Ext

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Contents

1 Ext using Injectives

If $\mathcal{A}$ is an abelian category, then $Hom(A,-)$ is a covariant, additive, kernel preserving functor $\mathcal{A} \longrightarrow \mathbf{Ab}$ and $Hom(-,B)$ is a contravariant, additive functor which maps cokernels to kernels. Throughout this section $\mathcal{A}$ will be an abelian category with enough injectives.

Definition 1. The right derived functors of $Hom(A,-)$ are called the *Ext* groups.

$$Ext^i(A,B) = R^i Hom(A,-)(B)$$

The functor $Ext^i(A,-) : \mathcal{A} \longrightarrow \mathbf{Ab}$ is additive and covariant for $i \geq 0$. Since $Hom(A,-)$ is left exact the functors $Ext^0(A,-)$ and $Hom(A,-)$ are naturally equivalent. We simply write $Ext(A,-)$ for $Ext^1(A,-)$.

The group $Ext^i(A,B)$ is only determined up to isomorphism, and to calculate it we find an injective resolution $0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \cdots$ and calculate the cohomology of the sequence

$$0 \longrightarrow Hom(A,I^0) \longrightarrow Hom(A,I^1) \longrightarrow Hom(A,I^2) \longrightarrow \cdots$$

We think of Ext^i as assigning to any pair of objects A,B an isomorphism class of abelian groups, which has the following properties:

- For any injective object I we have $Ext^i(A,I) = 0$ for $i \neq 0$, since this is a property of any right derived functor.
- For any projective object P we have $Ext^i(P,B) = 0$ for $i \neq 0$, since the higher right derived functors of the exact functor $Hom(P,-)$ are zero.

For any exact sequence

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

there are canonical morphisms $\omega^0 : Hom(A, B'') \longrightarrow Ext(A, B')$ and $\omega^n : Ext^n(A, B'') \longrightarrow Ext^{n+1}(A, B')$ for $n > 1$ such that the following sequence is long exact

$$\begin{aligned}
0 \longrightarrow Hom(A, B') \longrightarrow Hom(A, B) \longrightarrow Hom(A, B'') \longrightarrow \\
\longrightarrow Ext(A, B') \longrightarrow Ext(A, B) \longrightarrow Ext(A, B'') \longrightarrow \\
\longrightarrow Ext^2(A, B') \longrightarrow Ext^2(A, B) \longrightarrow Ext^2(A, B'') \longrightarrow \cdots
\end{aligned}$$

This sequence is called the *long exact Ext sequence in the second variable.* It is natural, in the sense that if we have a commutative diagram with exact rows

$$\begin{CD}
0 @>>> B' @>>> B @>>> B'' @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> C' @>>> C @>>> C'' @>>> 0
\end{CD}$$

Then the following diagrams commute for $n \geq 1$

$$\begin{CD}
Hom(A, B'') @>>> Ext(A, B') \\
@VVV @VVV \\
Hom(A, C'') @>>> Ext(A, C')
\end{CD}
\qquad
\begin{CD}
Ext^n(A, B'') @>>> Ext^{n+1}(A, B') \\
@VVV @VVV \\
Ext^n(A, C'') @>>> Ext^{n+1}(A, C')
\end{CD}$$

Let $\alpha : A \longrightarrow A'$ be a morphism, and let α also denote the associated natural transformation $Hom(A', -) \longrightarrow Hom(A, -)$. Let $\mathcal{I}$ be a fixed assignment of injective resolutions. Then there is a natural transformation $R^n\alpha : R^n Hom(A', -) \longrightarrow R^n Hom(A, -)$ and we denote by $Ext^n(\alpha, B)$ the morphism $(R^n\alpha)_B : Ext^n(A', B) \longrightarrow Ext^n(A, B)$. Notice that for another morphism $\gamma : A' \longrightarrow A''$, $(R^n\alpha)(R^n\gamma) = R^n(\gamma\alpha)$ so for any object B

$$Ext^n(\alpha, B) Ext^n(\gamma, B) = Ext^n(\gamma\alpha, B)$$

This defines a contravariant additive functor $Ext^n(-, B) : \mathcal{A} \longrightarrow \mathbf{Ab}$. For any exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ we have the following commutative diagram for $n \geq 0$

$$\begin{CD}
\cdots @>>> Ext^n(A', B') @>>> Ext^n(A', B) @>>> Ext^n(A', B'') @>{\omega^n}>> Ext^{n+1}(A', B') @>>> \cdots \\
@. @V{Ext^n(\alpha, B')}VV @V{Ext^n(\alpha, B)}VV @V{Ext^n(\alpha, B'')}VV @V{Ext^{n+1}(\alpha, B')}VV @. \\
\cdots @>>> Ext^n(A, B') @>>> Ext^n(A, B) @>>> Ext^n(A, B'') @>>{\omega^n}> Ext^{n+1}(A, B') @>>> \cdots
\end{CD} \tag{1}$$

Proposition 1. *For $n \geq 0$ and morphisms $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$*

$$Ext^n(A, \beta) Ext^n(\alpha, B) = Ext^n(\alpha, B') Ext^n(A', \beta) \tag{2}$$

It follows that Ext^n defines a functor $\mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathbf{Ab}$ for $n \geq 0$, with $Ext^n(\alpha, \beta) : Ext^n(A', B) \longrightarrow Ext^n(A, B')$ given by the equivalent expressions in (2). The partial functors are the functors $Ext^n(A, -)$ and $Ext^n(-, B)$ defined above.

Proof. This follows for arbitrary α and monomorphisms (or epimorphisms) β by commutativity of (1). Since $\mathcal{A}$ has epi-mono factorisations it then follows for arbitrary β. The bifunctor Ext^n is defined relative to an assignment of injective resolutions $\mathcal{I}$. If $\mathcal{J}$ is another such assignment then the associated bifunctor is canonically naturally equivalent to the one defined for $\mathcal{I}$. $\square$

For a short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ the corresponding sequence of natural transformations $Hom(A'', -) \longrightarrow Hom(A, -) \longrightarrow Hom(A', -)$ is exact on injectives. So for $n \geq 0$ and any object B there are canonical connecting morphisms $\omega^n : Ext^n(A', B) \longrightarrow Ext^{n+1}(A'', B)$ fitting in to a long exact sequence

$$\cdots \longrightarrow Ext^n(A'', B) \longrightarrow Ext^n(A, B) \longrightarrow Ext^n(A', B) \longrightarrow Ext^{n+1}(A'', B) \longrightarrow \cdots$$

This sequence is called the *long exact Ext sequence in the first variable*. It is natural in both B and the exact sequence. For a morphism $\beta : B \longrightarrow B'$ the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Ext^n(A'', B) & \longrightarrow & Ext^n(A, B) & \longrightarrow & Ext^n(A', B) & \xrightarrow{\omega^n} & Ext^{n+1}(A'', B) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & Ext^n(A'', B') & \longrightarrow & Ext^n(A, B') & \longrightarrow & Ext^n(A', B') & \xrightarrow[\omega^n]{} & Ext^{n+1}(A'', B') & \longrightarrow \cdots
\end{array}$$

And for a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

The following diagram commutes for any object B

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Ext^n(A'', B) & \longrightarrow & Ext^n(A, B) & \longrightarrow & Ext^n(A', B) & \xrightarrow{\omega^n} & Ext^{n+1}(A'', B) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & Ext^n(C'', B) & \longrightarrow & Ext^n(C, B) & \longrightarrow & Ext^n(C', B) & \xrightarrow[\omega^n]{} & Ext^{n+1}(C'', B) & \longrightarrow \cdots
\end{array}$$

We have shown that for every assignment of injective resolutions $\mathcal{I}$ we obtain a bifunctor $Ext^n_{\mathcal{I}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Ab}$ for $n \geq 0$ with the property that short exact sequences in either variable lead to a long exact sequence which is natural with respect to morphisms of the exact sequence and morphisms in the remaining variable. The connecting morphisms for these sequences depend only on $\mathcal{I}$.

