Graded Rings and Modules

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In this note we develop the basic theory of graded rings and modules, at least as far as we need it for the theory of projective spaces in algebraic geometry. All of this material can be found in Grothendieck's EGA. All rings in this note are commutative.

Contents

1 Definitions

Definition 1. A *graded ring* is a ring S together with a set of subgroups $S_d, d \geq 0$ such that $S = \bigoplus_{d \geq 0} S_d$ as an abelian group, and $st \in S_{d+e}$ for all $s \in S_d, t \in S_e$. One can prove that $1 \in S_0$ and if S is a domain then any unit of S also belongs to S_0. A *homogenous ideal* of S is an ideal $\mathfrak{a}$ with the property that for any $f \in \mathfrak{a}$ we also have $f_d \in \mathfrak{a}$ for all $d \geq 0$. A *morphism* of graded rings is a morphism of rings which preserves degree.

Proposition 1. *Let S be a graded ring, and $T \subseteq S$ a multiplicatively closed set. A homogenous ideal maximal among the homogenous ideals not meeting T is prime.*

Proof. Let T be multiplicatively closed and suppose the set Z of all homogenous ideals with $\mathfrak{a} \cap T = \emptyset$ is nonempty. Let $\mathfrak{p}$ be maximal in this set with respect to inclusion. It suffices to show that if $a, b \in S$ are homogenous with $ab \in \mathfrak{p}$ then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. If neither belong to $\mathfrak{p}$ then $(a) + \mathfrak{p}, (b) + \mathfrak{p}$ are homogenous ideals properly containing $\mathfrak{p}$, and thus both must intersect T. Say $t = am+p, t' = bn+q$ with $t, t' \in T$ and $p, q \in \mathfrak{p}$. Then $tt' \in T$, but $tt' = ambn+amq+pbn+pq \in \mathfrak{p}$ since $ab \in \mathfrak{p}$. This contradiction shows that $\mathfrak{p}$ is a homogenous prime ideal. ∎

Corollary 2. *Let S be a graded ring, and $T \subseteq S$ a multiplicatively closed set. A homogenous ideal $\mathfrak{a}$ not meeting T can be expanded to a homogenous prime ideal $\mathfrak{p}$ not meeting T.*

In particular any proper homogenous ideal is contained in a homogenous prime ideal, maximal with respect to inclusion in other proper homogenous ideals.

Corollary 3. *In a graded ring S the radical of a homogenous ideal $\mathfrak{a}$ is the intersection of all* homogenous *prime ideals containing $\mathfrak{a}$.*

If S is a graded ring and $f, g \in S$ then it is easy to see that for $d \geq 0$

$$(fg)_d = \sum_{i=0}^{d} f_i g_{d-i} = \sum_{\substack{i,j \geq 0 \\ i+j=d}} f_i g_j$$

As usual the *degree* of an element $f \in S$ is the largest integer $d \geq 0$ with $f_d \neq 0$ (the degree of 0 isn't defined). Provided $f, g, fg \neq 0$ we have $deg(fg) \leq deg(f) + deg(g)$. For a graded domain it is clear that $deg(fg) = deg(f) + deg(g)$ for nonzero f, g.

Lemma 4. *In a graded domain S if f divides a nonzero homogenous element g then f is also homogenous.*

Proof. Let g be a nonzero homogenous element of degree $d \geq 0$ and suppose $fh = g$. If $d = 0$ then $deg(f) + deg(h) = 0$ so $deg(f) = 0$ and so f is homogenous. Now suppose $d > 0$ and let $m = deg(f)$ and $n = deg(h)$ so $m + n = d$. Let $0 \leq i \leq n$ be the smallest integer with $h_i \neq 0$ and $0 \leq j \leq m$ the smallest integer with $f_j \neq 0$. Then $f_j h_i = (fh)_{i+j} = g_{i+j}$. If $i + j < d$ then we would have $f_j h_i = 0$, which is a contradiction, so $i + j = d = m + n$ and so $i = n, j = m$, which shows that f and h are both homogenous. □

For example if $S = k[x_1, \ldots, x_n]$ for a field k, let $f \in S$ be a nonzero polynomial with factorisation $f = ug_1 \cdots g_r$ into a product of a unit u and irreducible polynomials g_i. Then the Lemma implies that if f is homogenous then so are all of the g_i.

Definition 2. If S is a graded ring then a *graded S-module* is an S-module M together with a set of subgroups $M_n, n \in \mathbb{Z}$ such that $M = \bigoplus_{n \in \mathbb{Z}} M_n$ as an abelian group, and $sm \in M_{n+d}$ for all $s \in S_d, m \in M_n$. A *morphism* of graded S-modules is a morphism of modules which preserves degree. We denote the category of graded S-modules by S**GrMod**, and let $Hom(M, N)$ be the set of graded morphisms $M \longrightarrow N$. If $\phi : M \longrightarrow N$ is a morphism of graded S-modules then $\phi_n : M_n \longrightarrow N_n$ denotes the induced morphism of abelian groups for $n \in \mathbb{Z}$.

A *graded submodule* of a graded module M is a graded module N which is a subset of M and for which the inclusion $N \subseteq M$ is a morphism of graded modules. That is, N is an S-submodule of M with the property that for any $m \in N$ we also have $m_n \in N$ for all $n \in \mathbb{Z}$.

