Linearised Categories

Daniel Murfet

October 5, 2006

In this note we study several ways of turning a small category into a ringoid. We refer to such processes as *linearisation*, for lack of a better name. Although this material is very abstract, our main application is concrete: the category $\mathfrak{Mod}(X)$ of sheaves of modules over a scheme X can be viewed as a localisation of a category of modules over a ringoid, very naturally constructed from the structure sheaf of the scheme.

We freely use the notation and concepts introduced in our notes on Algebra in a Category. We allow noncommutative rings and noncommutative sheaves of rings. Throughout all additive topologies are right additive topologies (notation of our Localisation of Ringoids notes).

Contents

1 Quivers

Definition 1. A *quiver* is a tuple (Q, M, d) where Q is a nonempty set of *vertices*, M a set of *arrows* and $d : M \longrightarrow Q \times Q$ is a function. A quiver is *finite* if it has a finite number of vertices.

If f is an arrow and $d(f) = (i, j)$ then we say f *begins* at i and *ends* at j, and write $f : i \longrightarrow j$. A *composite* arrow in the quiver Q is a nonempty sequence $m_1, \ldots, m_p$ of arrows such that for each i, m_i ends where m_{i+1} begins. We write this composite arrow as $m_p m_{p-1} \ldots m_1$, and say that it begins where m_1 beings and ends where m_p ends. The collection of all composite arrows beginning at i and ending at j is denoted by $\kappa(i, j)$.

Definition 2 (Path category). We associate to any quiver Q a small category $\mathcal{C}(Q)$ called the *path category* of Q. The objects of $\mathcal{C}(Q)$ are the vertices of Q and for vertices i, j the set of morphisms $i \longrightarrow j$ in $\mathcal{C}(Q)$ is defined by

$$Hom(i,j) = \begin{cases} \kappa(i,j) & i \neq j, \\ \kappa(i,i) \cup 1_i & i = j \end{cases}$$

Composition is defined by concatentation of sequences, with the $1_i, i \in Q$ acting as identities.

Example 1. Consider the following examples of quivers and their categories:

1. If Q has a single vertex and no arrows, then $\mathcal{C}(Q)$ is the category with a single identity morphism.

2. If Q has a single vertex and one arrow x, then $\mathcal{C}(Q)$ has a single object and endomorphisms

$$1, x, x^2, x^3, \ldots$$

with composition defined by $x^n x^m = x^{n+m}$. This is the monoid $\mathbb{N}$.

3. More generally, if Q has a single vertex and a set M of arrows, then $\mathcal{C}(Q)$ is the free monoid on the letters of M.

Definition 3. A *morphism* $F : Q \longrightarrow Q'$ between two quivers (Q, M, d) and (Q', M', d') is a pair $F = (f, f')$ of functions $f : Q \longrightarrow Q'$ and $f' : M \longrightarrow M'$ which maps any arrow $i \longrightarrow j$ to an arrow $f(i) \longrightarrow f(j)$. Composition of morphisms is defined by composition of the component functions, and this defines the category **Qvr** of quivers.

It is not difficult to check that the process $Q \mapsto \mathcal{C}(Q)$ defines a functor $\mathbf{Qvr} \longrightarrow \mathbf{Cat}$ from the category of quivers to the category of small categories, and that this functor is left adjoint to the forgetful functor $\mathbf{Cat} \longrightarrow \mathbf{Qvr}$.

2 Linearisation

Definition 4. Let R be a ring and $\mathcal{C}$ a small category. Define a category $R\mathcal{C}$ as follows: the objects of $R\mathcal{C}$ are the objects of $\mathcal{C}$, and for $p, q \in \mathcal{C}$ the set $Hom_{R\mathcal{C}}(p, q)$ is the free R-module on the set $Hom_{\mathcal{C}}(p, q)$, realised as functions $Hom_{\mathcal{C}}(p, q) \longrightarrow R$ with finite support. We denote the function with a single nonzero value $r \in R$ on a morphism f by $r \cdot f$.

Composition is defined by $(r \cdot g)(s \cdot f) = rs \cdot gf$ for $f : p \longrightarrow q, g : q \longrightarrow s$ and $r, s \in R$. That is, for $\alpha \in Hom_{R\mathcal{C}}(p, q)$ and $\beta \in Hom_{R\mathcal{C}}(q, s)$ we define $\beta\alpha \in Hom_{R\mathcal{C}}(p, s)$ by

$$\beta\alpha(n) = \sum_{\substack{f:p\longrightarrow q \\ g:q\longrightarrow s \\ gf=n}} \beta(g)\alpha(f)$$

For $q \in \mathcal{C}$ the function $1 \cdot 1_q$ is the identity in $R\mathcal{C}$, and it is tedious but not difficult to check that this composition is associative, and therefore that $R\mathcal{C}$ is a preadditive category. The map $f \mapsto 1 \cdot f$ gives a faithful functor $\mathcal{C} \longrightarrow R\mathcal{C}$ and we identify $\mathcal{C}$ with a subcategory of $R\mathcal{C}$ in this way. If R is commutative, then the canonical left R-module structure on the morphism sets makes $R\mathcal{C}$ into an R-algebroid.

Theorem 1. *Let R be a ring and $\mathcal{C}$ a small category. Then there is a canonical isomorphism of categories*

$$\Phi : \mathbf{Mod}R\mathcal{C} \longrightarrow (\mathbf{Mod}R)^{\mathcal{C}^{op}}$$
$$\Phi(F)(p) = F(p)$$

Proof. The abelian group $F(p)$ is a right R-module via $x \cdot r = F(r \cdot 1_p)(x)$. For a morphism $f \in Hom_{\mathcal{C}}(p, q)$ we define $\Phi(F)(f) = F(f)$, which is easily seen to be a morphism of R-modules. Therefore $\Phi(F)$ is a contravariant functor. For $\phi : F \longrightarrow F'$ in $\mathbf{Mod}R\mathcal{C}$ we define

$$\Phi(\phi) : \Phi(F) \longrightarrow \Phi(F'), \qquad \Phi(\phi)_p = \phi_p : F(p) \longrightarrow F'(p')$$

This is clearly a natural transformation, and it is a pointwise morphism of R-modules since

$$\phi_p(x \cdot r) = \phi_p(F(r \cdot 1_p)(x)) = F'(r \cdot 1_p)(\phi_p(x)) = \phi_p(x) \cdot r$$

This defines the functor Φ, which is easily seen to be faithful. To see that it is full, notice that for $\alpha \in Hom_{R\mathcal{C}}(p, q)$ and $F \in \mathbf{Mod}R\mathcal{C}$ we have

$$F(\alpha) = F(\sum \alpha(f) \cdot f) = \sum F(\alpha(f) \cdot f) = \sum F(\alpha(f) \cdot 1_p)F(f)$$

Since Φ is trivially distinct on objects, to complete the proof it suffices to show that Φ is onto objects. Let a contravariant functor $T : \mathcal{C} \longrightarrow \mathbf{Mod}R$ be given. Define $T' : R\mathcal{C} \longrightarrow \mathbf{Ab}$ by $T'(p) = T(p)$, and for $\alpha \in Hom_{R\mathcal{C}}(p, q)$ define $T'(\alpha) : T(q) \longrightarrow T(p)$ by

$$T'(\alpha)(x) = \sum_{f:p\longrightarrow q} T(f)(x) \cdot \alpha(f)$$

The only nontrivial property that needs checking is $T'(\beta\alpha) = T'(\alpha)T'(\beta)$ for $\beta \in Hom_{R\mathcal{C}}(q, r)$ and $\alpha \in Hom_{R\mathcal{C}}(p, q)$. This is notationally complicated but otherwise straightforward. Since $\Phi(T') = T$, this completes the proof. $\square$

Example 2. If G is a group considered as a category with one object, then $\mathbb{Z}G$ is a ring with elements $\sum_{g\in G} a_g g$ and multiplication $(\sum_g a_g g)(\sum_g b_g g) = \sum_h (\sum_{ij=k} a_i b_j)h$. In this case Theorem 1 reduces to the following familiar fact

$$\mathbf{Mod}\mathbb{Z}G = \mathbf{Ab}^{G^{\mathrm{op}}}$$

We now turn to a further generalisation of this construction which is treated briefly in [13], as is Theorem 2. In the above, there was a fixed ring R with respect to which all the morphism sets were free modules. More generally we can allow the ring to vary over the category $\mathcal{C}$. To make the notation clearer, if P is a presheaf on a category $\mathcal{C}$ and $f : C \longrightarrow D$ then for $x \in P(D)$ we write $x|_f$ instead of $P(f)(x)$.