If $\mathcal{J}$ is another assignment of resolutions then we obtain another bifunctor $Ext^n_{\mathcal{J}}(-,-)$ for $n \geq 0$ which is canonically naturally equivalent to $Ext^n_{\mathcal{I}}(-,-)$. The connecting morphisms for the two assignments $\mathcal{I}, \mathcal{J}$ agree in the following sense: for an object A and an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Ext^n_{\mathcal{I}}(A, B') & \longrightarrow & Ext^n_{\mathcal{I}}(A, B) & \longrightarrow & Ext^n_{\mathcal{I}}(A, B'') & \xrightarrow{\omega^n_{\mathcal{I}}} & Ext^{n+1}_{\mathcal{I}}(A, B') & \longrightarrow \cdots \\
 & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\
\cdots \longrightarrow & Ext^n_{\mathcal{J}}(A, B') & \longrightarrow & Ext^n_{\mathcal{J}}(A, B) & \longrightarrow & Ext^n_{\mathcal{J}}(A, B'') & \xrightarrow[\omega^n_{\mathcal{J}}]{} & Ext^{n+1}_{\mathcal{J}}(A, B') & \longrightarrow \cdots
\end{array}$$

Similarly for an object B and an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Ext^n_{\mathcal{I}}(A'', B) & \longrightarrow & Ext^n_{\mathcal{I}}(A, B) & \longrightarrow & Ext^n_{\mathcal{I}}(A', B) & \xrightarrow{\omega^n_{\mathcal{I}}} & Ext^{n+1}_{\mathcal{I}}(A'', B) & \longrightarrow \cdots \\
 & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\
\cdots \longrightarrow & Ext^n_{\mathcal{J}}(A'', B) & \longrightarrow & Ext^n_{\mathcal{J}}(A, B) & \longrightarrow & Ext^n_{\mathcal{J}}(A', B) & \xrightarrow[\omega^n_{\mathcal{J}}]{} & Ext^n_{\mathcal{J}}(A'', B) & \longrightarrow \cdots
\end{array}$$

Both these claims follow directly from our Derived Functor notes.

1.1 Calculations using Injective Presentations

Since $Hom(X,-)$ is left exact we can use our results truncated injective resolutions to show that the functor $Ext(X,-)$ is naturally equivalent to the functor E defined by the following procedure: pick for every object A an exact sequence

$$0 \longrightarrow A \longrightarrow I \overset{\mu}{\longrightarrow} C \longrightarrow 0$$

with I injective. Then $E(A)$ is the cokernel of $Hom(X,I) \longrightarrow Hom(X,C)$ and given a morphism $\alpha : A \longrightarrow B$ where B is assigned the sequence $0 \longrightarrow B \longrightarrow J \longrightarrow D \longrightarrow 0$ use injectivity of J to lift α to a morphism $\varphi : I \longrightarrow J$ and then induce α' fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & I & \overset{\mu}{\longrightarrow} & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow & & {\scriptstyle\alpha'}\downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & J & \underset{\tau}{\longrightarrow} & D & \longrightarrow & 0 \end{array}$$

Then $E(\alpha) : Hom(X,C)/ImT(\mu) \longrightarrow Hom(X,D)/ImT(\tau)$ is defined by composition with α'. It turns out that this gives a well-defined additive functor naturally equivalent to $Ext(X,-)$.

2 Ext using Projectives

Throughout this section $\mathcal{A}$ will be an abelian category with enough projectives. For an object A the functor $Hom(-,A)$ is contravariant, but considered as a functor $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ it is a left exact covariant functor.

Definition 2. The right derived functors of $Hom(-,B)$ are the $\underline{Ext}$ groups.

$$\underline{Ext}^i(A,B) = R^i Hom(-,B)(A)$$

The functor $\underline{Ext}^i(-,B) : \mathcal{A} \longrightarrow \mathbf{Ab}$ is additive and contravariant for $i \geq 0$. The functors $\underline{Ext}^0(-,B)$ and $Hom(-,B)$ are naturally equivalent. We simply write $\underline{Ext}(-,B)$ for $\underline{Ext}^1(-,B)$.

The group $\underline{Ext}^i(A,B)$ is only determined up to isomorphism, and to calculate it we find a projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and calculate the cohomology of the sequence

$$0 \longrightarrow Hom(P_0,B) \longrightarrow Hom(P_1,B) \longrightarrow Hom(P_2,B) \longrightarrow \cdots$$

We think of $\underline{Ext}^i$ as assigning to any pair of objects A, B an isomorphism class of abelian groups, which has the following properties:

- For any projective object P we have $\underline{Ext}^i(P,B) = 0$ for $i \neq 0$, since this is a property of any right derived functor (remember we are taking right derived functors in $\mathcal{A}^{\mathrm{op}}$, where P is injective).
- For any injective object I we have $\underline{Ext}^i(A,I) = 0$ for $i \neq 0$, since the higher right derived functors of the exact functor $Hom(-,I)$ are zero.

For any exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

there are canonical morphisms $\omega_0 : Hom(A',B) \longrightarrow \underline{Ext}(A'',B)$ and $\omega^n : \underline{Ext}^n(A',B) \longrightarrow \underline{Ext}^{n+1}(A'',B)$ for $n \geq 1$ such that the following sequence is long exact

$$\begin{aligned} 0 \longrightarrow Hom(A'',B) \longrightarrow Hom(A,B) \longrightarrow Hom(A',B) \longrightarrow \\ \longrightarrow \underline{Ext}(A'',B) \longrightarrow \underline{Ext}(A,B) \longrightarrow \underline{Ext}(A',B) \longrightarrow \\ \longrightarrow \underline{Ext}^2(A'',B) \longrightarrow \underline{Ext}^2(A,B) \longrightarrow \underline{Ext}^2(A',B) \longrightarrow \cdots \end{aligned}$$

This sequence is called the *long exact $\underline{Ext}$ sequence in the first variable.* It is natural, in the sense that if we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

Then the following diagrams commute for $n \geq 1$

$$\begin{array}{ccc} Hom(C',B) & \longrightarrow & \underline{Ext}(C'',B) \\ \downarrow & & \downarrow \\ Hom(A',B) & \longrightarrow & \underline{Ext}(A'',B) \end{array} \qquad \begin{array}{ccc} \underline{Ext}^n(C',B) & \longrightarrow & \underline{Ext}^{n+1}(C'',B) \\ \downarrow & & \downarrow \\ \underline{Ext}^n(A',B) & \longrightarrow & \underline{Ext}^{n+1}(A'',B) \end{array}$$

Let $\beta : B \longrightarrow B'$ be a morphism, and let β also denote the associated natural transformation $Hom(-,B) \longrightarrow Hom(-,B')$. Let $\mathcal{P}$ be a fixed assignment of projective resolutions. Then there is a natural transformation $R^n\beta : R^n Hom(-,B) \longrightarrow R^n Hom(-,B')$ and we denote by $\underline{Ext}^n(A,\beta)$ the morphism $(R^n\beta)_A : \underline{Ext}^n(A,B) \longrightarrow \underline{Ext}^n(A,B')$. Notice that for another morphism $\gamma : B' \longrightarrow B''$, $(R^n\gamma)(R^n\beta) = R^n(\gamma\beta)$ so for any object A

$$\underline{Ext}^n(A,\gamma)\underline{Ext}^n(A,\beta) = \underline{Ext}^n(A,\gamma\beta)$$

This defines a covariant additive functor $\underline{Ext}^n(A,-) : \mathcal{A} \longrightarrow \mathbf{Ab}$. For any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ the following diagram is commutative for $n \geq 0$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \underline{Ext}^n(A'',B) & \longrightarrow & \underline{Ext}^n(A,B) & \longrightarrow & \underline{Ext}^n(A',B) & \xrightarrow{\omega^n} & \underline{Ext}^{n+1}(A'',B) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \underline{Ext}^n(A'',\beta)} & & \downarrow{\scriptstyle \underline{Ext}^n(A,\beta)} & & \downarrow{\scriptstyle \underline{Ext}^n(A',\beta)} & & \downarrow{\scriptstyle \underline{Ext}^{n+1}(A'',\beta)} & & \\ \cdots & \longrightarrow & \underline{Ext}^n(A'',B') & \longrightarrow & \underline{Ext}^n(A,B') & \longrightarrow & \underline{Ext}^n(A',B') & \xrightarrow[\omega^n]{} & \underline{Ext}^{n+1}(A'',B') & \longrightarrow & \cdots \end{array} \tag{3}$$

Proposition 2. *For $n \geq 0$ and morphisms $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$*

$$\underline{Ext}^n(A,\beta)\underline{Ext}^n(\alpha,B) = \underline{Ext}^n(\alpha,B')\underline{Ext}^n(A',\beta) \tag{4}$$

It follows that $\underline{Ext}^n$ defines a functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Ab}$ for $n \geq 0$, with $\underline{Ext}^n(\alpha,\beta) : \underline{Ext}^n(A',B) \longrightarrow \underline{Ext}^n(A,B')$ given by the equivalent expressions in (4). The partial functors are the functors $\underline{Ext}^n(A,-)$ and $\underline{Ext}^n(-,B)$ defined above.