Lemma 5. *Let S be a graded ring and M a graded S-module. An S-submodule $N \subseteq M$ is a graded submodule if and only if it is generated as an S-module by a nonempty set of homogenous elements of M.*

Proof. If $N \subseteq M$ is a graded submodule, it is generated by the set of all its homogenous elements. For the converse, let $\{m_i\}_{i \in I}$ be a nonempty set of homogenous elements of M that generate N as an S-module. Let $n \in N$ be given, and write

$$n = s^1 m_1 + \cdots + s^n m_n$$

If m_i is homogenous of degree d_i for $1 \leq i \leq n$ then for $d \in \mathbb{Z}$ we have

$$\begin{aligned} n_d &= (s_1 m_1)_d + \cdots + (s_n m_n)_d \\ &= s^1_{d-d_1} m_1 + \cdots + s^n_{d-d_n} m_n \end{aligned}$$

Hence $n_d \in N$, as required. □

Definition 3. Let S be a graded ring, M a graded S-module. For $n \in \mathbb{Z}$ we denote by $M(n)$ the graded S-module which has the same underlying S-module structure as M, but whose grading is defined by $M(n)_i = M_{n+i}$. If $\phi : M \longrightarrow N$ is a morphism of graded S-modules let $\phi(n)$ denote the same function, which is a morphism of graded S-modules $\phi(n) : M(n) \longrightarrow N(n)$. This defines a functor $-(n) : S$**GrMod** $\longrightarrow S$**GrMod**.

2 Homogenous Localisation

Definition 4. Let S be a graded ring, $\mathfrak{p}$ a homogenous prime ideal. Let T be the set of all *homogenous* elements of $S \setminus \mathfrak{p}$. Let $S_{(\mathfrak{p})}$ denote the subring of $T^{-1}S$ consisting of all f/g where f, g are homogenous of the same degree. This is a local ring with maximal ideal $T^{-1}\mathfrak{p} \cap S_{(\mathfrak{p})}$.

If M is a graded S-module then let $M_{(\mathfrak{p})}$ denote the subgroup of $T^{-1}M$ consisting of all m/s where m, s are homogenous of the same degree. This becomes an $S_{(\mathfrak{p})}$-module with the obvious action. If $\phi : M \longrightarrow N$ is a morphism of graded S-modules then $m/s \mapsto \phi(m)/s$ defines a morphism of $S_{(\mathfrak{p})}$-modules $M_{(\mathfrak{p})} \longrightarrow N_{(\mathfrak{p})}$ and this defines an additive functor

$$(-)_{(\mathfrak{p})} : S\mathbf{GrMod} \longrightarrow S_{(\mathfrak{p})}\mathbf{Mod}$$

Definition 5. Let S be a graded ring and $f \in S_d$ a homogenous element of degree $d > 0$. Then S_f is a $\mathbb{Z}$-graded ring where for $n \in \mathbb{Z}$ the subgroup $(S_f)_n$ consists of all s/f^m where s is homogenous of degree $n + md$. The subring of degree zero elements is denoted $S_{(f)}$. If M is a graded S-module then $M_{(f)}$ is the subgroup of M_f consisting of all m/f^k where m is homogenous of degree kd. This becomes a $S_{(f)}$-module in the obvious way. If $\phi : M \longrightarrow N$ is a morphism of graded S-modules then $m/f^k \mapsto \phi(m)/f^k$ defines a morphism of $S_{(f)}$-modules $M_{(f)} \longrightarrow N_{(f)}$, and this defines an additive functor

$$(-)_{(f)} : S\mathbf{GrMod} \longrightarrow S_{(f)}\mathbf{Mod}$$

Observe that for $n \in \mathbb{Z}$ we have an *equality* of $S_{(f)}$-modules $S(n)_{(f)} = (S_f)_n$, where the former module is the functor applied to $S(n)$ and the latter is the degree n subgroup of the $\mathbb{Z}$-graded ring S_f, which is a module of the degree zero subring in the usual way.

Lemma 6. *Let S be a graded ring, and suppose we have an exact sequence of graded S-modules $M \longrightarrow N \longrightarrow L$. Then for any homogenous prime $\mathfrak{p}$ the sequence $M_{(\mathfrak{p})} \longrightarrow N_{(\mathfrak{p})} \longrightarrow L_{(\mathfrak{p})}$ of $S_{(\mathfrak{p})}$-modules is exact.*

Proof. Let $\phi : M \longrightarrow N, \psi : N \longrightarrow L$ be morphisms of graded S-modules forming an exact sequence. Clearly $Im(\phi_{(\mathfrak{p})}) \subseteq Ker(\psi_{(\mathfrak{p})})$. On the other hand, suppose n, t are homogenous of the same degree k with $n \in N, t \notin \mathfrak{p}$ and $\psi(n)/t = \psi_{(\mathfrak{p})}(n/t) = 0$. Then $q\psi(n) = 0$ for some homogenous $q \notin \mathfrak{p}$, say of degree j. Thus $\psi(qn) = 0$ and so $qn = \phi(m)$ for some homogenous $m \in M$ of degree $j + k$. Then in $N_{(\mathfrak{p})}$ we have $n/t = qn/tq = \phi(m)/tq = \phi_{(\mathfrak{p})}(m/tq)$ so the sequence is exact, as required. □

3 Properties of Graded Rings

Lemma 7. *Let S be a graded ring. A nonempty set E of homogenous elements of S_+ generates S as an S_0-algebra if and only if E generates S_+ as an S-module.*

Proof. The implication ($\Rightarrow$) is trivial. For the implication ($\Leftarrow$) let F be the S_0-subalgebra of S generated by E. We show by induction that $S_n \subseteq F$ for all $n \geq 0$. The case $n = 0$ is trivial. Assume $n > 0$ and that the claim is true for all $j < n$. If $a \in S_n$ then for $e_i \in E$ and homogenous $s_i \in S$ we can write

$$a = s_1 e_1 + \cdots + s_n e_n$$

Since the s_i must all be of degree $< n$ they must belong to F by the inductive hypothesis, and therefore so does a. Therefore $F = S$ and the proof is complete. □

Corollary 8. *Let S be a graded ring. The ideal S_+ is finitely generated if and only if S is a finitely generated S_0-algebra.*

Proposition 9. *Let S be a graded ring. Then S is noetherian if and only if S_0 is a noetherian ring and S is finitely generated as an S_0-algebra.*

Proof. See Atiyah & Macdonald Proposition 10.7 or use the above results. □

Definition 6. Let S be a graded ring and $d > 0$. We denote by $S^{[d]}$ the subring $\oplus_{m \geq 0} S_{md}$ of S with vanishing graded pieces in degrees not divisible by d. So $S^{[d]}_0 = S_0, S^{[d]}_d = S_d, \ldots$ but all other degrees are zero. We denote by $S^{(d)}$ the same ring with the grading $S^{(d)}_m = S_{md}$ for $m \geq 0$. These are both graded rings. If $\phi : S \longrightarrow T$ is a morphism of graded rings, then the restriction gives morphisms of graded rings $\phi^{[d]} : S^{[d]} \longrightarrow T^{[d]}$ and $\phi^{(d)} : S^{(d)} \longrightarrow T^{(d)}$, so this construction is functorial.

