Localisation of Ringoids

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The localisation of commutative rings is a central part of commutative geometry. The localisation of noncommutative rings is more difficult, since the noncommutativity introduces numerous technical difficulties into the process of formally inverting ring elements. A more elegant way to treat localisation is by using additive grothendieck topologies. In this note we give a proof of the Gabriel-Popescu theorem, which is widely used in algebra.

Contents

1 Additive Topologies

Throughout we use the notation of our notes on Rings with Several Objects. In particular a *ringoid* $\mathcal{A}$ is a small preadditive category. In this note we fix a nonempty ringoid $\mathcal{A}$, and all modules are right $\mathcal{A}$-modules. Recall that a *sieve* at an object $C \in \mathcal{A}$ is a subfunctor of the set-valued functor $H_C : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$.

Definition 1. Let $\mathcal{A}$ be a ringoid. A *right ideal* at an object C is a submodule of the right $\mathcal{A}$-module $H_C : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$. Equivalently, a right ideal at C is a collection $\mathfrak{a}$ of morphisms with codomain C, which satisfies

(i) For all $D \in \mathcal{A}$ the zero morphism $0 : D \longrightarrow C$ is in $\mathfrak{a}$.

(ii) If $f, g : D \longrightarrow C$ are in $\mathfrak{a}$, so is $f - g$.

(iii) If $f : D \longrightarrow C$ is in $\mathfrak{a}$, and $h : X \longrightarrow D$ is any other morphism, then fh is in $\mathfrak{a}$.

A right ideal $\mathfrak{a}$ at A is *proper* if it is not equal to H_A and is *maximal* if it is proper and is not properly contained in any other proper right ideal at A. These definitions restrict to the usual ones when the preadditive category is a ring. By analogy with the single-object case, ideals are typically labelled $\mathfrak{a}, \mathfrak{b}$, etc.

Definition 2. Let $\mathcal{A}$ be a ringoid. A *left ideal* at an object C is a submodule of the left $\mathcal{A}$-module $H^C : \mathcal{A}^{\text{op}} \longrightarrow \mathbf{Ab}$. Equivalently, a left ideal at C is a collection $\mathfrak{a}$ of morphisms with domain C, which satisfies

(i) For all $D \in \mathcal{A}$ the zero morphism $0 : C \longrightarrow D$ is in $\mathfrak{a}$.

(ii) If $f, g : C \longrightarrow D$ are in $\mathfrak{a}$, so is $f - g$.

(iii) If $f : C \longrightarrow D$ is in $\mathfrak{a}$, and $h : D \longrightarrow X$ is any other morphism, then hf is in $\mathfrak{a}$.

A left ideal $\mathfrak{a}$ at A is *proper* if it is not equal to H^A and is *maximal* if it is proper and is not properly contained in any other proper left ideal at A. These definitions restrict to the usual ones when the preadditive category is a ring. Note that $\mathfrak{a}$ is a left ideal in $\mathcal{A}$ iff. it is a right ideal in $\mathcal{A}^{\text{op}}$, so without loss of generality we can restrict our attention to right ideals.

Example 1. The following are examples of right ideals in a ringoid:

1. Let F be a right $\mathcal{A}$-module, with $x \in F(A)$ for some $A \in \mathcal{A}$. We define the following right ideal at A, called the *annihilator* of x:

$$Ann(x) = \{\alpha : D \longrightarrow A \,|\, x\alpha = 0\}$$

   Corresponding to $x \in F(A)$ is a morphism $H_A \longrightarrow F$, and $Ann(x)$ is precisely the kernel of this morphism.

2. The collection of all zero morphisms with codomain A is a right ideal at A, called the *zero ideal* and denoted 0_A or just 0.

3. Let $f : B \longrightarrow A$ be a morphism of $\mathcal{A}$, and let $\mathfrak{a}$ be the submodule of H_A defined by

$$\mathfrak{a}(C) = \{fh \,|\, h : C \longrightarrow B\}$$

   That is, $\mathfrak{a}$ consists of all the morphisms in $\mathcal{A}$ which factor through f. It is easily checked that this is a right ideal at A, which we call the *principal ideal* generated by f, and denote fH_B or (f).

It is helpful to keep in mind that in the category $\mathbf{Mod}\mathcal{A}$ any subobject of a module F is isomorphic (as a subobject) to a submodule of F. Subfunctors precede each other as subobjects iff. pointwise they are *contained* in each other. In particular, when talking about subobjects of representable functors H_A it suffices to talk about the right ideals at A.

Definition 3. Let $\mathcal{A}$ be a ringoid and $\mathfrak{a}$ a right ideal at C. For a morphism $h : D \longrightarrow C$, we define the *pullback* of $\mathfrak{a}$ to be the following right ideal at D

$$h^*\mathfrak{a} = \{f : X \longrightarrow D \,|\, hf \in \mathfrak{a}\}$$

In the category of right $\mathcal{A}$-modules, $h^*\mathfrak{a}$ is the pullback of the submodule $\mathfrak{a}$ of H_C. In the case where $\mathcal{A}$ is a ring, this ideal is more commonly denoted by $(\mathfrak{a} : h)$. If $\mathfrak{b}$ is a left ideal at D then we define the *pushout* of $\mathfrak{b}$ to be the following left ideal at C

$$h_*\mathfrak{b} = \{f : C \longrightarrow X \,|\, fh \in \mathfrak{b}\}$$

Under the equality $\mathcal{A}\mathbf{Mod} = \mathbf{Mod}\mathcal{A}^{\text{op}}$ left ideals at C are identified with right ideals at C, and $h_*\mathfrak{b}$ is identified with $h^*\mathfrak{b}$.

We now define *additive grothendieck topologies* on $\mathcal{A}$. These topologies are different to the normal grothendieck topologies on $\mathcal{A}$, but the only difference is in the so-called "transitivity" axiom, where only right ideals are eligible for consideration.

Definition 4. Let $\mathcal{A}$ be a ringoid. A *right additive topology* on $\mathcal{A}$ is a function J assigning to each object C a set $J(C)$ of right ideals at C, which satisfies

(i) The maximal right ideal H_C is always in $J(C)$.

(ii) If $\mathfrak{a} \in J(C)$ and $h : D \longrightarrow C$, then $h^*\mathfrak{a} \in J(D)$.

(iii) If $\mathfrak{a} \in J(C)$ and $\mathfrak{b}$ is any right ideal at C, and if $f^*\mathfrak{b} \in J(D)$ for every $f : D \longrightarrow C \in \mathfrak{a}$, then $\mathfrak{b} \in J(C)$.

If $\mathfrak{a} \in J(A)$ then we say that $\mathfrak{a}$ is an *additive cover* of A. If J, K are two right additive topologies such that $J(C) \subseteq K(C)$ for all $C \in \mathcal{A}$, then we write $J \leq K$. This defines the partially ordered set of right additive topologies on $\mathcal{A}$.

Definition 5. Let $\mathcal{A}$ be a ringoid. A *left additive topology* on $\mathcal{A}$ is a function J assigning to each object C a set $J(C)$ of left ideals at C, which satisfies

(i) The maximal left ideal H^C is always in $J(C)$.

(ii) If $\mathfrak{a} \in J(C)$ and $h : C \longrightarrow D$, then $h_*\mathfrak{a} \in J(D)$.

(iii) If $\mathfrak{a} \in J(C)$ and $\mathfrak{b}$ is any left ideal at C, and if $f_*\mathfrak{b} \in J(D)$ for every $f : C \longrightarrow D \in \mathfrak{a}$, then $\mathfrak{b} \in J(C)$.

If $\mathfrak{a} \in J(A)$ then we say that $\mathfrak{a}$ is an *additive cover* of A. If J, K are two left additive topologies such that $J(C) \subseteq K(C)$ for all $C \in \mathcal{A}$, then we write $J \leq K$. This defines the partially ordered set of left additive topologies on $\mathcal{A}$.

Definition 6. A pair $(\mathcal{A}, J)$ consisting of a ringoid and a right additive topology is called a *small additive site*. Since an assignment of left ideals J is a left additive topology iff. it is a right additive topology on $\mathcal{A}^{\mathrm{op}}$, without loss of generality we can restrict our attention to right additive topologies. In the rest of this note, an *additive topology* (or sometimes just *topology*) is a right additive topology unless specified otherwise.

If A is a (not necessarily commutative) nonzero ring and $\mathcal{A}$ the corresponding ringoid, there is a bijection between right ideals of A and right ideals at the single object of $\mathcal{A}$. A collection of right ideals is a topology in the above sense if and only if it is a gabriel topology on the ring A as defined in [33], [14].

Example 2. The smallest possible topology on $\mathcal{A}$ is the topology J_0 defined by $J_0(C) = \{H_C\}$, and the largest topology is the topology J_1 defined by letting $J_1(C)$ be the collection of all right ideals at C. Clearly every topology on $\mathcal{A}$ contains J_0 and is contained in J_1, so that J_0 and J_1 are respectively the initial and terminal objects of the poset of additive topologies on $\mathcal{A}$.

One of the simplest classes of additive topologies are those determined by a multiplicatively closed subset:

Definition 7. A *multiplicatively closed subset* of a ringoid $\mathcal{A}$ is a collection S of morphisms in $\mathcal{A}$ with the property that if $f, g \in S$ and the composition makes sense, then also $fg \in S$. We say that S is an *Ore set* if it is multiplicatively closed, and satisfies the following two conditions:

(i) For $s : A \longrightarrow B$ in S and any morphism $a : C \longrightarrow B$ of $\mathcal{A}$, we can find $t : Z \longrightarrow C$ in S and $b : Z \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} & B & \\ A \arrow[ur, "s"] & & C \arrow[ul, "a"'] \\ & Z \arrow[ul, "b"] \arrow[ur, "t"'] & \end{tikzcd}$$

(ii) For any $A \in \mathcal{A}$ there is a morphism in S with codomain A.

Lemma 1. *If S is an Ore set, then*

$$J_S(A) = \{\mathfrak{a} \,|\, \mathfrak{a} \cap S \neq \emptyset\}$$

defines an additive topology on $\mathcal{A}$.

Proof. The fact that $H_A \in J_S(A)$ for all $A \in \mathcal{A}$ is the purpose of condition (ii) above. For the stability axiom, if $a : C \longrightarrow B$ is any morphism and $\mathfrak{a} \in J_S(B)$, say $s : A \longrightarrow B$ belongs to $\mathfrak{a} \cap S$, then apply (i) above to find t, z such that $t \in S$ and $sb = at$. Certainly $t \in a^*\mathfrak{a} \cap S$, as required.

The transitivity axiom follows directly from the fact that S is multiplicatively closed. $\square$

If $\mathfrak{a}$ is an additive cover of $C \in \mathcal{A}$ and $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ is any functor (not necessarily additive), a natural transformation $\phi : \mathfrak{a} \longrightarrow F$ of $\mathbf{Ab}$-valued functors is a family of elements $x_f \in F(D)$ indexed by the $f : D \longrightarrow C \in \mathfrak{a}$, with the following properties:

(i) $x_{f+g} = x_f + x_g$ for every pair $f, g : D \longrightarrow C$ in $\mathfrak{a}$.

(ii) $x_f \cdot g = x_{fg}$ whenever $f \in \mathfrak{a}$ and g is composable with f.

Notice that (i) implies $x_0 = 0$ for any zero morphism with codomain C. We call such collections $\{x_f \,|\, f \in \mathfrak{a}\}$ (equivalently, morphisms $\mathfrak{a} \longrightarrow F$) *additive matching families for* $\mathfrak{a}$ in F. We only consider additive matching families defined for additive covers $\mathfrak{a} \in J(C)$. An *amalgamation* for such an additive matching family is an element $x \in F(C)$ such that $x \cdot f = x_f$ for all $f \in \mathfrak{a}$.

Lemma 2. *Let $\mathcal{A}$ be a ringoid with additive topology J. Then*

(i) If $\mathfrak{a}, \mathfrak{b} \in J(C)$ then $\mathfrak{a} \cap \mathfrak{b} \in J(C)$.

(ii) If $f : D \longrightarrow C$ is in $\mathfrak{a} \in J(C)$ then $f^*\mathfrak{a} = H_C$.

(iii) If $\mathfrak{a} \in J(C)$ then any larger right ideal $\mathfrak{b} \supseteq \mathfrak{a}$ is also in $J(C)$.

(iv) If $f : D \longrightarrow C$ and $\mathfrak{b}$ is a right ideal at D, then

$$f\mathfrak{b} = \{fb \,|\, b \in \mathfrak{b}\}$$

is a right ideal at C.

(v) If $\mathfrak{a} \in J(C)$ and for each $f : D \longrightarrow C \in \mathfrak{a}$ we have $\mathfrak{b}_f \in J(D)$ then the ideal $\sum f\mathfrak{b}_f$ belongs to $J(C)$.

Proof. The proofs of $(i) - (iv)$ are trivial. In (v), $\sum f\mathfrak{b}_f$ denotes the sum of the submodules $f\mathfrak{b}_f$ of H_C. That is, $\sum f\mathfrak{b}_f$ consists of morphisms $g : X \longrightarrow C$ of the form

$$g = \sum_{f \in \mathfrak{a}} f\gamma_f \qquad \gamma_f \in \mathfrak{b}_f$$

For $f : D_f \longrightarrow C$ in $\mathfrak{a}$,

$$f^*(\sum f\mathfrak{b}_f) = \{g : X \longrightarrow D_f \,|\, fg \in \sum f\mathfrak{b}_f\} \supseteq \mathfrak{b}_f$$

Hence $f^*(\sum f\mathfrak{b}_f) \in J(D_f)$, and so by the transitivity axiom we have $\sum f\mathfrak{b}_f \in J(C)$. $\square$

Lemma 3. *Let $(\mathcal{A}, J)$ be an additive site. For a right $\mathcal{A}$-module M we define*

$$\begin{aligned} t_J(M)(C) &= \{x \in M(C) \,|\, \exists \mathfrak{a} \in J(C)\ x \cdot \mathfrak{a} = 0\} \\ &= \{x \in M(C) \,|\, Ann(x) \in J(C)\} \end{aligned}$$

Thus defined $t_J(M)$ is a submodule of M.

Proof. First we show that for each $C \in \mathcal{A}$, $t_J(M)(C)$ is a subgroup of $M(C)$. Clearly $0 \in t_J(M)(C)$. Suppose that $x, y \in t_J(M)(C)$ and say $\mathfrak{a}, \mathfrak{b} \in J(C)$ with $x \cdot \mathfrak{a} = 0$ and $y \cdot \mathfrak{b} = 0$. Then $\mathfrak{a} \cap \mathfrak{b} \in J(C)$ and $(x - y) \cdot \mathfrak{a} \cap \mathfrak{b} = 0$. Hence $x - y \in t_J(M)(C)$.

To show that $t_J(M)$ is a submodule of M, we need to show that it is closed under the action of an arbitrary morphism $f : C \longrightarrow D$. Let $x \in t_J(M)(D)$, so there is $\mathfrak{a} \in J(D)$ with $x \cdot \mathfrak{a} = 0$. Then $(x \cdot f) \cdot f^*\mathfrak{a} = 0$, so $x \cdot f \in t_J(M)(C)$, as required. □

Definition 8. We call $t_J(M)$ the *J-torsion* submodule of M, and say $x \in M(A)$ is a *J-torsion element* if $x \in t_J(M)$. A module M is *J-torsion* if $t_J(M) = M$ and *J-torsion-free* if $t_J(M) = 0$.

