr{A}}$ for all objects $X\in ob(\mathscr{C})$. The associator and unit natural isomorphisms are similar induced by the underlying associator and unit natural isomorphisms of $\mathscr{A}$. $Fun(\mathscr{C},\mathscr{A})$ with this bifunctor, will then satisfy the pentagon and triangle axoims since $\otimes_{\mathscr{A}}$ and $\mathbf{1}_{\mathscr{A}}$ does. □

Our favourite algebraic structures, such as groups, algebras, and modules, can now be generalised in a similar manner. Our underlying objects are now longer necessarily sets but could be topological spaces or manifolds. This extends the notion of algebras from a vector space to other abstract objects.

Definition A.5. An ***associative unital algebra*** in a tensor category $\mathscr{C}$ is a triple (A,m,u), where A is an object of $\mathscr{C}$, and $m: A\otimes A\to A$ and $u:\mathbf{1}\to A$ are morphisms called mulitplication and unit satisfying the following commutative diagrams:

(a) **Associativity:**

$$\begin{array}{ccc} (A\otimes A)\otimes A & \xrightarrow{a_{A,A,A}} & A\otimes(A\otimes A) \\ \downarrow{\scriptstyle m\otimes id_A} & & \downarrow{\scriptstyle id_A\otimes m} \\ A\otimes A & & A\otimes A \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array}$$

(b) **Left unit**:

$$\begin{array}{ccc} \mathbf{1}\otimes A & \xrightarrow{l_A} & A \\ \downarrow{\scriptstyle u\otimes id_A} & & \downarrow{\scriptstyle id_A} \\ A\otimes A & \xrightarrow{m} & A \end{array}$$

(c) **Right unit:**

$$\begin{array}{ccc} A\otimes \mathbf{1} & \xrightarrow{r_A} & A \\ \downarrow{\scriptstyle id_A\otimes u} & & \downarrow{\scriptstyle id_A} \\ A\otimes A & \xrightarrow{m} & A \end{array}$$

Furthermore, an algebra in $\mathscr{C}$ is called **commutative** if given $\sigma : A \otimes A \to A \otimes A$, defined by $\sigma(a_1 \otimes a_2) = a_2 \otimes a_1$, then the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\sigma} & A \otimes A \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array}$$

Remark. If we take $\mathscr{C} = \mathbf{Vec}_k$, then we get the usual definition of an associative unital k-algebra. Additionally, the dual notion of a coalgebra and bialgebra can be similarly generalised.

Definition A.6. A ***coassociative and counital coalgebra*** in $\mathscr{C}$ is a triple (C, Δ, ε) where C is an object of $\mathscr{C}$, and $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbf{1}$ are morphisms called comulitplication and counit satisfying the following commutative diagrams:

(a) **Coassociativity:**

$$\begin{array}{ccc} (C \otimes C) \otimes C & \xleftarrow{a^{-1}_{CCC}} & C \otimes (C \otimes C) \\ {\scriptstyle \Delta \otimes id_C}\uparrow & & \uparrow{\scriptstyle id_C \otimes \Delta} \\ C \otimes C & & C \otimes C \\ & {\scriptstyle \Delta}\nwarrow \quad \nearrow{\scriptstyle \Delta} & \\ & C & \end{array}$$

(b) **Left counit**:

$$\begin{array}{ccc} \mathbf{1} \otimes C & \xleftarrow{l^{-1}_A} & C \\ {\scriptstyle \varepsilon \otimes id_C}\uparrow & & \uparrow{\scriptstyle id_C} \\ C \otimes C & \xleftarrow[\Delta]{} & C \end{array}$$

(c) **Right counit:**

$$\begin{array}{ccc} C \otimes \mathbf{1} & \xleftarrow{r^{-1}_C} & C \\ {\scriptstyle id_C \otimes \varepsilon}\uparrow & & \uparrow{\scriptstyle id_C} \\ C \otimes C & \xleftarrow[\Delta]{} & C \end{array}$$

Definition A.7. A **bialgebra** in $\mathscr{C}$ is a quintuple $(B, m, u, \Delta, \varepsilon)$ such that B is an object of $\mathscr{C}$, the triple (B, m, u) is an algebra in $\mathscr{C}$ and the triple (B, Δ, ε) is a colagebra in $\mathscr{C}$ with Δ, ε algebra morphism.