Proof. This follows for arbitrary β and monomorphisms (or epimorphisms) α by commutativity of (3). Since $\mathcal{A}$ has epi-mono factorisations it then follows for arbitrary α. If we use a different assignment of projective resolutions to calculate $\underline{Ext}^n$ then the results will be canonically naturally equivalent. □

For a short exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ the corresponding sequence of natural transformations $Hom(-,B') \longrightarrow Hom(-,B) \longrightarrow Hom(-,B'')$ is exact on injectives (considered as covariant functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$). So for $n \geq 0$ and any object A there are canonical connecting morphisms $\omega^n : \underline{Ext}^n(A,B'') \longrightarrow \underline{Ext}^{n+1}(A,B')$ fitting in to a long exact sequence

$$\cdots \longrightarrow \underline{Ext}^n(A,B') \longrightarrow \underline{Ext}^n(A,B) \longrightarrow \underline{Ext}^n(A,B'') \longrightarrow \underline{Ext}^{n+1}(A,B') \longrightarrow \cdots$$

This sequence is called the *long exact $\underline{Ext}$ sequence in the second variable.* It is natural in both A and the exact sequence. For a morphism $\alpha : A \longrightarrow A'$ the following diagram commutes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \underline{Ext}^n(A',B') & \longrightarrow & \underline{Ext}^n(A',B) & \longrightarrow & \underline{Ext}^n(A',B'') & \xrightarrow{\omega^n} & \underline{Ext}^{n+1}(A',B') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \underline{Ext}^n(A,B') & \longrightarrow & \underline{Ext}^n(A,B) & \longrightarrow & \underline{Ext}^n(A,B'') & \xrightarrow[\omega^n]{} & \underline{Ext}^{n+1}(A,B') & \longrightarrow & \cdots \end{array} \tag{5}$$

And for a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

The following diagram commutes for any object A

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \underline{Ext}^n(A,B') & \longrightarrow & \underline{Ext}^n(A,B) & \longrightarrow & \underline{Ext}^n(A,B'') & \xrightarrow{\omega^n} & \underline{Ext}^{n+1}(A,B') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \underline{Ext}^n(A,C') & \longrightarrow & \underline{Ext}^n(A,C) & \longrightarrow & \underline{Ext}^n(A,C'') & \xrightarrow[\omega^n]{} & \underline{Ext}^{n+1}(A,C') & \longrightarrow & \cdots \end{array}$$

We have shown that for every assignment of projective resolutions $\mathcal{P}$ we obtain a bifunctor $\underline{Ext}^n_{\mathcal{I}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Ab}$ for $n \geq 0$ with the property that short exact sequences in either variable lead to a long exact sequence which is natural with respect to morphisms of the exact sequence and morphisms in the remaining variable. The connecting morphisms for these sequences depend only on $\mathcal{P}$.

If $\mathcal{Q}$ is another assignment of resolutions then we obtain another bifunctor $\underline{Ext}^n_{\mathcal{Q}}(-,-)$ for $n \geq 0$ which is canonically naturally equivalent to $\underline{Ext}^n_{\mathcal{I}}(-,-)$. The connecting morphisms for the two assignments $\mathcal{P}, \mathcal{Q}$ agree in the following sense: for an object B and an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ the following diagram commutes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \underline{Ext}^n_{\mathcal{P}}(A'',B) & \longrightarrow & \underline{Ext}^n_{\mathcal{P}}(A,B) & \longrightarrow & \underline{Ext}^n_{\mathcal{P}}(A',B) & \xrightarrow{\omega^n_{\mathcal{P}}} & \underline{Ext}^{n+1}_{\mathcal{P}}(A'',B) & \longrightarrow & \cdots \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & \underline{Ext}^n_{\mathcal{Q}}(A'',B) & \longrightarrow & \underline{Ext}^n_{\mathcal{Q}}(A,B) & \longrightarrow & \underline{Ext}^n_{\mathcal{Q}}(A',B) & \xrightarrow[\omega^n_{\mathcal{Q}}]{} & \underline{Ext}^n_{\mathcal{Q}}(A'',B) & \longrightarrow & \cdots \end{array}$$

Similarly for an object A and an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ the following diagram commutes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \underline{Ext}^n_{\mathcal{P}}(A,B') & \longrightarrow & \underline{Ext}^n_{\mathcal{P}}(A,B) & \longrightarrow & \underline{Ext}^n_{\mathcal{P}}(A,B'') & \xrightarrow{\omega^n_{\mathcal{P}}} & \underline{Ext}^{n+1}_{\mathcal{P}}(A,B') & \longrightarrow & \cdots \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & \underline{Ext}^n_{\mathcal{Q}}(A,B') & \longrightarrow & \underline{Ext}^n_{\mathcal{Q}}(A,B) & \longrightarrow & \underline{Ext}^n_{\mathcal{Q}}(A,B'') & \xrightarrow[\omega^n_{\mathcal{Q}}]{} & \underline{Ext}^{n+1}_{\mathcal{Q}}(A,B') & \longrightarrow & \cdots \end{array}$$

Both these claims follow directly from our Derived Functor notes.

2.1 Calculations using Projective Presentations

Since $Hom(-,B) : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ is left exact we can use our results on truncated injective resolutions to show that $\underline{Ext}$ is naturally equivalent to the functor $\underline{E}$ defined by the following procedure: pick for every object A an exact sequence

$$0 \longrightarrow K \xrightarrow{\mu} P \longrightarrow A \longrightarrow 0$$

with P projective. Then $\underline{E}(A)$ is the cokernel of $Hom(P,B) \longrightarrow Hom(K,B)$ and given a morphism $\alpha : A \longrightarrow C$ where C is assigned the sequence $0 \longrightarrow M \longrightarrow Q \longrightarrow C \longrightarrow 0$ use projectivity of Q to lift α to a morphism $\varphi : P \longrightarrow Q$ and then induce α' fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\mu} & P & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle\alpha'}\downarrow & & \downarrow & & {\scriptstyle\alpha}\downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow[\tau]{} & Q & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Then $\underline{E}(\alpha) : \underline{E}(C) \longrightarrow \underline{E}(A)$, which is a map $Hom(M,B)/ImT(\tau) \longrightarrow Hom(K,B)/ImT(\tau)$ is defined by composition with α'. It turns out that this is a well-defined contravariant additive functor naturally equivalent to $\underline{Ext}(-,B)$.

In fact we have already studied the functor $\underline{E}$ for right modules over a ring in our Hilton & Stammbach notes, where we proved the following

- For any two right modules A, B over a ring there is a bijection $\underline{E}(A,B) \cong Y(A,B)$ where $Y(A,B)$ is the set of extensions of A by B (which are exact sequences $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$) modulo a certain equivalence relation. In particular $\underline{E}(A,B) = 0$ if and only if every exact sequence $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ splits.

3 Balancing Ext

Throughout this section $\mathcal{A}$ is an abelian category with enough injectives and projectives, and we choose once and for all assignments of resolutions $\mathcal{P}, \mathcal{I}$, with respect to which all derived functors are calculated. We have defined two bifunctors $\underline{Ext}^n(-,-)$ and $Ext^n(-,-)$ for $n \geq 0$. The first is calculated by taking the right derived functors of the contravariant functors $Hom(-,B)$ and the second by taking the right derived functors of the covariant functors $Hom(A,-)$. We claim that these two bifunctors are naturally equivalent. We begin with the case $n = 0$.