Example 3. If R is a commutative domain and $S = R \setminus \{0\}$, then an element x of an R-module M is torsion for the topology J_S defined in Lemma 1 iff. it is torsion in the usual sense. If J, K are two additive topologies with $J \leq K$ it is clear that $t_J(M) \leq t_K(M)$. In the case of the improper topologies J_0, J_1 defined above, we have

$$t_{J_0}(M) = 0$$
$$t_{J_1}(M) = M$$

Note that for any module M over a ringoid $\mathcal{A}$, the module $M/t_J(M)$ is J-torsion-free.

In the notation of topos theory, a presheaf P is *separated* if whenever a matching family has an amalgamation it is unique, and is a *sheaf* if every matching family has a unique amalgamation. Let $(\mathcal{A}, J)$ be an additive site and notice that a module M is "separated" for additive matching families and an additive topology J precisely when it is J-torsion-free. For an additive cover $\mathfrak{a} \in J(A)$ the morphism of modules $\mathfrak{a} \longrightarrow H_A$ gives rise to a morphism of groups

$$M(A) \cong Hom_{\mathcal{A}}(H_A, M) \longrightarrow Hom_{\mathcal{A}}(\mathfrak{a}, M) \tag{1}$$

This maps an element $x \in M(A)$ to the additive matching family for $\mathfrak{a}$ in M given by $x_f = x \cdot f$. Injectivity of this morphism for all $\mathfrak{a} \in J(A)$ and $A \in \mathcal{A}$ is also equivalent to M being J-torsion-free.

Definition 9. Let $(\mathcal{A}, J)$ be an additive site. A right $\mathcal{A}$-module M is *J-injective* if the morphism in (1) is an epimorphism for each $\mathfrak{a} \in J(A)$ and $A \in \mathcal{A}$. This is equivalent to the condition that every matching family in M has an amalgamation.

Finally, we define the additive version of a sheaf:

Definition 10. Let $(\mathcal{A}, J)$ be an additive site. A right $\mathcal{A}$-module M is *J-closed* if one the following equivalent conditions hold:

- The morphism in (1) is an isomorphism for each $\mathfrak{a} \in J(A)$ and $A \in \mathcal{A}$;
- M is J-injective and J-torsion-free;
- Every additive matching family in M has a unique amalgamation.

If J is a left additive topology on $\mathcal{A}$ and M is a left $\mathcal{A}$-module, then we say M is J-injective or J-closed if M as this property considered as a right $\mathcal{A}^{\mathrm{op}}$-module with right additive topology J on $\mathcal{A}^{\mathrm{op}}$.

Example 4. Let R be a right Ore domain. This is a ring without zero-divisors such that the set S of all nonzero elements is an Ore set, in the sense of Definition 7. For example, R could be any right noetherian ring without zero-divisors ([33], II 1.7). If J_S is the topology described in Lemma 1, then the right ideal (s) is an additive cover for any regular element $s \in R$. Since s is regular, it is not hard to check that $x_{sa} = a$ defines an additive matching family for the cover (s) in the R-module R.

If R is J_S-closed, there must be an amalgamation $s' \in R$ such that $x_{sa} = s'sa$ for all $sa \in (s)$. In particular $s's = 1$. It follows that if R is J_S-closed all the elements of S must be invertible. In this way "sheafifying" a ring with respect to an additive topology generalises the process of adding inverses.

Example 5. Every module M is trivially closed for the topology J_0, and M is closed for the topology J_1 iff. $M = 0$. If J, K are two additive topologies such that $J \leq K$, then it is clear that any K-closed module is also J-closed.

2 Additive Sheafification

In this section we investigate the process of turning a module into a closed module by adding unique amalgamations for additive matching families. In the non-additive case this goes under the name of the "plus construction" or "sheafification", so we will refer to the following process as the *additive plus construction* or *additive sheafification*. Throughout this section $(\mathcal{A}, J)$ will be a fixed additive site.

The plus construction associates with every right $\mathcal{A}$-module F another right $\mathcal{A}$-module F^+. As we will show, applying this construction twice gives a J-closed module. For an object C of $\mathcal{A}$ the ideals $\mathfrak{a} \in J(C)$ correspond to distinct submodules of H_C, and the abelian groups $Hom(\mathfrak{a}, F)$ form a directed family of abelian groups. For a right $\mathcal{A}$-module F we define

$$F^+(C) = \varinjlim_{\mathfrak{a} \in J(C)} Hom_{\mathcal{A}}(\mathfrak{a}, F)$$

We realise this direct limit in the following way: an element $x \in F^+(C)$ is an equivalence class of morphisms $\xi : \mathfrak{a} \longrightarrow F$, where ξ and $\phi : \mathfrak{b} \longrightarrow F$ are equal iff. they agree on some $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$, with $\mathfrak{c} \in J(C)$. Equivalently, elements of $F^+(C)$ are additive matching families $\{x_f \mid f \in \mathfrak{a}\}$ with $\mathfrak{a} \in J(C)$ under the equivalence relation which says that $\{x_f \mid f \in \mathfrak{a}\} \sim \{y_g \mid g \in \mathfrak{b}\}$ iff. there is $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$ belonging to $J(C)$ such that $x_h = y_h$ for all $h \in \mathfrak{c}$.

This is an abelian group under the operation given by $\xi + \phi : \mathfrak{a} \cap \mathfrak{b} \longrightarrow F$, $(\xi + \phi)_D(f) = \xi_D(f) + \phi_D(f)$, or equivalently $\{x_f \mid f \in \mathfrak{a}\} + \{y_g \mid g \in \mathfrak{b}\} = \{x_h + y_h \mid h \in \mathfrak{a} \cap \mathfrak{b}\}$. To define F^+ on morphisms, let $h : C \longrightarrow C'$ be given together with an element $\{x_f \mid f \in \mathfrak{a}\}$ of $F^+(C')$. We define

$$\{x_f \mid f \in \mathfrak{a}\} \cdot h = \{x_{hf'} \mid f' \in h^*\mathfrak{a}\} \in F^+(C)$$

To see this is well-defined, suppose that $\{x_f \mid f \in \mathfrak{a}\} \sim \{y_g \mid g \in \mathfrak{b}\}$, so that there is $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$ belonging to $J(C')$ such that for $n \in \mathfrak{c}$, $x_n = y_n$. Then $h^*\mathfrak{c} \subseteq h^*\mathfrak{a} \cap h^*\mathfrak{b}$ and for $m \in h^*\mathfrak{c}$ we have $hm \in \mathfrak{c}$ so $x_{hm} = y_{hm}$ and therefore $\{x_f \mid f \in \mathfrak{a}\} \cdot h \sim \{y_g \mid g \in \mathfrak{b}\} \cdot h$, as required. It is easy to check that this makes $F^+(h) = - \cdot h$ into a morphism of groups.

In terms of morphisms $\mathfrak{a} \longrightarrow F$, acting on ξ with $h : C \longrightarrow C'$ is carried out by composing ξ with the morphism of right ideals $h^*\mathfrak{a} \longrightarrow \mathfrak{a}$ which fits into the pullback diagram of modules:

$$\begin{tikzcd}
H_C \arrow[r, "H_h"] & H_{C'} \\
h^*\mathfrak{a} \arrow[u] \arrow[r] & \mathfrak{a} \arrow[u]
\end{tikzcd}$$

This defines a functor $F^+ : \mathcal{A}^{\text{op}} \longrightarrow \mathbf{Ab}$, which we will now show is additive.. If $h, g : D \longrightarrow C$ then for any additive matching family $\{x_f \mid f \in \mathfrak{a}\} \in F^+(C)$ we have

$$\begin{aligned}
\{x_f \mid f \in \mathfrak{a}\} \cdot (h + g) &= \{x_{(h+g)f'} \mid f' \in (h+g)^*\mathfrak{a}\} \\
&\sim \{x_{hf'} + x_{gf'} \mid f' \in h^*\mathfrak{a} \cap g^*\mathfrak{a}\} \\
&= \{x_f \mid f \in \mathfrak{a}\} \cdot h + \{x_f \mid f \in \mathfrak{a}\} \cdot g
\end{aligned}$$

where we have used the fact that $h^*\mathfrak{a} \cap g^*\mathfrak{a} \subseteq (h+g)^*\mathfrak{a}$.

Definition 11. Associated to every right $\mathcal{A}$-module M is a canonical right $\mathcal{A}$-module M^+. If $\varphi : M \longrightarrow N$ is a morphism of modules, then we define a morphism of modules

$$\varphi^+ : M^+ \longrightarrow N^+$$
$$(\varphi^+)_C(\{x_f \mid f \in \mathfrak{a}\}) = \{\varphi(x_f) \mid f \in \mathfrak{a}\}$$

This defines an additive functor $(-)^+ : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{A}$. There is a canonical morphism $\phi_M : M \longrightarrow M^+$ of modules defined by $\phi_{M,A}(x) = \{x \cdot g \,|\, g \in H_A\}$, which is clearly natural in M. For an object A of $\mathcal{A}$ the morphism of abelian groups $\phi_{M,A}$ is the composite

$$M(A) \cong Hom(H_A, M) \longrightarrow \varinjlim_{\mathfrak{a} \in J(C)} Hom(\mathfrak{a}, M)$$

It is clear that for any right $\mathcal{A}$-module M we have $\phi_{M^+} = \phi_M^+$.

Lemma 4. *If M is a right $\mathcal{A}$-module then $Ker\phi_M = t_J(M)$.*

Lemma 5. *Let M be a right $\mathcal{A}$-module. Then M is J-torsion-free iff. $\phi : M \longrightarrow M^+$ is a monomorphism, and is J-closed iff. $\phi : M \longrightarrow M^+$ is an isomorphism.*

Proof. Using the previous Lemma the first statement is obvious. To prove the second statement, suppose that M is J-closed. Since it is then J-torsion-free, the morphism ϕ is monic, and to show that it is an isomorphism it suffices to show that it is pointwise epimorphic. Let $\mathfrak{a} \in J(A)$ be an additive cover of $A \in \mathcal{A}$ and suppose $\xi : \mathfrak{a} \longrightarrow M$ represents an element of $M^+(A)$. Since M is J-closed we can amalgamate this matching family to an element $x \in M(A)$ with $\phi_A(x) = \xi$. Hence ϕ is an isomorphism.

Conversely suppose that ϕ is an isomorphism. Since M is then J-torsion-free it suffices to show that every additive matching family $\{x_f \,|\, f \in \mathfrak{a}\}$ at A has an amalgamation. Since ϕ_A is an epimorphism, let $x \in M(A)$ be such that $\{x \cdot g \,|\, g \in H_A\} \sim \{x_f \,|\, f \in \mathfrak{a}\}$ in $M^+(A)$. Then there is $\mathfrak{b} \in J(A)$ with $\mathfrak{b} \subseteq \mathfrak{a}$ and $x \cdot g = x_g$ for all $g : D \longrightarrow A$ in $\mathfrak{b}$. To show that M is closed, we have to show that $x \cdot f = x_f$ for all $f \in \mathfrak{a}$ which are *not* in $\mathfrak{b}$.

But if $f : D \longrightarrow C$ is in $\mathfrak{a}$ then $f^*\mathfrak{b}$ covers D and M is separated so it would suffice to show that $(x \cdot f) \cdot h = x_f \cdot h$ for all $h : X \longrightarrow D$ in $f^*\mathfrak{b}$. But $(x \cdot f) \cdot h = x \cdot (fh)$ and $x_f \cdot h = x_{fh}$. Since $fh \in \mathfrak{b}$, the proof is complete. □

Lemma 6. *A right $\mathcal{A}$-module module M is J-torsion if and only if $M^+ = 0$.*

Proof. If $M^+ = 0$ then $\phi : M \longrightarrow M^+$ has kernel $t_J(M) = M$, so M is J-torsion by Lemma 4. Conversely suppose that M is J-torsion, and let $\{x_f \,|\, f \in \mathfrak{a}\}$ represent an element of $M^+(A)$, $\mathfrak{a} \in J(A)$. For each $f : D \longrightarrow A$ in $\mathfrak{a}$, $x_f \in M(D)$ is J-torsion. That is, there is an additive cover $\mathfrak{a}_f \in J(D)$ such that $x_f \cdot \mathfrak{a}_f = 0$. By Lemma 2 (v) the right ideal $\sum f\mathfrak{a}_f$ is in $J(A)$.

To prove that the family $\{x_f \,|\, f \in \mathfrak{a}\}$ represents the zero element in $M^+(A)$, it suffices to show that it restricts to the zero family on some cover of A. But for $\sum f\gamma_f \in \sum f\mathfrak{a}_f$ we have

$$x_{\sum f\gamma_f} = \sum x_{f\gamma_f} = \sum x_f \cdot \gamma_f = 0$$

since $\gamma_f \in \mathfrak{a}_f$ and $x_f \cdot \mathfrak{a}_f = 0$. □

Proposition 7. *Let M, N be right $\mathcal{A}$-modules and suppose that M is J-closed. Then any morphism of modules $\theta : N \longrightarrow M$ factors uniquely through $\phi : N \longrightarrow N^+$, as in the diagram*

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[dr, "\theta"'] & N^+ \arrow[d, dotted] \\ & M \end{tikzcd}$$

Proof. An element of $N^+(C)$ is represented by an additive matching family $\{x_f \,|\, f \in \mathfrak{a}\}$ for $\mathfrak{a} \in J(C)$. For $h : D \longrightarrow C$ in $\mathfrak{a}$ we have

$$\phi_D(x_h) = \{x_h \cdot k \,|\, k \in H_D\}$$

Of course, we also have $\{x_f \,|\, f \in \mathfrak{a}\} \cdot h = \{x_{hf'} \,|\, f' \in h^*\mathfrak{a}\}$. But $h^*\mathfrak{a} = H_D$, so since the x_f match,

$$\phi_D(x_h) = \{x_f \,|\, f \in \mathfrak{a}\} \cdot h$$

It follows that if a factorisation $\widehat{\theta} : N^+ \longrightarrow M$ existed, then $\widehat{\theta}_C(\{x_f \,|\, f \in \mathfrak{a}\})$ would have to be the unique $y \in M(C)$ such that for all $h : D \longrightarrow A \in \mathfrak{a}$,

$$\begin{aligned} y \cdot h &= \widehat{\theta}_C(\{x_f \,|\, f \in \mathfrak{a}\}) \cdot h = \widehat{\theta}_D(\{x_f \,|\, f \in \mathfrak{a}\} \cdot h) \\ &= \widehat{\theta}_D \phi_D(x_h) = \theta_D(x_h) \end{aligned} \tag{2}$$

This implies that $\widehat{\theta}$ will be unique, once we have defined it. To define $\widehat{\theta}_C : N^+(C) \longrightarrow M(C)$, we note that for an additive matching family $\{x_f \,|\, f \in \mathfrak{a}\}$, the collection $\{\theta(x_f) \,|\, f \in \mathfrak{a}\}$ is an additive matching family for the closed module M, and thus has unique amalgamation $y \in M(C)$. To show that this assignment $\{x_f \,|\, f \in \mathfrak{a}\} \mapsto y$ is well-defined, suppose we had an additive matching family $\{x'_g \,|\, g \in \mathfrak{b}\}$ equivalent to $\{x_f \,|\, f \in \mathfrak{a}\}$: say they agree on $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$. If z is the amalgamation in $M(C)$ of the family $\{\theta_D(x'_g) \,|\, g \in \mathfrak{b}\}$, then for $s : X \longrightarrow C \in \mathfrak{c}$ we have

$$y \cdot s = \theta_X(x_s) = \theta_X(x'_s) = z \cdot s$$

Since $\mathfrak{c}$ is a cover and M is J-injective, this implies that $y = z$. Hence, as a morphism of sets, our map $\widehat{\theta}_C$ is well-defined. It is straightforward to check that each of these maps is a morphism of groups. So it only remains to show naturality.