Definition 7. Let S be a graded ring and M a graded S-module. For $d > 0$ and $0 \leq k \leq d - 1$ denote by $M^{(d,k)}$ the submodule $\oplus_{n \in \mathbb{Z}} M_{nd+k}$ of M. This is a graded $S^{(d)}$-module with the graded piece of degree n being M_{nd+k}. We write $M^{(d)}$ for $M^{(d,0)}$.

Lemma 10. *Let S be a graded ring generated by S_1 as an S_0-algebra. Then for any $d > 0$, $S^{(d)}$ is generated by $S^{(d)}_1$ as an $S^{(d)}_0$-algebra. If S is* finitely *generated by S_1 as an S_0-algebra then $S^{(d)}$ is finitely generated by $S^{(d)}_1$ as an $S^{(d)}_0$-algebra.*

Proof. By assumption elements of $S^{(d)}_n = S_{nd}$ can be written as linear combinations of degree nd monomials in elements of S_1 with coefficients from S_0. But any degree nd monomial can be written as the product of n degree d monomials, so it is not hard to see that $S^{(d)}$ has the desired property. If S is finitely generated by S_1 as an S_0-algebra, say by $f_1, \ldots, f_t$, then since there are only a finite number of degree d monomials in the f_i it is clear that $S^{(d)}$ has the desired property. ☐

Lemma 11. *Let S be a graded ring which is a finitely generated S_0-algebra. Let M be a finitely generated graded S-module. Then*

(i) M_n is a finitely generated S_0-module, and there exists $n_0 \in \mathbb{Z}$ such that $M_n = 0$ for all $n \leq n_0$.

(ii) There exists $n_1 > 0$ and $h > 0$ such that, for any $n \geq n_1$ we have $M_{n+h} = S_h M_n$.

(iii) For any pair of integers (d, k) with $d > 0$ and $0 \leq k < d - 1$, $M^{(d,k)}$ is a finitely generated $S^{(d)}$-module. In particular $M^{(d)}$ is a finitely generated $S^{(d)}$-module.

(iv) For any $d > 0$, $S^{(d)}$ is a finitely generated S_0-algebra.

(v) There exists an integer $h > 0$ such that $S_{mh} = (S_h)^m$ for all $m > 0$.

Proof. Suppose S is generated as an S_0-algebra by homogenous elements f_i of degrees $h_i \geq 1$ $(1 \leq i \leq r)$, and M is generated as an S-module by homogenous elements x_j of degrees k_j $(1 \leq j \leq s)$. It is clear that nonzero elements of M_n consist of linear combinations, with coefficients in S_0, of elements $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ where the $\alpha_i \geq 0$ are such that $k_j + \sum_i \alpha_i h_i = n$. Since there are only a finite number of these monomials in the f_i, this proves the first part of (i). To prove the second part, take n_0 smaller than all the k_j. (ii) Let $h \geq 1$ be a common multiple of all the h_i and set $g_i = f_i^{h/h_i}$ for $1 \leq i \leq r$. Clearly the g_i are all of degree h. Let Z be the set of elements of M of the form $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leq \alpha_i < h/h_i$ for $1 \leq i \leq r$ and $0 \leq j \leq s$. Then Z is a finite set and we let $n_1 \geq 1$ be larger than the degree of every element of Z. If $n \geq n_1$ then any element of M_{n+h} can be written as an S_0-linear combination of elements $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ as above. Given such an element, write $\alpha_i = (h/h_i)\beta_i + \gamma_i$ with $\gamma_i < h/h_i$ and consider

$$f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j = g_1^{\beta_1} \cdots g_r^{\beta_r} (f_1^{\gamma_1} \cdots f_r^{\gamma_r} x_j) = b_1 \cdots b_q z$$

where every b_i is homogenous of degree h and $z \in Z$ is homogenous of some degree t, and the last expression makes sense because $n + h > n_1 \geq t$ (so we must have extracted at least one g_i). Now it is clear that this element belongs to $S_h M_n$, and therefore $M_{n+h} = S_h M_n$.

(iii) For $d > 0$ it is easy to see that any nonzero element of $M^{(d,k)}$ is an S_0-linear combination of elements of the form $g^d f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leq \alpha_i < d$ and g a homogenous element of S. Therefore the finite set of elements $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leq \alpha_i < d$ and $\sum_i \alpha_i h_i + k_j - k$ divisible by d form a set of generators for the $S^{(d)}$-module $M^{(d,k)}$.

(iv) Set $M = S_+$. This is a finitely generated, graded S-module with $M^{(d)} = (S^{(d)})_+$ as $S^{(d)}$-modules. Therefore by (iii), $(S^{(d)})_+$ is a finitely generated $S^{(d)}$-module. Then Lemma 7 implies that $S^{(d)}$ is a finitely generated S_0-algebra.

