Ext

Daniel Murfet

May 16, 2006

Contents

1 Ext using Injectives 1
  1.1 Calculations using Injective Presentations ................. 4

2 Ext using Projectives 4
  2.1 Calculations using Projective Presentations ................. 6

3 Balancing Ext 7

4 Properties of Ext 9
  4.1 Ext for Linear Categories ................................... 9
  4.2 Dimension Shifting ............................................ 10
  4.3 Ext and Coproducts ........................................... 11

5 Ext for Commutative Rings 11
  5.1 Coextension .................................................. 12

6 Another Characterisation of Derived Functors 12

1 Ext using Injectives

If $A$ is an abelian category, then $\text{Hom}(A, -$) is a covariant, additive, kernel preserving functor $A \to \text{Ab}$ and $\text{Hom}(-, B)$ is a contravariant, additive functor which maps cokernels to kernels. Throughout this section $A$ will be an abelian category with enough injectives.

Definition 1. The right derived functors of $\text{Hom}(A, -$) are called the Ext groups.

$$\text{Ext}^i(A, B) = R^i \text{Hom}(A, -)(B)$$

The functor $\text{Ext}^i(A, -$) : $A \to \text{Ab}$ is additive and covariant for $i \geq 0$. Since $\text{Hom}(A, -$) is left exact the functors $\text{Ext}^0(A, -$) and $\text{Hom}(A, -$) are naturally equivalent. We simply write $\text{Ext}(A, -$) for $\text{Ext}^1(A, -$).

The group $\text{Ext}^i(A, B)$ is only determined up to isomorphism, and to calculate it we find an injective resolution $0 \to B \to I^0 \to I^1 \cdots$ and calculate the cohomology of the sequence

$$0 \to \text{Hom}(A, I^0) \to \text{Hom}(A, I^1) \to \text{Hom}(A, I^2) \to \cdots$$

We think of $\text{Ext}^i$ as assigning to any pair of objects $A, B$ an isomorphism class of abelian groups, which has the following properties:

- For any injective object $I$ we have $\text{Ext}^i(A, I) = 0$ for $i \neq 0$, since this is a property of any right derived functor.
- For any projective object $P$ we have $\text{Ext}^i(P, B) = 0$ for $i \neq 0$, since the higher right derived functors of the exact functor $\text{Hom}(P, -$) are zero.
For any exact sequence
\[ 0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0 \]
there are canonical morphisms \( \omega^0 : \text{Hom}(A, B') \rightarrow \text{Ext}(A, B') \) and \( \omega^n : \text{Ext}^n(A, B'') \rightarrow \text{Ext}^{n+1}(A, B') \) for \( n > 1 \) such that the following sequence is long exact
\[ 0 \rightarrow \text{Hom}(A, B') \rightarrow \text{Hom}(A, B) \rightarrow \text{Hom}(A, B'') \rightarrow \]
\[ \rightarrow \text{Ext}(A, B') \rightarrow \text{Ext}(A, B) \rightarrow \text{Ext}(A, B'') \rightarrow \]
\[ \rightarrow \text{Ext}^2(A, B') \rightarrow \text{Ext}^2(A, B) \rightarrow \text{Ext}^2(A, B'') \rightarrow \cdots \]
This sequence is called the long exact \( \text{Ext} \) sequence in the second variable. It is natural, in the sense that if we have a commutative diagram with exact rows
\[ 0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0 \]
\[ 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \]
Then the following diagrams commute for \( n \geq 1 \)
\[ \text{Hom}(A, B'') \rightarrow \text{Ext}(A, B') \]
\[ \text{Hom}(A, C'') \rightarrow \text{Ext}(A, C') \]
Let \( \alpha : A \rightarrow A' \) be a morphism, and let \( \alpha \) also denote the associated natural transformation \( \text{Hom}(A', -) \rightarrow \text{Hom}(A, -) \). Let \( \mathcal{I} \) be a fixed assignment of injective resolutions. Then there is a natural transformation \( R^n\alpha : R^n\text{Hom}(A', -) \rightarrow R^n\text{Hom}(A, -) \) and we denote by \( \text{Ext}^n(\alpha, B) \) the morphism \( (R^n\alpha)_B : \text{Ext}^n(A', B) \rightarrow \text{Ext}^n(A, B) \). Notice that for another morphism \( \gamma : A' \rightarrow A'' \), \( (R^n\alpha)(R^n\gamma) = R^n(\gamma\alpha) \) so for any object \( B \)
\[ \text{Ext}^n(\alpha, B) \text{Ext}^n(\gamma, B) = \text{Ext}^n(\gamma\alpha, B) \]
This defines a contravariant additive functor \( \text{Ext}^n(-, B) : A \rightarrow \text{Ab} \). For any exact sequence
\[ 0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0 \]
we have the following commutative diagram for \( n \geq 0 \)
\[ \cdots \rightarrow \text{Ext}^n(A', B') \rightarrow \text{Ext}^n(A', B) \rightarrow \text{Ext}^n(A', B'') \rightarrow \omega^n \rightarrow \text{Ext}^{n+1}(A', B') \rightarrow \cdots \]
\[ \text{Ext}^n(\alpha, B') \downarrow \quad \text{Ext}^n(\alpha, B) \downarrow \quad \text{Ext}^n(\alpha, B'') \downarrow \quad \text{Ext}^{n+1}(\alpha, B') \downarrow \]
\[ \cdots \rightarrow \text{Ext}^n(A', B') \rightarrow \text{Ext}^n(A', B) \rightarrow \text{Ext}^n(A', B'') \rightarrow \omega^n \rightarrow \text{Ext}^{n+1}(A', B') \rightarrow \cdots \] (1)

**Proposition 1.** For \( n \geq 0 \) and morphisms \( \alpha : A \rightarrow A' \) and \( \beta : B \rightarrow B' \)
\[ \text{Ext}^n(A, \beta) \text{Ext}^n(\alpha, B) = \text{Ext}^n(\alpha, B') \text{Ext}^n(A', \beta) \] (2)
It follows that \( \text{Ext}^n \) defines a functor \( A^{op} \times A \rightarrow \text{Ab} \) for \( n \geq 0 \), with \( \text{Ext}^n(\alpha, \beta) : \text{Ext}^n(A', B) \rightarrow \text{Ext}^n(A, B') \) given by the equivalent expressions in (2). The partial functors are the functors \( \text{Ext}^n(A, -) \) and \( \text{Ext}^n(-, B) \) defined above.

**Proof.** This follows for arbitrary \( \alpha \) and monomorphisms (or epimorphisms) \( \beta \) by commutativity of (1). Since \( A \) has epi-mono factorisations it then follows for arbitrary \( \beta \). The bifunctor \( \text{Ext}^n \) is defined relative to an assignment of injective resolutions \( \mathcal{I} \). If \( \mathcal{J} \) is another such assignment then the associated bifunctor is canonically naturally equivalent to the one defined for \( \mathcal{I} \). \( \square \)
For a short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ the corresponding sequence of natural transformations $\text{Hom}(A'',-): \text{Hom}(A,-) \rightarrow \text{Hom}(A',-)$ is exact on injectives. So for $n \geq 0$ and any object $B$ there are canonical connecting morphisms $\omega^n : \text{Ext}^n(A',B) \rightarrow \text{Ext}^{n+1}(A'',B)$ fitting in to a long exact sequence

$$\cdots \rightarrow \text{Ext}^n(A'',B) \rightarrow \text{Ext}^n(A,B) \rightarrow \text{Ext}^n(A',B) \rightarrow \text{Ext}^{n+1}(A'',B) \rightarrow \cdots$$

This sequence is called the *long exact Ext sequence in the first variable*. It is natural in both $B$ and the exact sequence. For a morphism $\beta : B \rightarrow B'$ the following diagram commutes

$$\cdots \rightarrow \text{Ext}^n(A'',B) \rightarrow \text{Ext}^n(A,B) \rightarrow \text{Ext}^n(A',B) \xrightarrow{\omega^n} \text{Ext}^{n+1}(A'',B) \rightarrow \cdots$$

$$\cdots \rightarrow \text{Ext}^n(A'',B') \rightarrow \text{Ext}^n(A',B') \rightarrow \text{Ext}^{n+1}(A'',B') \rightarrow \cdots$$

And for a commutative diagram with exact rows

$$\begin{array}{ccc}
0 & \rightarrow & A' \\
\downarrow & & \downarrow \\
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
0 & \rightarrow & A'' \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}$$

The following diagram commutes for any object $B$

$$\begin{array}{ccc}
\cdots & \rightarrow & \text{Ext}^n(A'',B) \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & \text{Ext}^n(A',B) \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & \text{Ext}^{n+1}(A'',B) \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & \text{Ext}^{n+1}(A',B) \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & \cdots
\end{array}$$

We have shown that for every assignment of injective resolutions $\mathcal{I}$ we obtain a bifunctor $\text{Ext}^n_\mathcal{I}(-,-) : \mathcal{A}^{\mathcal{I}} \times \mathcal{A} \rightarrow \text{Ab}$ for $n \geq 0$ with the property that short exact sequences in either variable lead to a long exact sequence which is natural with respect to morphisms of the exact sequence and morphisms in the remaining variable. The connecting morphisms for these sequences depend only on $\mathcal{I}$.

