Grothendieck Topologies on Quivers

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A *quiver* is a nonempty directed graph (all our quivers and graphs are finite). We refer to edges in a quiver as arrows. A *composite arrow* is a nonempty sequence of arrows in Q in which every arrow begins at the end of the arrow preceeding it. A *cycle* is a composite arrow which beings and ends at the same vertex. The *length* of a composite arrow is the number of involved arrows. In a quiver without cycles, a composite arrow involves no repeated vertices. If Q is a quiver, a *root* is a vertex r with no arrows ending at r. The set of root vertices is denoted $r(Q)$.

Lemma 1. *Let Q be a quiver without cycles. Then $r(Q)$ is nonempty.*

Proof. Associate to every vertex the integer which gives the maximum length of a composite arrow beginning at the vertex, which is finite since Q has no cycles (choose 0 if no such composite arrows exist). We can assume that some vertex is assigned a nonzero integer, since otherwise the result is trivial. If r is a vertex which maximises this integer, then r is a root vertex. $\square$

Given any arrow $a : v \longrightarrow w$ there is a composite arrow beginning at a root vertex and ending at w whose last arrow is a. So if v is any vertex which is not a root, there is a composite arrow beginning at a root vertex and ending at v. Thus the set $P(v)$ of all composite arrows beginning at a root vertex and ending at v is nonempty. The *distance* of a vertex v is the maximum of the lengths of composite arrows in $P(v)$.

Throughout Q will denote a quiver without cycles and $\mathcal{C}$ the path category of Q. For vertices v, w let $Q(v, w)$ be the set of arrows $v \longrightarrow w$ in Q and $\mathcal{C}(v, w)$ the morphism set (so if $v = w$ this consists of the empty path, and otherwise it is the set of all composite arrows beginning at v and ending at w). Since Q has no cycles, the category $\mathcal{C}$ has a finite number of morphisms.

Definition 1. A function J which assigns to every vertex v a sieve $J(v)$ at v in $\mathcal{C}$ is called *consistent* if for every vertex w either $J(w) = t_w$ or

$$J(v) = \bigcup_{\substack{v \in Q \\ a \in Q(v,w)}} aJ(v) \tag{1}$$

Here $aJ(v)$ denotes the sieve $\{af \mid f \in J(v)\}$ and t_w is the set of all morphisms in $\mathcal{C}$ with codomain w. If the index set in (1) is empty then the union is the empty set.

Proposition 2. *Let Q be a quiver without cycles. If Q has m vertices, then there are 2^m consistent assignments of sieves to vertices in Q.*

Proof. We set up a bijection between functions $Q \longrightarrow \{0, 1\}$ and consistent assignments of sieves. Let $g : Q \longrightarrow \{0, 1\}$ be given. For each root vertex r define

$$J_g(r) = \begin{cases} \{1_r\} & \text{if } g(r) = 0 \\ \{\} & \text{if } g(r) = 1 \end{cases}$$

If w is a vertex of distance 1, then the only arrows ending at w are of the form $a : r \longrightarrow w$ where r is a root vertex, so we can define

$$J_g(w) = \begin{cases} t_w & \text{if } g(w) = 0 \\ \bigcup_{\substack{v \in Q \\ a \in Q(v,w)}} aJ_g(v) & \text{if } g(w) = 1 \end{cases} \tag{2}$$

Once we have defined J_g on the vertices of distance n, we use (2) to extend J_g to the vertices of distance $n+1$, and in this way we define J_g on all of Q. It is clear that J_g is consistent.

Conversely, given a consistent assignment J define

$$g_J(w) = \begin{cases} 0 & \text{if } J(w) = t_w \\ 1 & \text{otherwise} \end{cases}$$

Next we show that $J_{g_J} = J$ and $g_{J_g} = g$. Firstly, if r is a root vertex then there are only two sieves at r: the empty sieve $\{\}$ and $t_r = \{1_r\}$. So it is clear that $J_{g_J}(r) = J(r)$. Then by induction on the distance of vertices we see that $J_{g_J}(w) = J(w)$ for all vertices w.

