Elements of Algebraic Geometry I

Preface

This work proposes to give a systematic exposition of the fundamentals of algebraic geometry. It is now generally acknowledged that to obtain a theory of algebraic geometry as general as possible, it is necessary to reconsider the notion of schemes: we have tried in this Introduction to rapidly describe the evolution which has led to this notion.

The following is a general plan for the treatise:

- Chapter I. Language of schemes
- Chapter II. Elementary global study of some classes of morphisms
- Chapter III. Cohomology of coherent algebraic sheaves. Applications.
- Chapter IV. Local study of schemes and the morphisms of schemes.
- Chapter V. Complements on projective morphisms.
- Chapter VI. Techniques of construction of schemes.
- Chapter VII. Schemes in groups. Principal fibred spaces.
- Chapter VIII. The Picard scheme.
- Chapter IX. The fundamental group.
- Chapter X. Resiudes and duality.
- Chapter XI. The theory of intersection, Chern classes, Riemann-Roch theorem.
- Chapter XII. Etale cohomology of schemes.

Chapters I, II and III play a central role in the whole treatise and are indepensible in the reading of all works of algebraic geometry founded on the theory of schemes: the following chapters (in particular starting from Chapter VII) study more specific questions. But it suits our purposes to underline the fact that this treatise is not an encyclopaedia and does not consider numerous questions of algebraic geometry, without taking into account their historical importance or a number of works they lead to today.

The reader who already knows classical algebraic geometry from, for example, the following books [3],[7],[11],[12],[13],[16],[17] of the Bibliography will recognise the old concepts in the new theory of schemes. It is hardly possible to begin the reading of this treatise without having a good knowledge of the following subjects

- Commutative algebra.
- Homological Algebra.
- Theory of sheaves.
- Finally it will be useful for the reader to have a certain familiarity with the functorial language.

For the convenience of the reader we give in a Chapter 0 (published in several joint parts of other chapters) of the diverse complements of commutative algebra, homological algebra, theory of sheaves, utilised throughout the treatise which are more or less well known, but for which it was not possible to give convenient references. It is recommended to the reader to not refer to Chapter 0 immediately but to look back on it as reading.

The references will be given following the decimal system: for example III 4.9.3 the number III indicates the chapter, the number 4 indicates the paragraph, 9 the section of paragraph. Inside the same chapter we drop the III. The number O_N refers to the part of Chapter 0 joined to Chapter N.

It is sometimes useful to group results in the same paragraph even though some cannot be proved except in a later Chapter: these results will always be placed between $*...*$ and the reader will be able to, in each case, verify that this is not a vicious circle. As for the examples, we don't compel ourselves to only use propositions that have only been proved in the text.

The passages printed in small characters can be omitted in the first reading: the same for Chapter 0, it is considered to not refer to the passages of which we will have need.

The first two chapters of the treatise constitute a reedition of the two chapters already published seperately Publ. Math. Inst. Hautes. Et. Scient.., no. 4 (1960) and no. 8 (1961). The principle changes with regard to this first edition consist above all of reordering of material, notably by the incorporation in these two chapters of questions studied firstly (previewed) in these chapters; particularly note the introduction to Chapter I, § 9 of schemes representing certain functors particularly important in the applications, whose existence is established by a uniform method.

Finally note, with regard to the first edition, an important change of terminology. The word scheme now designates that which was called a prescheme in the first edition, and the words separated scheme that which was called scheme. The meaning of the word constructible has also changed: we now designate by this that which was called locally constructible in the first edition, and globally constructible has the meaning of constructible in the first edition.

The references to Chapters III, IV of the first edition are given as below, with the addition of an asterix to the number of the Chapter.

Introduction

1. We propose in this introduction to try to show (without entering into details) how the modern point of view in algebraic geometry is freed in a fairly natural manner of the evolution of the fundamental problems posed by this branch of mathematics. For the convenience of the exposition we use the language of modern mathematics even to describe historical situations where it is evident that the language and the technique of the contemporary authors was strongly different to modern conceptions.

2. We can say that the historical origin and one of the essential aims of algebra since the Babylonians, Hindus and Diophante up until today is the study of the solutions of systems of polynomial equations. In order to define the problem we consider a commutative ring k with unit, and the ring

$$P_1 = P = k[(T_i)_{i\in I}] \tag{1}$$

of polynomials in the indeterminates T_i with coefficients in k. Recall that for each family $t = (t_i)_{i\in I}$ of elements of k, there is a k-homomorphism (by convention, taking unit to unit) from P to k taking each T_i to the element t_i: the image of a polynomial $F \in P$ by this homomorphism is written $F(t)$. We define here for each polynomial $F \in P$ a polynomial application $t \mapsto F(t)$ from k^I to k.

This being so, the considered problem consists of giving a family $(F_j)_{j\in J}$ of polynomials of P and of searching all systems $t = (t_i)_{i\in k^I}$ for which we have

$$F_j(t) = 0 \text{ for all } j \in J \tag{2}$$

we say that such a system $t = (t_i)_{i\in I}$ is a solution of the system of polynomial equations

$$F_j((T_i)_{i\in I}) = 0 \text{ for } j \in J \tag{3}$$

In order not to restrict later applications, the index sets I and J are allowed to be infinite.

3. In addition to the purely algebraic aspect of the previous problem, since the invention of that which we have called Analytical Geometry, a geometric of large interest, firstly for $k = \mathbb{R}$ and I reduces to 2 or 3 elements, where the collections of solutions of certain systems (2) are curves or surfaces studied since Antiquity, like for example quadrics or conics. Since around the middle of the 19th century, we have habituated ourselves little by little to using a geometry language inspired by the elementary geometry while k is an arbitrary ring and I a collection of arbitrary indices; it is in this way that k^I is often called an *affine* space, on k with elements $t = (t_i)_{i\in I}$ as points.

4. The first natural questions that we ask ourselves in the study of system of equations 2 concern the collection of the solutions in k^I: is this collection empty or not? Is it finite or not? If it is finite, can we give an estimation of the number of solutions? If it is infinite, can we give asymptotic estimations of the number of solutions satisfying supplementary inequalities where parameters figure, while these parameters tend towards certain limits, etc.

We can qualify this naive point of view from the arithmetic point of view (in a very large sense), because the arithmetic nature of the ring k plays an essential role here: the methods and the results will be very different depending on whether or not k is a field, or a ring such as $\mathbb{Z}$ (for example), or the ring of the entire field of algebraic numbers. In the same way, if k is a field the results will differ greatly depending if k is a field of algebraic numbers, or a finite field, or an algebraically closed field (for example the field $\mathbb{C}$), or the field of real numbers $\mathbb{R}$ (real algebraic geometry).

5. It is precisely the study of algebraic curves and surfaces in the real domain which led to a different point of view: since the beginning of the 18th century, and systematically from Monge and Poncelet, we associate to a system 2 with real coefficients the same system but where we won't restrict our coefficients to be real numbers, but in the complex space corresponding to $\mathbb{C}$, using the fact that $\mathbb{R}$ is a subfield of $\mathbb{C}$. This idea has shown itself to be very fertile. Since the study of the algebraic objects simplifies itself considerably when we extend the base field; in fact we can even say that this extension succeeds in a certain very good sense, because of the additional advantage of using, over the field $\mathbb{C}$, the powerful theory of analytic functions, because during the 19th century we practically ceased to consider systems 2 other than those with complex coefficients (or in the subfield of $\mathbb{C}$ such as the fields of algebraic numbers); which leads to loss of long held view of the fundamental idea of change of the base field in its general form (the only exception concerns the theory of congruences, where the idea of finding imaginary solutions leads to the theory of finite fields (Gauss, Galois) and to their use in the theory of linear groups (Jordan, Dickson)).

6. It is only from 1940 with the abstract algebraic geometry (this is to say, with base an arbitrary field k being able to be able to be of characteristic not equaling 0) developed above all by Weil, Chevalley and Zariski, that the idea of change of base takes importance in a more general context: it is in effect frequently necessary to pass, for example, to the algebraic closure of k, or (where k is a valued field) to the completion of k. However, there is not a systematic of this operation in the work of Chevalley or Zariski; while in the works of Weil, which uses it well on other occasions, its generality is something not well covered by the part taken. He restrains himself once and for all to only envisaging subfields of a fairly large algebraic closed field (the universal field), resting fairly closely in appearance to the classic point of view, where the field $\mathbb{C}$ took this role. It is only more recently, firstly with E. Kähler [8], then in the first edition of the present treatise, that the utility of admitting the extensions k' of k which are commutative k-algebras (even while k is a field) were recognised, and that such arbitrary changes of base became without doubt one of the most important processes of modern algebraic geometry. So that we can oppose an arithmetic point of view, we further describe the point of view that we can qualify as purely geometric; we here make an abstraction of the special properties of the solutions of the system 2 which derive from the particular space k^I from where we begin, in order to consider for *each* k-algebra k', the collection of the solutions from 2 within k'^I and the way in which this collection varies with k'; we will research in particular the properties of the system of equations 2 which are invariant while k' varies (or, as we say again, which are stable by the change of base).

7. The idea of variation of the base ring which we just introduced expresses mathematically, without much use of functorial language (of which the abscence explains without doubt the timidity of previous attempts). We can say, in effect, that we have a covariant functor

$$E^I : k' \mapsto k'^I \tag{4}$$

from the category of k-algebras to the category of sets, this functor, which takes a k-homomorphism $\varphi : k' \longrightarrow k''$ to the application $\varphi^I : k'^I \longrightarrow k''^I$; we say again that 4 is the functor *affine space* of dimension I on k. If S designates the family $(F_j)_{j\in J}$ of polynomials considered in no.2, note $V_S(k')$ the part of k'^I formed of the solutions of the system 2; we say again that these solutions are the points with values in k' of the variety on k defined by the system 2. It is immediate that

$$V_S : k' \mapsto V_s(k') \tag{5}$$

is a subfunctor of the functor E^I (the image of $V_S(k')$ by φ^I being contained in $V_S(k'')$). We therefore pose in principle that the study of the system of equations 2 from the point of view of algebraic geometry is the study of the functor V_S (from the category of k-algebras to the category of sets). This study is composed of two aspects: firstly the study of the functor $k' \mapsto V_S(k')$, independently of the way in which it is realised as a subfunctor of an affine space functor; then if the case presents itself, the study of the properties of an inclusion $V_S(k') \longrightarrow k'^I$. For most of the problems that we ask ourselves ordinarily in algebraic geometry, this second aspect is entirely trivial: all that is important are the intrinsic properties of the functor V_S, independent of the particular affine immersion

$$V_S(k') \longrightarrow k'^I \tag{6}$$

This justifies regarding two families S_1, S_2 of polynomials (with regard to two families of mutually distinct indeterminates) as essentially equivalent, if the corresponding V_{S_1} and V_{S_2} are isomorphic.