Lemma 3. *There are canonical natural equivalences of bifunctors $\underline{Ext}^0(-,-) \cong Hom(-,-)$ and $Hom(-,-) \cong Ext^0(-,-)$.*

Proof. Let the Ext functors be calculated with respect to some assignment $\mathcal{I}$ of injective resolutions. For an object A there is a canonical natural equivalence $Ext^0(A,-) \cong Hom(A,-)$, so we need only show these isomorphisms are also natural in B, which is not difficult. Similarly there is a canonical natural equivalence $\underline{Ext}^0(-,B) \cong Hom(-,B)$, which is also natural in the first variable. So all three functors are naturally equivalent. $\square$

Proposition 4. *For $n \geq 0$ there is a canonical natural equivalence of bifunctors $\Phi^n : Ext^n(-,-) \cong \underline{Ext}^n(-,-)$.*

Proof. We proceed by induction on n, having already proved the result for $n = 0$. Assume that there is a canonical natural equivalence Φ^n and let objects A, B be given. We have to define a canonical isomorphism $\Phi^{n+1}_{A,B}$ which is natural in A and B. Choose an injective presentation of B

$$0 \longrightarrow B \xrightarrow{\nu} I \xrightarrow{\eta} S \longrightarrow 0$$

We know that $Ext^i(A,I) = 0 = \underline{Ext}^i(A,I)$ for $i \neq 0$. Now we show how to define the isomorphism $\Phi^{n+1}_{A,B} : Ext^{n+1}(A,B) \longrightarrow \underline{Ext}^{n+1}(A,B)$. There are two cases: if $n = 1$ then the long exact sequence for Ext in the second variable and the long exact sequence for $\underline{Ext}$ in the second variable give a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Ext^0(A,B) & \longrightarrow & Ext^0(A,I) & \longrightarrow & Ext^0(A,S) & \xrightarrow{\omega^0} & Ext^1(A,B) & \longrightarrow & 0 \\
 & & \Downarrow & & \Downarrow & & \Downarrow{\scriptstyle \Phi^0_{A,S}} & & \vdots & & \\
0 & \longrightarrow & \underline{Ext}^0(A,B) & \longrightarrow & \underline{Ext}^0(A,I) & \longrightarrow & \underline{Ext}^0(A,S) & \xrightarrow[\omega^0]{} & \underline{Ext}^1(A,B) & \longrightarrow & 0
\end{array}$$

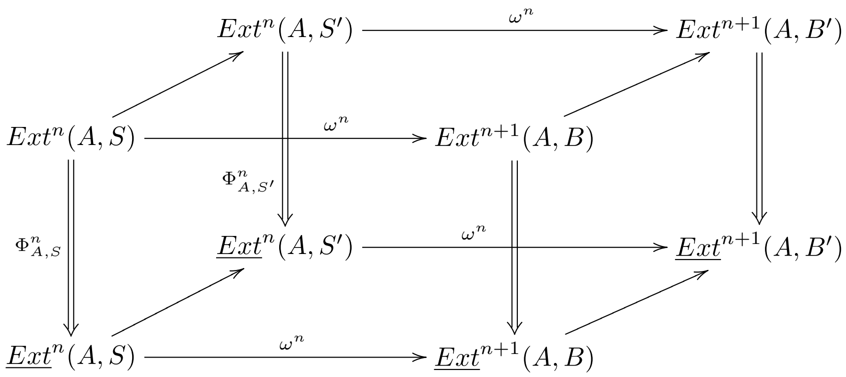
This induces an isomorphism $\Phi^1_{A,B} : Ext^1(A,B) \longrightarrow \underline{Ext}^1(A,B)$ making the diagram commute. For $n \geq 1$ the connecting morphisms $Ext^n(A,S) \longrightarrow Ext^{n+1}(A,B)$ and $\underline{Ext}^n(A,S) \longrightarrow \underline{Ext}^{n+1}(A,B)$ in the two sequences are isomorphisms, and we define $\Phi^{n+1}_{A,B}$ to be the unique morphism fitting into the following commutatie diagram

$$\begin{array}{ccc}
Ext^n(A,S) & \Longrightarrow & Ext^{n+1}(A,B) \\
{\scriptstyle \Phi^n_{A,S}}\Downarrow & & \vdots\, {\scriptstyle \Phi^{n+1}_{A,B}} \\
\underline{Ext}^n(A,S) & \Longrightarrow & \underline{Ext}^{n+1}(A,B)
\end{array}$$

Next we have to show that the isomorphism $\Phi^{n+1}_{A,B}$ does not depend on the chosen presentation. Suppose we have a commutative diagram with exact rows and the middle objects injective

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & I & \longrightarrow & S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & I' & \longrightarrow & S' & \longrightarrow & 0 \end{array}$$

Consider the following cube for $n \geq 0$



If we use the above technique to produce isomorphisms $Ext^{n+1}(A,B) \longrightarrow \underline{Ext}^{n+1}(A,B)$ and $Ext^{n+1}(A,B') \longrightarrow \underline{Ext}^{n+1}(A,B')$ using the given presentations then in either case ($n = 1$ or otherwise) these morphisms make the front and back squares on the cube commute. The left square commutes since by assumption Φ^n is natural, and the top and bottom squares commute by the naturality of the connecting morphism. Since $\omega^n : Ext^n(A,S) \longrightarrow Ext^{n+1}(A,B)$ is an epimorphism it follows that the right hand square also commutes.

If we are given two injective presentations of B then put $B = B'$ in the diagram and induce $I \longrightarrow I'$ and $S \longrightarrow S'$ making it commutative. Then the cube above shows that the resulting isomorphism $\Phi^{n+1}_{A,B}$ is the same in both cases. So we have constructed an isomorphism $\Phi^{n+1}_{A,B}$ that depends only on A, B, the assignments $\mathcal{P}, \mathcal{I}$ and the natural equivalence Φ^n. These isomorphisms are natural in B since we can lift $B \longrightarrow B'$ to a morphism of the injective presentations, and then use the cube.

To prove naturality in A we construct a cube similar to the one above, but with a fixed presentation and A varying. Using naturality of Φ^n in A and the diagrams (1) and (5) it is not hard to see that Φ^{n+1} is natural in A and is therefore a natural equivalence of bifunctors. Since by the inductive hypothesis Φ^n depends only on the assingment of resolutions $\mathcal{P}, \mathcal{I}$ it follows that this is true of Φ^{n+1} as well. $\square$

If $\mathcal{A}$ has both enough injectives and enough projectives and $\mathcal{I}, \mathcal{P}$ are assignments of injective and projective resolutions respectively, there is a natural equivalence of the bifunctors $Ext^n_{\mathcal{I}}(-,-)$ and $\underline{Ext}^n_{\mathcal{P}}(-,-)$ for $n \geq 0$. So every pair of objects A, B and integer $n \geq 0$ determines an isomorphism class of abelian groups. We can calculate a representative of this class in the following ways

- Choose a projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ of A and calculate the cohomology of the following cochain complex of abelian groups

$$0 \longrightarrow Hom(P_0, B) \longrightarrow Hom(P_1, B) \longrightarrow Hom(P_2, B) \longrightarrow \cdots$$

- Choose an injective resolution $0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ of B and calculate the cohomology of the following cochain complex of abelian groups

$$0 \longrightarrow Hom(A, I^0) \longrightarrow Hom(A, I^1) \longrightarrow Hom(A, I^2) \longrightarrow \cdots$$

If there is no chance of confusion we simply refer to any of these groups by $Ext^n(A, B)$ and drop $\underline{Ext}$ from the notation. But if $\mathcal{A}$ does not have both enough injectives and enough projectives, we will refer explicitly to the bifunctor Ext or $\underline{Ext}$ used.

In the case where $\mathcal{A} = \mathbf{Mod}R$ for a ring R, there is a bijection between elements of $Ext(A, B)$ and exact sequences $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ modulo a certain equivalence relation. In particular $Ext(A, B) = 0$ if and only if every such exact sequence is split.