Notice that in (2) if $g(w) = 1$ then $J_g(w)$ is a proper sieve. So $g(w) = 0$ iff. $J_g(w) = t_w$ iff. $g_{J_g}(w) = 0$, so $g = g_{J_g}$, completing the proof. □

Theorem 3. *There is a bijection between Grothendieck topologies on $\mathcal{C}$ and consistent assignments of sieves to vertices.*

Proof. Let G be a Grothendieck topology on $\mathcal{C}$. For every vertex v the covers at v are closed under finite intersections, so

$$G(v) = \{T \mid T \text{ is a sieve and } T \supseteq J_G(v)\}$$

where $J_G(v)$ is the intersection of all covers at v. We show that J_G is a consistent assignment of sieves. If r is a root vertex then either $G(r) = \{t_r\}$ in which case $J_G(r) = t_r$, or $G(r) = \{\{\}, t_r\}$ in which case $J_G(r) = \{\}$, so the consistency condition is satisfied for root vertices. Let w be any non-root vertex. If $G(w) = \{t_w\}$ then $J_G(w) = t_w$ and there is no problem. Otherwise, denote the sieve $\bigcup_{v \in Q, a \in Q(v,w)} aJ_G(v)$ by T, and notice that since $J_G(w)$ is a cover the pullback property gives $J_G(w) \supseteq T$. So to show the consistency condition holds for w, it suffices to show that T is a cover at w. For this we use the transitivity condition, and show that for all $f \in J_G(w)$ the pullback f^*T is a cover.

Since $J_G(w)$ is proper, we can assume that f is a composite arrow ending in $b : v \longrightarrow w$. If $a : q \longrightarrow w$ is any other arrow ending at w then $f^*(aJ_G(q)) = \{\}$, so

$$f^*T = \bigcup_{\substack{q \in Q \\ a \in Q(q,w)}} f^*\left(aJ_G(q)\right) = f^*\left(bJ_G(v)\right) = (f-b)^* J_G(v)$$

where $f - b$ denotes the composite arrow obtained by deleting b off the end (with $f - b = 1_v$ if $f = b$). Hence f^*T is a cover and J_G is a consistent assignment of sieves.

In the other direction, suppose we are given a consistent assignment of sieves J, and define

$$G_J(v) = \{T \mid T \text{ is a sieve and } T \supseteq J(v)\}$$

Clearly $G_J(v)$ contains the improper sieve for all v. Notice that since J is consistent, for any arrow $a : v \longrightarrow w$ we have $a^*J(w) \supseteq J(v)$. It follows that $f^*J(w) \supseteq J(v)$ for any morphism $f : v \longrightarrow w$ of $\mathcal{C}$, which establishes the pullback property for G_J.

To check transitivity, let a vertex w be given, $T \supseteq J(w)$ and S a sieve at $J(w)$ such that f^*S is a cover for all $f \in T$. If $J(w) = t_w$ or $J(w) = \{\}$ then S is trivially a cover. So we may assume $J(w)$ is proper and nonempty (hence w is not a root vertex) in which case by the consistency condition

$$J(w) = \bigcup_{\substack{v \in Q \\ a \in Q(v,w)}} aJ(v) \tag{3}$$

To show that S is a cover, we must show that $S \supseteq J(w)$. Let $f \in J(w)$ be given. It suffices to show that $f = hg$ for some $h : v \longrightarrow w \in T$ and $g \in J(v)$, since then $h^*S \supseteq J(v)$ implies that $f \in S$. Using (3) there is an arrow $a_1 : v_1 \longrightarrow w$ and $g_1 \in J(v_1)$ with $f = a_1 g_1$. If $J(v_1) = t_{v_1}$ then by consistency $a_1 \in J(w) \subseteq T$ and we are done. So we can assume that the equation (1) holds for $J(v_1)$, and there is an arrow $a_2 : v_2 \longrightarrow v_1$ and $g_2 \in J(v_2)$ with $g_1 = a_2 g_2$. Once again we are done if $J(v_2) = t_{v_2}$.

Proceeding in this way, we must eventually end up with $g_n = 1_{v_n}$, since there are only a finite number of arrows in f. But then $1_{v_n} \in J(v_n) \subseteq f^*S$ implies $f \in S$, as required. Hence G_J is a Grothendieck topology. It is easily checked that $G_{J_G} = J$ and $J_{G_J} = J$, so we have established the required bijection. $\square$

Corollary 4. *Let Q be a quiver without cycles with m vertices. If $\mathcal{C}$ is the path category of Q, then there are 2^m Grothendieck topologies on $\mathcal{C}$. Explicitly, there is a bijection between topologies G and subsets of the vertices of Q, given by*

$$S_G = \{v \in Q \mid G(v) = \{t_v\}\}$$