For $h : D \longrightarrow C$ in $\mathcal{A}$ we need to show that the following diagram commutes:

$$\begin{tikzcd} N^+(C) \arrow[r] \arrow[d, "\widehat{\theta}_C"'] & N^+(D) \arrow[d, "\widehat{\theta}_D"] \\ M(C) \arrow[r] & M(D) \end{tikzcd}$$

For $\{x_f \,|\, f \in \mathfrak{a}\}$ representing an element of $N^+(C)$ we have

$$\begin{aligned} \widehat{\theta}_D(\{x_f \,|\, f \in \mathfrak{a}\} \cdot h) &= \widehat{\theta}_D(\{x_{hf'} \,|\, f' \in h^*\mathfrak{a}\}) \\ &= y \in M(D) \text{ s.t. } y \cdot f' = \theta(x_{hf'}) \quad \forall f' \in h^*\mathfrak{a} \end{aligned}$$

But $\widehat{\theta}_C(\{x_f \,|\, f \in \mathfrak{a}\}) \cdot h$ is $y' \cdot h$ where y' is the unique element of $M(C)$ with $y' \cdot f = \theta(x_f)$ for all $f \in \mathfrak{a}$. Hence for $f' \in h^*\mathfrak{a}$

$$(y' \cdot h) \cdot f' = y' \cdot (hf') = \theta(x_{hf'})$$

Since y and $y' \cdot h$ agree on an additive cover of D and M is closed, they must be equal. This shows that $\widehat{\theta}$ is natural. □

Lemma 8. *For any right $\mathcal{A}$-module M, M^+ is J-torsion-free.*

Proof. Suppose $\{x_f \,|\, f \in \mathfrak{a}\}$ represents a torsion element of $M^+(C)$. That is, there is an additive cover $\mathfrak{b} \in J(C)$ with $\{x_f \,|\, f \in \mathfrak{a}\} \cdot g = 0$ for all $g \in \mathfrak{b}$. Since

$$\{x_f \,|\, f \in \mathfrak{a}\} \cdot g = \{x_{gf'} \,|\, f' \in g^*\mathfrak{a}\}$$

the fact that this element is torsion means that for each $g : D \longrightarrow C \in \mathfrak{b}$ there is $\mathfrak{c}_g \in J(D)$ with $\mathfrak{c}_g \subseteq g^*\mathfrak{a}$ and $x_{gf'} = 0$ for all $f' \in \mathfrak{c}_g$. By Lemma 2 (v) the right ideal $\sum_{g \in \mathfrak{a}} g\mathfrak{c}_g$ is a cover of C. If h is any morphism in this cover, say

$$h = \sum g\gamma_g \qquad \gamma_g \in \mathfrak{c}_g$$

Then we have

$$x_h = x_{\sum g\gamma_g} = \sum x_{g\gamma_g} = 0$$

Hence $\{x_f \,|\, f \in \mathfrak{a}\}$ represents the zero element, and so M^+ is J-torsion-free. □

Proposition 9. *If a right $\mathcal{A}$-module M is J-torsion-free, then M^+ is J-closed.*

Proof. By the previous Lemma M^+ is J-torsion-free, so it suffices to show that M^+ is J-injective. Let $\{\mathbf{x}_f \,|\, f \in \mathfrak{a}\}$ be an additive matching family for a cover $\mathfrak{a} \in J(C)$, with $\mathbf{x}_f \in M^+(D)$ for $f : D \longrightarrow C$ in $\mathfrak{a}$. That is, there is an additive cover $\mathfrak{a}_f \in J(D)$ with

$$\mathbf{x}_f = \{x_{f,g} \,|\, g \in \mathfrak{a}_f\} \qquad x_{f,g} \in M(E), g : E \longrightarrow D$$

Since $\{\mathbf{x}_f\}_f$ is a matching family $\mathbf{x}_f \cdot h = \mathbf{x}_{fh}$ for any $h : D' \longrightarrow D$. Thus there is an equivalence of matching families

$$\{x_{f,hg'} \,|\, g' \in h^*\mathfrak{a}_f\} \sim \{x_{fh,g} \,|\, g \in \mathfrak{a}_{fh}\}$$

Which implies that there is an additive cover $\mathfrak{a}_{f,h} \subseteq h^*\mathfrak{a}_f \cap \mathfrak{a}_{fh}$ with

$$x_{f,hg''} = x_{fh,g''} \qquad \forall g'' \in \mathfrak{a}_{f,h}$$

The fact that the $\mathbf{x}_f$ form an *additive* matching family means that $\mathbf{x}_f + \mathbf{x}_g = \mathbf{x}_{f+g}$ for any $f, g : D \longrightarrow C$ in $\mathfrak{a}$. That is, there is an equivalence of matching families

$$\{x_{f,h} + x_{g,h} \,|\, h \in \mathfrak{a}_f \cap \mathfrak{a}_g\} \sim \{x_{f+g,h} \,|\, h \in \mathfrak{a}_{f+g}\}$$

which means that there is an additive cover $\mathfrak{c}_{f,g} \subseteq \mathfrak{a}_f \cap \mathfrak{a}_g \cap \mathfrak{a}_{f+g}$ such that $x_{f,h} + x_{g,h} = x_{f+g,h}$ for all $h \in \mathfrak{c}_{f,g}$.

We define an amalgamation $\mathbf{y} \in M^+(C)$ of the family $\{\mathbf{x}_f \,|\, f \in \mathfrak{a}\}$ as follows: let $\mathbf{y}$ be defined on the additive cover $\sum_{f\in\mathfrak{a}} f\mathfrak{a}_f \in J(C)$ by

$$\begin{aligned} \mathbf{y} &= \{y_h \,|\, h \in \sum f\mathfrak{a}_f\} \\ y_{\sum f\gamma_f} &= \sum x_{f,\gamma_f} \end{aligned} \tag{3}$$

It is necessary to demonstrate that for $h \in \sum f\mathfrak{a}_f$ the element y_h is independent of how we write h as a sum $\sum f\gamma_f$. So suppose that two such sums are equal:

$$\sum_{i=1}^{n} f_i\gamma_i = \sum_{i=1}^{m} f'_i\gamma'_i$$

Define three additive covers of C:

$$\begin{aligned} \mathfrak{c} &= \mathfrak{c}_{f_1\gamma_1,f_2\gamma_2} \cap \mathfrak{c}_{f_1\gamma_1+f_2\gamma_2,f_3\gamma_3} \cap \ldots \cap \mathfrak{c}_{\sum_{i=1}^{n-1} f_i\gamma_i,f_n\gamma_n} \\ \mathfrak{c}' &= \mathfrak{c}_{f'_1\gamma'_1,f'_2\gamma'_2} \cap \mathfrak{c}_{f'_1\gamma'_1+f'_2\gamma'_2,f'_3\gamma'_3} \cap \ldots \cap \mathfrak{c}_{\sum_{i=1}^{n-1} f'_i\gamma'_i,f'_n\gamma'_n} \\ \mathfrak{d} &= \mathfrak{c} \cap \mathfrak{c}' \bigcap_{i=1}^{n} \mathfrak{a}_{f_i,\gamma_i} \bigcap_{i=1}^{m} \mathfrak{a}_{f'_i,\gamma'_i} \end{aligned}$$

Then for any $k \in \mathfrak{d}$

$$\begin{aligned} \left(\sum_{i=1}^{n} x_{f_i,\gamma_i}\right) \cdot k &= \sum_i x_{f_i,\gamma_i} \cdot k = \sum_i x_{f_i,\gamma_i k} \\ &= \sum_i x_{f_i\gamma_i,k} = x_{\sum_i f_i\gamma_i,k} \\ &= x_{\sum_i f'_i\gamma'_i,k} = \sum_i x_{f'_i\gamma'_i,k} \\ &= \sum_i x_{f'_i,\gamma'_i k} = \sum_i x_{f'_i,\gamma'_i} \cdot k \\ &= \left(\sum_{i=1}^{m} x_{f'_i,\gamma'_i}\right) \cdot k \end{aligned}$$

Since $\mathfrak{d}$ is a cover, this implies that $\sum_i x_{f_i,\gamma_i} = \sum_i x_{f'_i,\gamma'_i}$ and so the definition (3) is independent of how we write the morphism h.

It remains to show that $\mathbf{y}$ is an additive matching family and that it amalgamates the $\mathbf{x}_f$. Suppose $h = \sum f\gamma_f : D \longrightarrow C$ is an element of $\sum f\mathfrak{a}_f$. Then for any $z : D' \longrightarrow D$

$$\begin{aligned} y_h \cdot z &= \left(\sum x_{f,\gamma_f}\right) \cdot z \\ &= \sum x_{f,\gamma_f} \cdot z = \sum x_{f,\gamma_f z} \end{aligned}$$

But $y_{hz} = \sum x_{f,\gamma_f z}$ since $hz = \sum f(\gamma_f z)$ is a valid expansion of hz as an element of $\sum f\mathfrak{a}_f$. It is now easy to see that $\mathbf{y}$ is an additive matching family. It only remains to show that it amalgamates the family $\{\mathbf{x}_f \,|\, f \in \mathfrak{a}\}$. But for $g : D \longrightarrow C$ in $\mathfrak{a}$

$$\mathbf{y} \cdot g = \{y_h \,|\, h \in \sum f\mathfrak{a}_f\} \cdot g = \{y_{gh'} \,|\, h' \in g^* \sum f\mathfrak{a}_f\}$$

Now $\mathfrak{a}_g \subseteq g^* \sum f\mathfrak{a}_f$ by definition, and for $h' \in \mathfrak{a}_g$ it is clear that $y_{gh'} = x_{g,h'}$. This shows that the matching family $\mathbf{y} \cdot g \in M^+(D)$ is equivalent to the family $\mathbf{x}_g$, as required. □

Corollary 10. *For any right $\mathcal{A}$-module M, $(M^+)^+$ is J-closed.*

Definition 12. If $(\mathcal{A}, J)$ is a small additive site then $\mathbf{Mod}(\mathcal{A}, J)$ denotes the full subcategory of $\mathbf{Mod}\mathcal{A}$ consisting of all the J-closed modules, referred to as the *localisation of* $\mathbf{Mod}\mathcal{A}$ *with respect to* J. This is a replete subcategory of $\mathbf{Mod}\mathcal{A}$. If M is a right $\mathcal{A}$-module then we denote the J-closed module $(M^+)^+$ by M_J and refer to it as the *localisation of* M *with respect to* J. If $\varphi : M \longrightarrow N$ is a morphism of modules, then we have the following morphism of J-closed modules

$$\varphi_J = (\varphi^+)^+ : M_J \longrightarrow N_J$$
$$(\varphi_J)_C(\{\mathbf{x}_f \,|\, f \in \mathfrak{a}\}) = \{\varphi^+(\mathbf{x}_f) \,|\, f \in \mathfrak{a}\} = \{\{\varphi(x_{f,g}) \,|\, g \in \mathfrak{b}_f\} \,|\, f \in \mathfrak{a}\}$$

where $\mathbf{x}_f = \{x_{f,g} \,|\, g \in \mathfrak{b}_f\}$ for $f \in \mathfrak{a}$. This defines an additive functor

$$\mathbf{a} = (-)_J : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{A}, J)$$

Let $\mathbf{i} : \mathbf{Mod}(\mathcal{A}, J) \longrightarrow \mathbf{Mod}\mathcal{A}$ be the inclusion. Then there is a canonical natural transformation $\psi : 1 \longrightarrow \mathbf{ia}$ given for a module M by the composite $\phi_{M^+}\phi_M : M \longrightarrow M^+ \longrightarrow (M^+)^+$. That is,

$$\psi_M : M \longrightarrow M_J$$
$$\psi_{M,C}(x) = \{\{x \cdot hg \,|\, g \in H_D\} \,|\, h : D \longrightarrow C\}$$

Remark 1. Let M, N be distinct $\mathcal{A}$-modules. Then the $\mathcal{A}$-modules M^+ and N^+ are also distinct, since if $A \in \mathcal{A}$ the set $M^+(A)$ consists of equivalence classes of maps in $\mathbf{Mod}\mathcal{A}$ with codomain M, while $N^+(A)$ is a set of equivalence classes of maps with codomain N. It follows that if M, N are distinct so are the modules $\mathbf{a}M$ and $\mathbf{a}N$.

Lemma 11. *Let $\mathcal{A}$ be a ringoid and J, K additive topologies on $\mathcal{A}$. Then $J \leq K$ implies $\mathbf{Mod}(\mathcal{A}, K) \subseteq \mathbf{Mod}(\mathcal{A}, J)$.*

Proposition 12. *Let M, N be right $\mathcal{A}$-modules and suppose that M is J-closed. Then any morphism of modules $\theta : N \longrightarrow M$ factors uniquely through $\psi_N : N \longrightarrow N_J$.*

Proof. By Proposition 7 there is a unique factorisation $\theta' : N^+ \longrightarrow M$ of θ through $N \longrightarrow N^+$. Applying the Proposition again, we obtain a unique factorisation $\theta'' : N_J \longrightarrow M$ of θ' through $N^+ \longrightarrow N_J$, as in the diagram

$$\begin{tikzcd} N \arrow[r, "\theta"] \arrow[d] & M \\ N^+ \arrow[ur, "\theta'"] \arrow[dr] & \\ & N_J \arrow[uu, "\theta''"'] \end{tikzcd}$$

The morphism θ'' is then a unique factorisation of θ through ψ_N. □

Corollary 13. *If $(\mathcal{A}, J)$ is an additive site, then we have a pair of adjoint functors*

$$\begin{tikzcd} \mathbf{Mod}(\mathcal{A}, J) \arrow[r, bend left, "\mathbf{i}"] & \mathbf{Mod}\mathcal{A} \arrow[l, bend left, "\mathbf{a}"] \end{tikzcd} \qquad \mathbf{a} \dashv \mathbf{i}$$

The unit is the natural transformation $\psi : 1 \longrightarrow \mathbf{ia}$ defined above.

Lemma 14. *Let $(\mathcal{A}, J)$ be a small additive site and M a right $\mathcal{A}$-module. Then*

(i) M is J-torsion if and only if $M_J = 0$.

(ii) M is J-closed if and only if $\psi_M : M \longrightarrow M_J$ is an isomorphism.

(iii) $Ker\psi_M = t_J(M)$.