(iv) In (ii) we can assume that $h \geq n_1$ (since the proof works with h *any* common multiple of the h_i). And therefore with $S = M$ we have $S_{h+h} = S_h S_h$ and inductively $S_{mh} = (S_h)^m$ for all $m > 0$. □

Corollary 12. *If S is a noetherian graded ring, then so is $S^{(d)}$ for any $d > 0$.*

Proof. This is a result of Lemma 11 (iv) and Proposition 9. □

Corollary 13. *Let S be a graded ring which is a finitely generated S_0-algebra. Then for some $d > 0$ the graded ring $S^{(d)}$ is finitely generated by $S^{(d)}_1$ as an $S^{(d)}_0$-algebra.*

Proof. From Lemma 11(v) let $d > 0$ be such that $S_{md} = (S_d)^m$ for all $m > 0$. By Lemma 11(i) we can find a finite generating set $f_1, \ldots, f_s$ for S_d as an S_0-module. Then it is clear that the f_i generate $S^{(d)}$ as an $S_0 = S^{(d)}_0$-algebra, as required. □

Definition 8. Let S be a graded ring. We say S is *essentially reduced* if the ideal S_+ contains no nonzero nilpotents, and we say S is *essentially integral* if S_+ contains no nonzero zero-divisors. Essentially integral implies essentially reduced. If $\mathfrak{p}$ is a homogenous prime ideal then $S/\mathfrak{p}$ is clearly an essentially integral graded ring.

Proposition 14. *Let S be a graded ring, $d > 0$ an integer and $f \in S_d$. Then there is a canonical ring isomorphism $S_{(f)} \cong S^{(d)}/(f-1)S^{(d)}$. If we identify these rings by this isomorphism, then for a graded S-module M there is a canonical isomorphism of modules $M_{(f)} \cong M^{(d)}/(f-1)M^{(d)}$.*

Proof. We define the morphism as follows: an element x/f^n with $x \in S_{nd}$, is mapped to $x + (f - 1)S^{(d)}$. One checks that this is a well-defined morphism of rings. If $0 \neq x \in S_{nd}$ and $x = (f-1)y$ with $y = y_{hd} + y_{(h+1)d} + \cdots + y_{kd}$ with $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$ then necessarily $h = n$ and $x = -y_{hd}$. Then the relations $y_{(j+1)d} = f y_{jd}$ for $h \leq j \leq k-1$, $f y_{kd} = 0$ show that $f^{k-n}x = 0$. Therefore the map is injective. If $x \in S_{nd}$ then $x + (f-1)S^{(d)}$ is the image of x/f^n, so the map is an isomorphism. One proceeds in exactly the same way for M. □

Corollary 15. *Let S be a noetherian graded ring. Then the ring $S_{(f)}$ is noetherian for any homogenous $f \in S_+$ and if M is a finitely generated graded S-module, then $M_{(f)}$ is a finitely generated $S_{(f)}$-module.*

Proof. It follows from Proposition 14 and Corollary 12 that $S_{(f)}$ is noetherian. By Proposition 9, S is finitely generated as an S_0-algebra, so using Lemma 11 we see that $M^{(d)}$ is a finitely generated $S^{(d)}$-module. Therefore $M_{(f)}$ is a finitely generated $S_{(f)}$-module by Proposition 14. □

Lemma 16. *Let S be a graded ring, $\mathfrak{a}$ a homogenous ideal and $\mathfrak{p}$ a homogenous prime ideal of S with $\mathfrak{p} \not\supseteq S_+$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ if and only if there exists an integer $d_0 \geq 0$ with $\mathfrak{a}_d \subseteq \mathfrak{p}_d$ for all $d \geq d_0$.*

Proof. One implication is clear. We prove the other by recursion on d_0. If $d_0 = 0$ we are done. For the recursive step we assume $d_0 > 0$ and that $\mathfrak{a}_d \subseteq \mathfrak{p}_d$ for all $d \geq d_0$, and we show that $\mathfrak{a}_{d_0-1} \subseteq \mathfrak{p}$. Since $\mathfrak{p} \not\supseteq S_+$ there is some $e > 0$ and $f \in S_e$ with $f \notin \mathfrak{p}$. If $a \in \mathfrak{a}_{d_0-1}$ then $af \in \mathfrak{a}_{d_0-1+e} \subseteq \mathfrak{p}$. Since $f \notin \mathfrak{p}$ we have $a \in \mathfrak{p}$ as required. □

Lemma 17. *Let S be a graded ring. If $\mathfrak{p}, \mathfrak{q}$ are homogenous prime ideals with $\mathfrak{p} \not\supseteq S_+$ and $\mathfrak{q} \not\supseteq S_+$ then we have $\mathfrak{p} = \mathfrak{q}$ if and only if there exists an integer $d_0 \geq 0$ with $\mathfrak{p}_d = \mathfrak{q}_d$ for all $d \geq d_0$.*

In the next result result we make use of the following observations. Let S be a graded ring and $\mathfrak{p}$ a homogenous ideal of S with $\mathfrak{p} \not\supseteq S_+$. If $f \in S_+$ is not contained in $\mathfrak{p}$ then for any $x \in S$, $f^n x \in \mathfrak{p}$ is equivalent to $x \in \mathfrak{p}$. In particular if $f \in S_d$ for $d > 0$ and $x \in S_{m-nd}$ with $m \geq nd$ then $f^n x \in \mathfrak{p}_m$ if and only if $x \in \mathfrak{p}_{m-nd}$.

Proposition 18. *Let S be a graded ring. Suppose we are given an integer $n_0 > 0$ and for every $n \geq n_0$ a subgroup $\mathfrak{p}_n$ of S_n with the following properties*

1. $S_m \mathfrak{p}_n \subseteq \mathfrak{p}_{m+n}$ *for all* $m \geq 0$ *and* $n \geq n_0$.
2. *For* $m \geq n_0, n \geq n_0, f \in S_m, g \in S_n$ *the relation* $fg \in \mathfrak{p}_{m+n}$ *implies* $f \in \mathfrak{p}_m$ *or* $g \in \mathfrak{p}_n$.
3. $\mathfrak{p}_n \neq S_n$ *for at least one* $n \geq n_0$.

There exists a unique *homogenous prime ideal* $\mathfrak{p}$ *of* S *with* $\mathfrak{p} \not\supseteq S_+$ *and* $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ *for all* $n \geq n_0$.