8. We will define the structure of the functor 5; beginning from the observation that this functor does not change when we add to the given equations 2 all the equations of the form $F = 0$, where F is of the form

$$F = \Sigma_{j\in J} A_j F_j \tag{7}$$

The A_j being polynomials of the algebra $k[(T_i)_{i\in I}]$, zero except for a finite number of indices; the set of these polynomials F is none other than the *ideal* $\mathfrak{J}$ of the algebra P_I generated by the family $(F_j)_{j\in J}$. We can therefore always, for the study of the functors V_S, reduce to functors of the form $V_{\mathfrak{J}}$, with $\mathfrak{J}$ is an ideal of P_I.[1]

We introduce now the quotient k-algebra

$$A_{\mathfrak{J}} = P_I/\mathfrak{J} \tag{8}$$

and note that we have a functorial bijection in k'

$$k'^I \cong Hom_{k-\mathrm{alg}}(P_I, k') \tag{9}$$

associating to each point $t = (t_i)_{i\in I}$ of k'^I the k-homomorphism $F \mapsto F(t)$. By this bijection $V_{\mathfrak{J}}(k')$ corresponds to the set of homomorphisms of k-algebras $P_I \longrightarrow k'$ which are zero on $\mathfrak{J}$, or again to the set of homomorphisms of k-algebras from $A_{\mathfrak{J}}$ to k'. In other words, we obtain by restriction of 9 an isomorphism of functors in k'

$$V_{\mathfrak{J}}(k') \cong Hom_{k-\mathrm{alg}}(A_{\mathfrak{J}}, k') \tag{10}$$

Moreover, by the bijections 9 and 10, the canonical inclusion 6 is none other than the injection $Hom_{k-\mathrm{alg}}(A_{\mathfrak{J}}, k') \longrightarrow Hom_{k-\mathrm{alg}}(P_I, k')$ corresponding to the surjective canonical homomorphism

$$P_I \longrightarrow A_{\mathfrak{J}} = P_I/\mathfrak{J} \tag{11}$$

If we take into account the fact that all the commutative k-algebras A can be written in the form $P_I/\mathfrak{J}$ for suitable I and $\mathfrak{J}$, we therefore see that up to isomorphism the functors V_S are exactly the representable functors

$$V_A : k' \longrightarrow Hom_{k-\mathrm{alg}}(A, k') \tag{12}$$

An inclusion $V_A \longrightarrow E^I$ of such a functor (for a suitable set of indices I) is an injective map functorial in k'

$$V_A(k') \longrightarrow k'^I$$

[1]This shows that even while the set I of the indeterminates is finite and k is a field, it is not necessary to limit one's self to the systems of finite equations, as the most natural systems are those in which the set of indices I is an ideal of P (a set which is rarely finite). Even though while k is a field and I is finite, all the ideals $\mathfrak{J}$ of $P = k[(T_i)_{i\in I}]$ can be generated by a finite number of elements (Hilbert's Basis theorem), therefore the two possible definitions of the functor 5 (by a finite system S, or by an ideal of P) coincide, there only remains that the restriction of finiteness is *a priori* artificial, and technically awkward.

But, if we have a map functorial in k'

$$Hom_{k-\mathrm{alg}}(A,k') \longrightarrow Hom_{k-\mathrm{alg}}(P_I,k') \tag{13}$$

if we set $k'=A$, then under 13 the identity map 1_A is carried into the canonical surjection

$$\pi : P_I \longrightarrow A \tag{14}$$

and, by functoriality, the application 13 is none other than $\dot{a} \mapsto u \circ \pi$. As the homomorphism π is surjective, the application 13 is therefore injective; we limit ourselves ordinarily to the inclusions $V_A \longrightarrow E^I$ obtained as in 6 with the aid of a surjective homomorphism π. Giving such a homomorphism is equivalent, on the other hand, to giving the images of the T_i by this homomorphism, this is to say, a set of elements $(t_i)_{i\in I}$ of the k-algebra A having I as its set of indices.

9. These considerations show (taking into account the elementary properties of representable functors (see O,1)) that the category of functors

$$\mathbf{k-alg} \longrightarrow \mathbf{Sets}$$

associated to the systems of equations (2) (in other words, of the form V_S) is equivalent to the opposite category of the category of k-algebras $\mathbf{k-alg}$, associating to each k-algebra A the functor V_A defined in (12) (which depends on A in a contravariant fashion). We can therefore say that the study of the functors V_S independently of the immersions (6), the study that we have presented as being the initial aim of algebraic geometry on k, is very exactly equivalent to the study of the arbitrary k-algebras A. In this correspondence $A \leftrightarrow V_A$, to the k-algebras A of finite type corresponds the subfunctors of affine type space functors E^I of finite rank, this is to say that for those which I is finite. If we were, therefore (wrongly) limited to the finite families of indeterminates in (2), this would have as a consequence exclusively limited us to the study of the k-algebras of finite type. On the other hand, the study of the functors of the form V_S given the immersion (6) returns to the study of k-algebras A given again of a system of generators $(t_i)_{i\in I}$, or, in an equivalent manner, to the study of the ideals in the rings of polynomials P_I. We see in particular, for fixed I, that the correspondence $\mathfrak{J} \mapsto V_{\mathfrak{J}}$ of ideals of P_I with subfunctors of E^I of the form V_S is injective: an ideal $\mathfrak{J}$ is known when we know the subfunctor $V_{\mathfrak{J}}$ of E^I set of the solutions of the system of equations (2) defined by $\mathfrak{J}$, in any k-algebra k': In effect $\mathfrak{J}$ is the ideal of the polynomials F such that the corresponding polynomial function $t \mapsto F(t)$ is zero in $V_{\mathfrak{J}}(k')$ for each k-algebra k'. This shows more generally that for two ideals $\mathfrak{J}, \mathfrak{J}'$ of P_I we have the equivalence

$$(\mathfrak{J} \subset \mathfrak{J}') \Leftrightarrow (V_{\mathfrak{J}} \supset V_{\mathfrak{J}'}) \tag{15}$$

10. It is necessary to note well the difference, very important for the question of foundations, between these results, linked to the consideration of arbitrary k-algebras k', and that which occurs when we limit ourselves to the consideration of k-algebras k' which are fields, or more generally, which are reduced (that is to say, without nilpotent elements not equal to zero). Generally, whether $\mathcal{C}$ is a subcategory of **k-alg** formed of reduced algebras (for example, the category of all the reduced algebras, or of all the k-algebras which are fields, or the category having a single object which is a reduced algebra, or a field); note $V_{\mathfrak{J},\mathcal{C}}$, the *restriction* of $V_{\mathfrak{J}}$ to $\mathcal{C}$. It is immediate that $V_{\mathfrak{J},\mathcal{C}}$ is not modified when we replace the system of equations $F=0$, where F covers $\mathfrak{J}$, by the system of equations $F=0$, where F covers the set of polynomials such that there exists a suitable power F^n of F contained in $\mathfrak{J}$. The set of these polynomials, the preimage in P_I of the nilradical of $A_{\mathfrak{J}} = P_I/\mathfrak{J}$ is the *radical* $\sqrt{\mathfrak{J}}$ of $\mathfrak{J}$, and we therefore have

$$V_{\mathfrak{J},\mathcal{C}} = V_{\sqrt{\mathfrak{J}},\mathcal{C}} \tag{16}$$

As it is possible that $\sqrt{\mathfrak{J}} \neq \mathfrak{J}$, [2] the correspondence $\mathfrak{J} \mapsto V_{\mathfrak{J},\mathcal{C}}$ is no longer injective in general. If $\mathfrak{J} = \sqrt{\mathfrak{J}}$, and if $\mathcal{C}$ contains the fields of fractions of those k-algebras which are quotient domains of P_I,

[2] It suffices to take $P = k[x]$ and $\mathfrak{J}$ generated by x^2; we have $\sqrt{\mathfrak{J}} = (x)$.

thus the knowledge of $V_{\mathfrak{J},\mathcal{C}}$ completely determines $\mathfrak{J}$. In effect, $\mathfrak{J} = \sqrt{\mathfrak{J}}$ is the intersection of the prime ideals $\mathfrak{p}$ of P_I containing $\mathfrak{J}$ (Bourbaki, Alg. Comm., chap. II, § no.6, prop. 13); for all polynomials F in P_I not in $\mathfrak{J}$, there is therefore a prime ideal $\mathfrak{p} \supset \mathfrak{J}$ such that F is not in $\mathfrak{p}$, and if k' is the field of fractions of $P_I/\mathfrak{p}$, the polynomial function $t \mapsto F(t)$ in k'^I is not identically zero in $V_{\mathfrak{J}}(k')$, as the image of F in $P_I/\mathfrak{p}$ (and *a fortiori* in $P_I/\mathfrak{J}$) is not zero. We can therefore say that, with the preceding hypothesis on $\mathcal{C}$, *the map* $\mathfrak{J} \mapsto V_{\mathfrak{J},\mathcal{C}}$ *restricted to the set of the ideals equal to their radical, is injective.* More generally, the same reasoning shows that, under the same condition for $\mathcal{C}$, we have the equivalence

$$(\sqrt{\mathfrak{J}} \subset \sqrt{\mathfrak{J}'}) \Leftrightarrow (V_{\mathfrak{J},\mathcal{C}} \supset V_{\mathfrak{J}',\mathcal{C}}) \tag{17}$$

[3]

11. We can therefore say that the exclusive consideration of reduced k-algebras k', as rings of values, for the coordinates of the solutions of a system of polynomial equations (2), returns to develop an algebraic geometry in which we don't distinguish between an ideal $\mathfrak{J}$ (of an algebra of polynomials P_I) and its root $\sqrt{\mathfrak{J}}$; or in terms of the quotient ring $A = P_I/\mathfrak{J}$, where we don't distinguish between a k-algebra A and the quotient of A by its nilradical.