Definition 5. Let $\mathcal{C}$ be a small category and R a presheaf of rings on $\mathcal{C}$. Define a category $R\mathcal{C}$ as follows: the objects of $R\mathcal{C}$ are the objects of $\mathcal{C}$, and for $p, q \in \mathcal{C}$ the set $Hom_{R\mathcal{C}}(p, q)$ is the free $R(p)$-module on the set $Hom_{\mathcal{C}}(p, q)$, realised as functions $Hom_{\mathcal{C}}(p, q) \longrightarrow R(p)$ with finite support. We denote the function with a single nonzero value $r \in R(p)$ on a morphism $f : p \longrightarrow q$ by $r \cdot f$.

Composition is defined by $(r \cdot g)(s \cdot f) = r|_f s \cdot gf$ for $f : p \longrightarrow q, g : q \longrightarrow s$ and $r \in R(q), s \in R(p)$. That is, for $\alpha \in Hom_{R\mathcal{C}}(p, q)$ and $\beta \in Hom_{R\mathcal{C}}(q, s)$ we define $\beta\alpha \in Hom_{R\mathcal{C}}(p, s)$ by

$$\beta\alpha(n) = \sum_{\substack{f:p\longrightarrow q\\ g:q\longrightarrow s\\ gf=n}} \beta(g)|_f \alpha(f)$$

For $q \in \mathcal{C}$ the function $1 \cdot 1_q$ is the identity in $R\mathcal{C}$, and one checks that $R\mathcal{C}$ is a preadditive category in the same way as before. The map $f \mapsto 1 \cdot f$ gives a faithful functor $\mathcal{C} \longrightarrow R\mathcal{C}$ and we identify $\mathcal{C}$ with a subcategory of $R\mathcal{C}$ in this way. Notice that for $r \in R(q)$ and $f : p \longrightarrow q$ we have $(r \cdot 1_q)f = r|_f \cdot f = f(r|_f \cdot 1_p)$ and $(r \cdot 1_p)(s \cdot 1_p) = rs \cdot 1_p$ for $p \in \mathcal{C}$.

Theorem 2. *Let $\mathcal{C}$ be a small category and R a presheaf of rings on $\mathcal{C}$. Then there is a canonical isomorphism of categories*

$$\Psi : \mathbf{Mod}(P(\mathcal{C}); R) \longrightarrow \mathbf{Mod}R\mathcal{C}$$
$$\Psi(M)(p) = M(p)$$

Proof. Let M be a presheaf of right R-modules in $P(\mathcal{C})$. The contravariant functor $\Psi(M) : R\mathcal{C} \longrightarrow \mathbf{Ab}$ is given by $\Psi(M)(p) = M(p)$, where $M(p)$ has the canonical abelian group structure. For a morphism $\alpha \in Hom_{R\mathcal{C}}(p, q)$ we define

$$\Psi(M)(\alpha)(x) = \sum_{f:p\longrightarrow q} x|_f \cdot \alpha(f)$$

It is not difficult to check that $\Psi(M)(\alpha)$ is a morphism of abelian groups, and it is clear that $\Psi(M)(1_p) = 1$. Checking that $\Psi(M)(\beta\alpha) = \Psi(M)(\alpha)\Psi(M)(\beta)$ and $\Psi(M)(\alpha + \beta) = \Psi(M)(\alpha) + \Psi(M)(\beta)$ is tedious but straightforward. This defines the functor Ψ on objects. Now let $\phi : M \longrightarrow M'$ be a morphism of presheaves of right R-modules. This consists of a natural collection $\phi_p : M(p) \longrightarrow M'(p)$ of morphisms of $R(p)$-modules. Define $\Psi(\phi) : \Psi(M) \longrightarrow \Psi(M')$ by

$$(\Psi\phi)_p(x) = \phi_p(x)$$

It is not hard to check that $\Psi(\phi)$ is a morphism of right $R\mathcal{C}$-modules.

Therefore Ψ is a functor, which is clearly fully faithful and distinct on objects. To complete the proof that Ψ is an isomorphism of categories, we only need to show that it is onto objects. To this end, let $F : R\mathcal{C} \longrightarrow \mathbf{Ab}$ be additive and contravariant. We define $M \in \mathbf{Mod}(P(\mathcal{C}); R)$ in the obvious way: for $p \in \mathcal{C}$, $M(p)$ is $F(p)$ and for $f : p \longrightarrow q$ we set $M(f) = F(1 \cdot f)$. This makes M into a presheaf of abelian groups on $\mathcal{C}$. For $p \in \mathcal{C}$ we define the right $R(p)$-module structure on $M(p)$ by $x \cdot r = F(r \cdot 1_p)(x)$. It is easy to check this makes M into a presheaf of right R-modules, with $\Psi(M) = F$. Therefore Ψ is an isomorphism and the proof is complete. □

Corollary 3. *Let $\mathcal{C}$ be a small category and R a presheaf of rings on $\mathcal{C}$. Then* $\mathbf{Mod}(P(\mathcal{C}); R)$ *and* $(P(\mathcal{C}); R)\mathbf{Mod}$ *are complete grothendieck abelian categories. The structures on both categories are described as follows*

Zero *The zero object is the presheaf $Z(p) = 0$.*

Kernel *If $\phi : M \longrightarrow N$ is a morphism of presheaves of modules, then $K(p) = Ker(\phi_p)$ defines a presheaf of modules, and the inclusion $K \longrightarrow M$ is the kernel of ϕ.*

Cokernel *If $\phi : M \longrightarrow N$ is a morphism of presheaves of modules, then $C(p) = N(p)/Im(\phi_p)$ defines a presheaf of modules, and the projection $N \longrightarrow C$ is the cokernel of ϕ.*

Limits *If D is a diagram of presheaves of modules, then define $L(p)$ to be the limit of the diagram $D(p)$ of modules. This becomes a presheaf of modules and the projections $L \longrightarrow D_i$ are a limit for the diagram.*

Colimits *If D is a diagram of presheaves of modules, then define $C(p)$ to be the colimit of the diagram $D(p)$ of modules. This becomes a presheaf of modules and the injections $D_i \longrightarrow C$ are a colimit for the diagram.*

Proof. The fact that both categories are complete grothendieck abelian follows from Theorem 2 and (ALCAT,Lemma 6). Using the isomorphism of Theorem 2 and the fact that we know structures are computed pointwise in module categories over ringoids, it is easy to check the remaining items. □

Definition 6. Let $(\mathcal{C}, J)$ be a small site and R a sheaf of rings on $\mathcal{C}$. Let M be a sheaf of R-modules (right or left). A *submodule* of M is a monomorphism of sheaves of R-modules $\phi : N \longrightarrow M$ (right or left) with the property that for every $p \in \mathcal{C}$ the map $\phi_p : N(p) \longrightarrow M(p)$ is the inclusion of a subset. Every subobject of M is equivalent to a submodule. If N, N' are submodules of M then $N \leq N'$ if and only if $N(p) \subseteq N'(p)$ for every $p \in \mathcal{C}$.