If $\mathcal{J}$ is another assignment of resolutions then we obtain another bifunctor $\text{Ext}^n_\mathcal{J}(-,-)$ for $n \geq 0$ which is canonically naturally equivalent to $\text{Ext}^n_\mathcal{I}(-,-)$. The connecting morphisms for the two assignments $\mathcal{I}, \mathcal{J}$ agree in the following sense: for an object $A$ and an exact sequence $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ the following diagram commutes

$$\cdots \rightarrow \text{Ext}^n_\mathcal{I}(A',B') \rightarrow \text{Ext}^n_\mathcal{I}(A,B) \rightarrow \text{Ext}^n_\mathcal{I}(A',B') \rightarrow \cdots$$

Similarly for an object $B$ and an exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ the following diagram commutes

$$\cdots \rightarrow \text{Ext}^n_\mathcal{I}(A'',B') \rightarrow \text{Ext}^n_\mathcal{I}(A,B) \rightarrow \text{Ext}^n_\mathcal{I}(A',B') \rightarrow \cdots$$

Both these claims follow directly from our Derived Functor notes.
1.1 Calculations using Injective Presentations

Since $\text{Hom}(X, -)$ is left exact we can use our results truncated injective resolutions to show that the functor $\text{Ext}(X, -)$ is naturally equivalent to the functor $E$ defined by the following procedure: pick for every object $A$ an exact sequence

$$0 \longrightarrow A \longrightarrow I \overset{\mu}{\longrightarrow} C \longrightarrow 0$$

with $I$ injective. Then $E(A)$ is the cokernel of $\text{Hom}(X, I) \longrightarrow \text{Hom}(X, C)$ and given a morphism $\alpha : A \longrightarrow B$ where $B$ is assigned the sequence $0 \longrightarrow B \longrightarrow J \longrightarrow D \longrightarrow 0$ use injectivity of $J$ to lift $\alpha$ to a morphism $\varphi : I \longrightarrow J$ and then induce $\alpha'$ fitting into a commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow{\alpha} & & \downarrow{\alpha'} \\
0 & \longrightarrow & B \\
\end{array} \quad \begin{array}{ccc}
& & \\
\longrightarrow & \longrightarrow & \\
\alpha' & \longrightarrow & \alpha \\
\end{array} \quad \begin{array}{ccc}
0 & \longrightarrow & I \\
\downarrow{\tau} & & \downarrow{\tau} \\
0 & \longrightarrow & J \\
\end{array} \quad \begin{array}{ccc}
& & \\
\longrightarrow & \longrightarrow & \\
\mu & \longrightarrow & \mu \\
\end{array} \quad \begin{array}{ccc}
0 & \longrightarrow & C \\
\end{array} \longrightarrow 0$$

Then $E(\alpha) : \text{Hom}(X, C)/\text{Im}T(\mu) \longrightarrow \text{Hom}(X, D)/\text{Im}T(\tau)$ is defined by composition with $\alpha'$. It turns out that this gives a well-defined additive functor naturally equivalent to $\text{Ext}(X, -)$.

2 Ext using Projectives

Throughout this section $\mathcal{A}$ will be an abelian category with enough projectives. For an object $A$ the functor $\text{Hom}(-, A)$ is contravariant, but considered as a functor $\mathcal{A}^{\text{op}} \longrightarrow \text{Ab}$ it is a left exact covariant functor.

**Definition 2.** The right derived functors of $\text{Hom}(-, B)$ are the $\text{Ext}$ groups.

$$\text{Ext}^i(A, B) = R^i\text{Hom}(-, B)(A)$$

The functor $\text{Ext}^i(-, B) : \mathcal{A} \longrightarrow \text{Ab}$ is additive and contravariant for $i \geq 0$. The functors $\text{Ext}^0(-, B)$ and $\text{Hom}(-, B)$ are naturally equivalent. We simply write $\text{Ext}(-, B)$ for $\text{Ext}^0(-, B)$.

The group $\text{Ext}^i(A, B)$ is only determined up to isomorphism, and to calculate it we find a projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and calculate the cohomology of the sequence

$$0 \longrightarrow \text{Hom}(P_0, B) \longrightarrow \text{Hom}(P_1, B) \longrightarrow \text{Hom}(P_2, B) \longrightarrow \cdots$$

We think of $\text{Ext}^i$ as assigning to any pair of objects $A, B$ an isomorphism class of abelian groups, which has the following properties:

- For any projective object $P$ we have $\text{Ext}^i(P, B) = 0$ for $i \neq 0$, since this is a property of any right derived functor (remember we are taking right derived functors in $\mathcal{A}^{\text{op}}$, where $P$ is injective).

- For any injective object $I$ we have $\text{Ext}^i(A, I) = 0$ for $i \neq 0$, since the higher right derived functors of the exact functor $\text{Hom}(-, I)$ are zero.

For any exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

there are canonical morphisms $\omega_0 : \text{Hom}(A', B) \longrightarrow \text{Ext}(A'', B)$ and $\omega^n : \text{Ext}^n(A', B) \longrightarrow \text{Ext}^{n+1}(A'', B)$ for $n \geq 1$ such that the following sequence is long exact

$$0 \longrightarrow \text{Hom}(A', B) \longrightarrow \text{Hom}(A, B) \longrightarrow \text{Hom}(A', B) \longrightarrow \text{Ext}(A'', B) \longrightarrow \text{Ext}(A', B) \longrightarrow \text{Ext}^2(A'', B) \longrightarrow \text{Ext}^2(A', B) \longrightarrow \cdots$$
This sequence is called the long exact $\text{Ext}$ sequence in the first variable. It is natural, in the sense that if we have a commutative diagram with exact rows

\[ 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 \]
\[ 0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0 \]

Then the following diagrams commute for $n \geq 1$

\[
\begin{array}{ccc}
\text{Hom}(C', B) & \longrightarrow & \text{Ext}(C'', B) \\
\text{Hom}(A', B) & \longrightarrow & \text{Ext}(A'', B)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hom}(C', B) & \longrightarrow & \text{Ext}(C'', B) \\
\text{Hom}(A', B) & \longrightarrow & \text{Ext}(A'', B)
\end{array}
\]

This defines a covariant additive functor $\text{Ext}^n(A, -) : \mathcal{A} \rightarrow \text{Ab}$. For any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \rightarrow 0$ the following diagram is commutative for $n \geq 0$

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \text{Ext}^n(A', B) \\
\text{Ext}^n(A', \beta) & \longrightarrow & \text{Ext}^n(A, B) \\
\cdots & \longrightarrow & \text{Ext}^n(A', B') \\
\text{Ext}^n(A', \beta) & \longrightarrow & \text{Ext}^n(A, B')
\end{array}
\]

(3)

**Proposition 2.** For $n \geq 0$ and morphisms $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$

\[
\text{Ext}^n(A, \beta)\text{Ext}^n(\alpha, B) = \text{Ext}^n(\alpha, B')\text{Ext}^n(A', \beta)
\]

(4)

It follows that $\text{Ext}^n$ defines a functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \longrightarrow \text{Ab}$ for $n \geq 0$, with $\text{Ext}^n(\alpha, \beta) : \text{Ext}^n(A', B) \longrightarrow \text{Ext}^n(A, B')$ given by the equivalent expressions in (4). The partial functors are the functors $\text{Ext}^n(\alpha, -)$ and $\text{Ext}^n(-, \beta)$ defined above.