Such a point of view, would not only be a priori artificial, but it would appear today as a fact well established by experience that it would be inadequate for the description of a large number of important phenomenon in algebraic geometry (more particularly the phenomenon of infinitesimal nature), and for developing certain essential techniques (such as the technique of descent, or that of the passage of the formal geometry to the algebraic geometry [7]). We will notably see the appearance, throughout our treatise, the very important technical role of *Artinian local rings*, which intuitively represent the infinitesimal neighborhoods of points on algebraic varieties.

12. In the clasic point of view of algebraic geometry of the 19th century, where $k = \mathbb{C}$ and where we don't change the base field, the set $V(k) = V_S(k) \subset k^I$ is called the algebraic variety defined by the system (2) and interest is on concentrated the geometric properties (subvarieties, intersections with other included varieties in the affine space k^I, etc). We will see that in algebraic geometry conceived as we have exposed in no. 7, it is still possible to associate to a functor V_S (or V_A (equation (12))) that we study a well determined geometric object, which is part of the classical notion of algebraic variety and generalises it. As we want to study this functor independently of its possible inclusions (6), we must define this object (which will be called the spectrum of A) uniquely given the k-algebra A.

In classic algebraic geometry, the ring A would appear as the ring of the polynomial functions on the variety $V(k)$, the restrictions to this variety of the polynomial functions on k^I, and biunivocal correspondence (10) between the variety and the set $Hom_{\mathbf{k-alg}}(A, k)$ consists of associating to each point $t \in V(k)$ the homomorphism $F \mapsto F(t)$ associating to the function F its value at point t (an idea due initially to Dedekind and Weber) [4]. The same interpretation of the correspondence (10) is understood to be possible in the general case: if $f \in A_{\mathfrak{J}}$, for all polynomials $F \in P_I$ of which the canonical image in A is f the restriction to $V_{\mathfrak{J}}(k')$ of the polynomial $t \mapsto F(t)$ does not depend on the polynomial F chosen in the reciprocal image of f; it is therefore a function $f_{k'}$ on $V_{\mathfrak{J}}(k')$, well determined by f, and (for fixed k') the map

$$f \mapsto f_{k'}$$

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[3]The famous Nullstellensatz of Hilbert (Bourbaki, Alg. Comm., chap V, § 3, no 3, prop. 2) shows that the equivalence (17) is owing to the injectivity of $\mathfrak{J} \mapsto V_{\mathfrak{J},\mathcal{C}}$ on the set of the ideals $\mathfrak{J}$ equal to their radical, is again true where k is a field, I is finite, and we only suppose that $\mathcal{C}$ contains the finite extensions of k (where k is algebraically closed, $\mathcal{C}$ can therefore be reduced to k): in effect, the root of $\mathfrak{J}$ is therefore the intersection of the maximal ideals $\mathfrak{m}$ of P_I containing $\mathfrak{J}$ and these are such that $P_I/\mathfrak{m}$ is a finite extension of k).

[4]In virtue the Nullstellensatz and the fact that $k = \mathbb{C}$ is algebraically closed, there is also a biunivocal correspondence between $V(k)$ and the set maximal ideals (or maximal spectrums) of A (cf. chap. I appendix I)

of A to the k-algebra $k'^{V(k')}$ of maps of $V(k')$ to k', is a homomorphism (in general not injective) of k-algebras. To each point $t \in V(k')$ corresponds therefore the k-homomorphism

$$f \mapsto f_{k'}(t) \tag{18}$$

of A to k'. We are therefore led to simply write t for this homomorphism, so either $f(t)$ or $t(f)$ instead of $f_{k'}(t)$, and to call the elements of $Hom_{\mathbf{k-alg}}(A, k')$ the points of V_A with values in k' (or with coordinates in k')

13. We have thus well introduced a geometry language; however, we no longer have use for a well determined object as in the classical algebraic geometry, but instead for a family of variable objects with k'. In order to obtain the spectrum of A, we will firstly restrain our attention to the points of V_A with values in the k-algebras k' which are fields; we will say that these are the geometric points of V_A, and we will establish between these points (corresponding to different possible fields k') a relation of equivalence. We will say that two geometric points

$$t' : A \longrightarrow k', \quad t' : A \longrightarrow k''$$

are equivalent if there exists a third geometric point $s : A \longrightarrow K$ and the homomorphisms of k-algebras (necessarily injective)

$$f' : k' \longrightarrow K, \quad f'' : k'' \longrightarrow K$$

such that we have $s = f' \circ t = f'' \circ t'$, otherwise known as the diagram

$$\begin{array}{ccccc} k' & \xrightarrow{f'} & K & & \\ A\uparrow & & & \xrightarrow{t''} & k''\,f''\uparrow \end{array} \tag{19}$$

is commutative. Show that this relation is equivalent to the following: $t'^{-1}(0) = t''^{-1}(0)$ (this proves that it is an equivalence relation). In effect, since f' and f'' are injective, the commutativity of diagram (19) leads to the equality of the kernels of t' and t''. Inversely, note that since each subring of a field in an integral domain, the kernel of a homomorphism $A \longrightarrow k'$ is a prime ideal of A. If the kernels of t' and t'' are equal to the same prime ideal $\mathfrak{p}$, k' and k'' can be considered as two extensions of the same field $\kappa(\mathfrak{p})$, fields of the fractions of $A/\mathfrak{p}$, and we know (Bourabki, alg. chap. V, § 4, no. 2 prop. 2) that there exists a field K and two $\kappa(p)$-monomorphisms $f' : k \longrightarrow K$, $f'' : k'' \longrightarrow K$, which obviously render (19) commutative.

The equivalence classes for this relation (which we can again call the places of A) [5] are therefore in biunivocal correspondence with the prime ideals of A; in fact, we obtain in this way all the prime ideals because if $\mathfrak{p}$ is such an ideal and $\kappa(\mathfrak{p})$ the field of fractions of the integral ring $A/\mathfrak{p}$ the ideal $\mathfrak{p}$ corresponds to the equivalence class of the geometric point $A \longrightarrow A/\mathfrak{p} \longrightarrow \kappa(\mathfrak{p})$ where the two arrows are the canonical homomorphisms. We have therefore obtained in this way a canonical biunivocal correspondence between the set of places of the k-algebra A and the set of prime ideals of A. It is therefore the set $Spec(A)$ of the prime ideals of A that we take as the underlying set of the geometric object which must be the spectrum of A. In the classical case where $k = \mathbb{C}$ and A is a $\mathbb{C}$-algebra of finite type, this set contains the algebraic variety corresponding to A, the set of the geometric points of V_A with values in k.

14. The set $Spec(A)$ is naturally given from a topology linked to the generalisation of the notion of subvariety of an algebraic variety. Classically, a subvariety of an algebraic variety defined by a system of equations (2) is defined by a system of equations containing the system (2); in other words, if we consider the ring A as the ring of polynomial functions on the variety, a subvariety is defined as the set where some of these functions annihilate. In the general case, we are therefore led to the consider

[5] The well-informed reader will recognise in this language a reformulation of the general points or generics of the algebraic varieties, such that they intervene for example in the works of Zariski or A. Weil.

for a part S of the k-algebra A, for each k-algebra k', the set (part of $V_A(k')$) where all the functions $f_{k'}$ corresponding (no. 12) to the elements $f \in S$ annihilate. If we restrict ourselves as in no.13 to geometric points with values in fields, we are therefore led to the consider the set $V(S)$ of the places of those points (of the points mentioned) where $f_{k'}$ for $f \in S$ annihilate. It results from no.13 that this set corresponds biunivocally to the set of the prime ideals of A containing S; we say again that this part of *Spec*(A) is the algebraic set defined by S. We will remark that it does not change when we replace S by the radical of the ideal generated by S; and as this radical is precisely the intersection of the prime ideals of A containing S, we see that we obtain in this way a biunivocal correspondence between the set of the ideals of A equal to their radical, and the set of the parts $V(S)$ of *Spec*(A). We show (Bourbaki, Alg. comm. chap 2. § 4) that this set of parts is the set of closed parts for a topology on *Spec*(A), called te *spectral topology* or *topology of Zariski*. Moreover, (*loc.* cit.) the space X obtained in giving *Spec*(A) of this topology is quasi-compact and satisifes the axiom of Kolmogoroff, but possesses in general the nonclosed points (and a fortiori is not a separated space); for each element $f \in A$, the set $D(f) = X - V(f)$ is open in X, and the $D(f)$s (for $f \in A$) form a basis of the spectral topology.

15. If we wish that the object *Spec*(A), that we wish to associate to a ring A, inversely permits the reconstruction of the ring A, it does not suffice to take for such an object the topological space *Spec*(A) that we have just defined: for example, for all fields K we obtained the reduced space to a single point. In this last case it is clear that the consideration of this space does not provide anything new for the study of the field K.

It is therefore vain to hope to describe the spectrum of A in exclusively topological terms; we must provide the topological space $X =$ *Spec*(A) with a structure where the algebra A intervenes. The model on which we guide ourselves here is provided by the holomorph of varieties (of which the classic case of algebraic varieties without singularities ($k = \mathbb{C}$) provide particular cases); following the conception introduced by H. Cartan, since on such a variety X it is possible to define on each open $U \subset X$ the functions (complex) *holomorphic in* U, in associating to each open U the set $\mathcal{O}(U)$ of these functions, we define evidently a presheaf of rings on X and in fact this presheaf is a sheaf $\mathcal{O}_X$; in other words the holomorphic variety appears as the topological space given from a sheaf of rings, or, as we say again, a *ringed space*. In his fundamental work [FAC], J.P Serre showed in substance how we can transport this definition into algebraic geometry. Limit ourselves initially to the classic case ($k = \mathbb{C}$) and suppose that the algebraic variety X corresponds to a ring of polynomial functions A which are integral (where we say that the variety is irreducible, and to which we often limit ourselves in classical algebraic geometry). The field of fractions K of A is therefore called he field of rational functions on X; an element g/f of K, quotient of two polynomial functions ($f \neq 0$), is a function defined on the points $x \in X$, where $f(x) \neq 0$, but cannot be in general extended by continuity to the points where $f(x) = 0$ (poles, or points of indetermination). Since Riemann, these functions play traditionally, for an algebraic variety, the role of meromorphic functions on an analytical variety. We are therefore led to consider the presheaf $U \mapsto \mathcal{O}(U)$ on X, where, for each $U \subset X$, $\mathcal{O}(U)$ is the ring of rational functions defined in U.