Corollary 4. *Let $(\mathcal{C}, J)$ be a small site and R a sheaf of rings on $\mathcal{C}$. Then* $(Sh_J(\mathcal{C}); R)\mathbf{Mod}$ *and* $\mathbf{Mod}(Sh_J(\mathcal{C}); R)$ *are complete grothendieck abelian categories. The structures on both categories are described as follows*

Zero *The zero object is the presheaf $Z(p) = 0$.*

Kernel *If $\phi : M \longrightarrow N$ is a morphism of sheaves of modules, then $K(p) = Ker(\phi_p)$ defines a sheaf of modules, and the inclusion $K \longrightarrow M$ is the kernel of ϕ.*

Cokernel *If $\phi : M \longrightarrow N$ is a morphism of sheaves of modules, then $C(p) = N(p)/Im(\phi_p)$ defines a presheaf of modules, and the canonocal morphism $N \longrightarrow C \longrightarrow \mathbf{a}C$ is the cokernel of ϕ.*

Limits *If D is a diagram of presheaves of modules, then define $L(p)$ to be the limit of the diagram $D(p)$ of modules. This becomes a sheaf of modules and the projections $L \longrightarrow D_i$ are a limit for the diagram.*

Colimits *If D is a diagram of presheaves of modules, then define $C(p)$ to be the colimit of the diagram $D(p)$ of modules. This becomes a presheaf of modules and the morphisms $D_i \longrightarrow C \longrightarrow \mathbf{a}C$ are a colimit for the diagram.*

Image *Let $\phi : M \longrightarrow N$ be a morphism of sheaves of modules and let I be the submodule of N defined by $n \in I(p)$ if and only if there exists $T \in J(p)$ such that $n|_h \in Im(\phi_q)$ for every $h : q \longrightarrow p$ in T. Then $I \longrightarrow N$ is the image of ϕ.*

Inverse Image *Let $\phi : M \longrightarrow N$ be a morphism of sheaves of modules and $T \longrightarrow N$ a submodule of N. Then the inverse image $\phi^{-1}T$ is the submodule of M defined by $(\phi^{-1}T)(p) = \phi_p^{-1}(T(p))$.*

Proof. Since $(Sh_J(\mathcal{C}); R)\mathbf{Mod}$ is a Giraud subcategory of $(P(\mathcal{C}); R)\mathbf{Mod}$ and similarly for right modules (ALCAT,Section 2) the claims are all easily checked (AC,Section 3). □

Example 3. We now apply some of the results of our note on Rings with Several Objects to ringoids of the form $R\mathcal{C}$. Let $\mathcal{C}$ be a small category and R a presheaf of rings on $\mathcal{C}$. The injective cogenerator Q of $\mathbf{Mod}R\mathcal{C}$ is explicitly given in (RSO,Section 3). For $p \in \mathcal{C}$ this definition gives

$$\begin{aligned} Q(q) &= \prod_{p \in \mathcal{C}} Q_p(q) = \prod_p Hom_{\mathbf{Ab}}\left(Hom_{R\mathcal{C}}(p, q), \mathbb{Q}/\mathbb{Z}\right) \\ &= \prod_p Hom_{\mathbf{Ab}}(\oplus_{f:p \longrightarrow q} R(p), \mathbb{Q}/\mathbb{Z}) = \prod_{f:p \longrightarrow q} Hom_{\mathbf{Ab}}(R(p), \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

In particular if S is a quiver, k is a ring, and $\mathcal{C} = \mathcal{C}(S)$ then the category of k-representations of the quiver S is equivalent to the category of modules over the ringoid $k\mathcal{C}$. The above formula shows that the injective cogenerator of the latter category is defined for a vertex $q \in S$ by

$$Q(q) = \prod_{\text{Paths ending at } q} Hom_{\mathbf{Ab}}(k, \mathbb{Q}/\mathbb{Z})$$

Theorem 5. *Let $(\mathcal{C}, J)$ be a small site and R a presheaf of rings on $\mathcal{C}$. For $p \in \mathcal{C}$ define the following subset of the right ideals at p in the ringoid $R\mathcal{C}$*

$$J_R(p) = \{\mathfrak{a} \,|\, S \subseteq \mathfrak{a} \text{ for some } S \in J(p)\}$$

We claim that J_R is an additive topology on $R\mathcal{C}$, and that if R is a sheaf of rings there is a canonical isomorphism of categories

$$\begin{gathered} \Psi : \mathbf{Mod}(Sh_J(\mathcal{C}); R) \longrightarrow \mathbf{Mod}(R\mathcal{C}, J_R) \\ \Psi(M)(p) = M(p) \end{gathered}$$

Proof. If $\mathfrak{a}$ is a right ideal at p in $R\mathcal{C}$, and $f : p' \longrightarrow p$ is any morphism of $\mathcal{C}$ that belongs to $\mathfrak{a}$, then $r \cdot f$ belongs to $\mathfrak{a}$ for all $r \in R(p')$, since $r \cdot f = f(r \cdot 1)$. It is clear that $H_p \in J_R(p)$ for every $p \in \mathcal{C}$. To prove the stability condition, let $\alpha : p \longrightarrow q$ be a morphism in $R\mathcal{C}$ (which we may as well assume is nonzero) and suppose $\mathfrak{a} \in J_R(q)$, so there is $S \in J(q)$ with $S \subseteq \mathfrak{a}$. If $f_1, \dots, f_n \in Hom_{\mathcal{C}}(p, q)$ is the support of α, then for $g \in \mathcal{C}$ we have

$$\alpha g = (\sum_i \alpha(f_i) \cdot f_i) g = \alpha(f_1)|_g \cdot f_1 g + \cdots + \alpha(f_n)|_g \cdot f_n g \tag{1}$$

If $f_i^* S$ denotes the pullback in $\mathcal{C}$, then $f_i^* S \in J(p)$ for each i, since S is a cover and J is a topology. It follows from (1) that $\alpha^* \mathfrak{a} \supseteq \bigcap_{i=1}^n f_i^* S$ and therefore $\alpha^* \mathfrak{a} \in J_R(p)$, as required.

For the transitivity condition suppose that $\mathfrak{a}$ is a right ideal at q, while $\mathfrak{b} \in J_R(q)$ is such that $\alpha^* \mathfrak{a} \in J_R(p)$ for every $\alpha : p \longrightarrow q \in \mathfrak{b}$. Let $S \in J(q)$ be such that $S \subseteq \mathfrak{b}$, and for each $f : p \longrightarrow q \in S$ let $S_f \in J(p)$ be the cover contained in $f^* \mathfrak{a}$. Then $\cup_{f \in S} f S_f$ is a cover in $J(q)$ which is contained in $\mathfrak{a}$. This shows that $\mathfrak{a} \in J_R(q)$, which verifies the transitivity condition and shows that J_R is an additive topology.

By definition $\mathbf{Mod}(Sh_J(\mathcal{C}); R)$ is the full subcategory of $\mathbf{Mod}(P(\mathcal{C}); R)$ consisting of those presheaves of R-modules whose underlying presheaves of sets are J-sheaves. Using the isomorphism Ψ of Theorem 2, to complete the proof it suffices to show that $M \in \mathbf{Mod}(P(\mathcal{C}); R)$ is a J-sheaf iff. $\Psi(M)$ is J_R-closed. If $f \in \mathcal{C}$ then $-|_f$ denotes the action of $M(f)$ or $R(f)$, and if $\alpha \in R\mathcal{C}$ then $- \cdot \alpha$ denotes the action of $\Psi(M)(\alpha)$. If $r \in R(p)$ then $- \cdot r$ denotes the module action of r on $M(p)$.

