An Equivalence for Modules over Projective Schemes

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Let S be a graded ring generated by S_1 as an S_0-algebra and set $X = ProjS$. Then we have an adjoint pair (AAMPS,Proposition 2)

$$S\mathbf{GrMod} \underset{\Gamma_*(-)}{\overset{\widetilde{-}}{\rightleftarrows}} \mathfrak{Mod}(X) \qquad \widetilde{-} \longrightarrow\!\!\!\!\!\dashv \Gamma_*$$

with unit $\eta : M \longrightarrow \Gamma_*(\widetilde{M})$. This is the projective version of the adjunction $\widetilde{-} \dashv \Gamma(-)$, whose unit is always an isomorphism. In this note we show that under certain conditions the unit η is a *quasi-isomorphism* (GRM,Definition 10). But first, observe that if $\mathscr{F}$ is a coherent sheaf of modules on a noetherian affine scheme $X = SpecA$ then $\mathscr{F}(X)$ is a finitely generated A-module (MOS,Lemma 4). We want to prove the projective version of this result.

Corollary 1. *Let S be a noetherian graded ring finitely generated by S_1 as an S_0-algebra. If $\mathscr{F}$ is a coherent sheaf of modules on X then there exists an integer $n_0 > 0$ such that $\mathscr{F}(n)$ is generated by a finite number of global sections for $n \geq n_0$.*

Proof. It follows from (TPC,Lemma 19) that the structural morphism $ProjS \longrightarrow SpecS_0$ is projective, and the twisting sheaf $\mathcal{O}(1)$ on $ProjS$ is very ample relative to S_0 so we can apply (H,5.17) to reach the desired conclusion (GRM,Proposition 9). □

Proposition 2. *Let k be a field, S a finitely generated graded k-algebra that is also finitely generated by S_1 as an S_0-algebra and set $X = ProjS$. If $\mathscr{F}$ is a coherent sheaf of modules on X then $\Gamma_*(\mathscr{F})$ is a quasi-finitely generated graded S-module.*

Proof. It follows from the hypothesis that S is a noetherian graded ring finitely generated by S_1 as an S_0-algebra. Let $n > 0$ be an integer such that $\mathscr{F}(n)$ is generated by a finite set of global sections $m_1, \ldots, m_n \in \Gamma(X, \mathscr{F}(n))$. Let M be the graded S-submodule of $\Gamma_*(\mathscr{F})$ generated by the m_i. Let $i : M \longrightarrow \Gamma_*(\mathscr{F})$ be the inclusion and let ϕ denote the composite of $i^\sim : M^\sim \longrightarrow \Gamma_*(\mathscr{F})^\sim$ with the canonical isomorphism $\varepsilon : \Gamma_*(\mathscr{F})^\sim \longrightarrow \mathscr{F}$ (AAMPS,Proposition 13). Since the functor $\widetilde{-}$ is exact, ϕ is a monomorphism. Consider the following diagram

$$\begin{array}{ccccc} M(n)^\sim & \xrightarrow{i(n)^\sim} & \Gamma_*(\mathscr{F})(n)^\sim & & \\ \Big\Downarrow \rho & & \Big\Downarrow \rho & & \\ \widetilde{M}(n) & \xrightarrow[\widetilde{i}(n)]{} & \Gamma_*(\mathscr{F})^\sim(n) & \xRightarrow[\varepsilon(n)]{} & \mathscr{F}(n) \end{array}$$

We observed just after the proof of (AAMPS,Proposition 2) that $\varepsilon(n)_X \rho_X(\dot{m}_i) = m_i$ for $1 \leq i \leq n$. It follows that $\phi(n) = \varepsilon(n) i^\sim(n)$ is an epimorphism, since it is surjective on stalks. Twisting by $-n$ we see that ϕ is an isomorphism. Therefore $\Gamma_*(\mathscr{F}) \cong \Gamma_*(M^\sim)$ and we reduce to showing that if M is a finitely generated graded S-module then $\Gamma_*(M^\sim)$ is quasi-finitely generated.

This is trivial if $M = 0$, so assume $M \neq 0$. Then by (HSE,Proposition 1) there is a finite filtration

$$0 = M^0 \subset M^1 \subset \cdots \subset M^r = M$$

of M by graded submodules, where for each i we have an isomorphism of graded S-modules $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ for some homogenous prime ideal $\mathfrak{p}_i$ and $n_i \in \mathbb{Z}$. For each $i \geq 1$ we have an exact sequence

$$0 \longrightarrow M^{i-1} \longrightarrow M^i \longrightarrow (S/\mathfrak{p}_i)(n_i) \longrightarrow 0$$

and thus an exact sequence of sheaves of modules on X

$$0 \longrightarrow (M^{i-1})^{\sim} \longrightarrow (M^i)^{\sim} \longrightarrow (S/\mathfrak{p}_i)(n_i)^{\sim} \longrightarrow 0$$