Proof. This follows for arbitrary $\beta$ and monomorphisms (or epimorphisms) $\alpha$ by commutativity of (3). Since $\mathcal{A}$ has epi-mono factorisations it then follows for arbitrary $\alpha$. If we use a different assignment of projective resolutions to calculate $\text{Ext}^n$ then the results will be canonically naturally equivalent.

For a short exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ the corresponding sequence of natural transformations $\text{Hom}(-, B') \longrightarrow \text{Hom}(-, B) \longrightarrow \text{Hom}(-, B'')$ is exact on injectives (considered as covariant functors $\mathcal{A}^{\text{op}} \longrightarrow \text{Ab}$). So for $n \geq 0$ and any object $A$ there are canonical connecting morphisms $\omega^n : \text{Ext}^n(A, B'') \longrightarrow \text{Ext}^{n+1}(A, B')$ fitting in to a long exact sequence

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \text{Ext}^n(A, B) \\
\cdots & \longrightarrow & \text{Ext}^n(A', B')
\end{array}
\]

This sequence is called the long exact $\text{Ext}$ sequence in the second variable. It is natural in both $A$ and the exact sequence. For a morphism $\alpha : A \longrightarrow A'$ the following diagram commutes

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \text{Ext}^n(A', B') \\
\cdots & \longrightarrow & \text{Ext}^n(A, B)
\end{array}
\]

(5)
And for a commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow & B' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B'' \\
\end{array}
\]

The following diagram commutes for any object \(A\)

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \text{Ext}^n(A, B') \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{Ext}^n(A, B) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{Ext}^n(A, B'') \\
\end{array}
\]

We have shown that for every assignment of projective resolutions \(\mathcal{P}\) we obtain a bifunctor \(\text{Ext}^n_{\mathcal{P}}(-,-) : \text{Ab} \to \text{Ab}\) for \(n \geq 0\) which is canonically naturally equivalent to \(\text{Ext}^n_{\mathcal{Q}}(-,-)\). The connecting morphisms for these sequences depend only on \(\mathcal{P}\).

If \(\mathcal{Q}\) is another assignment of resolutions then we obtain another bifunctor \(\text{Ext}^n_{\mathcal{Q}}(-,-)\) for \(n \geq 0\) which is canonically naturally equivalent to \(\text{Ext}^n_{\mathcal{P}}(-,-)\). The two assignments \(\mathcal{P}, \mathcal{Q}\) agree in the following sense: for an object \(B\) and an exact sequence \(0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0\) the following diagram commutes

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \text{Ext}^n_{\mathcal{P}}(A'', B) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{Ext}^n_{\mathcal{P}}(A', B) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{Ext}^n_{\mathcal{P}}(A''', B) \\
\end{array}
\]

Similarly for an object \(A\) and an exact sequence \(0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0\) the following diagram commutes

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \text{Ext}^n_{\mathcal{Q}}(A'', B) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{Ext}^n_{\mathcal{Q}}(A', B) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{Ext}^n_{\mathcal{Q}}(A''', B) \\
\end{array}
\]

Both these claims follow directly from our Derived Functor notes.

2.1 Calculations using Projective Presentations

Since \(\text{Hom}(-, B) : \text{Ab} \to \text{Ab}\) is left exact we can use our results on truncated injective resolutions to show that \(\text{Ext}\) is naturally equivalent to the functor \(E\) defined by the following procedure: pick for every object \(A\) an exact sequence

\[
0 \longrightarrow K \xrightarrow{\mu} P \longrightarrow A \longrightarrow 0
\]

with \(P\) projective. Then \(E(A)\) is the cokernel of \(\text{Hom}(P, B) \longrightarrow \text{Hom}(K, B)\) and given a morphism \(\alpha : A \longrightarrow C\) where \(C\) is assigned the sequence \(0 \longrightarrow M \longrightarrow Q \longrightarrow C \longrightarrow 0\) use projectivity of \(Q\) to lift \(\alpha\) to a morphism \(\varphi : P \longrightarrow Q\) and then induce \(\alpha'\) fitting into a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K \\
\downarrow & \alpha' & \downarrow \\
0 & \longrightarrow & M \\
\downarrow & \tau & \downarrow \\
0 & \longrightarrow & Q \\
\end{array}
\]
Then $E'(a) : E'(C) \to E'(A)$, which is a map $\text{Hom}(M, B)/\text{Im}T(\tau) \to \text{Hom}(K, B)/\text{Im}T(\tau)$ is defined by composition with $\alpha'$. It turns out that this is a well-defined contravariant additive functor naturally equivalent to $\text{Ext}(-, B)$.

In fact we have already studied the functor $E'$ for right modules over a ring in our Hilton & Stammbach notes, where we proved the following

- For any two right modules $A, B$ over a ring there is a bijection $E'(A, B) \cong Y(A, B)$ where $Y(A, B)$ is the set of extensions of $A$ by $B$ (which are exact sequences $0 \to B \to E \to A \to 0$) modulo a certain equivalence relation. In particular $E(AB) = 0$ if and only if every exact sequence $0 \to B \to E \to A \to 0$ splits.

3 Balancing Ext

Throughout this section $A$ is an abelian category with enough injectives and projectives, and we choose once and for all assignments of resolutions $P, T$, with respect to which all derived functors are calculated. We have defined two bifunctors $\text{Ext}^n(-, -)$ and $\text{Ext}^n(-, -)$ for $n \geq 0$. The first is calculated by taking the right derived functors of the contravariant functors $\text{Hom}(-, B)$ and the second by taking the right derived functors of the covariant functors $\text{Hom}(A, -)$. We claim that these two bifunctors are naturally equivalent. We begin with the case $n = 0$.

Lemma 3. There are canonical natural equivalences of bifunctors $E\text{xt}^0(-, -) \cong \text{Hom}(-, -)$ and $\text{Hom}(-, -) \cong E\text{xt}^0(-, -)$.

Proof. Let the Ext functors be calculated with respect to some assignment $I$ of injective resolutions. For an object $A$ there is a canonical natural equivalence $E\text{xt}^0(A, -) \cong \text{Hom}(A, -)$, so we need only show these isomorphisms are also natural in $B$, which is not difficult. Similarly there is a canonical natural equivalence $E\text{xt}^0(-, B) \cong \text{Hom}(-, B)$, which is also natural in the first variable. So all three functors are naturally equivalent.

Proposition 4. For $n \geq 0$ there is a canonical natural equivalence of bifunctors $\Phi^n : \text{Ext}^n(-, -) \cong E\text{xt}^n(-, -)$.

Proof. We proceed by induction on $n$, having already proved the result for $n = 0$. Assume that there is a canonical natural equivalence $\Phi^n$ and let objects $A, B$ be given. We have to define a canonical isomorphism $\Phi_{A,B}^n$ which is natural in $A$ and $B$. Choose an injective presentation of $B$

$$0 \to B \xrightarrow{\nu} I \xrightarrow{\eta} S \to 0$$

We know that $\text{Ext}^1(A, I) = 0 = E\text{xt}^1(A, I)$ for $i \neq 0$. Now we show how to define the isomorphism $\Phi_{A,B}^{n+1} : E\text{xt}^{n+1}(A, B) \to E\text{xt}^n(A, B)$. There are two cases: if $n = 1$ then the long exact sequence for $\text{Ext}$ in the second variable and the long exact sequence for $E\text{xt}$ in the second variable give a commutative diagram with exact rows:

$$
\begin{array}{cccccccc}
0 & \to & E\text{xt}^0(A, B) & \to & E\text{xt}^0(A, I) & \to & E\text{xt}^0(A, S) & \xrightarrow{\omega^0} & E\text{xt}^1(A, B) & \to & 0 \\
& & \downarrow \Phi_{A,S}^0 & & \parallel & & \parallel & & \downarrow \Phi_{A,B}^1 & \\
0 & \to & E\text{xt}^0(A, B) & \to & E\text{xt}^0(A, I) & \to & E\text{xt}^0(A, S) & \xrightarrow{\omega^0} & E\text{xt}^1(A, B) & \to & 0 \\
\end{array}
$$