Suppose that M is a sheaf of right modules over R, let $\mathfrak{a} \in J_R(p)$ be an additive cover, and suppose we have an additive matching family $\{x_\alpha\}_{\alpha\in\mathfrak{a}}$ for $\Psi(M)$. Let $S \in J(p)$ be the cover contained in $\mathfrak{a}$. Then $\{x_h\}_{h\in S}$ is a matching family for M, and since M is a J-sheaf there is a unique $x \in M(p)$ such that $x|_h = x_h$ for $h \in S$. To show that x is an amalgamation of the family $\{x_\alpha\}$ we have to show that $x \cdot \alpha = x_\alpha$ for all $\alpha \in \mathfrak{a}$. This is trivial for $\alpha = 0$, so let α have support $f_1, \ldots, f_n \in Hom_{\mathcal{C}}(q,p)$. Since J is a grothendieck topology, the sieve $T = \bigcap_{i=1}^n f_i^* S$ is a cover of q in $\mathcal{C}$. For $g \in T$ we have $\alpha g = \sum_i \alpha(f_i)|_g \cdot f_i g$, so

$$\begin{aligned}(x\cdot\alpha)|_g = x\cdot\alpha g &= \sum_i x|_{f_i g}\cdot\alpha(f_i)|_g = \sum_i x_{f_i g}\cdot\alpha(f_i)|_g \\ &= \sum_i x_{f_i g}\cdot(\alpha(f_i)|_g\cdot 1) = \sum_i x_{\alpha(f_i)|_g\cdot f_i g} = x_{\alpha g} = x_\alpha|_g\end{aligned}$$

Since T is a cover of q and M is a sheaf for J, it follows that $x_\alpha = x \cdot \alpha$, as required. Uniqueness of x as an amalagamation of $\{x_\alpha\}$ is obvious, so we have shown that $\Psi(M)$ is J_R-closed.

Conversely suppose that $\Psi(M)$ is J_R-closed, and that $\{x_f\}_{f\in S}$ is a matching family for a cover $S \in J(p)$ in M. Let $\mathfrak{a}$ be the right ideal at p in $R\mathcal{C}$ consisting of all morphisms $\alpha \in Hom_{R\mathcal{C}}(q,p)$ whose support is contained in S. It is clear that $\mathfrak{a} \in J_R(p)$. For $\alpha : q \longrightarrow p$ in $\mathfrak{a}$ we define the following element of $M(q)$

$$x_\alpha = \sum_f x_f \cdot \alpha(f)$$

Note that for $f \in S$ this definition agrees with the matching family $\{x_f\}_{f\in S}$. We now check that the collection $\{x_\alpha\}_{\alpha\in\mathfrak{a}}$ is an additive matching family for $\Psi(M)$. It is clear that $x_{\alpha+\beta} = x_\alpha + x_\beta$, so suppose that $\alpha : q \longrightarrow p$ belongs to $\mathfrak{a}$ and let $\beta : s \longrightarrow q$ be any morphism of $R\mathcal{C}$. Then

$$\begin{aligned}x_\alpha\cdot\beta &= \sum_g x_\alpha|_g\cdot\beta(g) = \sum_{g,f}(x_f\cdot\alpha(f))|_g\cdot\beta(g) \\ &= \sum_{g,f}(x_f|_g\cdot\alpha(f)|_g)\cdot\beta(g) = \sum_{g,f} x_{fg}\cdot\alpha(f)|_g\beta(g) = x_{\alpha\beta}\end{aligned}$$

This shows that $\{x_\alpha\}_{\alpha\in\mathfrak{a}}$ is an additive matching family, which has a unique amalgamation $x \in M(p)$ since $\Psi(M)$ is J_R-closed. Clearly x is also an amalgamation for $\{x_f\}_{f\in S}$, which is unique since if $y|_f = x_f$ for all $f \in S$ then for $\alpha : q \longrightarrow p \in \mathfrak{a}$ we would have

$$y\cdot\alpha = \sum_f y|_f\cdot\alpha(f) = \sum_f x_f\cdot\alpha(f) = x_\alpha$$

which would imply that $y = x$. We have shown that M is J-closed, which completes the proof. □

3 Graded Linearisation

Throughout this section a $\mathbb{Z}$-*graded ring* is a not necessarily commutative ring S together with subgroups S_d for $d \in \mathbb{Z}$ such that $S = \bigoplus_{d\in\mathbb{Z}} S_d$ as an abelian group, and $S_d S_e \subseteq S_{d+e}$ for all $d, e \in \mathbb{Z}$ and $1 \in S_0$. See (LOR,Section 8) for the definition of left and right graded S-modules.

Definition 7. Let S be a $\mathbb{Z}$-graded ring and $\mathcal{C}$ a small category. Define a category $S \circ \mathcal{C}$ as follows: the objects of $S \circ \mathcal{C}$ are ordered pairs (d, p) where $d \in \mathbb{Z}$ and $p \in \mathcal{C}$, which we will often denote by p_d. For $p, q \in \mathcal{C}$ and $d, e \in \mathbb{Z}$ the set $Hom_{S\circ\mathcal{C}}(p_d, q_e)$ is the abelian group of functions $Hom_{\mathcal{C}}(p,q) \longrightarrow S_{e-d}$ with finite support and the pointwise operations. We denote the function with a single nonzero value $r \in S_{e-d}$ on a morphism f by $r \cdot f$.

Composition is defined by $(r\cdot g)(s\cdot f) = rs\cdot gf$ for $f : p \longrightarrow q, g : q \longrightarrow s$ and $r \in S_{a-e}, s \in S_{e-d}$. That is, for $\alpha \in Hom_{S\circ\mathcal{C}}(p_d, q_e)$ and $\beta \in Hom_{S\circ\mathcal{C}}(q_e, s_a)$ we define $\beta\alpha \in Hom_{S\circ\mathcal{C}}(p_d, s_a)$ by

$$\beta\alpha(n) = \sum_{\substack{f:p\longrightarrow q\\ g:q\longrightarrow s\\ gf=n}} \beta(g)\alpha(f)$$

For $q_d \in S \circ \mathcal{C}$ the function $1 \cdot 1_q$ is the identity, and it is not difficult to check that this composition is associative and that $S \circ \mathcal{C}$ is a preadditive category. For each morphism $f \in Hom_{\mathcal{C}}(p, q)$ and $e \in \mathbb{Z}$ we have the morphism $1 \cdot f \in Hom_{S\circ\mathcal{C}}(p_e, q_e)$, which we denote by f_e. For each $e \in \mathbb{Z}$ the map $f \mapsto f_e$ gives a faithful functor $\mathcal{C} \longrightarrow S \circ \mathcal{C}$.

Definition 8. Let $(\mathcal{C}, J)$ be a small site and S a sheaf of $\mathbb{Z}$-graded rings. Define a category $S \circ \mathcal{C}$ as follows: the objects of $S \circ \mathcal{C}$ are ordered pairs (d, p) where $d \in \mathbb{Z}$ and $p \in \mathcal{C}$, which we will often denote by p_d. For $p, q \in \mathcal{C}$ and $d, e \in \mathbb{Z}$ the set $Hom_{S\circ\mathcal{C}}(p_d, q_e)$ is the abelian group of functions $Hom_{\mathcal{C}}(p, q) \longrightarrow S_{e-d}(p)$ with finite support and the pointwise operations. We denote the function with a single nonzero value $r \in S_{e-d}(p)$ on a morphism f by $r \cdot f$.