Remark 1. One would like the natural equivalence of Ext and $\underline{Ext}$ to be compatible with the connecting morphisms for both bifunctors. One can get this in one variable (see Hilton & Stammbach), but it is not clear how to do it in the other variable.

4 Properties of Ext

4.1 Ext for Linear Categories

Definition 3. If R is a ring then an *R-linear* abelian category is an abelian category $\mathcal{A}$ together with a left R-module structure on all the morphism groups $Hom_{\mathcal{A}}(A, B)$ such that composition is bilinear. That is,

$$\gamma(r \cdot \alpha) = r \cdot (\gamma\alpha)$$
$$(r \cdot \alpha)\gamma = r \cdot (\alpha\gamma)$$

whenever $r \in R$ and the composition makes sense. Then for every object A, we have a covariant, additive, kernel preserving functor $Hom(A, -) : \mathcal{A} \longrightarrow R\mathbf{Mod}$ and a contravariant, additive functor $Hom(-, A) : \mathcal{A} \longrightarrow R\mathbf{Mod}$ which maps cokernels to kernels.

Let $U : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ be the forgetful functor, which is faithful and exact. This functor maps the canonical kernels, cokernels, images, zero and biproducts of $R\mathbf{Mod}$ to the corresponding canonical structure on $\mathbf{Ab}$. So if X is a (co)chain complex in $R\mathbf{Mod}$ then the (co)homology modules have as underlying groups the (co)homology groups of the sequence considered as a complex of groups.

For an object A let S be the functor $Hom(A, -) : \mathcal{A} \longrightarrow R\mathbf{Mod}$ and let T be $Hom(A, -) : \mathcal{A} \longrightarrow \mathbf{Ab}$. Then $T = US$ so for $n \geq 0$ and an assignment of injective resolutions $\mathcal{I}$ the functors R^nT and $U \circ R^nS$ are *equal*. So for an object B the Ext group $Ext^n(A, B)$ becomes an R-module in a canonical way, and for $\beta : B \longrightarrow B'$ the morphism of groups $Ext^n(A, \beta) : Ext^n(A, B) \longrightarrow Ext^n(A, B')$ is a morphism of these modules. Similarly if $\alpha : A \longrightarrow A'$ is a morphism of modules then the morphism of groups $Ext^n(A', B) \longrightarrow Ext^n(A, B)$ is a morphism of modules, so $Ext^n(-, B)$ lifts to a contravariant additive functor $\mathcal{A} \longrightarrow R\mathbf{Mod}$. Also $Ext^0(A, -) : \mathcal{A} \longrightarrow R\mathbf{Mod}$ is canonically naturally equivalent to $Hom(A, -)$.

So for a fixed assignment of injective resolutions $\mathcal{I}$ the bifunctor $Ext^n(-, -)$ becomes a bifunctor $Ext^n(-, -) : \mathcal{A}^{\text{op}} \times \mathcal{A} \longrightarrow R\mathbf{Mod}$. If $\mathcal{J}$ is another assignment of injective resolutions then the resulting bifunctors (with values in $R\mathbf{Mod}$) are canonically naturally equivalent.

Given an assignment of resolutions $\mathcal{I}$ and an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ the connecting morphisms $Ext^n(A, B'') \longrightarrow Ext^{n+1}(A, B')$ for $n \geq 0$ are all module morphisms, so the long exact sequence of Ext in the second variable

$$\cdots \longrightarrow Ext^n(A, B') \longrightarrow Ext^n(A, B) \longrightarrow Ext^n(A, B'') \longrightarrow Ext^{n+1}(A, B') \longrightarrow \cdots$$

is a long exact sequence of modules. Similarly if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence then the connecting morphisms $Ext^n(A', B) \longrightarrow Ext^{n+1}(A'', B)$ are module morphisms and the long exact sequence of Ext in the first variable

$$\cdots \longrightarrow Ext^n(A'', B) \longrightarrow Ext^n(A, B) \longrightarrow Ext^n(A', B) \longrightarrow Ext^{n+1}(A'', B) \longrightarrow \cdots$$

is a long exact sequence of modules.

Similarly for an object B let S be the functor $Hom(-,B) : \mathcal{A} \longrightarrow R\mathbf{Mod}$ and let T be $Hom(-,B) : \mathcal{A} \longrightarrow R\mathbf{Mod}$. Then $T = US$ so for $n \geq 0$ and an assignment of projective resolutions $\mathcal{P}$ the functors $R^n T$ and $U \circ R^n S$ are *equal*. So the functors $\underline{Ext}^n(-,B)$ and $\underline{Ext}^n(A,-)$ lift to module valued functors and $\underline{Ext}^0(-,B) : \mathcal{A} \longrightarrow R\mathbf{Mod}$ is naturally equivalent to $Hom(-,B)$. For a fixed assignment of projective resolutions $\mathcal{P}$ the bifunctor $\underline{Ext}^n(-,-)$ becomes a bifunctor $\underline{Ext}^n(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow R\mathbf{Mod}$. If $\mathcal{Q}$ is another assignment of projective resolutions then the resulting bifunctors (with values in $R\mathbf{Mod}$) are canonically naturally equivalent. The two long exact sequences for $\underline{Ext}^n$ are sequences of modules and module morphisms.

Now suppose $\mathcal{A}$ has enough projectives and injectives, and let $\mathcal{P}$ and $\mathcal{I}$ be assignments of projective and injective resolutions, respectively. The canonical natural equivalences $\underline{Ext}^0(-,-) \cong Hom(-,-)$ and $Hom(-,-) \cong Ext^0(-,-)$ give natural equivalences of the module-valued bifunctors. Then our earlier proof shows that for $n \geq 0$ there is a canonical natural equivalence $Ext^n(-,-) \cong \underline{Ext}^n(-,-)$ of bifunctors $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow R\mathbf{Mod}$.

So associated to any pair of objects A, B is an isomorphism class of R-modules $Ext^n(A,B)$. If $\mathcal{A}$ has enough projectives, we can find a representative of this class by choosing a projective resolution P of A and calculating the cohomology modules of $0 \longrightarrow Hom(P_0,B) \longrightarrow Hom(P_1,B) \longrightarrow \cdots$. If $\mathcal{A}$ has enough injectives, we can find a representative by choosing an injective resolution I of B and calculating the cohomology modules of $0 \longrightarrow Hom(A,I^0) \longrightarrow Hom(A,I^1) \longrightarrow \cdots$.

4.2 Dimension Shifting

The following two results are immediate consequences of our notes on dimension shifting.

Proposition 5. *Let $\mathcal{A}$ be an abelian category with enough injectives. Suppose we have an exact sequence in $\mathcal{A}$ with all I^i injective and $m \geq 0$*

$$0 \longrightarrow B \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{m-1} \longrightarrow I^m \longrightarrow M \longrightarrow 0$$

Then for any object A there are canonical isomorphisms $\rho^n : Ext^n(A,M) \longrightarrow Ext^{n+m+1}(A,B)$ for $n \geq 1$, and an exact sequence

$$Hom(A,I^m) \longrightarrow Hom(A,M) \longrightarrow Ext^{m+1}(A,B) \longrightarrow 0$$

These are both natural in A, in the sense that for a morphism $A \longrightarrow A'$ the following two diagrams commute for $n \geq 1$ and $m \geq 0$

$$\begin{array}{ccc} Ext^n(A',M) & \longrightarrow & Ext^{n+m+1}(A',B) \\ \downarrow & & \downarrow \\ Ext^n(A,M) & \longrightarrow & Ext^{n+m+1}(A,B) \end{array}$$

$$\begin{array}{ccccccc} Hom(A',I^m) & \longrightarrow & Hom(A',M) & \longrightarrow & Ext^{m+1}(A',B) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Hom(A,I^m) & \longrightarrow & Hom(A,M) & \longrightarrow & Ext^{m+1}(A,B) & \longrightarrow & 0 \end{array}$$

Proposition 6. *Let $\mathcal{A}$ be an abelian category with enough projectives. Suppose we have an exact sequence in $\mathcal{A}$ with all P_i projective and $m \geq 0$*

$$0 \longrightarrow M \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

Then for any object B there are canonical isomorphisms $\rho^n : \underline{Ext}^n(M,B) \longrightarrow \underline{Ext}^{n+m+1}(A,B)$ for $n \geq 1$ and an exact sequence

$$Hom(P_m,B) \longrightarrow Hom(M,B) \longrightarrow \underline{Ext}^{m+1}(A,B) \longrightarrow 0$$