But, we can again give an analgous definition where A is an arbitrary integral ring, X the topological space *Spec*(A), and where we restrict open U to those of base D_f (for $0 \neq f \in A$), and where we take for $\mathcal{O}(U)$ the ring A_f of the elements of K of the form g/f^n (arbitrary integer $n \geq 0$, $g \in A$). But in fact, it is also not useful to make no hypothesis on the ring A, and it is possible for all $f \in A$ (divisor of 0 or not) to define the ring A_f (Bourbaki, alg. comm. Chap 2. § 5 no.1); we therefore show that the map $D(f) \mapsto A_f$ defines a sheaf of k-algebras on X, written as $\tilde{A}$ or $\mathcal{O}_X$, and the object *spectrum of* A associated with A is finally the ringed space $(X, \mathcal{O}_X)$. The detailed demonstrations will be given in Chapter I § 1; we will there see among others, that the fibres of the sheaf of rings $\mathcal{O}_X$ are the local rings $A_{\mathfrak{p}}$, localised from A to the prime ideals of A, such that $(X, \mathcal{O}_X)$ is a locally k-ringed space; the ring A can be recovered (up to isomorphism) from its spectrum by the relation $\Gamma(X, \mathcal{O}_X) \cong A$.

We will also see in Chapter I, § 1 how, for each homomorphism

$$\varphi : A \longrightarrow A' \tag{20}$$

of k-algebras there is associated in a functorial fashion a morphism of locally k-ringed spaces

$$Spec(\varphi) : X' = Spec(A') \longrightarrow X = Spec(A) \tag{21}$$

such that the corresponding homomorphism of k-algebras

$$\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X', \mathcal{O}_{X'})$$

identifies itself (taking into account the canonical isomorphisms $\Gamma(X, \mathcal{O}_X) \cong A$, $\Gamma(X', \mathcal{O}_{X'}) \cong A'$) with the homomorphism given in (20). This implies in particular that the map

$$\varphi \mapsto Spec(\varphi) : Hom_{\mathbf{k-alg}}(A, A') \longrightarrow Hom_k(Spec(A'), Spec(A)) \tag{22}$$

(where the second term is the collection of morphisms of locally k-ringed spaces (0,4.1.12)) is injective. In fact, we will even prove that this map is bijective (I, 1.6.3); in other words, *the contravariant functor* $A \mapsto Spec(A)$ *from the category of k-algebras to the category of locally k-ringed spaces, is fully faithful.* This therefore permits (taking into account the usual reversal of arrows when we pass from a category to its opposite) the identification in practice of the category of k-algebras with a full subcategory of the category of locally k-ringed spaces, knowing this to be formed of those spaces isomorphic with $Spec(A)$ for a suitable k-algebra A; this locally k-ringed spaces are called *affine schemes on k*. The initial problem of algebraic geometry on k, which we identified in the study of k-algebras, is therefore also equivalent to the study of algebro-topological objects which are the affine schemes on k. As for a functor V_A defined by (12), it expresses itself simply in terms of $Spec(A)$, taking into account the bijectivity of (22), by the formula

$$V_A(k') \cong Hom_k(Spec(k'), Spec(A)) \tag{23}$$

where the second term has the same meaning as in (22).

16. It may seem at first glance that the equivalence of preceding categories can not lead to replacing the study of an object of fairly simple definition like a k-algebra by that of the much more complicated object which is its spectrum. In fact, even in the studies of local algebras, the translation of the properties in terms of the theory of affine schemes, in giving them a geometric interpretation, renders them often less abstract and more accessible to a sort of intuition which facilitates their manipulation, (c.f Chapter IV) even though by definition the demonstrations must always lead to some purely algebraic properties. But the decisive advantage provided by the geometric language, based on the introduction of the spectrum of a ring, is that the language applies without the effort of leaving the frame of commutative algebra, which is indispensible if we wish to develop modern algebraic geometry on the model of the classical theory. In effect, since the beginning of the 19th century, we have perceived that the study of the systems of polynomial equations of type (2) (which is that which we can call affine algebraic geometry) only gives simple and striking expressions when we place it in a more vast frame, that of the projective algebraic geometry [6]. We know that, classically, the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is obtained by the joining of $n+1$ affine spaces, the hyperplanes $X_j = 1$, in the space $\mathbb{C}^{n+1}$, by identifying points with the ray which joins them through the origin. But, in other domains of mathematics, such processes of rejoining intervene in some rather more general contexts, as it is in this way that we now define the diverse notions of variety: topological, differential, analytical, et cetera. In each of these domains, the crucial point in the operation of rejoining is the rejoining of topologies, those of the additional structures which hereby deduce themselves without trouble; it is

[6] It is moreover without doubt fortuitous that it is the same men, Monge and Poncelet, who are at the same time at the origin of this enlargement and of the passage of the field of real to the field of complex

without doubt H. Cartan who first saw the reason for this fact in observing that the diverse structures of which we have just spoken can all be defined as structures of ringed space, and that the operation of rejoining leads itself therefore in each case to the general operation of rejoining of sheaves.

But, once we have led the commutative algebra to a study of particular ringed spaces, it suffices to apply mutatis mutandis this general operation to arrive in the end at a fundamental notion of modern algebraic geometry, that of the scheme on k: this would simply be a space X locally ringed, which admits a recovery (X_α) by the opens which are (for the structure of the inferred ringed space) the affine schemes on k. Such an object X defines again a functor

$$k' \mapsto X(k') : \mathbf{k-alg} \longrightarrow \mathbf{Sets} \tag{24}$$

by the formula (generalising (23))

$$X(k') = Hom_k(Spec(k'), X) \tag{25}$$

(points of X with values in k'). We can moreover show (I. 2.3.6), that the knowledge of the functor (24) regives the scheme X up to a unique isomorphism, and more precisely that the functor $X \mapsto X(.)$ from the category of schemes on k to the category of functors $\mathbf{k-alg} \longrightarrow \mathbf{Sets}$, defined by the formula (25), is fully faithful; in other words, it permits the identification of the category of schemes with a full subcategory of the category of functors

$$\mathbf{k-alg} \longrightarrow \mathbf{Sets}$$

[7]

17. It remains to indicate that the base ring k has not played a trivial role in all that has preceeded. The ringed space $Spec(A)$, for a given k-algebra, only depends in fact on the structure of the ring of A, and the given homomorphism $k \longrightarrow A$ defining the structure of the k-algebra is simply equivalent to the given structure of k-algebra on the sheaf of rings $\mathcal{O}_X$, where again (according to that which we have seen in no.15 in assuming $k = \mathbb{Z}$) to the given morphism of locally ringed spaces

$$Spec(A) \longrightarrow Spec(k)$$

There is therefore interest in defining firstly the notion of schemes in the absolute sense (i.e a scheme on $\mathbb{Z}$) and in thus defining a scheme on k (or k-scheme) like a scheme X in the sheaf of rings $\mathcal{O}_X$ is given from a structure of k-algebra, i.e. defined by the given homomorphism of rings $k \longrightarrow \Gamma(X, \mathcal{O}_X)$, or again (and preferably) by the given morphism of locally ringed spaces

$$X \longrightarrow Spec(k)$$

This last point of view has the considerable advantage of lending itself to the substitution of the ring k (or even better, of the affine scheme $Spec(k)$) for an arbitrary scheme Y and of leading in this way to the notion of scheme X underlying scheme Y (or Y-scheme)(c.f I, 2.6.1), which will be studied in a detailed fashion in our treatise, and which intuitively is completely analgous to the notion of fibred space (or more generally of topological space X below a topological space Y, that is to say, given from a continuous map $f : X \longrightarrow Y$) fluently used by topologists.

The object of algebraic geometry in the sense we understand it in this treatise, is therefore the study of schemes, locally ringed spaces of a particular type; or in the same fashion, the study of functors (24) *to which they give birth.*

[7] In this sense, we can consider that the introduction of the structures of the locally ringed space is above all a technical artifice, permitting the formulation in a particularly convenient and intuitive fashion a procedure of rejoining of affine functors. For more details on relations between ringed spaces and functors $\mathbf{k-alg} \longrightarrow \mathbf{Sets}$ see [3], Chapter I. § 1, which also contains an exposition, excellent from a technical point of view, of the definition of schemes.

CHAPTER 1

Preliminaries

1. Representable Functors

1.1. Represtable Functors.

1.1.1. We designate by **Sets**the category of sets. Let $\mathcal{C}$ be a category [1]; for two objects X, Y of $\mathcal{C}$, we pose $h_X(Y) = Hom(Y, X)$; for each morphism $u : Y \longrightarrow Y'$ in $\mathcal{C}$, we designate by $h_X(u)$ the map $v \mapsto vu$ from $Hom(Y', X)$ to $Hom(Y, X)$. It is immediate that with these definitions, $h_X : \mathcal{C}^\star \longrightarrow$ **Sets** is a contravariant functor, that is to say an object of the category, written $Hom(\mathcal{C}^\star, \mathbf{Sets})$ of the covariant functors of the category $\mathcal{C}^\star$, opposite to the category $\mathcal{C}$, to the category **Sets**((T, 1.7, d) and SGA, 3.I).