Composition is defined by $(r \cdot g)(s \cdot f) = r|_f s \cdot gf$ for $f : p \longrightarrow q, g : q \longrightarrow s$ and $r \in S_{a-e}(q), s \in S_{e-d}(p)$. That is, for $\alpha \in Hom_{S\circ\mathcal{C}}(p_d, q_e)$ and $\beta \in Hom_{S\circ\mathcal{C}}(q_e, s_a)$ we define $\beta\alpha \in Hom_{S\circ\mathcal{C}}(p_d, s_a)$ by

$$\beta\alpha(n) = \sum_{\substack{f:p\longrightarrow q \\ g:q\longrightarrow s \\ gf=n}} \beta(g)|_f \alpha(f)$$

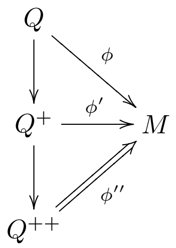
For $q \in \mathcal{C}$ the function $1 \cdot 1_q$ is the identity in $S \circ \mathcal{C}$, and one checks that $S \circ \mathcal{C}$ is a preadditive category. For each morphism $f \in Hom_{\mathcal{C}}(p, q)$ and $e \in \mathbb{Z}$ we have the morphism $1 \cdot f \in Hom_{S\circ\mathcal{C}}(p_e, q_e)$, which we denote by f_e. The map $f \mapsto f_e$ gives a faithful functor $\mathcal{C} \longrightarrow S \circ \mathcal{C}$ for every $e \in \mathbb{Z}$. Notice that for $r \in S_d(q)$ and $f : p \longrightarrow q$ we have $(r \cdot 1_q)f_e = r|_f \cdot f = f_{d+e}(r|_f \cdot 1_p)$ and $(r \cdot 1_p)(s \cdot 1_p) = rs \cdot 1_p$ for $p \in \mathcal{C}$ and homogenous r, s. We also have $u \cdot f = f_d(u \cdot 1_p)$ where $u \in S_{d-e}(p)$ and $u \cdot f : p_e \longrightarrow q_d$.

Lemma 6. *Let $(\mathcal{C}, J)$ be a small site and M a sheaf of abelian groups together with subsheaves of abelian groups $M_n, n \in \mathbb{Z}$ such that the morphisms $M_n \longrightarrow M$ are a coproduct of sheaves of abelian groups. Then*

(i) For $p \in \mathcal{C}$ the induced morphism $\bigoplus_{n\in\mathbb{Z}} M_n(p) \longrightarrow M(p)$ is injective.

(ii) For $p \in \mathcal{C}$ and $x \in M(p)$ there is $T \in J(p)$ such that for every $h : q \longrightarrow p \in T$, we have $x|_h \in \sum_{n\in\mathbb{Z}} M_n(q)$.

Proof. Let Q denote the canonical pointwise coproduct of the presheaves of abelian groups M_n, so that $Q(p) = \bigoplus_{n\in\mathbb{Z}} M_n(p)$. Let $\phi : Q \longrightarrow M$ be the morphism induced by the inclusions $M_n \longrightarrow M$, so that for $q \in \mathcal{C}$ we have $Im(\phi_q) = \sum_n M_n(q)$. The coproduct $\bigoplus_n M_n$ in $\mathbf{Ab}(Sh_J(\mathcal{C}))$ is the plus construction applied twice to Q. Let $\phi' : Q^+ \longrightarrow M$ and $\phi'' : Q^{++} \longrightarrow M$ be the unique morphisms of presheaves of abelian groups making the respective triangles in the following diagram commute



By assumption ϕ'' is an isomorphism, and since Q is a separated sheaf the morphism $Q \longrightarrow Q^{++}$ is a monomorphism of presheaves of abelian groups, and therefore so is the composite ϕ which proves (i).

(ii) If $x \in M(p)$ there is $\{\mathbf{x}_f \mid f \in S\}$ in $Q^{++}(p)$ mapping to x. Using the explicit definition of ϕ'' this means that $x|_f = \phi'(\mathbf{x}_f)$ for every $f \in S$. Applying the definition again, for every $f : q \longrightarrow p$ in S there is $S_f \in J(q)$ such that $x|_{fg} = \phi(x_{f,g})$ for every $g : s \longrightarrow q \in S_f$, where $x_{f,g}$ is some element of $Q(s)$. Set $T = \cup_{f\in S} f S_f$. Then $T \in J(p)$ has the required property. $\square$

Theorem 7. *Let* $(\mathcal{C}, J)$ *be a small site and* S *a sheaf of* $\mathbb{Z}$*-graded rings on* $\mathcal{C}$*. For* $p \in \mathcal{C}$ *and* $d \in \mathbb{Z}$ *define the following subset of the right ideals at* p_d *in the ringoid* $S \circ \mathcal{C}$

$$J_S(p_d) = \{\mathfrak{a} \,|\, T_d \subseteq \mathfrak{a} \text{ for some } T \in J(p)\}$$

where $T_d = \{f_d \,|\, f \in T\}$*. We claim that* J_S *is an additive topology on* $S \circ \mathcal{C}$*, and that there is a canonical equivalence of categories*

$$\Phi : \mathbf{GrMod}(Sh_J(\mathcal{C}); S) \longrightarrow \mathbf{Mod}(S \circ \mathcal{C}, J_S)$$
$$\Phi(M)(p_d) = M_{-d}(p)$$

Proof. Note that if $\mathfrak{a}$ is a right ideal at p_d in $S \circ \mathcal{C}$, and if $f : q \longrightarrow p$ is a morphism of $\mathcal{C}$ with $f_d \in \mathfrak{a}$, then for $r \in S_{d-e}(q)$ the morphism $r \cdot f$ belongs to $Hom_{S \circ \mathcal{C}}(q_e, p_d)$ since $r \cdot f = f_d(r \cdot 1_q)$. First we show that J_S defines an additive topology on $S \circ \mathcal{C}$.

It is clear that J_S contains the maximal right ideal at p_d. To check the stability condition, let $\alpha : q_e \longrightarrow p_d$ be a morphism in $S \circ \mathcal{C}$, which we may as well assume is nonzero with support $f_1, \dots, f_n \in Hom_{\mathcal{C}}(q, p)$, and let $\mathfrak{a} \in J_S(p_d)$ contain T_d for some $T \in J(p)$. For $g \in \mathcal{C}$ we have

$$\alpha g_e = (\sum_i \alpha(f_i) \cdot f_i) g_e = \alpha(f_1)|_g \cdot f_1 g + \dots + \alpha(f_n)|_g \cdot f_n g \tag{2}$$

If $f_i^* T$ denotes the pullback in $\mathcal{C}$, then $f_i^* T \in J(q)$ for each i, since T is a cover and J is a topology. Therefore $Q = \bigcap_{i=1}^n f_i^* T$ belongs to $J(q)$. It follows from (2) that $\alpha^* \mathfrak{a} \supseteq Q_e$ and therefore $\alpha^* \mathfrak{a} \in J_S(q_e)$, as required.

For the transitivity condition, suppose that $\mathfrak{a}$ is a right ideal at p_d, while $\mathfrak{b} \in J_S(p_d)$ is such that $\alpha^* \mathfrak{a} \in J_S(q_e)$ for every $\alpha : q_e \longrightarrow p_d \in \mathfrak{b}$. Let $T \in J(p)$ be such that $T_d \subseteq \mathfrak{b}$, and for each $f : q \longrightarrow p \in T$ let $T_f \in J(q)$ be the cover with $(T_f)_d \subseteq f_d^* \mathfrak{a}$. Then $Q = \cup_{f \in T} f T_f$ is a cover in $J(p)$ with $Q_d \subseteq \mathfrak{a}$. Therefore $\mathfrak{a} \in J_S(p_d)$, which shows that J_S is an additive topology.