This induces an isomorphism $\Phi_{A,B}^1 : E\text{xt}^1(A, B) \to E\text{xt}^1(A, B)$ making the diagram commute. For $n \geq 1$ the connecting morphisms $E\text{xt}^n(A, S) \to E\text{xt}^{n+1}(A, B)$ and $E\text{xt}^n(A, S) \to E\text{xt}^{n+1}(A, B)$ in the two sequences are isomorphisms, and we define $\Phi_{A,B}^{n+1}$ to be the unique morphism fitting into the following commutative diagram

$$
\begin{array}{cccccccc}
E\text{xt}^n(A, S) & \xrightarrow{\Phi_{A,S}^n} & E\text{xt}^{n+1}(A, B) \\
\Phi_{A,S}^n & \parallel & \Phi_{A,B}^{n+1} & \parallel \\
E\text{xt}^n(A, S) & \to & E\text{xt}^{n+1}(A, B) \\
\end{array}
$$

7
Next we have to show that the isomorphism $\Phi_{A,B}^{n+1}$ does not depend on the chosen presentation. Suppose we have a commutative diagram with exact rows and the middle objects injective

$$
\begin{array}{ccccccc}
0 & \rightarrow & B & \rightarrow & I & \rightarrow & S & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B' & \rightarrow & I' & \rightarrow & S' & \rightarrow & 0
\end{array}
$$

Consider the following cube for $n \geq 0$

$$
\begin{array}{ccccccc}
\text{Ext}^n(A, S') & \xrightarrow{\omega^n} & \text{Ext}^{n+1}(A, B') \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
\text{Ext}^n(A, S) & \xrightarrow{\omega^n} & \text{Ext}^{n+1}(A, B) & \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
\Phi_{A,S}^n & \xrightarrow{\Phi_{A,S'}^n} & \text{Ext}^n(A, S') & \xrightarrow{\omega^n} & \text{Ext}^{n+1}(A, B') \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
\text{Ext}^n(A, S) & \xrightarrow{\omega^n} & \text{Ext}^{n+1}(A, B) & \\
\end{array}
$$

If we use the above technique to produce isomorphisms $\text{Ext}^{n+1}(A, B) \rightarrow \text{Ext}^{n+1}(A, B)$ and $\text{Ext}^{n+1}(A, B') \rightarrow \text{Ext}^{n+1}(A, B')$ using the given presentations then in either case ($n = 1$ or otherwise) these morphisms make the front and back squares on the cube commute. The left square commutes since by assumption $\Phi^n$ is natural, and the top and bottom squares commute by the naturality of the connecting morphism. Since $\omega^n : \text{Ext}^n(A, S) \rightarrow \text{Ext}^{n+1}(A, B)$ is an epimorphism it follows that the right hand square also commutes.

If we are given two injective presentations of $B$ then put $B = B'$ in the diagram and induce $I \rightarrow I'$ and $S \rightarrow S'$ making it commutative. Then the cube above shows that the resulting isomorphism $\Phi_{A,B}^{n+1}$ is the same in both cases. So we have constructed an isomorphism $\Phi_{A,B}^{n+1}$ that depends only on $A, B$, the assignments $\mathcal{P}, \mathcal{I}$ and the natural equivalence $\Phi^n$. These isomorphisms are natural in $B$ since we can lift $B \rightarrow B'$ to a morphism of the injective presentations, and then use the cube.

To prove naturality in $A$ we construct a cube similar to the one above, but with a fixed presentation and $A$ varying. Using naturality of $\Phi^n$ in $A$ and the diagrams (1) and (5) it is not hard to see that $\Phi^{n+1}$ is natural in $A$ and is therefore a natural equivalence of bifunctors. Since by the inductive hypothesis $\Phi^n$ depends only on the assignment of resolutions $\mathcal{P}, \mathcal{I}$ it follows that this is true of $\Phi^{n+1}$ as well. \hfill \Box

If $A$ has both enough injectives and enough projectives and $\mathcal{I}, \mathcal{P}$ are assignments of injective and projective resolutions respectively, there is a natural equivalence of the bifunctors $\text{Ext}_{\mathcal{I}}^n(-, -)$ and $\text{Ext}_{\mathcal{P}}^n(-, -)$ for $n \geq 0$. So every pair of objects $A, B$ and integer $n \geq 0$ determines an isomorphism class of abelian groups. We can calculate a representative of this class in the following ways

- Choose a projective resolution $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ of $A$ and calculate the cohomology of the following cochain complex of abelian groups

  $$
  0 \rightarrow \text{Hom}(P_0, B) \rightarrow \text{Hom}(P_1, B) \rightarrow \text{Hom}(P_2, B) \rightarrow \cdots
  $$

- Choose an injective resolution $0 \rightarrow B \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ of $B$ and calculate the cohomology of the following cochain complex of abelian groups

  $$
  0 \rightarrow \text{Hom}(A, I^0) \rightarrow \text{Hom}(A, I^1) \rightarrow \text{Hom}(A, I^2) \rightarrow \cdots
  $$
If there is no chance of confusion we simply refer to any of these groups by \( \text{Ext}^n(A, B) \) and drop \( \text{Ext} \) from the notation. But if \( \mathcal{A} \) does not have both enough injectives and enough projectives, we will refer explicitly to the bifunctor \( \text{Ext} \) or \( \text{Ext} \) used.

In the case where \( \mathcal{A} = \text{Mod}_R \) for a ring \( R \), there is a bijection between elements of \( \text{Ext}(A, B) \) and exact sequences \( 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0 \) modulo a certain equivalence relation. In particular \( \text{Ext}(A, B) = 0 \) if and only if every such exact sequence is split.

**Remark 1.** One would like the natural equivalence of \( \text{Ext} \) and \( \text{Ext} \) to be compatible with the connecting morphisms for both bifunctors. One can get this in one variable (see Hilton & Stammbach), but it is not clear how to do it in the other variable.

### 4 Properties of \( \text{Ext} \)

#### 4.1 Ext for Linear Categories

**Definition 3.** If \( R \) is a ring then an \( R \)-linear abelian category is an abelian category \( \mathcal{A} \) together with a left \( R \)-module structure on all the morphism groups \( \text{Hom}_\mathcal{A}(A, B) \) such that composition is bilinear. That is,

\[
\gamma(r \cdot \alpha) = r \cdot (\gamma \alpha) \\
(r \cdot \alpha) \gamma = r \cdot (\alpha \gamma)
\]

whenever \( r \in R \) and the composition makes sense. Then for every object \( A \), we have a covariant, additive, kernel preserving functor \( \text{Hom}(A, -) : \mathcal{A} \rightarrow \text{RMod} \) and a contravariant, additive functor \( \text{Hom}(-, A) : \mathcal{A} \rightarrow \text{RMod} \) which maps cokernels to kernels.

Let \( U : \text{RMod} \rightarrow \text{Ab} \) be the forgetful functor, which is faithful and exact. This functor maps the canonical kernels, cokernels, images, zero and biproducts of \( \text{RMod} \) to the corresponding canonical structure on \( \text{Ab} \). So if \( X \) is a (co)chain complex in \( \text{RMod} \) then the (co)homology modules have as underlying groups the (co)homology groups of the sequence considered as a complex of groups.

For an object \( A \) let \( S \) be the functor \( \text{Hom}(A, -) : \mathcal{A} \rightarrow \text{RMod} \) and let \( T \) be \( \text{Hom}(A, -) : \mathcal{A} \rightarrow \text{Ab} \). Then \( T = US \) so for \( n \geq 0 \) and an assignment of injective resolutions \( I \) the functors \( R^n T \) and \( U \circ R^n S \) are equal. So for an object \( B \) the Ext group \( \text{Ext}^n(A, B) \) becomes an \( R \)-module in a canonical way, and for \( \beta : B \rightarrow B' \) the morphism of groups \( \text{Ext}^n(A, \beta) : \text{Ext}^n(A, B) \rightarrow \text{Ext}^n(A, B') \) is a morphism of these modules. Similarly if \( \alpha : A \rightarrow A' \) is a morphism of modules then the morphism of groups \( \text{Ext}^n(A', B) \rightarrow \text{Ext}^n(A, B) \) is a morphism of modules, so \( \text{Ext}^n(-, B) \) lifts to a contravariant additive functor \( \mathcal{A} \rightarrow \text{RMod} \). Also \( \text{Ext}^0(A, -) : \mathcal{A} \rightarrow \text{RMod} \) is canonically naturally equivalent to \( \text{Hom}(A, -) \).