These are both natural in B, in the sense that for a morphism $B \longrightarrow B'$ the following two diagrams commute for $n \geq 1$ and $m \geq 0$

$$\begin{array}{ccc} \underline{Ext}^n(M,B) & \longrightarrow & \underline{Ext}^{n+m+1}(A,B) \\ \downarrow & & \downarrow \\ \underline{Ext}^n(M,B') & \longrightarrow & \underline{Ext}^{n+m+1}(A,B') \end{array}$$

$$\begin{array}{ccccccc} Hom(P_m,B) & \longrightarrow & Hom(M,B) & \longrightarrow & \underline{Ext}^{m+1}(A,B) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Hom(P_m,B') & \longrightarrow & Hom(M,B') & \longrightarrow & \underline{Ext}^{m+1}(A,B') & \longrightarrow & 0 \end{array}$$

4.3 Ext and Coproducts

Proposition 7. *Let $\mathcal{A}$ be an infinite complete abelian category with exact products and enough injectives. For an object A, the functor $Ext^n(A,-) : \mathcal{A} \longrightarrow$ **Ab** preserves products.*

Proof. The functor $Hom(A,-) : \mathcal{A} \longrightarrow$ **Ab** preserves products, so this follows immediately from our Derived Functor notes. □

Proposition 8. *Let $\mathcal{A}$ be an infinite cocomplete abelian category with exact coproducts and enough projectives. For an object B, the contravariant functor $\underline{Ext}^n(-,B) : \mathcal{A} \longrightarrow$ **Ab** maps coproducts to products.*

Proof. By assumption $\mathcal{A}^{\text{op}}$ is a complete abelian category with exact products and enough injectives, and the functors $\underline{Ext}^n(-,B)$ are the right derived functors of the covariant additive functor $Hom(-,B) : \mathcal{A}^{\text{op}} \longrightarrow$ **Ab**. So once again the result follows from our Derived Functor notes. □

In particular both results apply in the case where $\mathcal{A}$ is **Ab**, R**Mod** or **Mod**R for a ring R. If $\mathcal{A}$ is R-linear for some ring R then the above results also apply to the functors $Ext^n(A,-) : \mathcal{A} \longrightarrow R$**Mod** and $\underline{Ext}^n(-,B) : \mathcal{A} \longrightarrow R$**Mod**. That is, the first preserves products and the second maps coproducts to products.

5 Ext for Commutative Rings

If R is a commutative ring and A, B are R-modules, then the group $Ext^n(A,B)$ doesn't depend on whether you consider A, B as left or right modules over R. That is, the calculations in the abelian categories R**Mod** and **Mod**R yield isomorphic groups.

For a commutative ring R the abelian category $\mathcal{A} = R$**Mod** is R-linear in the sense of Section 4.1. Each group $Hom_R(M,N)$ becomes an R-module via $(r \cdot \varphi)(x) = r \cdot \varphi(x)$ and this defines an R-linear structure on $\mathcal{A}$. For $r \in R$ let $\alpha : M \longrightarrow M, \beta : N \longrightarrow N$ be the endomorphisms defined by left multiplication by r. Then $r \cdot \varphi = \beta\varphi = \varphi\alpha$. So associated to two left R-modules M, N and an integer $i \geq 0$ is an isomorphism class of left R-modules, and the following procedures will calculate a representative

- Pick a projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and calculate the cohomology of the sequence of R-modules

$$0 \longrightarrow Hom(P_0,B) \longrightarrow Hom(P_1,B) \longrightarrow Hom(P_2,B) \longrightarrow \cdots$$

- Pick an injective resolution $0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ and calculate the cohomology of the sequence of R-modules

$$0 \longrightarrow Hom(A,I^0) \longrightarrow Hom(A,I^1) \longrightarrow Hom(A,I^2) \longrightarrow \cdots$$

It is not hard to check that for $r \in R$ left multiplication by r is given by $Ext^n_R(M, \beta) = Ext^n_R(\alpha, N)$.

Proposition 9. *Let R be a commutative noetherian ring and suppose A, B are finitely generated R-modules. Then $Ext^i_R(A, B)$ is a finitely generated R-module.*

Proof. Since R is noetherian and A is finitely generated we can find a projective resolution $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ with all the F_i finite free modules. Then in the following sequence every module is finitely generated (see our Module notes)

$$0 \longrightarrow Hom(F_0, B) \longrightarrow Hom(F_1, B) \longrightarrow Hom(F_2, B) \longrightarrow \cdots$$

So the cohomology modules $Ext^i_R(A, B)$ will also be finitely generated. $\square$

Recall that a module M over a commutative domain R is *divisible* if for every $0 \neq r \in R$ and $x \in M$ there is $y \in M$ such that $r \cdot y = x$. Any injective module is divisible. A commutative integral domain R is a Dedekind domain if and only if every divisible module is injective. Since a quotient of a divisible module is clearly divisible, it follows that over a Dedekind domain the quotient of an injective module is injective.

Proposition 10. *For any Dedekind domain R we have $Ext^n_R(A, B) = 0$ for $n \geq 2$.*

Proof. Find an injective module I and a monomorphism $B \longrightarrow I$. The quotient is divisible, hence injective, so $0 \longrightarrow B \longrightarrow I \longrightarrow J \longrightarrow 0 \longrightarrow \cdots$ is an injective resolution of B. It follows that $Ext^R_n(A, B)$ is the cohomology of the sequence

$$0 \longrightarrow Hom(A, I) \longrightarrow Hom(A, J) \longrightarrow 0 \longrightarrow \cdots$$

So it is clear that $Ext^R_n(A, B) = 0$ for $n \geq 2$. $\square$

5.1 Coextension

Let $\varphi : R \longrightarrow S$ be a morphism of commutative rings. Then for an R-modules A the R-module $Hom_R(S, A)$ has a canonical S-module structure, and this defines the *coextension functor* $P = Hom_R(S, -) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$. Let $U : S\mathbf{Mod} \longrightarrow \mathbf{Ab}$ be the forgetful functor and $Q = Hom_R(S, -) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ the usual functor. Then $Q = UP$ so for $n \geq 0$ and an assignment of injective resolutions $\mathcal{I}$ the functors $R^n Q$ and $U \circ R^n P$ are *equal*. So for an R-module A the Ext group $Ext^n_R(S, A)$ becomes an S-module in a canonical way, and for a morphism of R-modules $\beta : A \longrightarrow A'$ the morphism of groups $Ext^n_R(S, \beta) : Ext^n_R(S, A) \longrightarrow Ext^n_R(S, A')$ is a morphism of these modules. So the additive functor $Ext^n_R(S, -) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ lifts to an additive functor $R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$.

6 Another Characterisation of Derived Functors

Throughout this section $\mathcal{A}$ is an abelian category. If we say T is an additive functor, we mean it is an additive covariant functor $\mathcal{A} \longrightarrow \mathbf{Ab}$. Given two additive functors $T, T' : \mathcal{A} \longrightarrow \mathbf{Ab}$ we let $[T, T']$ denote the class of natural transformations $T \longrightarrow T'$. It is clear that this becomes a "large" abelian group (an abelian group whose underlying class may not be a set).

Suppose we have for every object A an additive functor $\Omega_A : \mathcal{A} \longrightarrow \mathbf{Ab}$ and for every morphism $\alpha : A \longrightarrow B$ a natural transformation $\Omega_\alpha : \Omega_B \longrightarrow \Omega_A$, such that $\Omega_\alpha \Omega_\gamma = \Omega_{\gamma\alpha}, \Omega_{\alpha+\gamma} = \Omega_\alpha + \Omega_\gamma$ and $\Omega_1 = 1$. We call this a *representation* of $\mathcal{A}$ in the additive functors $\mathcal{A} \longrightarrow \mathbf{Ab}$. We say it is a *small* representation if $[\Omega_A, T]$ is a set for any object A and additive functor $T : \mathcal{A} \longrightarrow \mathbf{Ab}$.