Now we have to define the functor Φ. Let M be a sheaf of graded right R-modules and define a right $S \circ \mathcal{C}$-module $\Phi(M)$ on objects by $\Phi(M)(p_d) = M_{-d}(p)$ (note the inverted "grading"). For a morphism $\alpha \in Hom_{S \circ \mathcal{C}}(q_e, p_d)$ we define

$$\Phi(M)(\alpha) : M_{-d}(p) \longrightarrow M_{-e}(q)$$
$$\Phi(M)(\alpha)(x) = \sum_{f : q \longrightarrow p} x|_f \cdot \alpha(f)$$

It is not hard to check that with this definition, $\Phi(M)$ is a right $S \circ \mathcal{C}$-module. Notice that $\Phi(M)(f_d)(x) = x|_f$ for $f : q \longrightarrow p$. If $\psi : M \longrightarrow N$ is a morphism of sheaves of graded right R-modules, define

$$\Phi(\psi) : \Phi(M) \longrightarrow \Phi(N)$$
$$\Phi(\psi)_{p_d} : M_{-d}(p) \longrightarrow N_{-d}(p)$$
$$x \mapsto \psi_p(x)$$

This is a morphism of right $S \circ \mathcal{C}$-modules, and to show that Φ is a well-defined functor it only remains to show that $\Phi(M)$ is J_S-closed. As in the proof of Theorem 5 we denote the presheaf restriction on M, R by $-|_f$ and write $- \cdot \alpha$ and $- \cdot r$ for the module action of $S \circ \mathcal{C}$ on $\Phi(M)$ and R on M respectively.

Let $\mathfrak{a} \in J_S(p_d)$ and suppose we have an additive matching family $\{x_\alpha\}_{\alpha \in \mathfrak{a}}$ for $\Phi(M)$. To be clear, note that if $\alpha : q_e \longrightarrow p_d$ then $x_\alpha \in M_{-e}(q)$. Let $T \in J(p)$ be such that $T_d \subseteq \mathfrak{a}$ and set $x_h = x_{h_d}$ for $h \in T$. Then $\{x_h\}_{h \in T}$ is a matching family for the sheaf of abelian groups M_{-d}, which therefore has a unique amalgamation $x \in M_{-d}(p) = \Phi(M)(p_d)$. To show that x is the unique amalgamation of the family $\{x_\alpha\}$ we have to show that $x \cdot \alpha = x_\alpha$ for all $\alpha \in \mathfrak{a}$. This follows in the same way as in the proof of Theorem 5. Therefore $\Phi(M)$ is J_S-closed.

It only remains to show that Φ is an equivalence. It is easy to check that Φ is faithful. To see that it is full, let $\gamma : \Phi(M) \longrightarrow \Phi(N)$ be a morphism of right $S \circ \mathcal{C}$-modules. For every $d \in \mathbb{Z}$ and

$p \in \mathcal{C}$, $\gamma_{p_{-d}} : M_d(p) \longrightarrow N_d(p)$ is a morphism of abelian groups. Together these give a morphism of sheaves of abelian groups $\Gamma_d : M_d \longrightarrow N_d$, which induce a morphism of sheaves of abelian groups $\Gamma = \bigoplus_{d \in \mathbb{Z}} \Gamma_d : M \longrightarrow N$. To show this is a morphism of sheaves of graded right S-modules it suffices to show it is a morphism of right S-modules. Using Lemma 6 we reduce to checking that $\Gamma_{n+d,p}(m \cdot s) = \Gamma_{n,p}(m) \cdot s$ for $p \in \mathcal{C}$, $m \in M_n(p)$ and $s \in S_d(p)$. But

$$\begin{aligned}\Gamma_{n+d,p}(m \cdot s) &= \gamma_{p_{-n-d}}(m \cdot s) = \gamma_{p_{-n-d}}(m \cdot (s \cdot 1)) \\ &= \gamma_{p_{-n}}(m) \cdot (s \cdot 1) = \gamma_{p_{-n}}(m) \cdot s = \Gamma_{n,p}(m) \cdot s\end{aligned}$$

It is clear that $\Phi(\Gamma) = \gamma$, so Φ is full.

To complete the proof, it only remains to show that Φ is representative. Let F be a J_S-closed right $S \circ \mathcal{C}$-module and for fixed $n \in \mathbb{Z}$ define a presheaf of abelian groups M_n on $\mathcal{C}$ by

$$\begin{aligned}M_n(p) &= F(p_{-n}) \\ m|_f &= m \cdot f_{-n}\end{aligned}$$

we claim that M_n is a sheaf of abelian groups. Let $S \in J(p)$ be given, together with a matching family $\{x_f\}_{f \in S}$ for M_n. Let $\mathfrak{a} \in J_S(p_{-n})$ be the right ideal consisting of all morphisms whose support belongs to S. For $\alpha : q_e \longrightarrow p_{-n} \in \mathfrak{a}$ we define

$$x_\alpha = \sum_{\substack{f: q \longrightarrow p \\ f \in S}} x_f \cdot (\alpha(f) \cdot 1_q)$$

This defines an additive matching family $\{x_\alpha\}_{\alpha \in \mathfrak{a}}$ which agrees with $\{x_f\}$ on morphisms $\alpha = h_n$ for $h \in S$. Since F is J_S-closed, there is a unique amalgamation $x \in F(p_{-n}) = M_n(p)$ which is a unique amalgamation for $\{x_f\}_{f \in S}$ as well. This shows that M_n is a J-sheaf. Let M be the coproduct of presheaves of abelian groups

$$\begin{aligned}M &= \bigoplus_{n \in \mathbb{Z}} M_n \\ M(p) &= \bigoplus_{n \in \mathbb{Z}} F(p_{-n})\end{aligned}$$

Make the coproduct of presheaves of abelian groups $A = \bigoplus_{n \in \mathbb{Z}} S_n$ into a presheaf of rings in the canonical way (see (ALCAT,Section 2)). Then M becomes a presheaf of right A-modules by defining for $p \in \mathcal{C}$, $(m_n) \in M(p)$ and $s \in A(p)$

$$\{(m_n) \cdot (s_n)\}_i = \sum_{x+y=i} m_x \cdot (s_y \cdot 1_p)$$

The sheaf of abelian groups $\mathbf{a}M$ is then canonically a sheaf of right modules over $\mathbf{a}A$, which is a sheaf of $\mathbb{Z}$-graded rings isomorphic as a sheaf of $\mathbb{Z}$-graded rings to S (ALCAT,Section 2). Since $\mathbf{Ab}(Sh_J(\mathcal{C}))$ is a Giraud subcategory of $\mathbf{Ab}(P(\mathcal{C}))$, the morphisms $M_n \longrightarrow M \longrightarrow \mathbf{a}M$ are a coproduct in $\mathbf{Ab}(Sh_J(\mathcal{C}))$ (ALCAT,Section 2), (AC,Proposition 63) and with these subsheaves of abelian groups, $\mathbf{a}M$ becomes a sheaf of graded right S-modules via $\mathbf{a}A \cong S$. It is not difficult to check that $\Phi(\mathbf{a}M) \cong F$, which completes the proof. □

Definition 9. Let $(\mathcal{C}, J)$ be a small site and S a sheaf of $\mathbb{Z}$-graded rings on $\mathcal{C}$. If M is a sheaf of graded S-modules (right or left) then a *graded submodule* of M is an S-submodule $\phi : N \longrightarrow M$ with the property that the collection of subsheaves of abelian groups $\phi^{-1}M_n \longrightarrow N$ is a coproduct of sheaves of abelian groups. Together with these subsheaves it is clear that N is a sheaf of graded S-modules and ϕ a morphism of sheaves of graded S-modules.