So for a fixed assignment of injective resolutions \( I \) the bifunctor \( \text{Ext}^n(\cdot, \cdot) : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{RMod} \). If \( J \) is another assignment of injective resolutions then the resulting bifunctors (with values in \( \text{RMod} \)) are canonically naturally equivalent.

Given an assignment of resolutions \( I \) and an exact sequence \( 0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0 \) the connecting morphisms \( \text{Ext}^n(A, B') \rightarrow \text{Ext}^{n+1}(A, B) \) for \( n \geq 0 \) are all module morphisms, so the long exact sequence of Ext in the second variable

\[
\cdots \rightarrow \text{Ext}^n(A, B') \rightarrow \text{Ext}^n(A, B) \rightarrow \text{Ext}^n(A, B'') \rightarrow \text{Ext}^{n+1}(A, B') \rightarrow \cdots
\]

is a long exact sequence of modules. Similarly if \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \) is an exact sequence then the connecting morphisms \( \text{Ext}^n(A', B) \rightarrow \text{Ext}^{n+1}(A'', B) \) are module morphisms and the long exact sequence of Ext in the first variable

\[
\cdots \rightarrow \text{Ext}^n(A'', B) \rightarrow \text{Ext}^n(A, B) \rightarrow \text{Ext}^n(A', B) \rightarrow \text{Ext}^{n+1}(A'', B) \rightarrow \cdots
\]

is a long exact sequence of modules.
Similarly for an object $B$ let $S$ be the functor $\text{Hom}(-, B) : \mathcal{A} \rightarrow \text{RMod}$ and let $T$ be $\text{Hom}(-, B) : \mathcal{A} \rightarrow \text{RMod}$. Then $T = US$ so for $n \geq 0$ and an assignment of projective resolutions $\mathcal{P}$ the functors $\text{Ext}^n(-, B)$ and $\text{Ext}^n(A, -)$ lift to module valued functors and $\text{Ext}^n(-, B) : \mathcal{A} \rightarrow \text{RMod}$ is naturally equivalent to $\text{Hom}(-, B)$.

For a fixed assignment of projective resolutions $\mathcal{P}$ the bifunctor $\text{Ext}^n(-, -)$ becomes a bifunctor $\text{Ext}^n : \mathcal{A}^{op} \times \mathcal{A} \rightarrow \text{RMod}$. If $Q$ is another assignment of projective resolutions then the resulting bifunctors (with values in $\text{RMod}$) are canonically naturally equivalent. The two long exact sequences for $\text{Ext}^n$ are sequences of modules and module morphisms.

Now suppose $\mathcal{A}$ has enough projectives and injectives, and let $\mathcal{P}$ and $I$ be assignments of projective and injective resolutions, respectively. The canonical natural equivalences $\text{Ext}^0(-, -) \cong \text{Hom}(-, -)$ and $\text{Hom}(-, -) \cong \text{Ext}^0(-, -)$ give natural equivalences of the module-valued bifunctors. Then our earlier proof shows that for $n \geq 0$ there is a canonical natural equivalence $\text{Ext}^n(-, -) \cong \text{Ext}^n(-, -)$ of bifunctors $\mathcal{A}^{op} \times \mathcal{A} \rightarrow \text{RMod}$.

So associated to any pair of objects $A, B$ is an isomorphism class of $R$-modules $\text{Ext}^n(A, B)$. If $\mathcal{A}$ has enough projectives, we can find a representative of this class by choosing a projective resolution $P$ of $A$ and calculating the cohomology modules of $0 \rightarrow \text{Hom}(P_0, B) \rightarrow \text{Hom}(P_1, B) \rightarrow \cdots$. If $\mathcal{A}$ has enough injectives, we can find a representative by choosing an injective resolution $I$ of $B$ and calculating the cohomology modules of $0 \rightarrow \text{Hom}(A, I^0) \rightarrow \text{Hom}(A, I^1) \rightarrow \cdots$.

### 4.2 Dimension Shifting

The following two results are immediate consequences of our notes on dimension shifting.

**Proposition 5.** Let $\mathcal{A}$ be an abelian category with enough injectives. Suppose we have an exact sequence in $\mathcal{A}$ with all $I^i$ injective and $m \geq 0$

$$0 \rightarrow B \rightarrow I^0 \rightarrow \cdots \rightarrow I^{m-1} \rightarrow I^m \rightarrow M \rightarrow 0$$

Then for any object $A$ there are canonical isomorphisms $\rho^n : \text{Ext}^n(A, M) \rightarrow \text{Ext}^{n+m+1}(A, B)$ for $n \geq 1$, and an exact sequence

$$\text{Hom}(A, I^m) \rightarrow \text{Hom}(A, M) \rightarrow \text{Ext}^{m+1}(A, B) \rightarrow 0$$

These are both natural in $A$, in the sense that for a morphism $A \rightarrow A'$ the following two diagrams commute for $n \geq 1$ and $m \geq 0$.

\[
\begin{align*}
\text{Ext}^n(A', M) & \longrightarrow \text{Ext}^{n+m+1}(A', B) \\
\downarrow & \quad \downarrow \\
\text{Ext}^n(A, M) & \longrightarrow \text{Ext}^{n+m+1}(A, B) \\
\text{Hom}(A', I^m) & \longrightarrow \text{Hom}(A', M) \longrightarrow \text{Ext}^{m+1}(A', B) \longrightarrow 0 \\
\downarrow & \quad \downarrow \\
\text{Hom}(A, I^m) & \longrightarrow \text{Hom}(A, M) \longrightarrow \text{Ext}^{m+1}(A, B) \longrightarrow 0
\end{align*}
\]

**Proposition 6.** Let $\mathcal{A}$ be an abelian category with enough projectives. Suppose we have an exact sequence in $\mathcal{A}$ with all $P_i$ projective and $m \geq 0$

$$0 \rightarrow M \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

Then for any object $B$ there are canonical isomorphisms $\rho^n : \text{Ext}^n(M, B) \rightarrow \text{Ext}^{n+m+1}(A, B)$ for $n \geq 1$ and an exact sequence

$$\text{Hom}(P_m, B) \rightarrow \text{Hom}(M, B) \rightarrow \text{Ext}^{m+1}(A, B) \rightarrow 0$$
These are both natural in \( B \), in the sense that for a morphism \( B \rightarrow B' \) the following two diagrams commute for \( n \geq 1 \) and \( m \geq 0 \):

\[
\begin{array}{ccc}
\text{Ext}^n(M, B) & \rightarrow & \text{Ext}^{n+m+1}(A, B) \\
\downarrow & & \downarrow \\
\text{Ext}^n(M, B') & \rightarrow & \text{Ext}^{n+m+1}(A, B') \\
\downarrow & & \downarrow \\
\text{Hom}(P_m, B) & \rightarrow & \text{Hom}(M, B) & \rightarrow & \text{Ext}^{m+1}(A, B) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Hom}(P_m, B') & \rightarrow & \text{Hom}(M, B') & \rightarrow & \text{Ext}^{m+1}(A, B') & \rightarrow & 0
\end{array}
\]

### 4.3 Ext and Coproducts

**Proposition 7.** Let \( \mathcal{A} \) be an infinite complete abelian category with exact products and enough injectives. For an object \( A \), the functor \( \text{Ext}^n(A, -) : \mathcal{A} \rightarrow \text{Ab} \) preserves products.

**Proof.** The functor \( \text{Hom}(A, -) : \mathcal{A} \rightarrow \text{Ab} \) preserves products, so this follows immediately from our Derived Functor notes. \(\square\)

**Proposition 8.** Let \( \mathcal{A} \) be an infinite cocomplete abelian category with exact coproducts and enough projectives. For an object \( B \), the contravariant functor \( \text{Ext}^n(-, B) : \mathcal{A} \rightarrow \text{Ab} \) maps coproducts to products.