Proof. (i) If M is J-torsion then by Lemma 6, $M^+ = 0$ and so $M_J = 0$. Conversely, if $M_J = 0$ then again by Lemma 6, M^+ must be J-torsion. Since M^+ is always J-torsion-free by Lemma 8 we must have $M^+ = 0$, which implies that M is J-torsion. (ii) is clear from Lemma 5. (iii) follows immediately from Lemma 4 and Lemma 5. □

Lemma 15. *If $(\mathcal{A}, J)$ is a small additive site and M a right $\mathcal{A}$-module then $\psi_{M_J} = (\psi_M)_J$ and this morphism is an isomorphism.*

Proof. The equality follows directly from the fact that $\psi : 1 \longrightarrow \mathbf{ia}$ is a natural transformation. By Lemma 14 the morphism ψ_{M_J} is an isomorphism, so the proof is complete. □

Proposition 16. *If $(\mathcal{A}, J)$ is an additive site, then the functor $\mathbf{a} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{A}, J)$ preserves finite limits.*

Proof. By Mitchell II 6.5 it suffices to show that $\mathbf{a}$ preserves kernels. Let $\varphi : M \longrightarrow N$ be a morphism of $\mathcal{A}$-modules with kernel $i : K \longrightarrow M$ and suppose that there is $C \in \mathcal{A}$ and $\mathbf{x} \in M_J(C)$ such that $(\varphi_J)_C(\mathbf{x}) = 0$, where $\mathbf{x} = \{\mathbf{x}_f \,|\, f \in \mathfrak{a}\}$ and $\mathbf{x}_f = \{x_{f,g} \,|\, g \in \mathfrak{b}_f\}$ for every $f \in \mathfrak{a}$. Therefore

$$\{\{\varphi(x_{f,g}) \,|\, g \in \mathfrak{b}_f\} \,|\, f \in \mathfrak{a}\} = 0$$

Let $\mathfrak{c} \subseteq \mathfrak{a}$ be such that $\mathfrak{c} \in J(C)$ and $\{\varphi(x_{f,g}) \,|\, g \in \mathfrak{b}_f\} = 0$ in $N^+(D)$ for every $f : D \longrightarrow C$ in $\mathfrak{c}$. Then there is $\mathfrak{c}_f \subseteq \mathfrak{b}_f$ with $\mathfrak{c}_f \in J(D)$ and $\varphi(x_{f,g}) = 0$ for all $g \in \mathfrak{c}_f$. It is not difficult to check that $\mathbf{y} = \{\{x_{f,g} \,|\, g \in \mathfrak{c}_f\} \,|\, f \in \mathfrak{c}\}$ defines an element of $K_J(C)$ with $(i_J)_C(\mathbf{y}) = \mathbf{x}$. In particular if φ is a monomorphism then $K_J = 0$ so φ_J must be a monomorphism of $\mathcal{A}$-modules. In particular $i_J : K_J \longrightarrow M_J$ is a monomorphism, and further we have just shown that i_J is the kernel (in $\mathbf{Mod}\mathcal{A}$) of the morphism $M_J \longrightarrow N_J$. It is therefore trivially the kernel in $\mathbf{Mod}(\mathcal{A}, J)$, as required. □

Corollary 17. *If $(\mathcal{A}, J)$ is an additive site then $\mathbf{Mod}(\mathcal{A}, J)$ is a giraud subcategory of $\mathbf{Mod}\mathcal{A}$, and is therefore a complete grothendieck abelian category.*

Proof. The first claim is immediate from Proposition 16, and it then follows from (AC,Corollary 65) and (AC,Proposition 62) that $\mathbf{Mod}(\mathcal{A}, J)$ is a complete grothendieck abelian category. □

Definition 13. Let $\mathcal{A}$ be a ringoid and J a left additive topology. Then $(\mathcal{A}, J)\mathbf{Mod}$ denotes the full subcategory of $\mathcal{A}\mathbf{Mod}$ consisting of the J-closed modules. It corresponds to $\mathbf{Mod}(\mathcal{A}^{\mathrm{op}}, J)$ under the equality $\mathcal{A}\mathbf{Mod} = \mathbf{Mod}\mathcal{A}^{\mathrm{op}}$. Therefore $(\mathcal{A}, J)\mathbf{Mod}$ is a complete grothendieck abelian category, which is a giraud subcategory of $\mathcal{A}\mathbf{Mod}$.

2.1 Alternative Approach

One problem with the functor $\mathbf{a}$ is that the module $\mathbf{a}M$ is difficult to work with explicitly - elements of $\mathbf{a}M(A)$ are matching families of matching families! Fortunately there is another way to express this localisation. For any $\mathcal{A}$-module M there is an exact sequence

$$0 \longrightarrow t_J(M) \longrightarrow M \xrightarrow{\mu_M} M/t_J(M) \longrightarrow 0$$

Since $t_J(M)$ is J-torsion, by applying the exact functor $\mathbf{a}$ to this exact sequence we see that $\mathbf{a}\mu_M$ is an isomorphism $\mathbf{a}M \cong \mathbf{a}(M/t_J(M))$. As $M/t_J(M)$ is J-torsion-free $(M/t_J(M))^+$ is J-closed by Proposition 9. Hence the canonical morphism

$$(M/t_J(M))^+ \longrightarrow (M/t_J(M))^{++}$$

is an isomorphism by Lemma 5. We define a functor $\mathbf{c}: \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{A}, J)$ by $\mathbf{c}(M) = (M/t_J(M))^+$. For $\alpha : M \longrightarrow N$ we induce $\alpha' : M/t_J(M) \longrightarrow N/t_J(N)$ and then by Proposition 7 a unique morphism $\mathbf{c}\alpha$ making the following diagram commute

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\alpha"'] & M/t_J(M) \arrow[r] \arrow[d, "\alpha'"] & (M/t_J(M))^+ \arrow[d, "\mathbf{c}\alpha"] \\
N \arrow[r] & N/t_J(N) \arrow[r] & (N/t_J(N))^+
\end{tikzcd}$$

It is easily checked that thus defined $\mathbf{c}$ is a functor. Moreover, the isomorphism $\mathbf{a}\mu_M$ gives rise to a natural equivalence $\mathbf{a} \cong \mathbf{c}$. Hence $\mathbf{c}$ is an additive, exact left adjoint to the inclusion $\mathbf{i} : \mathbf{Mod}(\mathcal{A}, J) \longrightarrow \mathbf{Mod}\mathcal{A}$. The unit of this adjunction is the canonical map $M \longrightarrow (M/t_J(M))^+$.

3 The Gabriel-Popescu Theorem

To each additive topology J on a ringoid $\mathcal{A}$ we have associated a giraud subcategory $\mathbf{Mod}(\mathcal{A}, J)$ of $\mathbf{Mod}\mathcal{A}$. Conversely, we may show that any giraud subcategory of $\mathbf{Mod}\mathcal{A}$ is the localisation of $\mathbf{Mod}\mathcal{A}$ at an additive topology.

Let $\mathcal{A}$ be a ringoid and let $\mathcal{D} \subseteq \mathbf{Mod}\mathcal{A}$ be a giraud subcategory, with inclusion $\mathbf{i} : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{A}$ and reflection $\mathbf{a} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{D}$. Let $\theta : 1 \longrightarrow \mathbf{ia}$ be the unit of the adjunction. As usual (AC,Section 3) we may assume that $\mathbf{a}D = D$ and $\theta_D = 1_D$ for $D \in \mathcal{D}$. Also recall that the zero module belongs to any reflective subcategory. We define the following function of the objects of $\mathcal{A}$:

$$J(A) = \{\mathfrak{a} \,|\, \mathfrak{a} \text{ is a right ideal at } A, \text{ and } \mathbf{a}(A/\mathfrak{a}) = 0\} \tag{4}$$

The next result says that if a module is J-torsion (in the obvious sense), its reflection is zero.

Lemma 18. *Let M be a right $\mathcal{A}$-module. If $Ann(x) \in J(A)$ for every $A \in \mathcal{A}$ and $x \in M(A)$, then $\mathbf{a}M = 0$.*

Proof. For $x \in M(A)$ the submodule (x) of M is the image of the canonical morphism $x : H_A \longrightarrow M$, and by definition the following sequence is exact:

$$0 \longrightarrow Ann(x) \longrightarrow H_A \xrightarrow{x} (x) \longrightarrow 0$$

It follows that $\mathbf{a}(x) = 0$ whenever $Ann(x) \in J(A)$. The inclusions $(x) \longrightarrow M$ for $x \in M(A)$ and $A \in \mathcal{A}$ induce an epimorphism

$$\phi : \bigoplus_{\substack{x \in M(A) \\ A \in \mathcal{A}}} (x) \longrightarrow M$$

Since $\mathbf{a}$ has a right adjoint it preserves colimits and epimorphisms, implying that $\mathbf{a}\phi$ is an epimorphism and that $\mathbf{a}(\bigoplus_x (x)) = \bigoplus_x \mathbf{a}(x) = 0$. Hence $\mathbf{a}M = 0$, as required. $\square$

Proposition 19. *The function J defined by (4) is an additive topology on $\mathcal{A}$.*

Proof. It is clear that $H_A \in J(A)$ for all $A \in \mathcal{A}$. To check the stability condition, let $h : D \longrightarrow C$ be a morphism of $\mathcal{A}$, and $\mathfrak{a} \in J(D)$. By definition $h^*\mathfrak{a}$ is the pullback of the module $\mathfrak{a}$ along $h : H_D \longrightarrow H_C$. Hence in the following commutative diagram the rows are exact and the left square is a pullback:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d] & H_D \arrow[r] \arrow[d, "h"'] & H_D/\mathfrak{a} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & h^*\mathfrak{a} \arrow[r] & H_C \arrow[r] & H_C/h^*\mathfrak{a} \arrow[r] & 0
\end{tikzcd}$$

Since $\mathbf{a}$ is exact it preserves finite limits and in particular pullbacks, so applying $\mathbf{a}$ to the above diagram and using (AC,Proposition 37) we see that $\mathbf{a}(H_C/h^*\mathfrak{a}) = 0$. Thus $h^*\mathfrak{a} \in J(C)$, as required.

To check the transitivity axiom, let $\mathfrak{b} \in J(A)$ and $\mathfrak{a}$ be a right ideal at A, such that $h^*\mathfrak{a} \in J(D)$ for all $h : D \longrightarrow A$ in $\mathfrak{b}$. Then there is an exact sequence

$$0 \longrightarrow (\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \longrightarrow H_A/\mathfrak{a} \longrightarrow H_A/(\mathfrak{a}+\mathfrak{b}) \longrightarrow 0$$

Since $H_A/(\mathfrak{a}+\mathfrak{b})$ is also a quotient of $H_A/\mathfrak{b}$ and $\mathbf{a}$ preserves epis, it follows that $\mathbf{a}(H_A/(\mathfrak{a}+\mathfrak{b})) = 0$. By (AC,Proposition 40), $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$. Lemma 18 implies that $\mathbf{a}(\mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})) = 0$, since for $h : D \longrightarrow A \in \mathfrak{b}$

$$Ann(h) = \{g : D' \longrightarrow D \mid hg \in \mathfrak{a}\cap\mathfrak{b}\} = h^*(\mathfrak{a}\cap\mathfrak{b}) = h^*\mathfrak{a} \in J(D)$$

Hence also $\mathbf{a}((\mathfrak{a}+\mathfrak{b})/\mathfrak{a}) = 0$, and by applying $\mathbf{a}$ to the above exact sequence we find that $\mathbf{a}(H_A/\mathfrak{a}) = 0$, as required. $\square$

Recall from Lemma 14 that in the case where $\mathcal{D}$ is the localisation of $\mathbf{Mod}\mathcal{A}$ at a topology J, the kernel of the unit $\psi_M : M \longrightarrow M_J = \mathbf{a}M$ is the J-torsion submodule of M.

Proposition 20. *Let $\mathcal{D}$ be a giraud subcategory of* $\mathbf{Mod}\mathcal{A}$ *with inclusion* $\mathbf{i}$, *reflection* $\mathbf{a}$ *and associated topology J. If $\theta : 1 \longrightarrow$* $\mathbf{ia}$ *is the unit of the adjunction, then for any module M*

$$Ker\theta_M = t_J(M)$$

Consequently, M is J-torsion if and only if $\mathbf{a}M = 0$.

Proof. For $x \in M(A)$ set $\mathfrak{a} = Ann(x) = Ker(H_A \longrightarrow M)$. Then $\mathbf{ia}\mathfrak{a} \longrightarrow \mathbf{ia}A$ is the kernel of $\mathbf{ia}H_A \longrightarrow \mathbf{ia}M$ since both $\mathbf{i}$ and $\mathbf{a}$ are left exact. We have the following diagram with exact rows:

$$\begin{tikzcd}
 & & 0 \arrow[r] & \mathbf{ia}\mathfrak{a} \arrow[r] & \mathbf{ia}H_A \arrow[r] \arrow[d] & \mathbf{ia}H_A/\mathfrak{a} \\
 & & & & \mathbf{ia}M & \\
0 \arrow[r] & \mathfrak{a} \arrow[r] & H_A \arrow[r] \arrow[d, "x"'] \arrow[uurr] & H_A/\mathfrak{a} \arrow[r] \arrow[ur, "\theta_M"] & 0 & \\
 & & M \arrow[ur, dash] & & &
\end{tikzcd}$$

If $\theta_M(x) = 0$ then $H_A \longrightarrow \mathbf{ia}H_A$ factors uniquely through $\mathbf{ia}\mathfrak{a} \longrightarrow \mathbf{ia}H_A$. Using the universal property of the unit, there is a morphism $\mathbf{a}H_A \longrightarrow \mathbf{a}\mathfrak{a}$ such that the composite $\mathbf{a}H_A \longrightarrow \mathbf{a}\mathfrak{a} \longrightarrow \mathbf{a}H_A$ is the identity. Hence $\mathbf{a}\mathfrak{a} \longrightarrow \mathbf{a}H_A$ is an isomorphism, since it is both a retraction and a monomorphism. It follows that $\mathbf{a}(H_A/\mathfrak{a}) = 0$, so $Ann(x) \in J(A)$ and consequently $x \in t_J(M)(A)$.

Conversely if $x \in t_J(M)(A)$ and $\mathfrak{a} = Ann(x) \in J(A)$, then $\mathbf{a}(H_A/\mathfrak{a}) = 0$ and so $\mathbf{a}\mathfrak{a} \longrightarrow \mathbf{a}H_A$ are $\mathbf{ia}\mathfrak{a} \longrightarrow \mathbf{ia}H_A$ isomorphisms, implying that $\mathbf{ia}H_A \longrightarrow \mathbf{ia}M$ is the zero morphism and so $\theta_M(x) = 0$. This proves that $t_J(M) = Ker\theta_M$.

If M is J-torsion, then $Ker\theta_M = t_J(M) = M$ so $\theta_M = 0$. Since $\mathbf{a}\theta_M = 1_{\mathbf{a}M}$, this implies that $\mathbf{a}M = 0$. Conversely if $\mathbf{a}M = 0$ then $\mathbf{ia}M = 0$, so $\theta_M = 0$ and consequently $t_J(M) = M$, so M is J-torsion. $\square$

The following Theorem generalises a well-known result for modules over a ring, originally due to Gabriel [14] (see also [33], Theorem 2.1).