The primary example is $A \mapsto Hom(A, -), \alpha \mapsto Hom(\alpha, -)$, which is small since by the Yoneda Lemma there is an isomorphism of abelian groups $[Hom(A, -), T] \cong T(A)$. This isomorphism is also natural in A: given any morphism $\alpha : A \longrightarrow B$, composition with Ω_α defines a morphism of groups $[Hom(A, -), T] \longrightarrow [Hom(B, -), T]$ which fits into a commutative diagram:

$$\begin{array}{ccc} [Hom(A, -), T] & \Longrightarrow & T(A) \\ \downarrow & & \downarrow{\scriptstyle T(\alpha)} \\ [Hom(B, -), T] & \Longrightarrow & T(B) \end{array}$$

It follows that we can recover the functor T (up to natural equivalence) from the representation $A \mapsto Hom(A,-)$ and the morphisms from these objects to T. In detail: given an additive functor T define $S(A) = [Hom(A,-),T]$. For a morphism $\alpha : A \longrightarrow B$ let $S(A) \longrightarrow S(B)$ act by composition with Ω_α. Then this defines an additive functor S naturally equivalent to T. This motivates the following definition

Definition 4. Let $\mathcal{A}$ be an abelian category, Ω a small representation of $\mathcal{A}$ in the additive functors $\mathcal{A} \longrightarrow \mathbf{Ab}$. Given an additive functor T let ΩT denote the following additive functor: $(\Omega T)(A) = [\Omega_A, T]$ and for $\alpha : A \longrightarrow B$ we define

$$(\Omega T)(\alpha) : [\Omega_A, T] \longrightarrow [\Omega_B, T]$$
$$\psi \mapsto \psi\Omega_\alpha$$

Now assume $\mathcal{A}$ is an abelian category with enough injectives and let $\mathcal{I}$ be a fixed assignment of injective resolutions, with respect to which all right derived functors are calculated. To every object A and $n \geq 0$ we have associated an additive functor $Ext^n(A,-) : \mathcal{A} \longrightarrow \mathbf{Ab}$ and to a morphism $\alpha : A \longrightarrow B$ we have associated a natural transformation $Ext^n(\alpha,-) : Ext^n(B,-) \longrightarrow Ext^n(A,-)$. We have already checked that $A \mapsto Ext^n(A,-)$ defines a representation of $\mathcal{A}$ (n fixed).

Similarly if $\mathcal{A}$ is an abelian category with enough projectives, $A \mapsto \underline{Ext}^n(A,-)$ and $\alpha \mapsto \underline{Ext}^n(\alpha,-)$ defines a representation. If $\mathcal{A}$ has both enough injectives and projectives then for every A there is a canonical natural equivalence $\underline{Ext}^n(A,-) \cong Ext^n(A,-)$ with the property that the following diagram commutes for any morphism $\alpha : A \longrightarrow B$

$$\begin{array}{ccc} \underline{Ext}^n(B,-) & \Longrightarrow & Ext^n(B,-) \\ {\scriptstyle \underline{Ext}^n(\alpha,-)}\downarrow & & \downarrow{\scriptstyle Ext^n(\alpha,-)} \\ \underline{Ext}^n(A,-) & \Longrightarrow & Ext^n(A,-) \end{array}$$

Lemma 11. *Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. For $n \geq 0$ the representations $\Omega : A \mapsto Ext^n(A,-)$ and $\underline{\Omega} : A \mapsto \underline{Ext}^n(A,-)$ are small.*

Proof. For $n = 0$ there is a natural equivalence $\underline{Ext}^0(A,-) \cong Hom(A,-) \cong Ext^0(A,-)$ so both representations are trivially small. For $n \geq 1$ there is a natural equivalence $\underline{Ext}^n(A,-) \cong Ext^n(A,-)$ so it suffices to show that $\underline{\Omega}$ is small. Fix $n \geq 1$, an additive functor T and an object A. Let P be the resolution assigned to A, $\mu : K_A \longrightarrow P_{n-1}$ be the image of $P_n \longrightarrow P_{n-1}$ and consider the exact sequence

$$0 \longrightarrow K_A \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

We show Ω is small by establishing an isomorphism $KerT(\mu) \cong [Ext^n(A,-),T]$.For an object B calculating $\underline{Ext}^n(-,B)$ we can use the corresponding truncations of the duals of the projective resolutions chosen by $\mathcal{P}$, so by Proposition 20 of our Derived Functor notes there is an exact sequence

$$Hom(P_{n-1},B) \longrightarrow Hom(K_A,B) \longrightarrow \underline{Ext}^n(A,B) \longrightarrow 0 \qquad (6)$$

The morphism $Hom(K_A,B) \longrightarrow \underline{Ext}^n(A,B)$ is canonical and natural in A. If $e : P_n \longrightarrow K_A$ is the factorisation of ∂_n through μ then this map is defined by $x \mapsto \overline{xe}$. It is also natural in B, in the sense that for any $\beta : B \longrightarrow B'$ the following diagram commutes

$$\begin{array}{ccccccc} Hom(P_{n-1},B) & \longrightarrow & Hom(K_A,B) & \longrightarrow & \underline{Ext}^n(A,B) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Hom(P_{n-1},B') & \longrightarrow & Hom(K_A,B') & \longrightarrow & \underline{Ext}^n(A,B') & \longrightarrow & 0 \end{array}$$

In particular the following diagram is commutative with exact rows

$$\begin{array}{ccccccc}
Hom(P_{n-1},K_A) & \longrightarrow & Hom(K_A,K_A) & \longrightarrow & \underline{Ext}^n(A,K_A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \underline{Ext}^n(A,\mu)} & & \\
Hom(P_{n-1},P_{n-1}) & \longrightarrow & Hom(K_A,P_{n-1}) & \longrightarrow & \underline{Ext}^n(A,P_{n-1}) & \longrightarrow & 0
\end{array}$$

Let η be the image in $\underline{Ext}^n(A,K_A)$ of 1_{K_A}. Commutativity of the diagram shows that $\underline{Ext}^n(A,\mu)(\eta)$ is zero. Let $\Phi : \underline{Ext}^n(A,-) \longrightarrow T$ be a natural transformation. Consider the following commutative diagram

$$\begin{array}{ccc}
\underline{Ext}^n(A,K_A) & \longrightarrow & T(K_A) \\
\downarrow & & \downarrow{\scriptstyle T(\mu)} \\
\underline{Ext}^n(A,P_{n-1}) & \longrightarrow & T(P_{n-1})
\end{array}$$

Since $\underline{Ext}^n(A,\mu)(\eta) = 0$ it follows that the image in $T(K_A)$ of η belongs to $KerT(\mu)$. This assigns to any natural transformation Φ an element $\xi = \Phi_{K_A}(\eta) \in KerT(\mu)$. Next we show that this assignment is injective, by showing that any other natural transformation Θ with $\Theta_{K_A}(\eta) = \xi$ must be equal to Φ.

Let $\sigma : K_A \longrightarrow B$ be any morphism. We claim that the image of σ in $\underline{Ext}^n(A,B)$ under the canonical morphism $Hom(K_A,B) \longrightarrow \underline{Ext}^n(A,B)$ defined above is $\underline{Ext}^n(A,\sigma)(\eta)$. The morphism σ induces a natural transformation $Hom(-,K_A) \longrightarrow Hom(-,B)$ and therefore a cochain morphism of the image of the dual of P under these two functors. The induced maps on cohomology at n is $\underline{Ext}^n(A,\sigma)$. So the claim is not too hard to check.

Let B be any object and let $\rho \in \underline{Ext}^n(A,B)$. The exact sequence (6) shows that ρ is the image of some morphism $\sigma : K_A \longrightarrow B$. Since Θ is natural the following square must commute

$$\begin{array}{ccc}
\underline{Ext}^n(A,K_A) & \xrightarrow{\Theta_{K_A}} & T(K_A) \\
\downarrow & & \downarrow{\scriptstyle T(\sigma)} \\
\underline{Ext}^n(A,B) & \xrightarrow[\Theta_B]{} & T(B)
\end{array}$$

So $\Theta_B(\rho) = \Theta_B \underline{Ext}^n(A,\sigma)(\eta) = T(\sigma)(\xi)$. Since B and ρ were arbitrary it follows that $\Theta = \Phi$.