**Proof.** By assumption \( \mathcal{A}^{\text{op}} \) is a complete abelian category with exact products and enough injectives, and the functors \( \text{Ext}^n(-, B) \) are the right derived functors of the covariant additive functor \( \text{Hom}(-, B) : \mathcal{A}^{\text{op}} \rightarrow \text{Ab} \). So once again the result follows from our Derived Functor notes. \(\square\)

In particular both results apply in the case where \( \mathcal{A} \) is \( \text{Ab}, \mathcal{RMod} \) or \( \text{Mod} \mathcal{R} \) for a ring \( \mathcal{R} \). If \( \mathcal{A} \) is \( \mathcal{R} \)-linear for some ring \( \mathcal{R} \) then the above results also apply to the functors \( \text{Ext}^n(A, -) : \mathcal{A} \rightarrow \mathcal{RMod} \) and \( \text{Ext}^n(-, B) : \mathcal{A} \rightarrow \mathcal{RMod} \). That is, the first preserves products and the second maps coproducts to products.

### 5 Ext for Commutative Rings

If \( \mathcal{R} \) is a commutative ring and \( \mathcal{A}, \mathcal{B} \) are \( \mathcal{R} \)-modules, then the group \( \text{Ext}^n(A, B) \) doesn’t depend on whether you consider \( A, B \) as left or right modules over \( \mathcal{R} \). That is, the calculations in the abelian categories \( \mathcal{RMod} \) and \( \text{Mod} \mathcal{R} \) yield isomorphic groups.

For a commutative ring \( \mathcal{R} \) the abelian category \( \mathcal{A} = \mathcal{RMod} \) is \( \mathcal{R} \)-linear in the sense of Section 4.1. Each group \( \text{Hom}_\mathcal{R}(M, N) \) becomes an \( \mathcal{R} \)-module via \( (r \cdot \varphi)(x) = r \cdot \varphi(x) \) and this defines an \( \mathcal{R} \)-linear structure on \( \mathcal{A} \). For \( r \in \mathcal{R} \) let \( \alpha : M \rightarrow M, \beta : N \rightarrow N \) be the endomorphisms defined by left multiplication by \( r \). Then \( r \cdot \varphi = \beta \varphi = \varphi \alpha \). So associated to two left \( \mathcal{R} \)-modules \( M, N \) and an integer \( i \geq 0 \) is an isomorphism class of left \( \mathcal{R} \)-modules, and the following procedures will calculate a representative

- Pick a projective resolution \( \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0 \) and calculate the cohomology of the sequence of \( \mathcal{R} \)-modules
  
  \[
  0 \rightarrow \text{Hom}(P_0, B) \rightarrow \text{Hom}(P_1, B) \rightarrow \text{Hom}(P_2, B) \rightarrow \cdots
  \]

- Pick an injective resolution \( 0 \rightarrow B \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \) and calculate the cohomology of the sequence of \( \mathcal{R} \)-modules
  
  \[
  0 \rightarrow \text{Hom}(A, I^0) \rightarrow \text{Hom}(A, I^1) \rightarrow \text{Hom}(A, I^2) \rightarrow \cdots
  \]
It is not hard to check that for \( r \in R \) left multiplication by \( r \) is given by \( \text{Ext}_R^n(M, \beta) = \text{Ext}_R^n(\alpha, N) \).

**Proposition 9.** Let \( R \) be a commutative noetherian ring and suppose \( A, B \) are finitely generated \( R \)-modules. Then \( \text{Ext}_R^n(A, B) \) is a finitely generated \( R \)-module.

**Proof.** Since \( R \) is noetherian and \( A \) is finitely generated we can find a projective resolution \( \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0 \) with all the \( F_i \) finite free modules. Then in the following sequence every module is finitely generated (see our Module notes)

\[
0 \rightarrow \text{Hom}(F_0, B) \rightarrow \text{Hom}(F_1, B) \rightarrow \text{Hom}(F_2, B) \rightarrow \cdots
\]

So the cohomology modules \( \text{Ext}_R^n(A, B) \) will also be finitely generated. □

Recall that a module \( M \) over a commutative domain \( R \) is *divisible* if for every \( 0 \neq r \in R \) and \( x \in M \) there is \( y \in M \) such that \( r \cdot y = x \). Any injective module is divisible. A commutative integral domain \( R \) is a Dedekind domain if and only if every divisible module is injective. Since a quotient of a divisible module is clearly divisible, it follows that over a Dedekind domain the quotient of an injective module is injective.

**Proposition 10.** For any Dedekind domain \( R \) we have \( \text{Ext}_R^n(A, B) = 0 \) for \( n \geq 2 \).

**Proof.** Find an injective module \( I \) and a monomorphism \( B \rightarrow I \). The quotient is divisible, hence injective, so \( 0 \rightarrow B \rightarrow I \rightarrow J \rightarrow 0 \rightarrow \cdots \) is an injective resolution of \( B \). It follows that \( \text{Ext}_R^n(A, B) \) is the cohomology of the sequence

\[
0 \rightarrow \text{Hom}(A, I) \rightarrow \text{Hom}(A, J) \rightarrow 0 \rightarrow \cdots
\]

So it is clear that \( \text{Ext}_R^n(A, B) = 0 \) for \( n \geq 2 \). □

### 5.1 Coextension

Let \( \varphi : R \rightarrow S \) be a morphism of commutative rings. Then for an \( R \)-modules \( A \) the \( R \)-module \( \text{Hom}_R(S, A) \) has a canonical \( S \)-module structure, and this defines the *coextension functor* \( P = \text{Hom}_R(S, -) : R\text{-Mod} \rightarrow S\text{-Mod} \). Let \( U : S\text{-Mod} \rightarrow \text{Ab} \) be the forgetful functor and \( Q = \text{Hom}_R(S, -) : R\text{-Mod} \rightarrow \text{Ab} \) the usual functor. Then \( Q \cong UP \) so for \( n \geq 0 \) and an assignment of injective resolutions \( I \) the functors \( R^nQ \) and \( U \circ R^nP \) are equal. So for an \( R \)-module \( A \) the Ext group \( \text{Ext}_R^n(S, A) \) becomes an \( S \)-module in a canonical way, and for a morphism of \( R \)-modules \( \beta : A \rightarrow A' \) the morphism of groups \( \text{Ext}_R^n(S, \beta) : \text{Ext}_R^n(S, A) \rightarrow \text{Ext}_R^n(S, A') \) is a morphism of these modules. So the additive functor \( \text{Ext}_R^n(S, -) : R\text{-Mod} \rightarrow \text{Ab} \) lifts to an additive functor \( R\text{-Mod} \rightarrow S\text{-Mod} \).

### 6 Another Characterisation of Derived Functors

Throughout this section \( A \) is an abelian category. If we say \( T \) is an additive functor, we mean it is an additive covariant functor \( A \rightarrow \text{Ab} \). Given two additive functors \( T, T' : A \rightarrow \text{Ab} \) we let \( [T, T'] \) denote the class of natural transformations \( T \rightarrow T' \). It is clear that this becomes a “large” abelian group (an abelian group whose underlying class may not be a set).

Suppose we have for every object \( A \) an additive functor \( \Omega_A : A \rightarrow \text{Ab} \) and for every morphism \( \alpha : A \rightarrow B \) a natural transformation \( \Omega_\alpha : \Omega_B \rightarrow \Omega_A \), such that \( \Omega_{\alpha \beta} = \Omega_\alpha \Omega_\beta, \Omega_{\alpha + \gamma} = \Omega_\alpha + \Omega_\gamma \) and \( \Omega_1 = 1 \). We call this a *representation* of \( A \) in the additive functors \( A \rightarrow \text{Ab} \). We say it is a small representation if \( \{ \Omega_A, T \} \) is a set for any object \( A \) and additive functor \( T : A \rightarrow \text{Ab} \).