Since the functor $\Gamma_* : \mathfrak{Mod}(X) \longrightarrow S\mathbf{GrMod}$ is left exact (AAMPS,Corollary 3) we have an exact sequence of graded S-modules

$$0 \longrightarrow \Gamma_*(M^{i-1\sim}) \longrightarrow \Gamma_*(M^{i\sim}) \longrightarrow \Gamma_*((S/\mathfrak{p}_i)(n_i)^{\sim}) \tag{1}$$

Suppose we could show that $\Gamma_*((S/\mathfrak{p})(n)^{\sim})$ was a quasi-finitely generated graded S-module for every homogenous prime $\mathfrak{p}$ and $n \in \mathbb{Z}$. Then we prove that $\Gamma_*(M^{i\sim})$ is quasi-finitely generated by induction on i. The case $i = 0$ is trivial. Suppose that $\Gamma_*(M^{i-1\sim})$ is quasi-finitely generated for some $i \geq 1$. Since S is noetherian we can use (GRM,Lemma 25) and (1) to see that $\Gamma_*(M^{i\sim})$ is quasi-finitely generated. Once we reach $i = r$ the proof is complete.

So it remains to show that $\Gamma_*((S/\mathfrak{p})(n)^{\sim})$ is a quasi-finitely generated graded S-module for every homogenous prime $\mathfrak{p}$ and $n \in \mathbb{Z}$. Let $\phi : ProjS/\mathfrak{p} \longrightarrow ProjS$ be the canonical closed immersion. If we write $Y = ProjS/\mathfrak{p}$ then by (AAMPS,Lemma 7) we have a canonical isomorphism of graded S-modules

$$\Gamma_*(\mathcal{O}_Y(n)) \cong \Gamma_*((S/\mathfrak{p})(n)^{\sim})$$

Using (GRM,Lemma 27) we reduce to the case where S is a finitely generated graded k-domain finitely generated by S_1 as an S_0-algebra, and we have to show that $\Gamma_*(\mathcal{O}_X(n))$ is quasi-finitely generated for every $n \in \mathbb{Z}$. But $\Gamma_*(\mathcal{O}_X(n)) \cong \Gamma_*(\mathcal{O}_X)(n)$ (AAMPS,Lemma 14) so by (GRM,Lemma 23) it suffices to show that $\Gamma_*(\mathcal{O}_X)$ is quasi-finitely generated. We may as well assume X is nonempty, in which case we can identify S with a subring of the graded domain $\Gamma_*(\mathcal{O}_X)' = \Gamma_*(\mathcal{O}_X)\{0\}$, with $\Gamma_*(\mathcal{O}_X)'$ integral over S (AAMPS,Proposition 17). So to complete the proof we need only show that $\Gamma_*(\mathcal{O}_X)'$ is a finitely generated S-module. Let Q be the quotient field of S. It follows from the proof of (AAMPS,Proposition 17) that we can $\Gamma_*(\mathcal{O}_X)'$ with a subring of Q containing S. Therefore $S \subseteq \Gamma_*(\mathcal{O}_X)' \subseteq C$ where C is the integral closure of S in Q. By (H,Ch.1 3.9A) C is a finitely generated S-module, and since S is noetherian it follows that $\Gamma_*(\mathcal{O}_X)'$ is also finitely generated, completing the proof. □

Corollary 3. *Let k be a field, S a finitely generated graded k-algebra that is also finitely generated by S_1 as an S_0-algebra and set $X = ProjS$. If M is a quasi-finitely generated graded S-module then the unit $\eta : M \longrightarrow \Gamma_*(M^{\sim})$ is a quasi-isomorphism. That is, there is an integer $d \geq 0$ such that for every $n \geq d$ we have an isomorphism of S_0-modules*

$$\eta_n : M_n \longrightarrow \Gamma(X, M(n)^{\sim})$$
$$m \mapsto \dot{m/1}$$

Proof. Combining Proposition 2 and (MPS,Proposition 20) we see that the graded S-module $\Gamma_*(M^{\sim})$ is quasi-finitely generated. Therefore to show that η is a quasi-isomorphism it suffices to show that $\eta^{\sim}$ is an isomorphism of sheaves of modules (MPS,Corollary 21). But the composite of $\widetilde{\eta}$ with the counit $\Gamma_*(M^{\sim})^{\sim} \longrightarrow M^{\sim}$ is the identity, so this follows immediately from (AAMPS,Proposition 13). □

Corollary 4. *Let k be a field, S a finitely generated graded k-algebra that is also finitely generated by S_1 as an S_0-algebra and set $X = ProjS$. The morphism of graded rings $\eta : S \longrightarrow \Gamma_*(\mathcal{O}_X)'$ is a quasi-isomorphism. That is, there is an integer $d \geq 0$ such that for every $n \geq d$ there is an isomorphism of S_0-modules*

$$\eta_n : S_n \longrightarrow \Gamma(X, \mathcal{O}_X(n))$$
$$s \mapsto \dot{s}$$