Theorem 21. *For a ringoid $\mathcal{A}$ there is a bijective correspondence between additive topologies on $\mathcal{A}$ and Giraud subcategories of $\mathbf{Mod}\mathcal{A}$:*

$$\begin{tikzcd} \textit{Additive topologies} \arrow[r, bend left, "\Gamma"] & \textit{Giraud subcategories} \arrow[l, bend left, "\Phi"] \end{tikzcd}$$

$$\Gamma(J) = \mathbf{Mod}(A, J)$$
$$\Phi(\mathcal{D}, \mathbf{a} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{D})(A) = \{\mathfrak{a} \,|\, \mathbf{a}(H_A/\mathfrak{a}) = 0\}$$

In particular, the collection of Giraud subcategories of $\mathbf{Mod}\mathcal{A}$ is a set.

Proof. Corollary 17 shows that $\mathbf{Mod}(\mathcal{A}, J)$ is a giraud subcategory of $\mathbf{Mod}\mathcal{A}$. Conversely, we have just seen that a giraud subcategory of $\mathbf{Mod}\mathcal{A}$ determines an additive topology J on $\mathcal{A}$ where the additive covers are those ideals $\mathfrak{a}$ for which the reflection of $H_A/\mathfrak{a}$ is zero. It only remains to verify that these two maps are inverse. If J is an additive topology then $\Phi\Gamma(J) = J$, since

$$\begin{aligned} \Phi\Gamma(J)(A) &= \{\mathfrak{a} \,|\, \mathbf{a}(H_A/\mathfrak{a}) = 0\} \\ &= \{\mathfrak{a} \,|\, H_A/\mathfrak{a} \text{ is } J\text{-torsion }\} \\ &= \{\mathfrak{a} \,|\, \forall f : D \longrightarrow A, f^*\mathfrak{a} \in J(D)\} = J(A) \end{aligned}$$

Conversely, let $\mathcal{D}$ be a giraud subcategory of $\mathbf{Mod}\mathcal{A}$. Let $\mathbf{i} : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{A}$ be the inclusion with exact left adjoint $\mathbf{a} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{D}$, and unit $\theta : 1 \longrightarrow \mathbf{ia}$. Let J be the induced topology $\Phi(\mathcal{D})$

$$J(A) = \{\mathfrak{a} \,|\, \mathbf{a}(H_A/\mathfrak{a}) = 0\}$$

Let $\mathbf{i}_J : \mathbf{Mod}(A, J) \longrightarrow \mathbf{Mod}\mathcal{A}$ be the inclusion of the localisation at J, with exact left adjoint $\mathbf{a}_J$, and unit $\psi : 1 \longrightarrow \mathbf{i}_J\mathbf{a}_J$. We have to show that $\mathcal{D} = \mathbf{Mod}(A, J)$. We establish the inclusion $\mathcal{D} \subseteq \mathbf{Mod}(A, J)$ by showing that each $M \in \mathcal{D}$ is J-closed. Indeed if $\mathfrak{a} \in J(A)$ then $\mathbf{a}(H_A/\mathfrak{a}) = 0$, and since $\mathbf{a}$ is exact the following sequence is exact:

$$0 \longrightarrow \mathbf{a}\mathfrak{a} \longrightarrow \mathbf{a}H_A \longrightarrow \mathbf{a}(H_A/\mathfrak{a}) \longrightarrow 0$$

It follows that $\mathbf{a}\mathfrak{a} \cong \mathbf{a}A$. Using the commutative square

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d] & H_A \arrow[d] \\ \mathbf{ia}\mathfrak{a} \arrow[r, Rightarrow] & \mathbf{ia}H_A \end{tikzcd}$$

and the universal property of the unit $1 \longrightarrow \mathbf{ia}$, it is straightforward to check that $Hom_A(H_A, M) \cong Hom_A(\mathfrak{a}, M)$ and hence that $D \in \mathbf{Mod}(A, J)$.

We now have to establish the reverse inclusion $\mathbf{Mod}(\mathcal{A}, J) \subseteq \mathcal{D}$. So suppose that M is a J-closed $\mathcal{A}$-module. Consider the following exact sequence:

$$0 \longrightarrow Ker\theta_M \longrightarrow M \xrightarrow{\theta_M} \mathbf{ia}M \longrightarrow Coker\theta_M \longrightarrow 0$$

Since $\mathbf{a}\theta_M$ is the identity, $\mathbf{a}Ker\theta_M$ and $\mathbf{a}Coker\theta_M$ are both zero. The previous Proposition implies that $Ker\theta_M, Coker\theta_M$ are J-torsion. Using Lemma 6 it follows that $\mathbf{a}_J Ker\theta_M$ and $\mathbf{a}_J Coker\theta_M$ are both zero, and thus $\mathbf{a}_J\theta_M : \mathbf{a}_J M \longrightarrow \mathbf{a}_J\mathbf{ia}M$ is an isomorphism.

Since M and $\mathbf{ia}M$ are both J-closed (since we already know $D \subseteq \mathbf{Mod}(\mathcal{A}, J)$) this is an isomorphism of M with $\mathbf{ia}M$. Using the fact that $\mathcal{D}$ is replete, we have $M \in \mathcal{D}$. This establishes the required equality $\mathbf{Mod}(\mathcal{A}, J) = \mathcal{D}$. □

Corollary 22. *Let $\mathcal{A}$ be a ringoid and J, K additive topologies on $\mathcal{A}$. Then $J \leq K$ if and only if $\mathbf{Mod}(\mathcal{A}, K) \subseteq \mathbf{Mod}(\mathcal{A}, J)$.*

Proof. We proved one implication in Lemma 11. Suppose $\mathbf{Mod}(\mathcal{A}, K) \subseteq \mathbf{Mod}(\mathcal{A}, J)$ and $\mathfrak{a} \in J(C)$ for some object C of $\mathcal{A}$. It is not hard to see that $\mathbf{Mod}(\mathcal{A}, K)$ is a giraud subcategory of $\mathbf{Mod}(\mathcal{A}, J)$, with reflection $\mathbf{r}$ given by the restriction of $\mathbf{a}_K$. So we have a diagram of functors

$$\begin{tikzcd} & \mathbf{Mod}\mathcal{A} \arrow[dl, bend right, "\mathbf{a}_J"'] \arrow[dr, bend left, "\mathbf{a}_K"] & \\ \mathbf{Mod}(\mathcal{A}, J) \arrow[ur, bend right] \arrow[rr, bend right, "\mathbf{r}"'] & & \mathbf{Mod}(\mathcal{A}, K) \arrow[ul, bend left] \arrow[ll, bend right] \end{tikzcd} \tag{5}$$

By composing adjoints and using the uniqueness of left adjoints, we see that there is a natural equivalence $\mathbf{r}\mathbf{a}_J \cong \mathbf{a}_K$. By Theorem 21 we have $\mathbf{a}_J(H_C/\mathfrak{a}) = 0$, and therefore $\mathbf{a}_K(H_C/\mathfrak{a}) \cong \mathbf{r}\mathbf{a}_J(H_C/\mathfrak{a}) = 0$. This shows that $\mathfrak{a} \in K(C)$, as required. □

Corollary 23. *Let $\mathcal{A}$ be a ringoid and $J \leq K$ additive topologies. For an $\mathcal{A}$-module M there is a canonical isomorphism of modules $(M_J)_K \cong M_K$ natural in M.*

Proof. This follows immediately from commutativity of (5) up to canonical natural equivalence. □

Next we generalise a theorem originally due to Gabriel and Popescu [18], which classifies grothendieck categories as localisations of module categories. A proof for modules over a ring can be found in Stenström's book [33]. Our generalisation is based on an alternative proof, which combines a suggestion of Barry Mitchell [27] together with (RSO,Theorem 11).

Let $\mathcal{C}$ be a grothendieck abelian category $\mathcal{C}$ with a family of generators $\{U_i\}_I$. Let $\mathcal{A}$ be the small, full, additive subcategory of $\mathcal{C}$ consisting of the objects U_i and the morphisms between them. As usual, we may define the additive functor $H^{\mathcal{A}} : \mathcal{C} \longrightarrow \mathbf{Mod}\mathcal{A}$ by

$$H^{\mathcal{A}}(C)(U) = Hom_{\mathcal{C}}(U, C)$$
$$H^{\mathcal{A}}(f)_U(\phi) = f\phi$$

The following Lemma gives the connection between morphisms $H^{\mathcal{A}}(B) \longrightarrow H^{\mathcal{A}}(C)$ of $\mathcal{A}$-modules and morphisms $B \longrightarrow C$ in $\mathcal{C}$. It is motivated by a Lemma from [27], which deals with the case of a ring.

Lemma 24. *Let B, C be objects of $\mathcal{C}$ and M a submodule of the right $\mathcal{A}$-module $H^{\mathcal{A}}(B)$. Let $f : H^{\mathcal{A}}(B) \longrightarrow H^{\mathcal{A}}(C)$ be a morphism of $\mathcal{A}$-modules. The elements $m : U \longrightarrow B \in M(U)$ and $f_U(m) : U \longrightarrow C$ for $U \in \mathcal{A}$ induce morphisms out of the coproduct in $\mathcal{C}$:*

$$\begin{tikzcd} \bigoplus_U {}^{M(U)}U \arrow[r, "\psi"] \arrow[d, "\phi"'] & B \\ C & \end{tikzcd} \qquad \psi u^U_m = m, \quad \phi u^U_m = f_U(m)$$

We claim that ϕ factors through $Im\psi$.

Proof. Let $\mu : K \longrightarrow \bigoplus_U {}^{M(U)}U$ be the kernel of ψ. The assertion is equivalent to $\phi\mu = 0$. Picking a finite subcoproduct consists of picking a subset $F(U)$ (possibly empty) of $M(U)$ for each U, such that $\bigcup_U F(U)$ is a finite set. We denote such a collection simply by F. The coproduct $\bigoplus_U {}^{M(U)}U$ is the union of the direct family of subobjects consisting of the finite subcoproducts. Since $\mathcal{C}$ is grothendieck abelian inverse images commute with unions, so it suffices to show $\phi\mu\lambda_F = 0$ for all the finite F, where λ_F is the morphism in the following pullback diagram:

$$\begin{tikzcd}
K' \arrow[r, "\mu'"] \arrow[d, "\lambda_F"'] & \bigoplus_U {}^{F(U)}U \arrow[d, "\sum_{U, m\in F(U)} u_m^U \widehat{p}_m^U"] \\
K \arrow[r, "\mu"'] & \bigoplus_U {}^{M}(U)U
\end{tikzcd}$$

Here $\widehat{p}_m^U$ denotes the projection from the finite coproduct. Since the objects of $\mathcal{A}$ form a generating family, we only have to show that $\phi\mu\lambda_F\alpha = 0$ for any $V \in \mathcal{A}$ and $\alpha : V \longrightarrow K'$. But this follows from the fact that f is a morphism of $\mathcal{A}$-modules:

$$\begin{aligned}
\phi\mu\lambda_F\alpha &= \phi\sum_{U,m} u_m^U\widehat{p}_m^U\mu'\alpha = \sum_{U,m} f_U(m)\widehat{p}_m^U\mu'\alpha = \sum_{U,m} f_V(m\widehat{p}_m^U\mu'\alpha) \\
&= f_V(\sum_{U,m} m\widehat{p}_m^U\mu'\alpha) = f_V(\sum_{U,m}\psi u_m^U\widehat{p}_m^U\mu'\alpha) = f_V(\psi\mu\lambda_F\alpha) \\
&= f_V(0) = 0
\end{aligned}$$

□

Theorem 25 (Gabriel-Popescu). *Let $\mathcal{C}$ be a grothendieck abelian category with a family of generators $\mathcal{A}$. Let $H^{\mathcal{A}} : \mathcal{C} \longrightarrow \mathbf{Mod}\mathcal{A}$ be $H^{\mathcal{A}}(C)(U) = Hom_{\mathcal{C}}(U, C)$. Then $H^{\mathcal{A}}$ is a full embedding and has an exact left adjoint.*

Proof. Since the objects of $\mathcal{A}$ form a family of generators it is easy to see that $H^{\mathcal{A}}$ is faithful, and similarly that $H^{\mathcal{A}}$ is distinct on objects. To see that $H^{\mathcal{A}}$ is full, let $f : H^{\mathcal{A}}(B) \longrightarrow H^{\mathcal{A}}(C)$ be any morphism of $\mathcal{A}$-modules. Put $M = H^{\mathcal{A}}(B)$ in the Lemma. Then ψ is an epimorphism, so ϕ factors through ψ, say $\phi = \theta\psi$, and $f = H^{\mathcal{A}}(\theta)$. This proves that T is full.

By (RSO,Theorem 12), $H^{\mathcal{A}}$ has a left adjoint $-\otimes\mathcal{A} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{C}$. Since a grothendieck abelian category has enough injectives, to show that this left adjoint is exact it suffices by (AC,Theorem 26) to show that $H^{\mathcal{A}}$ preserves injectives. So let E be injective in $\mathcal{C}$, and suppose that $\mathfrak{a}$ is a submodule of $H^{\mathcal{A}}(U)$ for $U \in \mathcal{A}$. Let $f : \mathfrak{a} \longrightarrow H^{\mathcal{A}}(E)$ be any morphism. Put $B = H^{\mathcal{A}}(U), M = \mathfrak{a}$ and use the Lemma and injectivity of E to see that ϕ factors through ψ, say $\phi = \theta\psi$. Then it is easily checked that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{a} \arrow[r] \arrow[d, "f"'] & H^{\mathcal{A}}(U) \arrow[dl, "H^{\mathcal{A}}(\theta)"] \\
H^{\mathcal{A}}(E) &
\end{tikzcd}$$

Since by definition $\{H^{\mathcal{A}}(U)\}_{U\in\mathcal{A}}$ is a family of generators for $\mathbf{Mod}\mathcal{A}$, it follows from (AC,Proposition 50) that $H^{\mathcal{A}}(E)$ is injective. Hence $H^{\mathcal{A}}$ has an exact left adjoint $-\otimes\mathcal{A}$, as required. □

We denote by $\theta : 1 \longrightarrow H^{\mathcal{A}}(-\otimes\mathcal{A})$ the unit of the above adjunction. The functor $H^{\mathcal{A}}$ gives an isomorphism of $\mathcal{C}$ with the following full subcategory of $\mathbf{Mod}\mathcal{A}$:

$$\mathcal{D}' = \{H^{\mathcal{A}}(C)\,|\,C \in \mathcal{C}\}$$

Let $\mathbf{i}' : \mathcal{D}' \longrightarrow \mathbf{Mod}\mathcal{A}$ be the inclusion, and $\mathbf{a}' = H^{\mathcal{A}}(-\otimes\mathcal{A}) : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{D}'$ the reflection, where $H^{\mathcal{A}}$ is considered as a functor into $\mathcal{D}'$. The morphisms $\theta_F : F \longrightarrow H^{\mathcal{A}}(F\otimes\mathcal{A})$ for $F \in \mathbf{Mod}\mathcal{A}$

establish that $\mathbf{i}'$ is left adjoint to $\mathbf{a}'$, since if $\gamma : F \longrightarrow H^{\mathcal{A}}(C)$ then there is a unique $\mu : F \otimes \mathcal{A} \longrightarrow C$ in $\mathcal{C}$ making the following diagram commute:

$$\begin{tikzcd} F \arrow[r, "\gamma"] \arrow[d, "\theta_F"'] & H^{\mathcal{A}}(C) \arrow[dl, "H^{\mathcal{A}}(\mu)"] \\ H^{\mathcal{A}}(F \otimes \mathcal{A}) & \end{tikzcd}$$

Since $H^{\mathcal{A}}$ is fully faithful $H^{\mathcal{A}}(\mu)$ is also the unique morphism in $\mathcal{D}$ making this diagram commute, implying that $\mathbf{a}' \dashv \mathbf{i}'$. The functor $\mathbf{a}'$ is exact since $H^{\mathcal{A}}$ and $- \otimes \mathcal{A}$ are both left exact, and if a morphism is monic in $\mathbf{Mod}\mathcal{A}$ then it is certainly monic in any subcategory.