The primary example is \( A \rightarrow \text{Hom}(A, -) \), \( \alpha \mapsto \text{Hom}(\alpha, -) \), which is small since by the Yoneda Lemma there is an isomorphism of abelian groups \( [\text{Hom}(A, -), T] \cong T(A) \). This isomorphism is also natural in \( A \): given any morphism \( \alpha : A \rightarrow B \), composition with \( \Omega_\alpha \) defines a morphism of groups \( [\text{Hom}(A, -), T] \rightarrow [\text{Hom}(B, -), T] \) which fits into a commutative diagram:

\[
\begin{array}{ccc}
[\text{Hom}(A, -), T] & \longrightarrow & T(A) \\
\downarrow & & \downarrow T(\alpha) \\
[\text{Hom}(B, -), T] & \longrightarrow & T(B)
\end{array}
\]
It follows that we can recover the functor $T$ (up to natural equivalence) from the representation $A \mapsto \text{Hom}(A,-)$ and the morphisms from these objects to $T$. In detail: given an additive functor $T$ define $S(A) = \text{[Hom}(A,-),T]$. For a morphism $\alpha : A \rightarrow B$ let $S(A) \rightarrow S(B)$ act by composition with $\Omega_\alpha$. Then this defines an additive functor $S$ naturally equivalent to $T$. This motivates the following definition

**Definition 4.** Let $\mathcal{A}$ be an abelian category, $\Omega$ a small representation of $\mathcal{A}$ in the additive functors $\mathcal{A} \rightarrow \text{Ab}$. Given an additive functor $T$ let $\Omega T$ denote the following additive functor: $(\Omega T)(A) = [\Omega_A, T]$ and for $\alpha : A \rightarrow B$ we define

$$\psi \mapsto \psi_{\Omega_\alpha}$$

Now assume $\mathcal{A}$ is an abelian category with enough injectives and projectives. For an object $A$ and $n \geq 0$ we have associated an additive functor $\text{Ext}^n(A,-) : \mathcal{A} \rightarrow \text{Ab}$ and to a morphism $\alpha : A \rightarrow B$ we have associated a natural transformation $\text{Ext}^n(\alpha,-) : \text{Ext}^n(B,-) \rightarrow \text{Ext}^n(A,-)$. We have already checked that $A \mapsto \text{Ext}^n(A,-)$ defines a representation of $\mathcal{A}$ ($n$ fixed).

Similarly if $\mathcal{A}$ is an abelian category with enough projectives, $A \mapsto \text{Ext}^n(A,-)$ and $\alpha \mapsto \text{Ext}^n(\alpha,-)$ defines a representation. If $\mathcal{A}$ has both enough injectives and projectives then for every $A$ there is a canonical natural equivalence $\text{Ext}^n(A,-) \cong \text{Ext}^n(A,-)$ with the property that the following diagram commutes for any morphism $\alpha : A \rightarrow B$

$$\text{Ext}^n(B,-) \xrightarrow{\text{Ext}^n(\alpha,-)} \text{Ext}^n(A,-) \xrightarrow{\text{Ext}^n(\alpha,-)}$$

**Lemma 11.** Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. For $n \geq 0$ the representations $\Omega : A \mapsto \text{Ext}^n(A,-)$ and $\Omega : A \mapsto \text{Ext}^n(A,-)$ are small.

**Proof.** For $n = 0$ there is a natural equivalence $\text{Ext}^0(A,-) \cong \text{Hom}(A,-) \cong \text{Ext}^0(A,-)$ so both representations are trivially small. For $n \geq 1$ there is a natural equivalence $\text{Ext}^n(A,-) \cong \text{Ext}^n(A,-)$ so it suffices to show that $\Omega$ is small. Fix $n \geq 1$, an additive functor $T$ and an object $A$. Let $P$ be the resolution assigned to $A$, $\mu : K_A \rightarrow P_{n-1}$ be the image of $P_n \rightarrow P_{n-1}$ and consider the exact sequence

$$0 \rightarrow K_A \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

We show $\Omega$ is small by establishing an isomorphism $\text{Ker}T(\mu) \cong [\text{Ext}^n(A,-), T]$. For an object $B$ calculating $\text{Ext}^n(-,B)$ we can use the corresponding truncations of the duals of the projective resolutions chosen by $\mathcal{P}$, so by Proposition 20 of our Derived Functor notes there is an exact sequence

$$\text{Hom}(P_{n-1}, B) \rightarrow \text{Hom}(K_A, B) \rightarrow \text{Ext}^n(A,B) \rightarrow 0$$

(6)

The morphism $\text{Hom}(K_A, B) \rightarrow \text{Ext}^n(A,B)$ is canonical and natural in $A$. If $e : P_n \rightarrow K_A$ is the factorisation of $\partial_n$ through $\mu$ then this map is defined by $x \mapsto \pi x$. It is also natural in $B$, in the sense that for any $\beta : B \rightarrow B'$ the following diagram commutes

$$\begin{array}{ccc}
\text{Hom}(P_{n-1}, B) & \rightarrow & \text{Hom}(K_A, B) & \rightarrow & \text{Ext}^n(A,B) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Hom}(P_{n-1}, B') & \rightarrow & \text{Hom}(K_A, B') & \rightarrow & \text{Ext}^n(A,B') & \rightarrow & 0
\end{array}$$
In particular the following diagram is commutative with exact rows

\[
\begin{array}{c}
\text{Hom}(P_{n-1}, K_A) \\
\downarrow \\
\text{Hom}(P_{n-1}, P_{n-1})
\end{array}
\begin{array}{c}
\text{Hom}(K_A, K_A) \\
\downarrow \\
\text{Hom}(K_A, P_{n-1})
\end{array}
\begin{array}{c}
\text{Ext}^n(A, K_A) \\
\downarrow \\
\text{Ext}^n(A, P_{n-1})
\end{array}
\begin{array}{c}
0 \\
\end{array}
\begin{array}{c}
\text{Ext}^n(A, K_A) \\
\downarrow \\
\text{Ext}^n(A, P_{n-1})
\end{array}
\begin{array}{c}
0
\end{array}
\]

Let $\eta$ be the image in $\text{Ext}^n(A, K_A)$ of $1_{K_A}$. Commutativity of the diagram shows that $\text{Ext}^n(A, \mu)(\eta)$ is zero. Let $\Phi : \text{Ext}^n(A, -) \to T$ be a natural transformation. Consider the following commutative diagram

\[
\begin{array}{c}
\text{Ext}^n(A, K_A) \\
\downarrow T(\mu) \\
\text{Ext}^n(A, P_{n-1})
\end{array}
\begin{array}{c}
T(K_A) \\
\downarrow T(\mu) \\
T(P_{n-1})
\end{array}
\]

Since $\text{Ext}^n(A, \mu)(\eta) = 0$ it follows that the image in $T(K_A)$ of $\eta$ belongs to $\text{Ker}T(\mu)$. This assigns to any natural transformation $\Phi$ an element $\xi = \Phi_{K_A}(\eta) \in \text{Ker}T(\mu)$. Next we show that this assignment is injective, by showing that any other natural transformation $\Theta$ with $\Theta_{K_A}(\eta) = \xi$ must be equal to $\Phi$.

Let $\sigma : K_A \to B$ be any morphism. We claim that the image of $\sigma$ in $\text{Ext}^n(A, B)$ under the canonical morphism $\text{Hom}(K_A, B) \to \text{Ext}^n(A, B)$ defined above is $\text{Ext}^n(A, \sigma)(\eta)$. The morphism $\sigma$ induces a natural transformation $\text{Hom}(\sigma, K_A) \to \text{Hom}(\sigma, B)$ and therefore a cochain morphism of the image of the dual of $P$ under these two functors. The induced maps on cohomology at $n$ is $\text{Ext}^n(A, \sigma)$. So the claim is not too hard to check.

Let $B$ be any object and let $\rho \in \text{Ext}^n(A, B)$. The exact sequence (6) shows that $\rho$ is the image of some morphism $\sigma : K_A \to B$. Since $\Theta$ is natural the following square must commute

\[
\begin{array}{c}
\text{Ext}^n(A, K_A) \\
\downarrow T(\sigma) \\
\text{Ext}^n(A, B)
\end{array}
\begin{array}{c}
\Theta_{K_A} \\
\Theta_{\sigma} \\
T(\sigma)
\end{array}
\begin{array}{c}
T(K_A) \\
T(B)
\end{array}
\]

So $\Theta_B(\rho) = \Theta_B \text{Ext}^n(A, \sigma)(\eta) = T(\sigma)(\xi)$. Since $B$ and $\rho$ were arbitrary it follows that $\Theta = \Phi$.