Proof. Given subgroups $\mathfrak{p}_n$ for $n \geq n_0$ satisfying $1, 2, 3$ we construct a homogenous prime ideal $\mathfrak{p}$ of S with $\mathfrak{p} \not\supseteq S_+$ and $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ for all $n \geq n_0$. It follows from Lemma 16 that if such a prime exists, it is unique. Throughout the proof we fix some $f \notin \mathfrak{p}$ with $f \in S_d$ and $d \geq n_0$. For an integer $m < n_0$ we define recursively (i.e. beginning with $n_0 - 1$)

$$\mathfrak{p}_m = \{x \in S_m \,|\, f^r x \in \mathfrak{p}_{m+rd} \text{ for for all } r > 0\} \tag{1}$$

One checks recursively that each $\mathfrak{p}_m$ is a subgroup of S_n, and it follows from $1, 2$ that for $m \geq n_0$ the definition in (1) agrees with the given subgroup $\mathfrak{p}_m$. We define

$$\mathfrak{p} = \sum_{n=0}^{\infty} \mathfrak{p}_n$$

One checks recursively that if $g \in S_m$ and $x \in \mathfrak{p}_n$ then $gx \in \mathfrak{p}_{m+n}$, so $\mathfrak{p}$ is a homogenous ideal with $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ for all $n \geq 0$. To show that $\mathfrak{p}$ is a homogenous prime ideal, we prove the following statement by "reverse" induction on the integer $N = min\{m, n\}$

> Let $x \in S_m, y \in S_n$ with $m, n \geq 0$ be such that $xy \in \mathfrak{p}_{m+n}$. Then $x \in \mathfrak{p}_m$ or $y \in \mathfrak{p}_n$.

If $N \geq n_0$ this follows from condition 2. So assume that $0 \leq N < n_0$ and that the statement is true for $m, n > N$. Suppose that $xy \in \mathfrak{p}_{m+n}$ but $x \notin \mathfrak{p}_m$, so that $f^r x \notin \mathfrak{p}_{m+rd}$ for some $r > 0$. Given $s > 0$ observe that $f^s y f^r x = f^{r+s} xy \in \mathfrak{p}_{rd+sd+m+n}$. But the elements $f^r x, f^s y$ are homogenous of degrees greater than N, so it follows from the inductive hypothesis that $f^s y \in \mathfrak{p}_{n+sd}$. This shows that $y \in \mathfrak{p}_n$, as required. We have now constructed a homogenous prime ideal $\mathfrak{p}$ with the necessary properties (clearly $\mathfrak{p} \not\supseteq S_+$ since otherwise we contradict 3). □

4 The Category of Graded Modules

If we define addition of morphisms in the usual pointwise manner, then S**GrMod** is a preadditive category with zero object given by the zero module $\{0\}$ which is trivially graded.

Lemma 19 (Graded Yoneda Lemma)**.** *Let S be a graded ring, M a graded S-module. Then for any $n \in \mathbb{Z}$ there is an isomorphism of groups:*

$$\gamma : Hom(S(n), M) \longrightarrow M_{-n}$$

Moreover this isomorphism is natural in M.

Proof. Given a morphism of graded S-modules $f : S(n) \longrightarrow M$ we define $\gamma(f) = f(1)$. Clearly $f(1) \in M_{-n}$ so this is well-defined morphism of groups. A morphism $S(n) \longrightarrow M$ is in particular a morphism of S-modules, so by the usual Yoneda lemma the map γ is injective. To see it is surjective, let $m \in M_{-n}$ be given and let $f : S \longrightarrow M$ be the morphism of S-modules mapping 1 to m. It is easily checked that f gives a morphism of graded S-modules $S(n) \longrightarrow M$, so γ is surjective. Naturality with respect to a morphism of graded modules $M \longrightarrow M'$ is easily checked. □

If N is a graded submodule of M, then the S-module M/N is a graded S-module with grading $(M/N)_n = M_n + N$, and the morphism $M \longrightarrow M/N$ is a morphism of graded modules. If $f : M \longrightarrow N$ is a morphism of graded modules, then the image $f(M)$ is a graded submodule of N.

Lemma 20. *For any graded ring S the following statements hold for a morphism $f : M \longrightarrow N$ of S***GrMod***:*

(i) f is a monomorphism $\Leftrightarrow$ f is injective $\Leftrightarrow$ $f_n : M_n \longrightarrow N_n$ is injective for all $n \in \mathbb{Z}$.

(ii) f is an epimorphism $\Leftrightarrow$ f is surjective $\Leftrightarrow$ $f_n : M_n \longrightarrow N_n$ is surjective for all $n \in \mathbb{Z}$.

(iii) The set $K = \{m \,|\, f(m) = 0\}$ is a graded submodule of M, and $K \longrightarrow M$ is the kernel of f.

(iv) The morphism $N \to N/f(M)$ is the cokernel of f.

(v) The morphism $f(M) \longrightarrow N$ is the image of f.

Proof. (i) It is clear that an injection is a monomorphism. Conversely, suppose $f : M \longrightarrow N$ is a monomorphism of graded S-modules and that $f(x) = f(y)$ for $x, y \in M$. It follows that $f(x_n) = g(y_n)$ for all $n \in \mathbb{Z}$, so we can reduce to the case where x, y are homogenous of common degree n. Let $x', y' : S(-n) \longrightarrow M$ be the morphisms corresponding to x, y respectively. Then $fx' = fy'$ implies $x' = y'$ and thus $x = y$. Hence f is injective. It is clear that f is injective iff. f_n is injective for every n.

(ii) It is clear that a surjection is an epimorphism. Conversely, suppose $f : M \longrightarrow N$ is an epimorphism. The image of f is a graded submodule of N, so we can form the graded module $N/f(M)$. Let $g, h : N \longrightarrow N/f(M)$ be the projection and zero morphism respectively. Then $gf = hf$ implies that $g = h$, so $f(M) = N$ and so f is surjective. It is clear that f is surjective iff. f_n is surjective for every n.