Lemma 8. *Let $(\mathcal{C}, J)$ be a small site and S a sheaf of $\mathbb{Z}$-graded rings on $\mathcal{C}$. Let $\phi : N \longrightarrow M$ be a morphism of sheaves of graded S-modules (right or left). Then N is a graded submodule of M if and only if it is a submodule of M.*

Proof. One implication is trivial. Suppose that N is a submodule of M. To show it is a *graded* submodule we need only show that $N_n = \phi^{-1}M_n$ for every $n \in \mathbb{Z}$. It is clear that $N_n \subseteq \phi^{-1}M_n$. So suppose that $p \in \mathcal{C}$ and $x \in N(p)$ is such that $\phi_p(x) \in M_n(p)$. By Lemma 6 there is $T \in J(p)$ such that for every $h : q \longrightarrow p$ in T we have $x|_h \in \sum_i N_i(q)$. Since the sum $\sum_i N_i(q)$ is direct by Lemma 6 (i) there are well-defined $x_{h,i} \in N_i(q)$ such that $x|_h = \sum_i x_{h,i}$. Then by assumption

$$\sum_i \phi_q(x_{h,i}) = \phi_p(x)|_h \in M_n(q)$$

Part (i) of Lemma 6 and the fact that ϕ is a monomorphism implies that $x_{h,i} = 0$ for $i \neq n$, so $x|_h \in N_n(q)$. Since N_n is a subsheaf of N and T a cover of p, this shows that $x \in N_n(p)$ as required. □

Remark 1. The argument of the above proof shows further that if $\phi : N \longrightarrow M$ is any morphism of sheaves of graded S-modules (right or left) and $y \in M_n(p)$ belongs to the image of $\phi_p : N(p) \longrightarrow M(p)$, then there is $T \in J(p)$ such that for every $h : q \longrightarrow p$ in T we have $y|_h = \phi_q(x)$ for some $x \in N_n(q)$.

Corollary 9. *Let $(\mathcal{C}, J)$ be a small site and S a sheaf of $\mathbb{Z}$-graded rings on $\mathcal{C}$. Then the internal graded module categories $(Sh_J(\mathcal{C}); S)$**GrMod** and **GrMod**$(Sh_J(\mathcal{C}); S)$ are complete grothendieck abelian. The structures on both categories are described as follows*

Zero *The zero object is the sheaf $Z(p) = 0$ with $Z_n = 0$ for all $n \in \mathbb{Z}$.*

Kernel *If $\phi : M \longrightarrow N$ is a morphism of sheaves of graded S-modules then the subgroups $K_n(p) = Ker(\phi_p) \cap M_n(p)$ define a subsheaf of abelian groups $K_n \subseteq K$ of the kernel $K \longrightarrow M$ of sheaves of modules, and these subsheaves make K a sheaf of graded S-modules and $K \longrightarrow M$ the kernel of ϕ.*

Cokernel *If $\phi : M \longrightarrow N$ is a morphism of sheaves of graded S-modules, let $\mu : N \longrightarrow Q$ be the cokernel of sheaves of modules. Let Q_n be the subsheaf of abelian groups given by the image of the composite $N_n \longrightarrow N \longrightarrow Q$. Then Q is a sheaf of graded S-modules and μ is the cokernel of ϕ.*

Coproduct *Let $\{M_i\}_{i \in I}$ be a nonempty family of sheaves of graded S-modules and $\bigoplus_i M_i$ the coproduct of sheaves of S-modules. The morphisms $(M_i)_n \longrightarrow M_i \longrightarrow \bigoplus_i M_i$ are a coproduct of sheaves of abelian groups, and $\bigoplus_i M_i$ is a sheaf of graded S-modules with degree n subsheaf the image of $\bigoplus_i (M_i)_n \longrightarrow \bigoplus_i M_i$.*

Image *If $\phi : M \longrightarrow N$ is a morphism of sheaves of graded S-modules then the image $I \longrightarrow N$ of sheaves of modules is a graded submodule of N, and is the image of ϕ.*

Exact Sequence *A sequence of sheaves of graded S-modules $M' \longrightarrow M \longrightarrow M''$ is exact if and only if it is exact as a sequence of sheaves of S-modules.*

Proof. It follows immediately from (LOR,Corollary 17), Theorem 7 and (ALCAT,Lemma 7) that both categories are complete grothendieck abelian. In what follows, fix a morphism $\phi : M \longrightarrow N$ of sheaves of graded S-modules.

Let $K_n(p) = Ker(\phi_p) \cap M_n(p)$ be the sheaf of abelian groups described above. We claim that the kernel K of $\phi : M \longrightarrow N$ considered as a morphism of sheaves of S-modules, together with the subsheaves K_n, is a sheaf of graded S-modules. This follows from the fact that **Ab**$(Sh_J(\mathcal{C}))$ is grothendieck abelian, so taking coproducts preserves kernels. It is now easy to see that $K \longrightarrow M$ is a morphism of sheaves of graded S-modules and is the kernel of ϕ.

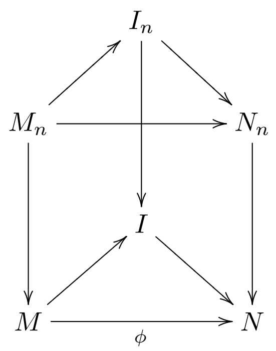
Now for the cokernel. The cokernel $\mu : N \longrightarrow Q$ of sheaves of modules is also the cokernel of sheaves of abelian groups. For each n let $\mu_n : N_n \longrightarrow N_n/M_n$ be the cokernel of $\phi_n : M_n \longrightarrow N_n$ in the category of sheaves of abelian groups. Then for every n we have a commutative diagram with exact rows in $\mathbf{Ab}(Sh_J(\mathcal{C}))$

$$\begin{array}{ccccccc} M_n & \longrightarrow & N_n & \longrightarrow & N_n/M_n & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Since coproducts preserve cokernels it follows that the morphisms $N_n/M_n \longrightarrow Q$ are a coproduct of sheaves of abelian groups, and it is not difficult to check that Q together with the images of these subsheaves is a sheaf of graded S-modules, and moreover that $\mu : N \longrightarrow Q$ is the cokernel of ϕ.

For the coproduct, it is clear that the morphisms $(M_i)_n \longrightarrow M_n \longrightarrow \bigoplus_i M_i$ are a coproduct of sheaves of abelian groups, hence so are the induced morphisms $\bigoplus_i (M_i)_n \longrightarrow \bigoplus_i M_i$ for $n \in \mathbb{Z}$. Let $(\bigoplus_i M_i)_n$ denote the image of this morphism of sheaves of abelian groups. Then it is not hard to check the morphisms $M_i \longrightarrow \bigoplus_i M_i$ are a coproduct of sheaves of graded S-modules.

For the image, let $I \longrightarrow N$ be the image of ϕ considered as a morphism of sheaves of S-modules. Since the epimorphism $M \longrightarrow I$ is the cokernel of the kernel of ϕ, it follows from the above construction of the cokernel that I together with the subsheaves of abelian groups given by the images of $M_n \longrightarrow M \longrightarrow I$ is a sheaf of graded S-modules and $M \longrightarrow I$ is an epimorphism of sheaves of graded S-modules. For $n \in \mathbb{Z}$ denote by I_n the image of $\phi_n : M_n \longrightarrow N_n$. It follows from Remark 1 that $I_n = I \cap N_n$. So we have a commutative diagram



(3)

Therefore $I \longrightarrow N$ is a monomorphism of sheaves of graded S-modules and we have shown that this morphism is the image of ϕ. $\square$

Remark 2. Let $(\mathcal{C}, J)$ be a small site and S a sheaf of $\mathbb{Z}$-graded rings on $\mathcal{C}$. If M is a sheaf of graded S-modules (right or left) then every subobject of M is equivalent to a graded submodule. If N, N' are graded submodules of M then $N \leq N'$ if and only if $N(p) \subseteq N'(p)$ for every $p \in \mathcal{C}$.