1.1.2. Now let $w : X \longrightarrow X'$ be a morphism in $\mathcal{C}$; for each $Y \in \mathcal{C}$ and each $v \in Hom(Y, X) = h_X(Y)$, we have $wv \in Hom(Y, X') = h_{X'}(Y)$; designate by $h_w(Y)$ the map $v \mapsto wv$ from $h_X(Y)$ to $h_{X'}(Y)$. It is immediate that for each homomorphism $u : Y \longrightarrow Y'$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} h_X(Y') & \xrightarrow{h_X(u)} & h_X(Y) \\ {\scriptstyle h_w(Y')}\downarrow & & \downarrow{\scriptstyle h_w(Y)} \\ h_{X'}(Y) & \xrightarrow[h_{X'}(u)]{} & h_{X'}(Y) \end{array}$$

is commutative; in other words, h_w is a functorial morphism $h_X \longrightarrow h_{X'}$ (T, 1.2), or again a homomorphism in the category $Hom(\mathcal{C}^\star, \mathbf{Sets})$ (T, 1.7, d). The definitions of h_X and of h_w consititute therefore the definition of a *canonical covariant functor*

$$h : \mathcal{C} \longrightarrow Hom(\mathcal{C}^\star, \mathbf{Sets} \tag{26}$$

1.1.3. Let X be an object of $\mathcal{C}$, F a contravariant functor from $\mathcal{C}$ to **Sets**(object of $Hom(\mathcal{C}^\star, \mathbf{Sets})$). Let $g : h_X \longrightarrow F$ be a functorial morphism: for each $Y \in \mathcal{C}$, $g(Y)$ is therefore a map $h_X(Y) \longrightarrow F(Y)$ such that for each morphism $u : Y \longrightarrow Y'$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} h_X(Y') & \xrightarrow[h_X(u)]{} & h_X(Y) \\ {\scriptstyle g(Y')}\downarrow & & {\scriptstyle g(Y)}\downarrow \\ F(Y') & \xrightarrow{F(u)} & F(Y) \end{array} \tag{27}$$

is commutative. In particular, we have a map

$$g(X) : h_X(X) = Hom(X, X) \longrightarrow F(X)$$

from which an element

$$\alpha(g) = (g(X))(1_X) \in F(X) \tag{28}$$

[1] We consider the categories from a naive point of view, as if they consisted of sets and refer to SAG, 4, I for questions of logic linked to the theory of categories, and the justification of the language which we use.

and by consequence a canonical application

$$\alpha : Hom(h_X, F) \longrightarrow F(X) \tag{29}$$

Inversely, consider an element $\xi \in F(X)$; for each morphism $v : Y \longrightarrow X$ in $\mathcal{C}$, $F(v)$ is a map $F(X) \longrightarrow F(Y)$; consider the map

$$v \mapsto (F(v))(\xi) \tag{30}$$

from $h_X(Y)$ to $F(Y)$; if we designate by $(\beta(\xi))(Y)$ this map,

$$\beta(\xi) : h_X \longrightarrow F \tag{31}$$

is a functorial morphism, because we have for each morphism $u : Y \longrightarrow Y'$ in $\mathcal{C}$, $(F(vu))(\xi) = (F(V) \circ F(u))(\xi)$, which verifies the commutativity of (27) for $g = \beta(\xi)$. We have in this way defined a canonical map

$$\textbf{??}\beta : F(X) \longrightarrow Hom(h_X, F) \tag{32}$$

Proposition 1. *The map α and β are mutually inverse bijections.*

Recall that a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is defined by the condition that these objects are objects of $\mathcal{C}$, and that if X', Y' are two objects of $\mathcal{C}'$ the set $Hom_{\mathcal{C}'}(X', Y')$ of the morphisms X' to Y' in $\mathcal{C}$, the canonical map of composition of morphisms

$$Hom_{\mathcal{C}'}(X', Y') \times Hom_{\mathcal{C}'}(Y', Z') \times Hom_{\mathcal{C}'}(X', Z')$$

being the restriction of the canonical map

$$Hom_{\mathcal{C}'}(X', Y') \times Hom_{\mathcal{C}}(Y', Z') \longrightarrow Hom_{\mathcal{C}}(X', Z')$$

We see that $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ if $Hom_{\mathcal{C}'}(X', Y') = Hom_{\mathcal{C}}(X', Y')$ for all X', Y' in $\mathcal{C}'$. The subcategory $\mathcal{C}''$ of $\mathcal{C}$ formed of objects of $\mathcal{C}$ isomorphic to the objects of $\mathcal{C}'$ is again therefore a full subcategory of $\mathcal{C}$ equivalent (T, 1.2) to $\mathcal{C}'$.

A covariant functor $F : \mathcal{C}_\infty \longrightarrow \mathcal{C}_\in$ is called *fully faithful* if, for each couple of objects X_1, Y_1 of $\mathcal{C}_\infty$, the map $u \mapsto F(u)$ from $Hom(X_1, Y_1)$ to $Hom(F(X_1), F(Y_1))$ is bijective; this leads to the fact that the subcategory $F(\mathcal{C}_\infty)$ of $\mathcal{C}_\in$ is *full*. Moreover, if two objects X_1, X_1' have the same image X_2, there exists a unique isomorphism $u : X_1 \longrightarrow X_1'$ such that $F(u) = 1_{X_2}$. For each object X_2 of $F(\mathcal{C}_\infty)$ let therefore $G(X_2)$ be one of the objects X_1 of $\mathcal{C}_\infty$ such that $F(X_1) = X_2$ (G being defined by the axiom of choice); for each morphism $v : X_2 \longrightarrow Y_2$ in $F(\mathcal{C}_\infty)$, $G(v)$ will be the unique morphism $u : G(X_2) \longrightarrow G(Y_2)$ such that $F(u) = v$; G is therefore a functor of $F(\mathcal{C}_\infty)$ to $\mathcal{C}_\infty$; FG is the identity functor on $F(\mathcal{C}_\infty)$, and that which preceeds shows that there exists an isomorphism of functors $\varphi : 1_{\mathcal{C}_\infty} \cong GF$ such that F, G, φ is the identity $1_{F(\mathcal{C}_\infty)} \cong FG$ defines an equivalence of the category $\mathcal{C}_\infty$ and of the full subcategory $F(\mathcal{C}_\infty)$ of $\mathcal{C}_\in$ (T, 1.2).

1.1.4. Apply the prop. (1.1.4) to the case where the functor F is $h_{X'}$, X' being an arbitrary object of $\mathcal{C}$; the map

$$\beta : Hom(X, X') \longrightarrow Hom(h_X, h_{X'})$$

is none other than the map $w \mapsto h_w$ defined in (1.1.2); this map being bijective, we see, with the terminology of (1.1.5), that:

Proposition 2. *The canonical functor $h : \mathcal{C} \longrightarrow Hom(\mathcal{C}^\star, \textbf{Sets})$ is fully faithful.*

We will very often utilise this fact in order to prove the results of the morphism of the category $\mathcal{C}$: in order to show, for example, that a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array}$$

of morphisms of $\mathcal{C}$ is *commutative*, it suffices to prove that for *each object* $Y \in \mathcal{C}$ the diagram

$$\begin{array}{ccc} h_A(Y) & \xrightarrow[h_f(Y)]{} & h_B(Y) \\ {\scriptstyle h_u(Y)}\downarrow & & {\scriptstyle h_v(Y)}\downarrow \\ h_C(Y) & \xrightarrow{h_g(Y)} & h_D(Y) \end{array}$$

is commutative, because the relation $h_v \circ h_f = h_g \circ h_u$ is equivalent in virtue of (1.1.7) to $v \circ f = g \circ u$. We are in this way led to verify the commutativity of diagrams in **Sets**, which is in general much easier.

1.1.5. Let F be a contravariant functor from $\mathcal{C}$ to **Sets**; we say that F is representable if there exists an object X in $\mathcal{C}$ such that F is isomorphic to h_X; it results from (1.1.7) that the given X in $\mathcal{C}$ and the given isomorphism of functors $g : h_X \longrightarrow F$ determines X up to a unique isomorphism. The prop. (1.1.7) signifies again that h defines an equivalence of $\mathcal{C}$ and of the full subcategory $Hom(\mathcal{C}^\star, \mathbf{Sets})$ formed from the contravariant representable functors. It results, moreover, from (1.1.4) that the given functorial morphism $g : h_X \longrightarrow F$ is equivalent to that of an element $\xi \in F(X)$: to say that g is an isomorphism is equivalent for ξ to the following condition: *for each object Y of $\mathcal{C}$ the map $v \mapsto F(v)(\xi)$ from $Hom(Y, X)$ to $F(Y)$ is bijective.* Where ξ verifies this condition we say that the couple (X, ξ) represents the representable functor F. By abuse of language, we would also say that the object X in $\mathcal{C}$ represents F if there exists $\xi \in F(X)$ such that (X, ξ) represents F, in other words if h_X is isomorphic to F. Let F, F' be two representable contravariant functors from $\mathcal{C}$ to **Sets**, let $h_X \longrightarrow F$ and $h_{X'} \longrightarrow F'$ two isomorphisms of functors. Therefore it results from (1.1.6) that there is a canonical biunivocal correspondence between $Hom(X, X')$ and the set $Hom(F, F')$ of functorial morphisms $F \longrightarrow F'$.

1.1.6. *Examples I: Projective limits.* The notion of representable contravariant functor covers in particular the dual notion of the usual notion of solution of a universal problem. More generally, we will see that the notion of projective limit a particular case of that of a representable functor. Recall [2] that in a category $\mathcal{C}$ we define a projective system by the given preordered set I, of a family $(A_\alpha)_{\alpha \in I}$ of objects of $\mathcal{C}$, and for each couple of indices (α, β) such that $\alpha \leq \beta$, of a morphism $u_{\alpha\beta} : A_\beta \longrightarrow A_\alpha$ with $u_{\alpha\gamma} = u_{\alpha\beta} \circ u_{\beta\gamma}$ for $\alpha \leq \beta \leq \gamma$. A projective limit of this system in $\mathcal{C}$ is consitituted by an object B of $\mathcal{C}$ (written as $\varprojlim A_\alpha$), and for each $\alpha \in I$ a morphism $u_\alpha : B \longrightarrow A_\alpha$ such that:

1 $u_\alpha = u_\alpha \beta u_\beta$ for $\alpha \leq \beta$;
2 For each object X of $\mathcal{C}$ and each family $(v_\alpha)_{\alpha \in I}$ of morphisms $v_\alpha : X \longrightarrow A_\alpha$, such that $v_\alpha = u_{\alpha\beta} v_\beta$ for $\alpha \leq \beta$

There exists a unique morphism $v : X \longrightarrow B$ (written as $\varprojlim v_\alpha$) such that $v_\alpha = u_\alpha v$ for each α in I (T, 1.8). This is interpreted in the following fashion: the $u_{\alpha\beta}$ canonically define the map

$$Hom(1_X, u_{\alpha\beta}) = \bar{u}_{\alpha\beta} : Hom(X, A_\beta) \longrightarrow Hom(X, A_\alpha)$$

which define a *projective system* of sets $(Hom(X, A_\alpha), \bar{u}_{\alpha\beta})$, and (v_α) is by definition an element of the set $\varprojlim_\alpha Hom(X, A_\alpha)$; it is clear that $X \mapsto \varprojlim Hom(X, A_\alpha)$ is a *contravariant functor* from $\mathcal{C}$ to **Sets**, and the existence of the projective limit B is equivalent to saying that $(v_\alpha) \mapsto \varprojlim v_\alpha$ is an isomorphism of functors in X

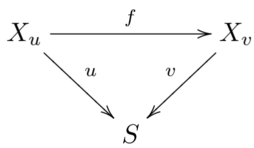
$$\varprojlim Hom(X, A_\alpha) \cong Hom(X, B) \tag{33}$$

in other words the functor $X \mapsto \varprojlim Hom(X, A_\alpha)$ is *representable.* If each projective system of objects in $\mathcal{C}$ has a projective limit, we say that $\mathcal{C}$ has projective limits.