Morphisms between algebras and coalgebras are just morphisms in the category $\mathscr{C}$ that make the diagrams defined in chapter 3 commute. To define analogies of an algebra action on a set, we can define an algebra action on the underlying objects in the monoidal category.

Definition A.8. Suppose $\mathscr{C}$ is a tensor category and (A,m,u) is an algebra in $\mathscr{C}$. A **right module** over an algebra A in $\mathscr{C}$ is pair (M,p), where M is an object in $\mathscr{C}$, and $p: M\otimes A \to M$ is a morphism satisfying the following commutative diagrams:

(a)

$$\begin{array}{ccccc} (M\otimes A)\otimes A & & \xrightarrow{\quad a_{MAA} \quad} & & M\otimes(A\otimes A) \\ \downarrow{\scriptstyle p\otimes id_A} & & & & \downarrow{\scriptstyle id_M\otimes m} \\ M\otimes A & & & & M\otimes A \\ & \searrow^{p} & & {}^{p}\swarrow & \\ & & M & & \end{array}$$

(b)

$$\begin{array}{ccc} M\otimes \mathbf{1} & \xrightarrow{r_M} & M \\ \downarrow{\scriptstyle id_M\otimes u} & & \downarrow{\scriptstyle id_M} \\ M\otimes A & \xrightarrow{p} & M \end{array}$$

A left A-module in $\mathscr{C}$ can be similarly defined. If A is a commutative algebra in $\mathscr{C}$ then M is a left A-module if and only if it is a right A-module. In this case, we can simply refer to M as an A-module in $\mathscr{C}$. Given two A-modules M_1, M_2 in $\mathscr{C}$, module homomorphisms between M_1 and M_2 form a subspace of the set $Hom_{\mathscr{C}}(M_1,M_2)$. This subspace can be viewed as the categorical analogy of A-linear homomorphisms between two A-modules. Denote this subspace $Hom_A(M_1,M_2)$. Taking all A-modules in $\mathscr{C}$ as objects and the A-module homomorphisms as morphisms, forms a category, which we denote $\mathbf{Mod}_{\mathscr{C}}(A)$. In a similar way we can define bialgebra-modules and coalgebra-comodules.

Nomenclature

Algebraic Symbols

$(\ )^\circ$ Sweedler dual

$*$ Convolution

$\beta(H,A)$ Universal measuring algebra

Δ Comultiplication

ε Counit

η Unit

$\mathbb{1}$ Vacuum vector

$\mathscr{A}$ Additive symmetric monoidal category

$\mathscr{C}$ Symmetric monoidal category

$\mathscr{F}$ A collection of mutually local fields

$\mathscr{H}$ Heisenberg Lie algebra

∇ Multiplication

Fin The category of finite sets

$\mathbf{Fin}^{\not\equiv}$ The category of finite sets with an inequivalence relation

A Associative unital algebra

$A[[z]]$ Formal power series

$A[[z^{\pm 1}]]$ Formal distributions

$A[z]$	Formal polynomials
C	Coassociative counital coalgebra
$G(C)$	The set of group-like elements of C
H	Hopf algebra
I, J	Finite sets
$J_\infty A$	Infinite jet scheme
$J_\infty X$	Infinite jet space
k	Field
L	Lattice
M_0	Verma module
$P(C)$	The set of all primitive elements of C
r	Bicharacter
S	Singular commutative algebra
T	Translation operator
$T^*(H)(I)$	The tensor product $\bigotimes_{i\in I} H$
$T_*(M)(I)$	The tensor product $\bigotimes_{i\in I} M$
V_L	Lattice vertex algebra
Y	State-field correspondence