Let $\mathcal{D}$ be the replete closure of $\mathcal{D}'$ (this is defined in our Abelian Category notes). Denote the inclusion by $\mathbf{i} : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{A}$, and let $\mathbf{a}$ be the composite of $\mathbf{a}'$ with the inclusion $\mathcal{D} \longrightarrow \mathcal{D}'$. Then as noted in (AC,Lemma 66) $\mathbf{a}$ is left adjoint to $\mathbf{i}$ and $\mathbf{a}$ is exact. Hence $\mathcal{D}$ is a giraud subcategory of $\mathbf{Mod}\mathcal{A}$.

Corollary 26. *Let $\mathcal{C}$ be a grothendieck abelian category with a family of generators $\mathcal{A}$. Define $H^{\mathcal{A}} : \mathcal{C} \longrightarrow \mathbf{Mod}\mathcal{A}$ and $- \otimes \mathcal{A} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{C}$ as above, and let J be the following additive topology on $\mathcal{A}$:*

$$J(A) = \{\mathfrak{a} \,|\, H_A/\mathfrak{a} \otimes \mathcal{A} = 0\}$$

Then $H^{\mathcal{A}}$ gives an equivalence of $\mathcal{C}$ with the giraud subcategory $\mathbf{Mod}(\mathcal{A}, J)$ of $\mathbf{Mod}\mathcal{A}$.

Proof. The above discussion shows that $H^{\mathcal{A}}$ defines an equivalence of $\mathcal{C}$ with the giraud subcategory $\mathcal{D}$ of $\mathbf{Mod}\mathcal{A}$. By Theorem 21, $\mathcal{D}$ is $\mathbf{Mod}(\mathcal{A}, J)$ for the topology

$$J(A) = \{\mathfrak{a} \,|\, \mathbf{a}(H_A/\mathfrak{a}) = 0\}$$

But

$$\mathbf{a}(H_A/\mathfrak{a})(U) = Hom_{\mathcal{C}}(U, H_A/\mathfrak{a} \otimes \mathcal{A})$$

So the $\mathcal{A}$-module $\mathbf{a}(H_A/\mathfrak{a})$ is zero iff. for every U in $\mathcal{A}$ the only morphism $U \longrightarrow H_A/\mathfrak{a} \otimes \mathcal{A}$ is the zero morphism. But the objects of $\mathcal{A}$ form a generating family, so this can only happen if $H_A/\mathfrak{a} \otimes \mathcal{A}$ is zero in $\mathcal{C}$. □

Corollary 27. *Any grothendieck abelian category is complete.*

The result (RSO,Theorem 13) describes explicitly any additive colimit preserving functor $\mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$ for ringoids $\mathcal{A}, \mathcal{B}$. We now extend this result to Giraud subcategories of $\mathbf{Mod}\mathcal{B}$.

Lemma 28. *Let $\mathcal{A}, \mathcal{B}$ be ringoids, K an additive topology on $\mathcal{B}$ and Q an $\mathcal{A}$-$\mathcal{B}$-bimodule. Let $- \otimes_{\mathcal{A}} Q : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$ be the induced functor. For an $\mathcal{A}$-module M let*

$$\Omega_Q(M) = M \otimes_{\mathcal{A}} Q / t_K(M \otimes_{\mathcal{A}} Q)$$

Then for $B \in \mathcal{B}$, $\Omega_Q(M)(B)$ is generated as an abelian group by elements

$$x \otimes f \qquad A \in \mathcal{A}, x \in M(A), f \in Q(A)(B)$$

which satisfy the relations

$$\begin{aligned} (x + x') \otimes f &= x \otimes f + x \otimes f & & x, x' \in F(A), f \in Q(A)(B) \\ x \otimes (f + f') &= x \otimes f + x \otimes f' & & x \in F(A), f, f' \in Q(A)(B) \\ (x \cdot \alpha) \otimes f &= x \otimes Q(\alpha)_B(f) & & \alpha : A' \longrightarrow A, x \in F(A), f \in Q(A')(B) \\ x \otimes f &= 0 & & x \in F(A), f \in Q(A)(B) \text{ with } f \text{ torsion} \end{aligned}$$

For $\beta : B \longrightarrow B'$ and $x \otimes f \in \Omega(M)(B')$ we have $(x \otimes f) \cdot \beta = x \otimes (\beta \cdot f) = x \otimes Q(A)(\beta)(f)$. If $\phi : M \longrightarrow M'$ is a morphism of $\mathcal{A}$-modules, there is a morphism of $\mathcal{B}$-modules $\Omega_Q(\phi) : \Omega_Q(M) \longrightarrow \Omega_Q(M')$ defined by

$$\Omega_Q(\phi)_B(x \otimes f) = \phi_A(x) \otimes f \qquad x \in M(A), f \in Q(A)(B)$$

Proof. By (RSO,Theorem 13) the abelian group $(M \otimes_{\mathcal{A}} Q)(B)$ is the free abelian group on the symbols $x \otimes f$ modulo the first four classes of relations above. The torsion submodule of $M \otimes_{\mathcal{A}} Q$ does not seem to be generated by any nice class of relations, so when we quotient out by it, the best we can say is that the resulting abelian group is generated by the $x \otimes f$, which satisfy the fifth type of relation since clearly if f is torsion, $x \otimes f \in t_K(M \otimes_{\mathcal{A}} Q)(B)$. □

Proposition 29. *Let $\mathcal{A}, \mathcal{B}$ be ringoids and let K be an additive topology on $\mathcal{B}$. Let $Q : \mathcal{A} \longrightarrow$ $\mathbf{Mod}(\mathcal{B}, K)$ be an additive covariant functor and let*

$$- \otimes_{\mathcal{A}} Q : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{B}, K)$$

be the unique additive, colimit preserving functor extending Q. If

$$\mathbf{i} : \mathbf{Mod}(\mathcal{B}, K) \longrightarrow \mathbf{Mod}\mathcal{B}$$

is the canonical inclusion, then for an $\mathcal{A}$-module M, $M \otimes_{\mathcal{A}} Q$ is the $\mathcal{B}$-module $\Omega_{\mathbf{i}Q}(M)^+$. That is, for $B \in \mathcal{B}$, $(M \otimes_{\mathcal{A}} Q)(B)$ is the abelian group of all matching families $\{x_g \,|\, g \in \mathfrak{a}\}$ where $\mathfrak{a} \in K(B)$ and for $g : D \longrightarrow B \in \mathfrak{a}$, $x_g \in \Omega_{\mathbf{i}Q}(D)$. For $\beta : B \longrightarrow B'$ we have

$$\{x_g \,|\, g \in \mathfrak{a}\} \cdot \beta = \{x_{\beta h} \,|\, h \in \beta^* \mathfrak{a}\}$$

And for a morphism of $\mathcal{A}$-modules $\phi : M \longrightarrow M'$

$$(\phi \otimes_{\mathcal{A}} Q)_B(\{x_g \,|\, g \in \mathfrak{a}\}) = \{\Omega_{\mathbf{i}Q}(\phi)(x_g) \,|\, g \in \mathfrak{a}\}$$

Proof. The functor $\mathbf{i}Q : \mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$ is additive and extends uniquely to an additive, colimit preserving functor $- \otimes_{\mathcal{A}} \mathbf{i}Q$. Composing this functor with $\mathbf{c} : \mathbf{Mod}\mathcal{B} \longrightarrow \mathbf{Mod}(\mathcal{B}, K)$ leaves us with an additive, colimit preserving functor

$$\mathbf{c}(- \otimes_{\mathcal{A}} \mathbf{i}Q) : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{B}, K)$$

On $\mathcal{A}$ this functor is naturally equivalent to $\mathbf{ci}Q \cong Q$ (since $\mathbf{ci} \cong 1$). But it follows by uniqueness that $- \otimes_{\mathcal{A}} Q$ is naturally equivalent to $\mathbf{c}(- \otimes_{\mathcal{A}} \mathbf{i}Q)$. For an $\mathcal{A}$-module M,

$$\mathbf{c}(M \otimes_{\mathcal{A}} \mathbf{i}Q) = (M \otimes_{\mathcal{A}} \mathbf{i}Q / t_K(M \otimes_{\mathcal{A}} \mathbf{i}Q))^+ = \Omega_{\mathbf{i}Q}(M)^+$$

as required. □

Theorem 30. *Let $\mathcal{C}, \mathcal{D}$ be grothendieck abelian categories with families of generators $\mathcal{A}, \mathcal{B}$ respectively. Then there is an additive topology K on $\mathcal{B}$ with the property that any additive, colimit preserving functor $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ is characterised up to natural equivalence by its values on $\mathcal{A}$, as follows:*

Objects *For any $M \in \mathcal{C}$ and $B \in \mathcal{B}$ consider the free abelian group on the symbols*

$$x \otimes f \qquad A \in \mathcal{A}, x \in Hom_{\mathcal{C}}(A, M), f \in Hom_{\mathcal{D}}(B, \theta(A))$$

subject to the relations

$$(x + x') \otimes f = x \otimes f + x \otimes f$$
$$x \otimes (f + f') = x \otimes f + x \otimes f'$$
$$x\alpha \otimes f = x \otimes \theta(\alpha) f$$

Let $T_{M,B}$ be this group modulo those $\sum x_i \otimes f_i$ for which there is a cover $\mathfrak{a} \in K(B)$ with $\sum x_i \otimes f_i g = 0$ for all $g \in \mathfrak{a}$. Then $Hom_{\mathcal{D}}(B, \theta(M))$ is isomorphic to the group of matching families $\{x_g \,|\, g \in \mathfrak{a}\}$ with

$$x_g = \sum_i x_{g,i} \otimes f_{g,i} \in T_{M,D} \text{ for } g : D \longrightarrow B \in \mathfrak{a}.$$

Moreover for $\beta : B' \longrightarrow B$ and $\{x_g \,|\, g \in \mathfrak{a}\} : B \longrightarrow \theta(M)$

$$\{x_g \,|\, g \in \mathfrak{a}\} \circ \beta = \{x_{\beta h} \,|\, h \in \beta^* \mathfrak{a}\}$$

Morphisms *For* $\phi : M \longrightarrow M'$ *and* $B \in \mathcal{B}$ *the morphism* $\theta(\phi) : \theta(M) \longrightarrow \theta(M')$ *is defined by*

$$\theta(\phi) \circ \{x_g \,|\, g \in \mathfrak{a}\} = \Big\{ \sum_i \phi x_{g,i} \otimes f_{g,i} \,|\, g \in \mathfrak{a} \Big\}$$

Proof. Let

$$\Phi : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{A}, J), \quad \Psi : \mathcal{D} \longrightarrow \mathbf{Mod}(\mathcal{B}, K)$$

be the equivalences provided by Theorem 25. Let $\mathbf{i} : \mathbf{Mod}(\mathcal{A}, J) \longrightarrow \mathbf{Mod}\mathcal{A}$ and $\mathbf{i}' : \mathbf{Mod}(\mathcal{B}, K) \longrightarrow \mathbf{Mod}\mathcal{B}$ be the inclusions with $\mathbf{a}$ left adjoint to $\mathbf{i}$. Note that $\mathbf{ai} = 1$. Let $Q : \mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{B}, K)$ be the additive functor $\Psi\theta|_{\mathcal{A}}$. The composition

$$\Psi\theta\Phi^{-1}\mathbf{a} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{B}, K)$$

is additive, colimit preserving, and restricts to a functor on $\mathcal{A}$ naturally equivalent to Q. By uniqueness $\Psi\theta\Phi^{-1}\mathbf{a}$ is naturally equivalent to the functor $- \otimes_{\mathcal{A}} Q$ of Proposition 29. Hence

$$\Psi\theta \cong \Psi\theta\Phi^{-1}\mathbf{ai}\Phi \cong (- \otimes_{\mathcal{A}} Q)\mathbf{i}\Phi$$

Let $M \in \mathcal{C}$. Then for $B \in \mathcal{B}$

$$Hom_{\mathcal{D}}(B, \theta(M)) = \Psi\theta(M)(B) \cong (\mathbf{i}\Phi(M) \otimes_{\mathcal{A}} Q)(B)$$

Applying Proposition 29 and the naturality of this isomorphism in M, we establish the various assertions of the Theorem. □

4 The Submodule Classifier

Let $\mathcal{C}$ be a small category (not necessarily additive). The topos of presheaves $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ on $\mathcal{C}$ has a *subobject classifier* which is given by

$$\Omega(A) = \{S \,|\, S \text{ is a sieve at } C\} \tag{6}$$

Where Ω acts on sieves by pullback along morphisms of $\mathcal{A}$. We think of the elements of Ω as being "truth values".

For a subobject $\phi : P \longrightarrow F$ in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ and $x \in F(C)$, we can ask the question "does x belong to P"? For sets P, F the answer is "yes" or "no". In this more general context, the answer is all of those $f : D \longrightarrow C$ for which $x \cdot f \in P(D)$, so in particular the answer is the improper sieve H_C iff. x belongs to $P(C)$.

Each subobject P determines a collection of such answers as x varies over $F(C)$ and C varies over $\mathcal{C}$ (this collection is a natural transformation $F \longrightarrow \Omega$). Any "suitable" collection of answers determines a subobject of F by letting an element into the subobject iff. the corresponding answer is "completely true" (the improper sieve). Here "suitable" means that if you use $f : D \longrightarrow C$ to restrict $x \in F(C)$ to $x \in F(D)$, the answer for $x \cdot f$ is the pullback of the answer for x along f (i.e. naturality). This defines the bijection between subobjects of F and morphisms $F \longrightarrow \Omega$ in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$.

The above has an obvious analogue for modules. If M is a module over a ring R with a submodule L, then we can think of the elements $r \in R$ with $x \cdot r \in L$ as being answers to the question "does x belong to L"? The answer is only completely true if $x = x \cdot 1 \in L$ and only completely false if 0 is the only element of R sending x into L. We now develop this notion of a *submodule classifier* in detail. These ideas are explored for algebraic in theories in [5] and [6], and we adopt the notation used there.