[2] We limit ourselves here to projective (or inductive) limits following a preordered set, which will only be used until Chapter VI. For the extension of these notions to the case where the preordered set is replaced by a category, see [10] and SGA, 4,I,2.

1.1.7. *Example II: Final objects* Let $\mathcal{C}$ be a category, $\{a\}$ a reduced set with a single element. Consider the contravariant functor $F : \mathcal{C}^\star \longrightarrow \mathbf{Sets}$ which, to each object X of $\mathcal{C}$ corresponds the set $\{a\}$ and to each morphism $X \longrightarrow X'$ in $\mathcal{C}$ the unique map $\{a\} \longrightarrow \{a\}$. To say that this functor is representable signifies that there exists an object $e \in \mathcal{C}$ such that for each Y in $\mathcal{C}$, $Hom(Y, E) = h_E(Y)$ is reduced to an element; we say that E is a final object of $\mathcal{C}$, and it is clear that two final objects of $\mathcal{C}$ are isomorphic (which permits the definition, in general with the aid of the axiom of choice, *one* final object of $\mathcal{C}$ which we will write $e_{\mathcal{C}}$). For example, in the category of sets, the final objects are the sets reduced to an element; in the category of augmented algebras on a field K (where the morphisms are the homomorphisms of compatible algebras with augmentations), K is a final object.

1.1.8. For each category $\mathcal{C}$ and each $S \in \mathcal{C}$, we introduce a new category $\mathcal{C}/S$ called the S-objects of $\mathcal{C}$ in the following way; the objects of $\mathcal{C}/S$ are the morphisms (of $\mathcal{C}$) $u : X_u \longrightarrow S$ with codomain S; we call S-morphisms from X_u to X_v (for u, v in $\mathcal{C}/S$) a morphism $f : X_u \longrightarrow X_v$ of $\mathcal{C}$ such that the diagram



is commutative, the composition of S-morphisms being the same as in $\mathcal{C}$. We therefore take as the set $Hom_{\mathcal{C}/S}(U, V)$ of the morphisms from U to V the set of triplets (X_u, X_v, f) where $f : X_u \longrightarrow X_v$ is an S-morphism; the composite of the morphism (X_v, X_w, g) from v to w and of the morphism (X_u, X_v, f) from u to v is therefore $(X_u, X_w, g \circ f)$. In this category, 1_S is a final object, because for each morphism $u : X_u \longrightarrow S$, $Hom_{\mathcal{C}/S}(u, 1_S)$ is reduced by definition to a single morphism (X_u, S, u).

We have a functor $\mathcal{C}/S \longrightarrow \mathcal{C}$ which, to each morphism $u : X_u \longrightarrow S$ corresponds its source X_u, and to each morphism $f : u \longrightarrow v$ the morphism $f : X_u \longrightarrow X_v$ corresponds. Although $u \mapsto X_u$ is not in general injective, we often speak (by abuse of language) of X_u (instead of u) as an S-object, and we say that u is its *structural morphism.* The S-morphisms $s : S \longrightarrow X_u$, this is to say the morphisms of $\mathcal{C}$ such that $u \circ s = 1_S$ are called the S-sections of u (or of X_u); these are the monomorphisms in $\mathcal{C}/S$ as we will verify soon; their collection is written as $\Gamma(X/S)$.

Where the category $\mathcal{C}$ admits a final object e, $\mathcal{C}/e$ is identified canonically with $\mathcal{C}$.

1.1.9. For two objects X, Y of a category $\mathcal{C}$, pose $h'_X(Y) = hom(X, Y)$ and for each morphism $u : Y \longrightarrow Y'$, let, $h'_X(u)$ be the map $v \mapsto uv$ from $Hom(X, Y)$ to $Hom(X, Y')$; h'_X is therefore a covariant functor $\mathcal{C} \longrightarrow \mathbf{Sets}$, from where we deduce as in (1.1.2) the definition of a canonical covariant functor $h' : \mathcal{C}^\star \longrightarrow Hom(\mathcal{C}, \mathbf{Sets})$; a covariant functor $F : \mathcal{C} \longrightarrow \mathbf{Sets}$, in other words an object of $Hom(\mathcal{C}, \mathbf{Sets})$, is therefore said to be representable if there exists an object X in $\mathcal{C}$ (necessarily unique up to isomorphism) such that F is isomorphic to h'_X; we leave to the reader the development of the dual considerations of the precedence for this notion, which covers this time that of inductive limits, and in particular the usual notion of the solution of universal problems.

We also have a dual notion of this final object, that of the initial object of a category $\mathcal{C}$: it is an object e' in $\mathcal{C}$ such that $Hom(e', Y)$ is reduced to an element for all Y in $\mathcal{C}$.

1.1.10. Let $\mathcal{C}$ be an arbitrary category, and consider the category $Hom(\mathcal{C}, \mathbf{Sets})$ of the functors (covariant) from $\mathcal{C}$ to $\mathbf{Sets}$; this last category *admits projective and injective limits.* In effect let (F_α) be a projective system of the functors from $\mathcal{C}$ to $\mathbf{Sets}$; for $\alpha \leq \beta$, we have therefore a functorial morphism $u_{\alpha\beta} : F_\beta \longrightarrow F_\alpha$ with $u_{\alpha\gamma} = u_{\alpha\beta} \circ u_{\beta\gamma}$ for $\alpha \leq \beta \leq \gamma$. For each object X in $\mathcal{C}$ the sets $F_\alpha(X)$ form therefore a projective system of sets for the maps $u_{\alpha\beta}(X) : F_\beta(X) \longrightarrow F_\alpha(X)$, and have therefore a projective limit $F(X)$. Moreover, if Y is a second object of $\mathcal{C}$ and $v : X \longrightarrow Y$ is a morphism of $\mathcal{C}$, the map $F_\alpha(v) : F_\alpha(X) \longrightarrow F_\alpha(Y)$ form a projective system of maps, which has therefore a limit $F(v) : F(X) \longrightarrow F(Y)$. We will soon verify that we have in this way defined a functor F from $\mathcal{C}$ to $\mathbf{Sets}$, and that F is the projective limit of the projective limit (F_α). We also see that an inductive system (F_α)

of functors from $\mathcal{C}$ to **Sets**has an inductive limit defined by $F(X) = \varinjlim F_\alpha(X)$ and $F(v) = \varinjlim F_\alpha(v)$ (we therefore say that the inductive and projective limits in the category $Hom(\mathcal{C}, \mathbf{Sets})$ are calculated pointwise).

1.1.11. In the category $Hom(\mathcal{C}, \mathbf{Sets})$ there is also a final object, which we call P, that to each object $X \in \mathcal{C}$ corresponds a set $P(X)$ with an element and to each morphism $u : X \longrightarrow Y$ the unique map $P(u) : P(X) \longrightarrow P(Y)$.

1.2. Fibred products in a category.

1.2.1. Let $\mathcal{C}$ be a category, X, Y two objects of $\mathcal{C}$; we say that X and Y *admit a product* in $\mathcal{C}$ if the contravariant functor

$$F : T \mapsto Hom(T, X) \times Hom(T, Y)$$

from $\mathcal{C}$ to **Sets**is representable (for each morphism $v : T \longrightarrow T'$, $F(v)$ is the map $(f', g') \mapsto (f' \circ v, g' \circ v)$ from $Hom(T', X) \times Hom(T', Y)$ to $Hom(T, X) \times Hom(T, Y)$). An object representing this functor is therefore formed from an object Z in $\mathcal{C}$ and from a couple of morphisms $p_1 : Z \longrightarrow X$, $p_2 : Z \longrightarrow Y$, and these objects are determined up to a unique isomorphism (1.1.8). We say that Z is the *product* of X and Y in $\mathcal{C}$, p_1 and p_2 the *first* and the *second projection* of Z; we write $X \times Y$ most often in order to designate the product Z. The map

$$g \mapsto (p_1 \circ g, p_2 \circ g) \tag{34}$$

is an isomorphism of functors in T:

$$Hom(T, X \times Y) \cong Hom(T, X) \times Hom(T, Y) \tag{35}$$

By this map, to the morphism $1_{X \times Y}$ corresponds the couple (p_1, p_2).

It is clear that we can say that $X \times Y$ is the *projective limit* of the projective system where the preordinated set I is formed from two distinct elements α, β, the relation of order in I being the relation of equality, the family $(X_\lambda)_{\lambda \in I}$ is such that $X_\alpha = X$ and $X_\beta = Y$, the only morphisms $u_{\lambda\mu}$ being therefore the identities 1_X and 1_Y (1.1.9).

1.2.2. We consider now an object S of $\mathcal{C}$ and two morphisms

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1.2.3. *1.2.4.* We say that a diagram of morphisms of $\mathcal{C}$

$$\begin{array}{ccc} X & \xleftarrow{\ u\ } & Z \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle v} \\ S & \xleftarrow{\ \psi\ } & Y \end{array}$$

is *cartesian* if it is commutative and if, for each S-object T the map

$$g \mapsto (u \circ g, v \circ g)$$

is a bijective map from $Hom_S(T, Z)$ to $Hom_S(T, X) \times Hom_S(T, Y)$ (X, Y and Z being considered as S-objects by virtue of $\varphi, \psi and \theta = \varphi \circ u = \psi \circ v$ respectively); the S-prescheme Z is therefore equal to $X \times_S Y$ up to isomorphism.