(iii) and (iv) are easily checked. □

Every subobject of a graded module M in S**GrMod** is equivalent to the inclusion of a graded submodule, and every quotient object is equivalent to the quotient M/N for some graded submodule N. So it is clear that S**GrMod** is normal, conormal, has kernels and cokernels and epi-mono factorisations. Next we show that S**GrMod** has finite products. Since S**GrMod** has a zero object, we need only consider a nonempty finite family of graded S-modules $M_1, \ldots, M_n$. Take the product $\bigoplus_{i=1}^n M_i$ of S-modules and give it the following grading:

$$\left(\bigoplus_{i=1}^n M_i \right)_k = \{(m_1, \ldots, m_n) \,|\, m_i \in (M_i)_k \;\forall\, 1 \le i \le n\}$$

So $(m_1, \ldots, m_n)_k = ((m_1)_k, \ldots, (m_n)_k)$. It is easy to check that with this definition $\bigoplus_{i=1}^n M_i$ is a graded S-module, and is in fact a product in the category S**GrMod** with the canonical projections. More generally if $\{M_i\}_{i \in I}$ is any nonempty family of graded S-modules, the direct sum $\bigoplus_{i \in I} M_i$ is a graded S-module with the above grading, and the canonical injections make it into a coproduct in the category S**GrMod**.

Infinite products in S**GrMod** have to be approached a little differently. Let $\{M_i\}_{i \in I}$ be any nonempty family of graded S-modules. For $n \in \mathbb{Z}$ define $T_n = \prod_{i \in I} (M_i)_n$ as an abelian group. For $s \in S_d$ we define a morphism of abelian groups $s : T_n \longrightarrow T_{n+d}$ for every $n \in \mathbb{Z}$ by $(q_i) \mapsto (sq_i)$. Let T be the abelian group $\bigoplus_{n \in \mathbb{Z}} T_n$ and for $s \in S$ and $m \in T$ define

$$(s \cdot m)_n = \sum_{\substack{d \ge 0, j \in \mathbb{Z} \\ d+j=n}} s_d \cdot m_j$$

It is not hard to check with this definition that T is a graded S-module with T_n being the subgroup of degree n. We define morphisms of graded S-modules

$$p_i : T \longrightarrow M_i$$
$$p_i(m) = \sum_{n \in \mathbb{Z}} m_{n,i}$$

To see these morphisms are a product, let $\phi_i : K \longrightarrow M_i$ be morphisms of graded S-modules, and define $\phi : K \longrightarrow T$ by $\phi(k)_{n,i} = \phi_i(k_n)$. This is a well-defined morphism of graded S-modules, and it is clear that ϕ is unique with $p_i\phi = \phi_i$. Therefore T is the product of the M_i in $S\mathbf{GrMod}$ and we have

Proposition 21. *For a graded ring S the category $S\mathbf{GrMod}$ is a complete, cocomplete abelian category.*

A sequence $M \longrightarrow N \longrightarrow L$ of graded S-modules is exact in $S\mathbf{GrMod}$ iff. it is exact as a sequence of S-modules. We already know that a morphism is a monomorphism (epimorphism) in $S\mathbf{GrMod}$ iff. it is a monomorphism (epimorphism) in $S\mathbf{Mod}$.

5 Quasi-Structures

Definition 9. Let S be a graded ring and M a graded S-module. As usual write $ProjS$ for the set of all homogenous primes of S not containing S_+. For a homogenous element $m \in M$ the normal annihilator ideal $Ann(m) = \{s \in S \mid s \cdot m = 0\}$ is a homogenous ideal, as is the module annihilator $Ann(M) = \{s \in S \mid s \cdot M = 0\}$. We also define

$$Suph(M) = \{\mathfrak{p} \in ProjS \mid M_{(\mathfrak{p})} \neq 0\}$$

As usual $V(\mathfrak{a})$ denotes the set of all $\mathfrak{p} \in ProjS$ which contain $\mathfrak{a}$.

Definition 10. Let S be a graded ring and M a graded S-module. For $d \in \mathbb{Z}$ we define $M\{d\}$ to be the S-module $\bigoplus_{n \geq d} M_n$ with the grading $M\{d\}_n = M_{n+d}$ for $n \geq 0$ and $M\{d\}_n = 0$ for $n < 0$. This is the graded S-submodule of $M(d)$ obtained by removing all negative grades. If $\phi : M \longrightarrow N$ is a morphism of graded S-modules then ϕ restricts to give a morphism of graded S-modules $\phi\{d\} : M\{d\} \longrightarrow N\{d\}$. This defines an additive exact functor $-\{d\} : S\mathbf{GrMod} \longrightarrow S\mathbf{GrMod}$. If $e \geq 0$ and $d \in \mathbb{Z}$ then it is clear that $(-\{e\}) \circ (-\{d\}) = -\{d + e\}$. In particular if $d \geq 0$ then $(-\{d\}) \circ (-\{-d\}) = -\{0\}$.

We say two graded S-modules M, N are *quasi-isomorphic* and write $M \sim N$ if there exists an integer $d \geq 0$ such that $M\{d\} \cong N\{d\}$ as graded S-modules. This is an equivalence relation on the class of graded S-modules, and it is clear that if M, N are isomorphic as graded S-modules then they are quasi-isomorphic. Clearly $M \sim 0$ if and only if there exists $d \geq 0$ such that $M_n = 0$ for all $n \geq d$.

Let $\phi : M \longrightarrow N$ be a morphism of graded S-modules. We say ϕ is a *quasi-isomorphism* if $\phi\{d\}$ is an isomorphism in $S\mathbf{GrMod}$ for some $d \geq 0$, and similarly we define *quasi-monomorphisms* and *quasi-epimorphisms*. Clearly ϕ is a quasi-isomorphism iff. there exists $d \geq 0$ such that $\phi_n : M_n \longrightarrow N_n$ is bijective for all $n \geq d$, and similarly for quasi-monomorphisms and quasi-epimorphisms. Note that $M \sim N$ does *not* necessarily mean there exists a quasi-isomorphism $M \longrightarrow N$, although the converse is obviously true.

Remark 1. Let S be a graded ring and M a graded S-module. Then for $n, d \in \mathbb{Z}$ it is clear that $M(n)\{d\} = M\{n + d\}$. In fact we have an equality of functors $(-\{d\}) \circ (-(n)) = -\{n + d\}$.