Let $\mathcal{A}$ be a ringoid. Then in particular $\mathcal{A}$ is a small category, so we can consider the presheaf topos $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$ of contravariant functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$. Any additive functor $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$

becomes an element of $\mathbf{Sets}^{\mathcal{A}^{\rm op}}$ once we compose it with the forgetful functor $\mathbf{Ab} \longrightarrow \mathbf{Sets}$, and this defines a functor

$$\mathbf{f} : \mathbf{Ab}^{\mathcal{A}^{\rm op}} \longrightarrow \mathbf{Sets}^{\mathcal{A}^{\rm op}}$$

Note that this functor may identify two distinct modules by forgetting about the distinction between two different abelian groups on the same set. Rather than write $\mathbf{f}F$ all the time, we adopt the convention that whenever we talk about a morphism $F \longrightarrow G$ where F is a right $\mathcal{A}$-module and G is just a presheaf of sets, we really mean a morphism $\mathbf{f}F \longrightarrow G$. Since $\mathcal{A}$ is additive, we can define a subfunctor of $\Omega \in \mathbf{Sets}^{\mathcal{A}^{\rm op}}$ by

$$\Omega^a(A) = \{S \,|\, S \text{ is an right ideal at } A\}$$

The pullback of a right ideal is a right ideal, so Ω^a really is a subfunctor of Ω. Notice in particular that the pullback of any right ideal at A along a zero morphism $0 : A' \longrightarrow A$ is the improper right ideal $H_{A'}$. We also need the contravariant functor

$$Sub : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Sets}$$

Here $Sub(F)$ is the set of submodules of F in $\mathbf{Mod}\mathcal{A}$. For a morphism of modules $\mu : F \longrightarrow F'$, $Sub(\mu) : Sub(F') \longrightarrow Sub(F)$ acts by pullback.

We introduce additivity into our subobject classifiers in the following way. Suppose that F is a right $\mathcal{A}$-module with a submodule G. If $x \in F(A)$ then the answer to the question "does x belong to G" should be all those $f : D \longrightarrow A$ for which $x \cdot f \in G(D)$. These answers define a natural transformation $\theta : F \longrightarrow \Omega$ given by $\theta_A(x) = \{f : D \longrightarrow A \,|\, x \cdot f \in G(D)\}$. Beyond naturality, the collection of answers θ has the following properties:

(i) If for $x \in F(A)$ the answer to "does x belong to G" includes two morphisms $f, g : D \longrightarrow A$, then it includes their sum $f + g$.

(ii) If for $x, y \in F(A)$ the answers to "does x belong to G" and "does y belong to G" both include $f : D \longrightarrow A$, then so does the answer to "does $x + y$ belong to G".

In terms of θ, property (i) says that θ factors through $\Omega^a \longrightarrow \Omega$, and property (ii) says that $\theta_A(x) \cap \theta_A(y) \subseteq \phi_A(x + y)$. Hence in the additive situation we redefine our notion of a "suitable" collection of answers in the following way.

Definition 14. Let F be a right $\mathcal{A}$-module. A morphism $\theta : F \longrightarrow \Omega^a$ of presheaves of sets is a *characteristic morphism* if, for all $A \in \mathcal{A}$ and $x, y \in F(A)$

$$\theta_A(x) \cap \theta_A(y) \subseteq \theta_A(x + y)$$

The set of all characteristic morphisms $F \longrightarrow \Omega^a$ is denoted $Hom_{\mathbf{char}}(F, \Omega^a)$.

Notice that if $\theta : F \longrightarrow G$ is any morphism in $\mathbf{Sets}^{\mathcal{A}^{\rm op}}$ and $\psi : G \longrightarrow \Omega^a$ is a characteristic morphism, then $\psi\theta$ is also a characteristic morphism, so $Hom_{\mathbf{char}}(-, \Omega^a)$ defines a subfunctor of $Hom(-, \Omega^a)$.

Theorem 31. *There is a natural bijective correspondence between submodules of a right $\mathcal{A}$-module F and characteristic morphisms $F \longrightarrow \Omega^a$ (of presheaves of sets). Equivalently there is a bijection of sets*

$$\mathbf{char} : Sub(F) \longrightarrow Hom_{\mathbf{char}}(F, \Omega^a)$$

which is natural in F, provided we understand that the naturality involves morphisms $F \longrightarrow G$ of $\mathbf{Mod}\mathcal{A}$.

Proof. Let F be a right $\mathcal{A}$-module. Given an element $\phi : M \longrightarrow F$ of $Sub(F)$, which we assume is a submodule, define $\mathbf{char}\phi : F \longrightarrow \Omega^a$ as the pointwise morphism of sets:

$$\mathbf{char}\phi_A : F(A) \longrightarrow \Omega^a(A)$$
$$x \mapsto \{\alpha : D \longrightarrow A \,|\, x \cdot \alpha \in M(A)\}$$

Intuitively, $\mathbf{char}\phi$ answers the question "is x in M?" by responding with all those morphisms of the ringoid that send x *into* M. It is easy to check that the specified set is in fact a right ideal at A. To see that $\mathbf{char}\phi$ is natural, let $\alpha : A' \longrightarrow A$ be a morphism in $\mathcal{A}$. We need to show that the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "F(\alpha)"] \arrow[d, "\mathbf{char}\phi_A"'] & F(A') \arrow[d, "\mathbf{char}\phi_{A'}"] \\
\Omega^a(A) \arrow[r] & \Omega^a(A')
\end{tikzcd}$$

But for $x \in F(A)$,

$$\begin{aligned}
\Omega^a(\alpha)\,(\mathbf{char}\phi_A(x)) &= \{f : D \longrightarrow A' \,|\, \alpha f \in \mathbf{char}\phi_A(x)\} \\
&= \{f : D \longrightarrow A' \,|\, x \cdot \alpha f \in M(D)\} \\
&= \{f : D \longrightarrow A' \,|\, (x \cdot \alpha) \cdot f \in M(D)\} \\
&= \mathbf{char}\phi_{A'}(x \cdot \alpha)
\end{aligned}$$

In addition, $\mathbf{char}\phi$ is a characteristic morphism since if $\alpha : D \longrightarrow C$ is in $\mathbf{char}\phi_A(x)$ and $\mathbf{char}\phi_A(y)$, then by definition $x \cdot \alpha, y \cdot \alpha \in M(A)$. Since $M(A)$ is a subgroup of $F(A)$, it follows that $(x+y) \cdot \alpha = x \cdot \alpha + y \cdot \alpha \in M(A)$, so $\alpha \in \mathbf{char}\phi_A(x+y)$, as required.

Conversely, suppose we are given a characteristic morphism of presheaves $\mu : F \longrightarrow \Omega^a$. Define a submodule M_μ of F in $\mathbf{Mod}\mathcal{A}$ by

$$M_\mu(A) = \{x \in F(A) \,|\, \mu_A(x) = H_A\}$$

Naturality of μ for every zero morphism $0 : A \longrightarrow A$ implies that

$$\mu_A F(0) = \Omega^a(0)\mu_A$$

Since $\Omega^a(0)$ will give H_A on any element of $\Omega^a(A)$, this implies that $\mu_A(0) = H_A$ and hence for every $A \in \mathcal{A}$ we have $0 \in M_\mu(A)$. It remains to show that for each A, $M_\mu(A)$ is closed under addition. But if $x, y \in M_\mu(A)$ then by definition $\mu_A(x) = \mu_A(y) = H_A$, and hence since μ is characteristic morphism

$$\mu_A(x+y) \supseteq \mu_A(x) \cap \mu_A(y) = H_A$$

so $x + y \in M_\mu(A)$.

It is easy to see that these two assignments are mutually inverse, so $\mathbf{char}$ defines a bijection of the claimed sets. Naturality of $\mathbf{char}$ for a morphism $\eta : F \longrightarrow G$ in $\mathbf{Mod}\mathcal{A}$ means commutativity of the following diagram

$$\begin{tikzcd}
Sub(G) \arrow[r] \arrow[d] & Hom_{\mathbf{char}}(G, \Omega^a) \arrow[d] \\
Sub(F) \arrow[r] & Hom_{\mathbf{char}}(F, \Omega^a)
\end{tikzcd}$$

To check that this commutes, let $\phi : M \longrightarrow G$ be a submodule of G in $\mathbf{Mod}\mathcal{A}$. Then $Sub(\eta)(\phi)$ is the pullback of ϕ along η, and since ϕ is monic this is the pointwise inverse image under η_A of $M(A) \subseteq G(A)$. For $x \in F(A)$ we have

$$\begin{aligned}
\mathbf{char}(Sub(\eta)(\phi))_A(x) &= \{\alpha : D \longrightarrow A \,|\, x\alpha \in \eta_D^{-1}(M(D))\} \\
&= \{\alpha \,|\, \eta_D(x\alpha) \in M(D)\} \\
&= \{\alpha \,|\, \eta_A(x)\alpha \in M(D)\} \\
&= \mathbf{char}\phi_A \eta_A(x)
\end{aligned}$$

which proves naturality of $\mathbf{char}$ in F. □

Of course, a morphism $G \longrightarrow F$ is monic in $\mathbf{Mod}\mathcal{A}$ if and only if its image $\mathbf{f}G \longrightarrow \mathbf{f}F$ in $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$ is monic, so that the subobjects of F in $\mathbf{Mod}\mathcal{A}$ form a subset of the subobjects of F in $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$. Another way of stating the defining property of the subobject classifier Ω for $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$ is that every monic $\phi : P \longrightarrow F$ of presheaves is associated with a unique morphism $\mathbf{char}\phi : F \longrightarrow \Omega$ making the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\phi"'] & \mathbf{1} \arrow[d] \\ F \arrow[r, "\mathbf{char}\phi"'] & \Omega \end{tikzcd}$$

a pullback. Under this association, the subobjects of F in $\mathbf{Mod}\mathcal{A}$ correspond to the morphisms $F \longrightarrow \Omega$ which factor through Ω^a and have the characteristic property defined above.

Example 6. Let $\mathcal{A}$ be a ringoid and F a right $\mathcal{A}$-module. Consider the zero subobject $0 \longrightarrow F$. The characteristic map $\mathbf{char}0 : F \longrightarrow \Omega^a$ is defined by

$$\mathbf{char}0_A(x) = \{\alpha : D \longrightarrow A \,|\, x \cdot \alpha = 0\}$$

For $x \in F(A)$ this is clearly the annihilator ideal $Ann(x)$.

Let M be a right $\mathcal{A}$-module. For every additive topology J on $\mathcal{A}$ there is a submodule $t_J(M)$ of M and a corresponding characteristic morphism $\phi = \mathbf{char}t_J(M) : M \longrightarrow \Omega^a$, defined by

$$\begin{aligned} \phi_A(x) &= \{\alpha : D \longrightarrow A \,|\, x \cdot \alpha \in t_J(M)\} \\ &= \{\alpha : D \longrightarrow A \,|\, Ann(x \cdot \alpha) \in J(A)\} \end{aligned}$$

Since $Ann(x \cdot \alpha) = \alpha^* Ann(x)$,

$$\phi_A(x) = \{\alpha \,|\, \alpha^* Ann(x) \in J(A)\}$$

In the case where $J = J_0$, we recover the previous example.

5 The Matrix Ring of a Ringoid

In the case of a ring R and its category of modules $\mathbf{Mod}R$, we can recover R as the ring of endomorphisms of the small projective generator R of $\mathbf{Mod}R$. Moreover, if we can find a small projective generator X in any cocomplete abelian category, by (RSO,Theorem 15) that category is equivalent to the category modules over the endomorphism ring of X.

For a general ringoid $\mathcal{A}$ we have a generating family of small projectives $\{H_A\}_{A \in \mathcal{A}}$ whose coproduct $\bigoplus_{A \in \mathcal{A}} H_A$ is a projective generator for $\mathbf{Mod}\mathcal{A}$. We say a ringoid is *finite* when $\mathcal{A}$ has only finitely many objects, and in this case we can define:

Definition 15. Let $\mathcal{A}$ be a finite ringoid with n objects. The *matrix ring* of $\mathcal{A}$, denoted $R(\mathcal{A})$, is defined to be the endomorphism ring of $\bigoplus_A H_A$ in $\mathbf{Mod}\mathcal{A}$. This is the set of all $n \times n$ matrices Λ where $\Lambda_{BA} \in Hom_{\mathcal{A}}(A, B)$. Matrices are added and multiplied using addition and composition in $\mathcal{A}$, and in this way $R(\mathcal{A})$ becomes a ring with identity.

The reason why an element in the Bth row and Ath column belongs to $Hom_{\mathcal{A}}(A, B)$ and not $Hom_{\mathcal{A}}(B, A)$ is that we are working with *right* modules, where (denoting the injections and projections resp. by u_A, p_B) if Λ is a matrix in $R(\mathcal{A})$ corresponding to $\phi : \bigoplus_A H_A \longrightarrow \bigoplus_A H_A$ the element Λ_{BA} is $p_B \phi u_A : H_A \longrightarrow H_B$ which is an element of $Hom_{\mathcal{A}}(A, B)$. If we worked with left modules, $p_B \phi u_A : H^A \longrightarrow H^B$ would be a morphism from B to A.

There is a generalisation of this ring to the case where $\mathcal{A}$ is not finite, but we can no longer take $R(\mathcal{A})$ to be an endomorphism ring because the coproduct is no longer necessarily a biproduct. The interested reader is directed to [26], Section 33.

Example 7. For any ring R and positive integer n, define a ringoid $\mathcal{A}$ as follows: the objects are the integers $1, 2, \ldots, n$ with

$$Hom(i,j) = \begin{cases} 0 & i \neq j \\ R & i = j \end{cases}$$

Composition and addition of morphisms is defined using the structure of the ring R. It is clear that the ring $R(\mathcal{A})$ is just the product R^n.

The following Proposition can also be found in [26], but we give a much simpler proof.

Proposition 32. *If $\mathcal{A}$ is a finite ringoid then* $\mathbf{Mod}\mathcal{A}$ *is equivalent to the category of right modules over the ring $R(\mathcal{A})$, where the equivalence is given by*

$$Hom(\bigoplus_A H_A, -) : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}R(\mathcal{A})$$

Proof. Since a finite coproduct of small objects is small by (AC,Lemma 88), we see that $\bigoplus_A H_A$ is a small projective generator for $\mathbf{Mod}\mathcal{A}$. The result now follows by applying Theorem (RSO,Theorem 15). $\square$

Under this equivalence a right $\mathcal{A}$-module F is taken to the module $\prod_A F(A)$, so the correspondence turns a multi-object module into a normal module in the most trivial possible way. From one point of view, Proposition 32 says that there is no point in talking about modules over finite ringoids - we might as well just forget the extra generality and work with normal modules. However, in many cases there is a real advantage to working with the ringoid $\mathcal{A}$ rather than the ring $R(\mathcal{A})$, as we will see in Section 7.

6 Algebroids

For a commutative ring K, an associative K-algebra R can be defined in two equivalent ways. Firstly, as a ring which is also a K-module, so that the action of K and the multiplication are compatible. This is equivalent to giving a morphism of rings from K to $C(R)$, the center of the ring R. To define "algebras with several objects" over a commutative ring K, we first define the center of a ringoid.

Definition 16. Let $\mathcal{A}$ be a ringoid, and let $C(\mathcal{A})$ denote the ring of endomorphisms of the identity functor $1 : \mathcal{A} \longrightarrow \mathcal{A}$. Then $C(\mathcal{A})$ is a commutative ring with identity, which we call the *center* of $\mathcal{A}$.

Less abstractly, an element in the center of $\mathcal{A}$ consists of a sequence of endomorphisms $c_A \in End(A)$ in the center of $End(A)$ for each $A \in \mathcal{A}$, such that $c_{A'}\alpha = \alpha c_A$ for any morphism $\alpha : A \longrightarrow A'$ between objects.