1.2.4. Let X, Y, X', Y' be four S-objects; suppose that the products $X \times_S Y$ and $X' \times_S Y'$ exist and let p_1, p_2, p'_1, p'_2 be the canonical projections. Therefore, for each couple of S-morphisms $u : X' \longrightarrow X$, $v : Y' \longrightarrow Y$, we pose

$$u \times_S v = (u \circ p'_1, v \circ p'_2)_S : X' \times_S Y' \longrightarrow X \times_S Y \tag{36}$$

We also write $u \times v$ if there is no confusion.

PROPOSITION 3. *Suppose that the products exist in $\mathcal{C}/S$. If we pose $F(X, Y) = X \times_S Y$, $F(u, v) = u \times_S v$, F is a covariant bifunctor of $\mathcal{C}/S \times \mathcal{C}/S$ to $\mathcal{C}/S$.*

Proof. It suffices to remark that the diagram

$$\begin{array}{ccccc} X \times_S Y & \xrightarrow{f\times_S 1_Y} & X' \times_S Y & \xrightarrow{f'\times_S 1_Y} & X'' \times_S Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p''} \\ X & \xrightarrow[f]{} & X' & \xrightarrow[f']{} & X'' \end{array}$$

is commutative (the vertical arrows being the first projections) as results immediately from the definition (1.2.5.1). □

Proposition 4. *For each S-object X, the product $X \times_S S$ (resp. $S \times_S X$) exists; the first (resp. second) projection of $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism of $X \times_S S$ (resp. $S \times_S X$) on X, of which the reciprocal isomorphism is $(1_X, \varphi)_S$ (resp. $(\varphi, 1_X)_S$), in designating by $\varphi : X \longrightarrow S$ the structural morphism.*

In effect, for each S-object T, $Hom_S(T, S)$ is a set with an element, of which the object X represents the functor

$$T \mapsto Hom_S(T, X) \times Hom_S(T, S)$$

We can therefore write up to canonical isomorphism

$$X \times_S S = S \times_S X = X \tag{37}$$

Corollary 5. *Let X and Y be two S-objects, $\varphi : X \longrightarrow S$, $\psi : Y \longrightarrow S$ the structure morphisms. If we canonically identify X with $X \times_S S$ and Y with $S \times_S Y$, the first projection $p_1 : X \times_S Y \longrightarrow X$ is identified with $1_X \times_S \psi$ and the second projection $p_2 : X \times_S Y \longrightarrow Y$ with $\varphi \times_S 1_Y$.*

The verification results immediately from definitions.

Proposition 6. *Given a commutative diagram of morphisms of $\mathcal{C}$*

$$\begin{array}{ccccc} X & \xleftarrow[g]{} & X' & \xleftarrow[g']{} & X'' \\ {\scriptstyle \varphi}\downarrow & & {\scriptstyle \varphi'}\downarrow & & {\scriptstyle \varphi''}\downarrow \\ S & \xleftarrow{f} & S' & \xleftarrow{f'} & S'' \end{array} \tag{38}$$

suppose that the square in the left is cartesian (1.2.4). Therefore, in order that the right square of (1.2.9.1) is cartesian, it is necessary and sufficient that the composed square

$$\begin{array}{ccc} X & \xleftarrow[g\circ g']{} & X'' \\ {\scriptstyle \varphi}\downarrow & & \\ S & \xleftarrow{f\circ f'} & S'' \end{array} \tag{39}$$

is cartesian.

Proposition 7. *Let $f : X \longrightarrow S$, $g : Y \longrightarrow S$ be two morphisms of $\mathcal{C}$ such that the product $X \times_S Y$ exists; let $p_1 : X \times_S Y \longrightarrow X, p_2 : X \times_S Y \longrightarrow Y$ be canonical projections. If $s : S \longrightarrow Y$ is an S-section of g (1.1.11), therefore $s' = (1_X, s \circ f)_S$ is an X-section of p_1, and the square*

$$\begin{array}{ccc} X \times_S Y & \xleftarrow[s']{} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xleftarrow{s} & S \end{array}$$

is cartesian.

PROOF. We have $p_1 \circ s' = 1_X$ by definition, therefore it is certainly a section of p_1. The second assertion results from the map of (1.2.9) to the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times_S Y & \xleftarrow{s'} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ S & \xleftarrow{g} & Y & \xleftarrow{s} & S \end{array}$$

□

PROPOSITION 8. *Let X, Y be two S-objects such that $X \times_S Y$ exists; hence the map $f \mapsto \Gamma_f = (1_X, f)_S$ is a bijection*

$$Hom_S(X, Y) \cong Hom_X(X, X \times_S Y) = \Gamma(X \times_S Y/X)$$

of the set of S-morphisms from X to Y with the set of S-sections of $X \times_S Y$ (for the first projection $p_1 : X \times_S Y \longrightarrow X$).

PROOF. This is a particualr case of the fibred product definition (1.2.2). □

The X-section Γ_f of $X \times_S Y$ is called the *graph morphism* of the morphism f; it is therefore a monomorphism.

PROPOSITION 9. *Let $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ be two S-morphisms which are monomorphisms of $\mathcal{C}/S$ (T, I, 1.1); hence, if the products $X \times_S Y$ and $X' \times_S Y'$ are defined, $f \times_S g : X \times_S Y \longrightarrow X' \times_S Y'$ is a monomorphism of $\mathcal{C}/S$.*

PROOF. In effect, let p_1, p_2 be two projections of $X \times_S Y$, p'_1, p'_2 be the two projections of $X' \times_S Y'$; if u, v are two S-morphisms from T to $X \times_S Y$ such that $(f \times_S g) \circ u = (f \times_S g) \circ v$, we take from this $p'_1 \circ (f \times_S g) \circ u = p'_1 \circ (f \times_S g) \circ v$, in other words, $f \circ p_1 \circ u = f \circ p_1 \circ v$, and as f is a monomorphism, we take from this $p_1 \circ u = p_1 \circ v$; utilising as well the fact that g is a monomorphism, we obtain $p_2 \circ u = p_2 \circ v$, whence $u = v$. □

1.2.5. Suppose the category $\mathcal{C}$ such that all the fibred products $X \times_S Y$ exist. Hence, for n arbitrary S-objects $X_1, \ldots, X_n$, we can define by recurrence the fibred product

$$X_1 \times_S X_2 \times_S \ldots \times_S X_n = (X_1 \times_S \ldots \times_S X_{n-1}) \times_S X_n$$

also denoted by $\Pi_{1 \leq i \leq n} X_i$ and it is immediate that this object represents the functor

$$T \mapsto Hom_S(T, X_1) \times Hom_S(T, X_2) \times \ldots \times Hom_S(T, X_n)$$

The properties of associativity and of commutativity of the cartesian product of sets therefore gives the corresponding properties for the fibered products (1.1.7), which we leave to the reader to express. For example, if p_1, p_2, p_3 are the three canonical projections of $X_1 \times_S X_2 \times_S X_3$ in X_1, X_2, X_3 respectively, and if we canonically identify this product with $(X_1 \times_S X_2) \times_S X_3$, the first projection of this fibered product is identified with $(p_1, p_2)_S$. If $f_i : X_i \longrightarrow Y_i$ is an S-morphism, the morphism $f_1 \times_S f_2 \times_S \ldots \times_S f_n$ from $\prod_{S 1 \leq i \leq n} X_i$ to $\prod_{S 1 \leq i \leq n} Y_i$ this is also written $\prod_{S 1 \leq i \leq n} f_i$.

PROPOSITION 10. *Let $f : S \longrightarrow S'$ be a morphism of $\mathcal{C}$ that is a monomorphism $(T, I.1.1)$, X, Y two S-objects, $\varphi : X \longrightarrow S$, $\psi : Y \longrightarrow S$ the structural morphisms; the morphism $f \circ \varphi : X \longrightarrow S'$, $f \circ \psi : Y \longrightarrow S'$ define X and Y as S'-objects. Therefore each product of the S-objects X, Y is a product of the S'-objects X,Y and reciprocally.*

Proof. In effect, if T is an S'-object, $u : T \longrightarrow X$, $v : T \longrightarrow Y$ are two S'-morphisms, we have by definition $f \circ \varphi \circ u = f \circ \psi \circ v = \theta'$, structural morphism of T; the hypothesis on f leads to the fact that we have $\varphi \circ u = \psi \circ v$, and that we can therefore consider T as an S-object with structural morphism $\theta = \varphi \circ u = \psi \circ v$. The conclusion results immediately from this, and from the definition of a product. □

Proposition 11. *Suppose that in $\mathcal{C}$, products exist. Consider a family of morphisms $g_i : B_i \longrightarrow C_i$ $(1 \leq i \leq r)$, and pose $B = B_1 \times B_2 \times \ldots \times B_r$, $C = C_1 \times \ldots \times C_r$, $g = g_1 \times \ldots \times g_r : B \longrightarrow C$. For each i, let $q_i : A \longrightarrow B_i$ be a morphism, and let $j : A \longrightarrow C$ and $v : A \longrightarrow B$ be the (unique) morphisms which rend these diagrams commutative*

$$\begin{array}{ccc} A & \xrightarrow{q_i} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g_i} \\ C & \xrightarrow[p_i]{} & C_i \end{array} \qquad \begin{array}{ccccc} A & & \xrightarrow{v} & & B \\ & {\scriptstyle q_i}\searrow & & \swarrow{\scriptstyle p'_i} & \\ & & B_i & & \end{array}$$

for $1 \leq i \leq r$, where p_i and p'_i are the canonical projections (1.2.1). *Therefore j is factorised as*

$$j : A \xrightarrow{\quad v \quad} B \xrightarrow{\quad g \quad} C$$

Proof. In effect, we have for each i, $p_i \circ g \circ v = g_i \circ p'_i \circ v$ by definition of g(1.2.5), since $g \circ p'_i \circ v = g_i \circ q_i = p_i \circ j$, and the relation $p_i \circ (g \circ v) = p_i \circ j$ for each i leads to the fact that $g \circ v = j$ (1.2.1). □

Remark 12. The objects of the category opposite $\mathcal{C}^\star$ being the same as those of $\mathcal{C}$, we are therefore led to develop a dual terminology of the preceding in $\mathcal{C}$ corresponding to the preceding notions applied to the category $\mathcal{C}^\star$: which corresponds to the fibered product in $\mathcal{C}^\star$ relative to two morphisms from $\mathcal{C}$, $\varphi : S \longrightarrow X$, $\psi : S \longrightarrow Y$, is often called amalgamated sum of X and Y, and written $X \coprod_S Y$; is the object of $\mathcal{C}$ representing the covariant functor (1.1.11)

$$T \mapsto Hom(X, T) \times_{Hom(S,T)} Hom(Y, T)$$

where the setist fibered product is relative to the applications

$$Hom(\varphi, 1_T) : Hom(X, T) \longrightarrow Hom(S, T)$$

and

$$Hom(\psi, 1_T) : Hom(Y, T) \longrightarrow Hom(S, T)$$

For example, if $\mathcal{C}$ is the category of commutative k-algebras with unit (where k is a commutative ring with unit), the amalgamanted sum $B \coprod_A C$ is none other than the tensor product $B \otimes_A C$.