Next we show how to assign a natural transformation Φ to any $\xi \in KerT(\mu) \subseteq T(K_A)$. The obvious definition is the following: for $\rho \in \underline{Ext}^n(A,B)$ let $\sigma : K_A \longrightarrow B$ be any morphism mapping to ρ under $Hom(K_A,B) \longrightarrow \underline{Ext}^n(A,B)$ and let $\Phi_B(\rho) = T(\sigma)(\xi)$. We have to show that $T(\sigma)(\xi)$ does not depend on the morphism σ chosen in the preimage of ρ. If σ' is another such morphism, then $\sigma - \sigma'$ is in the kernel of $Hom(K_A,B) \longrightarrow \underline{Ext}^n(A,B)$ and since (6) is exact there is $\tau : P_{n-1} \longrightarrow B$ with $\sigma - \sigma' = \tau\mu$. Hence $T(\sigma - \sigma')(\xi) = 0$ since ξ is in the kernel of $T(\mu)$, and so $T(\sigma)(\xi) = T(\sigma')(\xi)$, as required. It is easy to check that Φ_B is a morphism of groups.

It is clear that $\Phi_{K_A}(\eta) = \xi$ so it only remains to show that Φ is natural. Suppose $\beta : B \longrightarrow B'$ is given and consider the diagram

$$\begin{array}{ccccc}
Hom(K_A,B) & \longrightarrow & \underline{Ext}^n(A,B) & \longrightarrow & T(B) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle T(\beta)} \\
Hom(K_A,B') & \longrightarrow & \underline{Ext}^n(A,B') & \longrightarrow & T(B')
\end{array}$$

The left hand square commutes by naturality of (6), so if we choose $\sigma : K_A \longrightarrow B$ to represent $\rho \in \underline{Ext}^n(A,B)$ then we can choose $\beta\sigma$ to represent $\underline{Ext}^n(A,\beta)(\rho)$. Hence

$$\Phi_{B'} \underline{Ext}^n(A,\beta)(\rho) = T(\beta\sigma)(\xi) = T(\beta)\Phi_B(\rho)$$

This finishes the construction of the bijection $[\underline{Ext}^n(A,-),T] \cong KerT(\mu)$. $\square$

Theorem 12. *Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. For $n \geq 1$ and any right exact functor T there is a canonical isomorphism natural in A and T*

$$[\underline{Ext}^n(A, -), T] \cong L_n T(A)$$

That is, there is a canonical natural equivalence $\underline{\Omega} T \cong L_n T$.

Proof. We assume all derived functors (including those making up the definition of Ω) are calculated relative to fixed assignments of injective and projective resolutions $\mathcal{I}, \mathcal{P}$. Assume $n \geq 1$ and for every object A with projective resolution P let $\mu_A : K_A \longrightarrow P_{n-1}$ be the canonical image of $P_n \longrightarrow P_{n-1}$. Let $\ell_n T(A)$ be $Ker T(\mu_A)$. For a morphism $\alpha : A \longrightarrow B$ let φ be a chain morphism lifting α, induce $\alpha' : K_A \longrightarrow K_B$ and define $\ell_N T(\alpha)$ by $x \mapsto T(\alpha')(x)$. As we showed in Section 3 of our Derived Functor notes, $\ell_n T$ is canonically naturally equivalent to $L_n T$ since T is right exact. But in the previous Lemma we defined a bijection $[\underline{Ext}^n(A, -), T] \cong \ell_n T(A)$ for arbitrary A by $\Phi \mapsto \Phi_{K_A}(\eta_A)$ where η_A was a special element of $\underline{Ext}^n(A, K_A)$. It is clear that this bijection is an isomorphism of abelian groups, and to show $\underline{\Omega} T$ is canonically naturally equivalent to $L_n T$ is only remains to show that this isomorphism is natural in A.

Let $\alpha : A \longrightarrow B$ be a morphism and consider the following diagram

$$\begin{array}{ccc}
[\underline{Ext}^n(A, -), T] & \longrightarrow & \ell_n T(A) \\
\downarrow & & \downarrow \\
[\underline{Ext}^n(B, -), T] & \longrightarrow & \ell_n T(B)
\end{array}$$

Lift α to a chain morphism $\varphi : P \longrightarrow Q$ of the chosen resolutions and let this induce a morphism $\alpha' : K_A \longrightarrow K_B$. Let $\Phi : \underline{Ext}^n(A, -) \longrightarrow T$ be a natural transformation. We have to show that $T(\alpha')\Phi_{K_A}(\eta_A) = (\Phi \underline{Ext}^n(\alpha, -))_{K_B}(\eta_B)$, which reduces to showing that $\underline{Ext}^n(A, \alpha')(\eta_A) = \underline{Ext}^n(\alpha, K_B)(\eta_B)$. So it would be enough to show that the following diagram commutes:

$$\begin{array}{ccc}
Hom(K_A, K_A) & \longrightarrow & \underline{Ext}^n(A, K_A) \\
\downarrow & & \downarrow {\scriptstyle \underline{Ext}^n(A, \alpha')} \\
Hom(K_A, K_B) & \longrightarrow & \underline{Ext}^n(A, K_B) \\
\uparrow & & \uparrow {\scriptstyle \underline{Ext}^n(\alpha, K_B)} \\
Hom(K_B, K_B) & \longrightarrow & \underline{Ext}^n(B, K_B)
\end{array}$$

But the top square commutes by naturality of the sequence (6) for A in the second variable and the bottom square commutes by naturality of the sequence (6) for B in the first variable, so the proof of naturality in A is complete.

Now suppose $\gamma : T \longrightarrow T'$ is a natural transformation. For any object A with chosen resolution P this gives rise to a chain morphism $\gamma_P : TP \longrightarrow T'P$ and we let $\ell_n T(A) \longrightarrow \ell_n T'(A)$ be defined by the restriction of γ_{K_A}. It is then clear that the left hand square in the following diagram commutes

$$\begin{array}{ccccc}
[\underline{Ext}^n(A, -), T] & \Longrightarrow & \ell_n T(A) & \Longrightarrow & L_n T(A) \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle (L_n \gamma)_A} \\
[\underline{Ext}^n(A, -), T'] & \Longrightarrow & \ell_n T'(A) & \Longrightarrow & L_n T'(A)
\end{array}$$

The natural transformation $L_n \gamma : L_n T \longrightarrow L_n T'$ is defined elsewhere in our notes. By definition $(L_n \gamma)_A : L_n T(A) \longrightarrow L_n T'(A)$ is the map $x + Im T(\partial_{n+1}) \mapsto \gamma_{P_n}(x) + Im T'(\partial_{n+1})$ which clearly makes the right hand diagram commute. This completes the proof. $\square$

Corollary 13. *For a ring R there is a canonical isomorphism natural in the right R-module A and the left R-module B*

$$[\underline{Ext}^n(A, -), - \otimes_R B] \cong Tor_n(A, B)$$

Proof. This is just $\mathcal{A} = \mathbf{Mod}R$, $T = - \otimes_R B$ and $L_n T = Tor_n(-, B)$ in the Theorem. Just to be perfectly clear what we mean by naturality: for any morphism $\alpha : A \longrightarrow A'$ of right R-modules the following diagram commutes

$$\begin{array}{ccc} [\underline{Ext}^n(A, -), - \otimes_R B] & \longrightarrow & Tor_n(A, B) \\ \downarrow & & \downarrow \\ [\underline{Ext}^n(A', -), - \otimes_R B] & \longrightarrow & Tor_n(A', B) \end{array}$$

For a morphism of right R-modules $\beta : B \longrightarrow B'$ the following diagram commutes

$$\begin{array}{ccc} [\underline{Ext}^n(A, -), - \otimes_R B] & \longrightarrow & Tor_n(A, B) \\ \downarrow & & \downarrow \\ [\underline{Ext}^n(A, -), - \otimes_R B'] & \longrightarrow & Tor_n(A, B') \end{array}$$

where the left hand vertical morphism acts by composition with the natural transformation $- \otimes_R B \longrightarrow - \otimes_R B'$ determined by β. $\square$