Lemma 33. *For a ringoid $\mathcal{A}$, the endomorphisms of the identity functor* $1 : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{A}$ *form a set, isomorphic as a ring to the opposite ring of $C(\mathcal{A})$.*

Of course, for normal rings $C(R)$ is the whole ring iff. R is commutative. The fact that a ringoid admits morphisms with distinct domain and codomain means that the obvious generalisation of commutativity is impossible. Instead,

Definition 17. A ringoid $\mathcal{A}$ is *commutative* if each endomorphism ring $End(A)$ is commutative for $A \in \mathcal{A}$, and if for distinct A, B we have $Hom_{\mathcal{A}}(A, B) = 0$.

Lemma 34. *For a finite ringoid $\mathcal{A}$ the following are equivalent:*

(i) The ringoid $\mathcal{A}$ is commutative.

(ii) The ring $R(\mathcal{A})$ is commutative.

(iii) Every family of endomorphisms $\{c_A : A \longrightarrow A\}_{A\in\mathcal{A}}$ is an element of $C(\mathcal{A})$.

Example 8. Let R be a commutative ring and n a positive integer. In Section 5 we defined a ringoid $\mathcal{A}$ having the integers $1, 2, \dots, n$ as objects, with each endomorphism ring equal to R and no other nonzero morphisms. This is obviously an example of a commutative ringoid, and in this case $C(\mathcal{A}) = R(\mathcal{A}) = R^n$.

Definition 18. Let K be a commutative ring. A K-*algebroid* is a ringoid $\mathcal{A}$ together with a ring morphism $K \longrightarrow C(\mathcal{A})$. Equivalently, a K-algebroid is a small category $\mathcal{A}$ with a K-module structure on each of its morphism sets so that composition is bilinear. A *morphism* of K-algebroids is a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ such that the induced map

$$Hom(A, A') \longrightarrow Hom(F(A), F(A'))$$

is a morphism of K-modules for every $A, A' \in \mathcal{A}$. The K-algebroids form a category, which we denote by K**Algb**.

Notice that the category of K-algebras K**Alg** forms a full subcategory of K**Algb**. If $\mathcal{A}$ is a K-algebroid, then a morphism of algebroids $K[x] \longrightarrow \mathcal{A}$ is determined by choosing an object $A \in \mathcal{A}$ and a morphism of normal K-algebras from $K[x]$ to the endomorphism ring of A (which we have agreed to also denote by A). Hence morphisms $K[x] \longrightarrow \mathcal{A}$ in K**Algb** are in one-to-one correspondence with endomorphisms in $\mathcal{A}$.

7 Application: Triangular Matrix Rings

Let A, B be rings and M an A-B-bimodule. Then we can define a ringoid $\mathcal{A}$ with two objects A, B whose endomorphism rings are the rings A and B respectively, and where $Hom_{\mathcal{A}}(B, A) = M$ and $Hom_{\mathcal{A}}(A, B) = 0$. The composition comes from ring multiplication and the action of A, B on M. Then the endomorphism ring of $H_A \oplus H_B$ in **Mod**$\mathcal{A}$ consists of 2×2 matrices

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$$

where $a \in A, b \in B$ and $m \in M$. Composition gives the following multiplication:

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}$$

This ring is more commonly known as the generalised matrix ring $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$. By Proposition 32 there is an equivalence

$$\Phi : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

Given for a right $\mathcal{A}$-module F by $\Phi(F) = F(A) \oplus F(B)$, and for $\mu : F \longrightarrow F'$ by $\Phi(\mu) : F(A) \oplus F(B) \longrightarrow F'(A) \oplus F'(B)$,

$$\Phi(\mu)(a, b) = (\mu_A(a), \mu_B(b))$$

Proposition 35. *The inclusions $\varphi : A \longrightarrow \mathcal{A}$ and $\psi : B \longrightarrow \mathcal{A}$ induce respective triples of adjoint functors*

$$\begin{tikzcd} \mathbf{Mod}A \arrow[r, bend left, "\varphi^*"] \arrow[r, bend right, "\varphi^!"'] & \mathbf{Mod}\mathcal{A} \arrow[l, "\varphi_*"'] \end{tikzcd} \qquad \varphi^* \dashv \varphi_* \dashv \varphi^!$$

and

$$\begin{tikzcd} \mathbf{Mod}B \arrow[r, bend left, "\psi^*"] \arrow[r, bend right, "\psi^!"'] & \mathbf{Mod}\mathcal{A} \arrow[l, "\psi_*"'] \end{tikzcd} \qquad \psi^* \dashv \psi_* \dashv \psi^!$$

Of course, the results of (RSO,Section 4) allow us to give these functors explicitly. For a right A-module L, a right B-module N and a right $\mathcal{A}$-module F,

$$\begin{aligned} &\varphi^*(L)(A) = L, \quad \varphi^*(L)(B) = L \otimes_A M \\ &\varphi^!(L)(A) = L, \quad \varphi^!(L)(B) = 0 \\ &\varphi_*(F) = F(A) \end{aligned}$$

and

$$\begin{aligned} &\psi^*(N)(A) = 0, \quad \psi^*(N)(B) = N \\ &\psi^!(N)(A) = Hom_B(M, N), \quad \psi^!(N)(B) = N \\ &\psi_*(F) = F(B) \end{aligned}$$

Let F be a right $\mathcal{A}$-module, and denote the right A-module $F(A)$ by L and the right B-module $F(B)$ by N. These two modules capture the value of the functor F on the objects and endomorphisms of $\mathcal{A}$. The only remaining data consists of the map $M \longrightarrow Hom_{\mathbf{Ab}}(N, L)$ given by $m \mapsto F(m)$. This allows us to characterise the $\mathcal{A}$-modules explicitly:

Lemma 36. *A right $\mathcal{A}$-module consists of the following data:*

- *A right A-module L;*
- *A right B-module N;*
- *A morphism $\theta : M \longrightarrow Hom_{\mathbf{Ab}}(N, L)$ of A-B-bimodules, where the group $Hom_{\mathbf{Ab}}(N, L)$ is given its canonical module structure.*

We can also characterise the ideals of $\mathcal{A}$: the right ideals at the object B correspond to right ideals of the ring B, and a right ideal at the object A is the union $\mathfrak{a} \cup N$ of a collection of endomorphisms $\mathfrak{a}$ of A and a collection N of morphisms $B \longrightarrow A$ belonging to M. The condition that $\mathfrak{a} \cup N$ be an ideal says precisely that $\mathfrak{a}$ is an ideal of A and N is a B-submodule of M such that $\mathfrak{a}M \subseteq N$. Proceeding in this way, one can characterise the additive topologies on $\mathcal{A}$ in terms of the gabriel topologies on the rings A, B.

The results of Section (RSO,Section 3) allow us to give explicitly the injective cogenerator of $\mathbf{Mod}\mathcal{A}$. The injective cogenerator is the product $Q_A \times Q_B$, where

$$Q_A(A) = Hom_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z}), \quad Q_A(B) = 0$$

$$Q_B(A) = Hom_{\mathbf{Ab}}(M, \mathbb{Q}/\mathbb{Z}), \quad Q_B(B) = Hom_{\mathbf{Ab}}(B, \mathbb{Q}/\mathbb{Z})$$

Hence the injective cogenerator for $\mathbf{Mod}\mathcal{A}$ is the module I defined by

$$I(A) = Hom_{\mathbf{Ab}}(A \oplus M, \mathbb{Q}/\mathbb{Z}), \quad I(B) = Hom_{\mathbf{Ab}}(B, \mathbb{Q}/\mathbb{Z})$$

In particular, this implies that the injective cogenerator for modules over the ring $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is the module

$$Hom_{\mathbf{Ab}}(A \oplus M \oplus B, \mathbb{Q}/\mathbb{Z})$$

In the previous section we saw an example of a commutative ringoid $\mathcal{B}$ whose center $C(\mathcal{B})$ was equal to the matrix ring $R(\mathcal{B})$, both of which were equal to a product ring R^n. It is easy to determine the center of the ringoid $\mathcal{A}$:

Lemma 37. *The ring $C(\mathcal{A})$ is the following subring of $A \times B$:*

$$C(\mathcal{A}) = \{(a, b) \in A \times B \,|\, a \cdot m = m \cdot b \;\; \forall m \in M\}$$

If $M = 0$, so that $\mathcal{A}$ is just the trivial ringoid derived from A and B, then $C(\mathcal{A}) = R(\mathcal{A}) = A \times B$. For $M \neq 0$ the rings $C(\mathcal{A})$ and $R(\mathcal{A})$ are generally distinct. In the particular case where $A = B$ is commutative and M is torsion-free, $C(\mathcal{A}) = \{(a, a) \,|\, a \in A\}$ is the diagonal.

Example 9. Take $A = \mathbb{Z}, B = \mathbb{Q}$ and $M = \mathbb{Q}$, and let $\mathcal{A}$ be the corresponding ringoid:

$$\mathbb{Z} \xleftarrow{\quad \mathbb{Q} \quad} \mathbb{Q}$$

There are two ideals at B: $0 \cup 0$ and $0 \cup \mathbb{Q}$ (the first 0 denoting the single morphism $A \longrightarrow B$). Apart from the trivial ideal $0 \cup 0$ at A, there is a family of ideals:

$$(n) \cup \mathbb{Q} \qquad n \neq 0$$

The injective cogenerator for the ring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right)$ is the module

$$Hom_{\mathbf{Ab}}(\mathbb{Z} \oplus \mathbb{Q}^2, \mathbb{Q}/\mathbb{Z})$$

8 Application: Graded rings

Definition 19. A $\mathbb{Z}$-*graded* ring is a ring A together with a decomposition of A into a direct sum of additive subgroups $A = \bigoplus_{d \in \mathbb{Z}} A_d$ in such a way that $A_d A_e \subseteq A_{d+e}$ and $1 \in A_0$. We do not require A to be commutative. Throughout this section a *graded ring* will denote a $\mathbb{Z}$-graded ring.

Definition 20. Let S be a graded ring. A graded left S-module is a left S-module M together with a set of subgroups $M_n, n \in \mathbb{Z}$ such that $M = \bigoplus_{n \in \mathbb{Z}} M_n$ as an abelian group, and $sm \in M_{n+d}$ for $s \in S_d, m \in M_n$. The preadditive category of graded left S-modules is denoted $S\mathbf{GrMod}$. A graded right S-module is a right S-module graded by subgroups M_n with $ms \in M_{n+d}$ for $s \in S_d, n \in M_n$. The preadditive category of graded right S-modules is denoted $\mathbf{GrMod}S$.

Example 10. Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k, then $A_i = 0$ for $i < 0$ and for $i \geq 0$ the abelian group A_i is the vector space over k spanned by all monomials of weight i. This makes A into a commutative graded ring.

Example 11. Let $A = k\langle x_1, \ldots, x_n \rangle$ be the free k-algebra in n variables over a field k. The ring A is the free k-algebra on the set of all sequences taken from the set $\{x_1, \ldots, x_n\}$ (including the empty word). We define the *length* of a sequence to be the number of symbols occurring in it (the length of the empty word is zero). Let $A_i = 0$ for $i < 0$ and for $i \geq 0$ let A_i be the vector space over k spanned by all sequences of length i. This makes A into a noncommutative graded ring.

Definition 21. Let A be a graded ring and define the ringoid $\mathbb{G}(A)$ as follows. The objects of $\mathbb{G}(A)$ are the integers $n \in \mathbb{Z}$ for $m, n \in \mathbb{Z}$ we define the morphism sets by

$$Hom_{\mathbb{G}(A)}(m, n) = A_{n-m}$$

Composition and addition of morphisms is defined using multiplication and addition in A, respectively, and it is easily checked that with these definitions $\mathbb{G}(A)$ is a ringoid. The identity at n is the morphism corresponding to $1 \in A_0$.

Example 12. Let Q be the quiver whose vertices are the integers and which has n arrows from i to $i+1$ for all $i \in \mathbb{Z}$ and no other arrows for some $n \geq 1$. In the notation of our notes on Linearised Categories, let $\mathcal{C}(Q)$ be the path category of Q. For $i < j$ the morphism set $Hom_{\mathcal{C}(Q)}(i, j)$ consists of all sequences taken from the set $\{x_1, \ldots, x_n\}$ of length $j - i$, since a path beginning at i and ending at j must choose one x_i to get from i to $i+1$, another to get from $i+1$ to $i+2$, and so on. If $i < j$ then the morphism set $Hom_{\mathcal{C}(Q)}(j, i)$ is empty. For all integers i the set $Hom_{\mathcal{C}(Q)}(i, i)$ is the singleton $\{1_i\}$. Let k be a field and apply the ringoid construction of our Linearised Category notes to $\mathcal{C}(Q)$ using k. Call this ringoid kQ. Let $A = k\langle x_1, \ldots, x_n \rangle$ be the free k-algebra in n variables. Then $\mathbb{G}(A)$ is isomorphic to kQ.

Proposition 38. *Let A be a graded ring and $\mathbb{G}(A)$ the associated ringoid. There are canonical equivalences of categories*

$$\Phi : \mathbf{Mod}\mathbb{G}(A) \longrightarrow \mathbf{GrMod}A$$
$$\Phi(M) = \bigoplus_{n\in\mathbb{Z}} M(-n)$$

and

$$\Psi : \mathbb{G}(A)\mathbf{Mod} \longrightarrow A\mathbf{GrMod}$$
$$\Psi(M) = \bigoplus_{n\in\mathbb{Z}} M(n)$$

Proof. Let M be a right $\mathbb{G}(A)$-module, let $\Phi(M)$ be the abelian group $\bigoplus_{n\in\mathbb{Z}} M(-n)$, and let $\Phi(M)_n$ denote the subgroup given by the image of $M(-n)$. The contravariant nature of the functor M means homogenous elements of A *reduce* degree instead of increasing it, so $\Phi(M)$ needs to take the inverted grading. If $a \in A_d$ is homogenous of degree $d \in \mathbb{Z}$ and $n \in \mathbb{Z}$ then a determines a morphism $(-n-d) \longrightarrow -n$ of $\mathbb{G}(A)$ and therefore $M(a)$ is a morphism $M(-n) \longrightarrow M(-n-d)$, or equivalently $\Phi(M)_n \longrightarrow \Phi(M)_{n+d}$. Define the A-module structure on $\Phi(M)$ by

$$((m_n) \cdot a)_i = \sum_{x+y=i} m_x \cdot a_y$$

It is not hard to check this makes $\Phi(M)$ into a graded right A-module with $\Phi(M)_n$ the subgroup of grade n. If $\tau : M \longrightarrow N$ is a morphism of right $\mathbb{G}(A)$-modules then $\oplus_{n\in\mathbb{Z}}\tau_{-n}$ gives a morphism of graded A-modules $\Phi(\tau) : \Phi(M) \longrightarrow \Phi(N)$ and this defines the additive functor Φ. Since Φ just repackages the information that defines a graded module, it is clear that Φ is an equivalence which is distinct on objects.

One defines the functor Ψ in the same way, but now there is no need to invert the grading. For a morphism $\tau : M \longrightarrow N$ of left $\mathbb{G}(A)$-modules we define $\Psi(\tau) = \oplus_{n\in\mathbb{Z}}\tau_n$. It is not hard to check this is an equivalence which is distinct on objects. $\square$

Corollary 39. *For any graded ring A the categories A**GrMod** and **GrMod**A are grothendieck abelian.*

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