Lemma 22. *Let S be a graded ring which is a finitely generated S_0-algebra, and let M be a finitely generated graded S-module. Then $M\{d\}$ is a finitely generated graded S-module for any $d \geq 0$.*

Proof. With the notation of the proof of Lemma 11, fix an integer $1 \leq j \leq s$ and let a tuple $\alpha = (\alpha_0, \ldots, \alpha_r)$ of integers $\alpha_i \geq 0$ be called *j-minimal* if $k_j + \sum_i \alpha_i h_i \geq d$ and if whenever we replace any α_i by a strictly smaller non-negative integer this sum is strictly smaller than d. There are a finite number of such tuples, each corresponding to a homogenous element $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ of $M\{d\}$. Taking the collection of these elements as j varies from 1 to s gives a finite set of homogenous generators for $M\{d\}$ as an S-module. ☐

Definition 11. Let S be a graded ring and M a graded S-module. We say that M is *quasi-finitely generated* if there exists $d \geq 0$ such that $M\{d\}$ is a finitely generated graded S-module. This property is stable under isomorphism.

Lemma 23. *Let S be a graded ring which is a finitely generated S_0-algebra, and let M be a graded S-module. Then*

(i) If $d \geq 0$ then M is quasi-finitely generated if and only if $M\{d\}$ is.

(ii) If $n \in \mathbb{Z}$ then M is quasi-finitely generated if and only if $M(n)$ is.

Proof. (i) Follows from Lemma 22 and (ii) follows from (i) and Remark 1. $\square$

Lemma 24. *Let S be a graded ring which is a finitely generated S_0-algebra, and let M be a graded S-module. Then M is quasi-finitely generated if and only if $M \sim N$ for some finitely generated graded S-module N.*

Proof. It follows immediately from Lemma 22 that if N is a finitely generated graded S-module with $M \sim N$ then M is quasi-finitely generated. Conversely suppose that M is quasi-finitely generated, say $d \geq 0$ is such that $M\{d\}$ is a finitely generated graded S-module. Then $M \sim M\{d\}\{-d\}$ and since $M\{d\} = M\{d\}\{-d\}$ as S-modules it is clear that $M\{d\}\{-d\}$ is finitely generated. $\square$

Lemma 25. *Let S be a noetherian graded ring and suppose we have an exact sequence of graded S-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Then M is quasi-finitely generated if and only if both M', M'' are quasi-finitely generated.

Proof. By Proposition 9, S is a finitely generated S_0-algebra, so the result follows Lemma 22 and exactness of the functors $-\{d\} : S\mathbf{GrMod} \longrightarrow S\mathbf{GrMod}$. $\square$

Lemma 26. *Let S be a graded ring and $\phi : M \longrightarrow N$ a morphism of graded S-modules. Then*

(i) ϕ is a quasi-monomorphism $\Leftrightarrow$ $Ker\phi \sim 0$.

(ii) ϕ is a quasi-epimorphism $\Leftrightarrow$ $Coker\phi \sim 0$.

(iii) ϕ is a quasi-isomorphism $\Leftrightarrow$ $Ker\phi \sim 0$ and $Coker\phi \sim 0$.

Proof. All three statements follow immediately from the fact that the functors $-\{d\}$ are exact. $\square$

Lemma 27. *Let $\phi : S \longrightarrow T$ be a morphism of graded rings. Then for $d \in \mathbb{Z}$ the following diagram commutes*

$$\begin{array}{ccc} T\mathbf{GrMod} & \longrightarrow & S\mathbf{GrMod} \\ {\scriptstyle -\{d\}}\downarrow & & \downarrow{\scriptstyle -\{d\}} \\ T\mathbf{GrMod} & \longrightarrow & S\mathbf{GrMod} \end{array}$$

In particular if M, N are graded T-modules and $M \sim N$ then ${}_SM \sim {}_SN$.

Lemma 28. *Let S be a graded ring generated by S_1 as an S_0-algebra and let M be a finitely generated graded S-module. Then $Suph(M) = V(Ann(M))$.*

Proof. Take homogenous generators $m_1, \ldots, m_n$ for M as an S-module (say $m_i \in M_{j_i}$) and note that

$$Ann(M) = \bigcap_{i=1}^{n} Ann(m_i)$$

Let $\mathfrak{p} \in ProjS$ be given, and find $f \in S_1$ with $f \notin \mathfrak{p}$. Then $m_i/f^{j_i} \in M_{(\mathfrak{p})}$ for $1 \leq j \leq n$ (if $j_i < 0$ we use $f^{-j_i}m_i/1$). Clearly $m_i/f^{j_i} = 0$ in $M_{(\mathfrak{p})}$ if and only if $Ann(m_i) \nsubseteq \mathfrak{p}$, so we see that $M_{(\mathfrak{p})} \neq 0$ if and only if $Ann(m_i) \subseteq \mathfrak{p}$ for some i. By the usual result on prime avoidance, this implies that $M_{(\mathfrak{p})} \neq 0$ if and only if $Ann(M) \subseteq \mathfrak{p}$, which is what we wanted. $\square$

Lemma 29. *Let S be a graded ring and M a graded S-module. Then $M = 0$ if and only if $M_{(\mathfrak{p})} = 0$ for every homogenous prime ideal $\mathfrak{p} \subseteq S$.*

Proof. One implication is obvious. Otherwise suppose $M \neq 0$ and let $0 \neq m \in M$ be homogenous with annihilator $Ann(m)$, which is a proper homogenous ideal of S. This ideal is contained in some homogenous prime $\mathfrak{p}$, in which case $M_{(\mathfrak{p})} \neq 0$. □

Proposition 30. *Let S be a graded ring finitely generated by S_1 as an S_0-algebra and let M be a finitely generated graded S-module. Then $M \sim 0$ if and only if $M_{(\mathfrak{p})} = 0$ for every homogenous prime ideal $\mathfrak{p} \in ProjS$.*

Proof. If $M \sim 0$ and $d > 0$ is such that $M_n = 0$ for all $n \geq d$, then given $\mathfrak{p} \in ProjS$ we can find $f \in S_1$ with $f \notin \mathfrak{p}$, and whenever $m/s \in M_{(\mathfrak{p})}$ we have $m/s = mf^d/sf^d = 0$ since $mf^d = 0$. Therefore $M \sim 0$ implies $M_{(\mathfrak{p})} = 0$ for every $\mathfrak{p} \in ProjS$.