Corollary 10. *Let $(\mathcal{C}, J)$ be a small site, S a sheaf of $\mathbb{Z}$-graded rings and $\phi : M \longrightarrow N$ a morphism of sheaves of graded S-modules (right or left). Then ϕ is a monomorphism, epimorphism or isomorphism if and only if it has this property as a morphism of sheaves of S-modules (equivalently, as a morphism of sheaves of abelian groups). In particular*

- *ϕ is a monomorphism $\Leftrightarrow$ ϕ_p is injective for all $p \in \mathcal{C}$.*
- *ϕ is an epimorphism $\Leftarrow$ ϕ_p is surjective for all $p \in \mathcal{C}$.*
- *ϕ is an isomorphism $\Leftrightarrow$ ϕ_p is bijective for all $p \in \mathcal{C}$.*

Proof. This is immediate from the fact that the category of sheaves of graded S-modules is abelian (therefore balanced) and the kernels and cokernels agree with those of sheaves of modules, which agree with the kernels and cokernels for sheaves of abelian groups. $\square$

Lemma 11. *Let $(\mathcal{C}, J)$ be a small site, S a sheaf of $\mathbb{Z}$-graded rings and $\phi : M \longrightarrow N$ a morphism of sheaves of graded S-modules (right or left). Then*

- *ϕ is a monomorphism $\Leftrightarrow \phi_n : M_n \longrightarrow N_n$ is a monomorphism for all $n \in \mathbb{Z}$.*
- *ϕ is an epimorphism $\Leftrightarrow \phi_n : M_n \longrightarrow N_n$ is a epimorphism for all $n \in \mathbb{Z}$.*
- *ϕ is an isomorphism $\Leftrightarrow \phi_n : M_n \longrightarrow N_n$ is a isomorphism for all $n \in \mathbb{Z}$.*

Proof. The implications $\Leftarrow$ all follow from Corollary 10 and the fact that coproducts in $\mathbf{Ab}(Sh_J(\mathcal{C}))$ are exact. If ϕ is a monomorphism (resp. isomorphism) it is not difficult to check that ϕ_n is a monomorphism (resp. isomorphism) of sheaves of abelian groups. Suppose that ϕ is an epimorphism. Then the image $I \longrightarrow N$ is an isomorphism. But using (3) we see that $I_n \longrightarrow N_n$ is an isomorphism for $n \in \mathbb{Z}$. Since this is the image of $\phi_n : M_n \longrightarrow N_n$ it follows that ϕ_n is an epimorphism for all $n \in \mathbb{Z}$. □

References

[1] M. Artin, "Théorèmes de représentabilité pour les espaces algébriques", *Les Presses de l'université de Montreal* (1973).

[2] M. Artin, "Geometry of quantum planes", *Contemporary Mathematics* Vol.124 (1992).

[3] M. Artin, J. J. Zhang, "Noncommutative projective schemes", *preprint* (1994).

[4] F. Borceux, "Handbook of Categorical Algebra", *Cambridge University Press* Vol. 1-3 (1994).

[5] F. Borceux, B. Veit, "Subobject Classifier for Algebraic Structures", *Journal of algebra* Vol. 112 (1988), 306-314.

[6] F. Borceux, G. Van den Bossche, "Algebra in a Localic Topos with Applications to Ring Theory", *Springer-Verlag*, Lecture Notes in Mathematics No. 1038 (1983).

[7] P. Cartier, "A Mad Day's Work: From Grothendieck to Connes and Kontsevich, The Evolution of Concepts of Space and Symmetry", *Bulletin of the American Mathematical Society* Vol. 38 No. 4, 389-408.

[8] A. Connes, "Noncommutative Geometry", *Academic Press* (1994).

[9] A. Connes, "Noncommutative differential geometry and the structure of spacetime", *Deformation Theory and Symplectic Geometry*, Edited by Sternheimer et al. (1997), 1-33.

[10] V.G. Drinfeld, "Quantum Groups", *Proc. Int. Cong. Math.* (1986), 798-820.

[11] B. Eckmann, A. Schopf, "Über injektive Moduln", *Arch. Math.* Vol. 4 (1953), 75-78.

[12] R. Fossum, P. Griffith, I. Reiten "Trivial extensions of Abelian categories: homological algebra of trivial extensions of abelian categories with applications to ring theory", *Springer* (1975).

[13] D. Howe, "Module categories over topoi", *Journal of Pure and Applied Algebra* Vol. 21 (1981), 161-165.

[14] P. Gabriel, "Des catégories abéliennes", *Bull. Soc. Math. France* Vol. 90 (1962), 323-449.

[15] I.M. Gelfand, M.A. Naimark, "On the imbedding of normed rings into the ring of operators in Hilbert space", *Mat.Sb.* Vol. 12 (1943), 197-213.

[16] A. Grothendieck, "Sur quelques points d'algébre homologique", *Tohoku Math. J.* (2) Vol. 9 (1957), 119-221

[17] A. Grothendieck, "Elements de geometrie algebrique", *Springer* (1971).

[18] N. Popescu, P. Gabriel, "Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes", *C.R. Acad. Sci. Paris* 258 (1964), 4188-4190.

[19] R. Hartshorne, "Algebraic Geometry", *Springer*, Graduate Texts in Mathematics No. 52 (1977).

[20] S. Mac Lane "Duality for Groups", *Bull. Amer. Math. Soc.* Vol. 56 (1950), 485-516.

[21] S. Mac Lane, I. Moerdijk, "Sheaves in Geometry and Logic: A first introduction to topos theory", *Springer-Verlag* (1991).

[22] Yuri I. Manin, "Quantum Groups and Noncommutative Geometry", *Publications du C.R.M*, Univ. de Montreal (1988).

[23] Yuri I. Manin, "Topics in Noncommutative Geometry", *Princeton University Press* (1991).

[24] E. Mendelson, "Introduction to Mathematical Logic", *Chapman & Hall* (1997)

[25] B. Mitchell, "Theory of Categories", *Academic Press* (1965).

[26] B. Mitchell, "Rings with Several Objects", *Advances in Mathematics* Vol. 8 (1972), 1-161.

[27] B. Mitchell, "A Quick Proof of the Gabriel-Popesco Theorem", *Journal of Pure and Applied Algebra* Vol. 20 (1981), 313-315.

[28] F. Van Oystaeyen, L. Willaert, "Grothendieck topology, Coherent sheaves and Serre's theorem for schematic algebras", *J. Pure and Applied Algebra* Vol. 104 (1995), 109-122.

[29] A. L. Rosenberg, "Reconstruction of Schemes", *MPI* (1996).

[30] H. Schubert, "Categories", *Springer-Verlag* (1972).

[31] J.P. Serre, "Faisceaux algébriques cohérents", *Annals of Math.* Vol. 62 (1955).

[32] P. Smith, "Noncommutative Algebraic Geometry", *Online notes:* http://www.math.washington.edu/∼smith/Research/nag.pdf

[33] B. Stenström, "Rings of Quotients: An Introduction to Methods of Rings Theory", *Springer-Verlag* (1975).

[34] R. Street, "Ideals, Radicals, and Structure of Additive Categories", *Applied Categorical Structures* Vol. 3 (1995), 139-149.

[35] S. Sun, A. Dhillon, "Prime spectra of additive categories", *Journal of Pure and Applied Algebra* Vol. 122 (1997), 135-157.

[36] A.B. Verevkin, "On a noncommutative analogue of the category of coherent sheaves on a projective scheme", *Amer. Math. Soc. Transl. (2)* Vol.151 (1992).

[37] A.B. Verevkin, "Serre injective sheaves", *Math. Zametki* Vol. 52 (1992), 35-41.