Next we show how to assign a natural transformation $\Phi$ to any $\xi \in \text{Ker}T(\mu) \subseteq T(K_A)$. The obvious definition is the following: for $\rho \in \text{Ext}^n(A, B)$ let $\sigma : K_A \to B$ be any morphism mapping to $\rho$ under $\text{Hom}(K_A, B) \to \text{Ext}^n(A, B)$ and let $\Phi_B(\rho) = T(\sigma)(\xi)$. We have to show that $T(\sigma)(\xi)$ does not depend on the morphism $\sigma$ chosen in the preimage of $\rho$. If $\sigma'$ is another such morphism, then $\sigma - \sigma'$ is in the kernel of $\text{Hom}(K_A, B) \to \text{Ext}^n(A, B)$ and since (6) is exact there is $\tau : P_{n-1} \to B$ with $\sigma - \sigma' = \tau \mu$. Hence $T(\sigma - \sigma')(\xi) = 0$ since $\xi$ is in the kernel of $T(\mu)$, and so $T(\sigma)(\xi) = T(\sigma')(\xi)$, as required. It is easy to check that $\Phi_B$ is a morphism of groups.

It is clear that $\Phi_{K_A}(\eta) = \xi$ so it only remains to show that $\Phi$ is natural. Suppose $\beta : B \to B'$ is given and consider the diagram

\[
\begin{array}{c}
\text{Hom}(K_A, B) \\
\downarrow \\
\text{Hom}(K_A, B')
\end{array}
\begin{array}{c}
\text{Ext}^n(A, B) \\
\downarrow T(\beta) \\
\text{Ext}^n(A, B')
\end{array}
\begin{array}{c}
T(B) \\
\downarrow T(\beta) \\
T(B')
\end{array}
\]

The left hand square commutes by naturality of (6), so if we choose $\sigma : K_A \to B$ to represent $\rho \in \text{Ext}^n(A, B)$ then we can choose $\beta \sigma$ to represent $\text{Ext}^n(A, \beta)(\rho)$. Hence

$\Phi_B' \text{Ext}^n(A, \beta)(\rho) = T(\beta \sigma)(\xi) = T(\beta)\Phi_B(\rho)$

This finishes the construction of the bijection $[\text{Ext}^n(A, -), T] \cong \text{Ker}T(\mu)$. \qed
Theorem 12. Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. For $n \geq 1$ and any right exact functor $T$ there is a canonical isomorphism natural in $A$ and $T$

$$[\text{Ext}^n(A, -), T] \cong L_n T(A)$$

That is, there is a canonical natural equivalence $\Omega T \cong L_n T$.

Proof. We assume all derived functors (including those making up the definition of $\Omega$) are calculated relative to fixed assignments of injective and projective resolutions $I$, $P$. Assume $n \geq 1$ and for every object $A$ with projective resolution $P$ let $\mu_A : K_A \rightarrow P_{n-1}$ be the canonical image of $P_n \rightarrow P_{n-1}$. Let $\ell_n T(A)$ be $\text{Ker} T(\mu_A)$. For a morphism $\alpha : A \rightarrow B$ let $\varphi$ be a chain morphism lifting $\alpha$, induce $\alpha' : K_A \rightarrow K_B$ and define $\ell_n T(\alpha)$ by $x \mapsto T(\alpha')(x)$. As we showed in Section 3 of our Derived Functor notes, $\ell_n T$ is canonically naturally equivalent to $L_n T$ since $T$ is right exact. But in the previous Lemma we defined a bijection $[\text{Ext}^n(A, -), T] \cong \ell_n T(A)$ for arbitrary $A$ by $\Phi \mapsto \Phi_{K_A}(\eta_A)$ where $\eta_A$ was a special element of $\text{Ext}^n(A, K_A)$. It is clear that this bijection is an isomorphism of abelian groups, and to show $\Omega T$ is canonically naturally equivalent to $L_n T$ is only remains to show that this isomorphism is natural in $A$.

Let $\alpha : A \rightarrow B$ be a morphism and consider the following diagram

$$
\begin{array}{ccc}
[\text{Ext}^n(A, -), T] & \cong & \ell_n T(A) \\
\downarrow & & \downarrow \\
[\text{Ext}^n(B, -), T] & \cong & \ell_n T(B)
\end{array}
$$

Lift $\alpha$ to a chain morphism $\varphi : P \rightarrow Q$ of the chosen resolutions and let this induce a morphism $\alpha' : K_A \rightarrow K_B$. Let $\Phi : \text{Ext}^n(A, -) \rightarrow T$ be a natural transformation. We have to show that $T(\alpha')\Phi_{K_A}(\eta_A) = (\Phi \text{Ext}^n(\alpha, -))K_n(\eta_B)$, which reduces to showing that $\text{Ext}^n(A, \alpha')(\eta_A) = \text{Ext}^n(A, \alpha)(\eta_B)$. So it would be enough to show that the following diagram commutes:

$$
\begin{array}{ccc}
\text{Hom}(K_A, K_A) & \rightarrow & \text{Ext}^n(A, K_A) \\
\downarrow & & \downarrow \\
\text{Hom}(K_A, K_B) & \rightarrow & \text{Ext}^n(A, K_B) \\
\downarrow & & \downarrow \\
\text{Hom}(K_B, K_B) & \rightarrow & \text{Ext}^n(B, K_B)
\end{array}
$$

But the top square commutes by naturality of the sequence (6) for $A$ in the second variable and the bottom square commutes by naturality of the sequence (6) for $B$ in the first variable, so the proof of naturality in $A$ is complete.

Now suppose $\gamma : T \rightarrow T'$ is a natural transformation. For any object $A$ with chosen resolution $P$ this gives rise to a chain morphism $\gamma_P : TP \rightarrow T'P$ and we let $\ell_n T(A) \rightarrow \ell_n T'(A)$ be defined by the restriction of $\gamma_{K_A}$. It is then clear that the left hand square in the following diagram commutes

$$
\begin{array}{ccc}
[\text{Ext}^n(A, -), T] & \cong & \ell_n T(A) \\
\downarrow & & \downarrow \\
[\text{Ext}^n(A, -), T'] & \cong & \ell_n T'(A)
\end{array}
$$

The natural transformation $L_n \gamma : L_n T \rightarrow L_n T'$ is defined elsewhere in our notes. By definition $(L_n \gamma)_A : L_n T(A) \rightarrow L_n T'(A)$ is the map $x + \text{Im} T(\partial_{n+1}) \rightarrow \gamma_{P_n}(x) + \text{Im} T'(\partial_{n+1})$ which clearly makes the right hand diagram commute. This completes the proof.

Corollary 13. For a ring $R$ there is a canonical isomorphism natural in the right $R$-module $A$ and the left $R$-module $B$

$$[\text{Ext}^n(A, -), - \otimes_R B] \cong \text{Tor}_n(A, B)$$
Proof. This is just $\mathcal{A} = \textbf{Mod}_R$, $T = - \otimes_R B$ and $L_nT = \text{Tor}_n(-, B)$ in the Theorem. Just to be perfectly clear what we mean by naturality: for any morphism $\alpha : A \to A'$ of right $R$-modules the following diagram commutes

\[
\begin{array}{ccc}
\text{Ext}^n(A, -), - \otimes_R B & \longrightarrow & \text{Tor}_n(A, B) \\
\downarrow & & \downarrow \\
\text{Ext}^n(A', -), - \otimes_R B & \longrightarrow & \text{Tor}_n(A', B)
\end{array}
\]

For a morphism of right $R$-modules $\beta : B \to B'$ the following diagram commutes

\[
\begin{array}{ccc}
\text{Ext}^n(A, -), - \otimes_R B & \longrightarrow & \text{Tor}_n(A, B) \\
\downarrow & & \downarrow \\
\text{Ext}^n(A, -), - \otimes_R B' & \longrightarrow & \text{Tor}_n(A, B')
\end{array}
\]

where the left hand vertical morphism acts by composition with the natural transformation $- \otimes_R B \to - \otimes_R B'$ determined by $\beta$. \qed