For the converse, suppose that $M_{(\mathfrak{p})} = 0$ for every $\mathfrak{p} \in ProjS$. If $M = 0$ then the result is trivial, so we can assume that $Ann(M)$ is proper, and is therefore contained in at least one homogenous prime of S. By Lemma 28, $V(Ann(M))$ is empty, so the homogenous primes containing $Ann(M)$ must all contain S_+. Then Corollary 3 implies that the radical of $Ann(M)$ contains S_+. Suppose that $f_1, \ldots, f_n \in S_1$ generate S as an S_0-algebra, and let $h > 0$ be such that $f_i^h \in Ann(M)$ for $1 \leq i \leq n$. Let $m_1, \ldots, m_t$ be homogenous generators of M as an S-module, with say m_j of degree k_j for $1 \leq j \leq t$. Let $k > 0$ be strictly greater than every k_j.

We showed in Lemma 11 that for $d \in \mathbb{Z}$, M_d is generated as an S_0-module by elements of the form $f_1^{\alpha_1} \cdots f_n^{\alpha_n} m_j$ where $k_j + \sum_i \alpha_i = d$. If we take $d > nh + k$ then some f_i must occur to the power h, and therefore $M_d = 0$. This shows that $M \sim 0$ and completes the proof. □

Proposition 31. *Let S be a graded ring. The graded modules $\{S(n)\}_{n \in \mathbb{Z}}$ form a generating family for S**GrMod**.*

Proof. Suppose $\alpha : M \longrightarrow N$ is a nonzero morphism of graded modules. Then $\alpha(m) \neq 0$ for some homogenous $m \in M$. If m is of degree n let $f : S(-n) \longrightarrow M$ be the morphism of graded modules corresponding to m in Lemma 19. Then $\alpha f \neq 0$ since $f(1) = m$, which completes the proof. □

6 Grading Tensor Products

Throughout this section let S be a graded ring and M, N graded S-modules. We claim that the tensor product $M \otimes_S N$ has a canonical structure as a graded S-module with

$$(M \otimes_S N)_n = \{\sum_i m_i \otimes n_i \,|\, deg(m_i) + deg(n_i) = n \;\forall i\} \qquad n \in \mathbb{Z} \tag{2}$$

where m_i, n_i are homogenous and the sums are nonempty. The set of all such sums is a subgroup of $M \otimes_S N$ we claim that together with these subgroups $M \otimes_S N$ is a graded S-module. First consider M, N as abelian groups and let B be the subgroup of the free abelian group A on the set $M \times N$ generated by the relations

$$(x + x', y) - (x, y) - (x', y), (x, y + y') - (x, y) - (x, y') \tag{3}$$
$$(s \cdot x, y) - (x, s \cdot y) \tag{4}$$

Then $B = P + L$ where P is the subgroup generated by the relations (3) and L is generated by the relations (4). By definition $M \otimes_{\mathbb{Z}} N = A/P$ and $M \otimes_S N = A/B$. Since the tensor product preserves direct sums, we have

$$M \otimes_{\mathbb{Z}} N = \bigoplus_{q \in \mathbb{Z}} \bigoplus_{m+n=q} M_m \otimes_{\mathbb{Z}} N_n$$

where a sequence $(a_{m,n,q} \otimes b_{m,n,q})_{m,n,q}$ is mapped to the sum of all the entries. In other words, $M \otimes_{\mathbb{Z}} N$ is the direct sum of the following subgroups

$$(M \otimes_{\mathbb{Z}} N)_q = \{\sum_i m_i \otimes n_i \,|\, deg(m_i) + deg(n_i) = q \;\forall i\} \qquad q \in \mathbb{Z} \tag{5}$$

There is a canonical morphism of abelian groups $\alpha : M \otimes_{\mathbb{Z}} N \longrightarrow M \otimes_S N$ given by $a \otimes b \mapsto a \otimes b$, which is the morphism $A/P \longrightarrow A/B$ induced by the inclusion $P \subseteq B$. The kernel of α is $L+P/P$, which is the subgroup of $M \otimes_{\mathbb{Z}} N$ generated by elements of the form $(s \cdot x) \otimes y - x \otimes (s \cdot y)$. In fact, $P' = Ker(\alpha)$ is generated as an abelian group by the subset of these elements with x, y, s all homogenous. Considering $M \otimes_{\mathbb{Z}} N$ as a graded $\mathbb{Z}$-module with the grading given by (5), P' is a graded submodule since it is generated by homogenous elements. With the notation of (2), (5) we have $\alpha((M \otimes_{\mathbb{Z}} N)_n) = (M \otimes_S N)_n$ and therefore since $(M \otimes_{\mathbb{Z}} N)/P' \cong M \otimes_S N$ as abelian groups we have

$$M \otimes_S N = \bigoplus_{n \in \mathbb{Z}} (M \otimes_S N)_n$$

So $M \otimes_S N$ is a graded S-module, as required.

If $\phi : M \longrightarrow M'$ is a morphism of graded S-modules then the morphism of S-modules $\phi \otimes N : M \otimes_S N \longrightarrow M' \otimes_S N$ preserves grade, so we have an additive functor

$$- \otimes_S N : S\mathbf{GrMod} \longrightarrow S\mathbf{GrMod}$$

Similarly if $\psi : N \longrightarrow N'$ is a morphism of graded S-modules then the morphism $M \otimes \psi : M \otimes_S N \longrightarrow M \otimes_S N'$ preserves grade and we have an additive functor

$$M \otimes_S - : S\mathbf{GrMod} \longrightarrow S\mathbf{GrMod}$$