1.3. Change of Base.

1.3.1. Suppose that $\mathcal{C}$ is a category in which *all the fibered products exist.* Let $\varphi : S' \longrightarrow S$ be a morphism of $\mathcal{C}$; for each S-object X of structural morphism π, $X \times_S S'$, provided from the projection morphism $\pi' : X \times_S S' \longrightarrow S'$, is an S'-object, which we also write as $X_{(S')}$ or $X_{(\varphi)}$, and which we say is *deduced from X by the change of base morphism φ*, or also *reciprocal image* of X by φ. For each S-morphism $f : X \longrightarrow Y$, we call $f_{(S')}$ the S-morphism $f \times_S 1_{S'} : X_{(S')} \longrightarrow Y_{(S')}$ and we say that $f_{(S')}$ is the reciprocal image of f by φ; we have therefore defined (taking into account (1.2.6)) a *covariant functor* $X \mapsto X_{(S')}$ from $\mathcal{C}/S$ to $\mathcal{C}/S'$. If f is an isomorphism, so is $f_{(S')}$.

Proposition 13 (Transitivity of change of base). *Let $\varphi : S' \longrightarrow S$, $\varphi' : S'' \longrightarrow S'$ be two morphisms of $\mathcal{C}$. Therefore for each S-object X, there exists a functorial canonical isomorphism from the S''-object $(X_{(\varphi)})_{(\varphi')}$ to the S''-object $X_{(\varphi\circ\varphi')}$.*

Proof. We must prove that in the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow[p]{} & X_{(\varphi)} & \xleftarrow[p']{} & (X_{(\varphi)})_{(\varphi')} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi''} \\ S & \xleftarrow{\varphi} & S' & \xleftarrow{\varphi'} & S'' \end{array}$$

the composed square is cartesian, knowing that the square on the left and the square on the right are, which results from (1.2.9). □

This result expresses in writing the equality (up to canonical isomorphism) $(X_{(S')})_{(S'')} = X_{(S'')}$, or again

$$(X \times_S S') \times_{S'} S'' = X \times_S S'' \tag{40}$$

the functorial character of the defined isomorphism in (1.3.2) is also expressed by a formula of *transitivity* of the reciprocal images of morphisms

$$(f_(S'))_{(S'')} = f_{(S'')} \tag{41}$$

for each S-morphism $f : X \longrightarrow Y$.

Corollary 14. *If X and Y are two S-objects there exists a functorial canonical isomorphism of the S-object $X_{(S')} \times_{S'} Y_{(S')}$ on the S'-object $(X \times_S Y)_{(S')}$.*

Proof. In effect, up the functorial canonical isomorphism, we have

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

taking into account (1.3.2.1) and the associativity of the fibered products (1.2.13). The functorial character of the isomorphism defined in (1.3.3) is expressed by the formula

$$(u_{(S')}, v_{(S')})_{S'} = ((u, v)_S)_{(S')} \tag{42}$$

for each couple of S-morphisms $u : T \longrightarrow X$, $v : T \longrightarrow Y$. □

In other terms, the reciprocal image functor $X \mapsto X_{(S')}$ *commutes with the formation of products* in the category $\mathcal{C}/S$ and $\mathcal{C}/S'$ respectively.

Corollary 15. *Let Y be an S-object, $f : X \longrightarrow Y$ a morphism, by which X becomes a Y-object, and consequently also an S-object by means of the composed morphism $X \longrightarrow^f Y \longrightarrow^\psi S$, where ψ is the structural morphism. Therefore $X_{(S')}$ is identified with the product $X \times_Y Y_{(S')}$, the projection*

$$X \times_Y Y_{(S')} \longrightarrow Y_{(S')}$$

being identified with $f_{(S')}$.

Proof. It is again an application of (1.2.9) where we replace S, S', S'', X, X', X'' respectively by $S, Y, X, S', Y_{(S')}, X_{(S')}$. □

Corollary 16. *If X, Y, Z are three S-objects, $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ two S-morphisms, $X \times_Z Y$, the relative fibred product of f and g, we have up to canonical isomorphism*

$$(X \times_Z Y)_{(S')} = X_{(S')} \times_{Z_{(S')}} Y_{(S')}$$

Proof. In effect, in virtue of (1.3.4), $(X \times_Z Y)_{(S')}$ is identified with $(X \times_Z Y)_{Z_{(S')}}$ and it suffices to apply (1.3.3). □

Proposition 17. *If the S-morphism $f : X \longrightarrow Y$ is a monomorphism, $f_{(S')} : X_{(S')} \longrightarrow Y_{(S')}$ is a monomorphism.*

Proof. It is a particular case of (1.1.12). □

Proposition 18. *The map* $f \mapsto (f, 1_{S'})_S$ *is a canonical bijection*

$$Hom_S(S', X) \cong Hom_{S'}(S', X_{(S')}) = \Gamma(X_{(S')}/S') \tag{43}$$

of the set of S*-morphisms from* S' *to* X*, in the set of* S'*-sections* (1.1.10) *of* $X_{(S')}$*, functorial in* X *and* S'*.*

Proof. This is none other than a manner of expressing (1.2.11). If $p : X_{(S')} \longrightarrow X$ is the canonical projection, the reciprocal bijection of (1.3.7.1) is $f' \longrightarrow p \circ f'$. □

Recall that the section $f' = (f, 1_{S'})_S$ is the *morphism graph of* f and is written Γ_f (1.2.11).

Proposition 19. *Let* X, Y *be two* S*-objects,* $f : X \longrightarrow Y$ *an* S*-morphism,* $\Gamma_f = (1_X, f)_S$ *the morphism graph of* f*; for each morphism* $g : S' \longrightarrow S$*, we have* $\Gamma_{f(S')} = (\Gamma_f)_{(S')}$*.*

Proof. It is a particular case of (1.3.3.1). □

Remark 20. Consider a property P of the morphisms of the category $\mathcal{C}$ and the two following propositions:

(i) If $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ are two S-morphisms possessing the property P, $f \times_S g$ possesses the property P.

(ii) If $f : X \longrightarrow Y$ is an S-morphism possessing the property P, each S-morphism $f_{(S')} : X_{(S')} \longrightarrow Y_{(S')}$, deduced from f by a change of base $S' \longrightarrow S$ possesses the property P.

Like $f_{(S')} = f \times_S 1_{S'}$, we that if for each object X of $\mathcal{C}$ the identity 1_X possesses the property P, (i) implies (ii); as on the other hand $f \times_S g$ is the composed morphism

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y'$$

we see that if the composite of two morphisms possessing the property P, also possesses this property, therefore (ii) implies (i).

Proposition 21. *Let* $F : T \mapsto F(T)$ *be a contravariant functor from the category* $\mathcal{C}/S$ *to the category of sets. If this functor is representable by a couple* (X, ξ) *where* $X \in \mathcal{C}/S$ *and* $\xi \in F(X)$*, therefore, for each morphism change of base* $g : S' \longrightarrow S$*, the functor* $T' \mapsto F(T')$*, the restriction to* $\mathcal{C}/S$ *of* F*, is representable by the couple* (X', ξ')*, where* $X' = X \times_S S'$ *and* $\xi' = F(p_1)(\xi)$*, where* $p_1 : X' \longrightarrow X$ *is the first projection.*

Proof. In effect, we have two functorial bijections in T'

$$Hom_{S'}(T', X') \cong Hom_S(T', X) \cong F(T')$$

where the first is written $f' \mapsto p_1 \circ f'$ and the second $f \mapsto F(f)(\xi)$; it suffices to compose them. □

1.4. Kernels; diagonal morphism.

1.4.1. Let E, F be two sets; and consider two maps $u_1 : E \longrightarrow F$,$u_2 : E \longrightarrow F$; we call the kernel of u_1 and u_2 (or set of coincidences of u_1 and u_2) the set of $x \in E$ such that $u_1(x) = u_2(x)$, and we call it $Ker(u_1, u_2)$. We say that a *diagram of maps*

$$N \xrightarrow[j]{} E \underset{u_2}{\overset{u_1}{\rightrightarrows}} F$$

is *exact* if j is injective and if $j(N) = Ker(u_1, u_2)$; this diagram is obviously therefore commutative.

1.4.2. Let $\mathcal{C}$ now be an arbitrary category; X, Y two objects of $\mathcal{C}$, u_1, u_2 two morphisms from X to Y; we define a contravariant functor $F : \mathcal{C}^\star \longrightarrow$ **Sets** in posing, for each object $T \in \mathcal{C}$

$$F(T) = Ker(Hom(1_T, u_1), Hom(1_T, u_2)) \subset Hom(T, X)$$

and, for each morphism $w : T \longrightarrow T'$, in taking for $F(w)$ the restriction to $F(T')$ of $Hom(w, 1_X)$ (we verify immediately that the image from $F(T')$ of $Hom(w, 1_X)$ is contained in $F(T)$, in reason of the definition of the kernel of two maps). If the functor F is representable, an object which represents it (determined up to unique isomorphism) is called a kernel of u_1 and u_2, or an object of coincidences of u_1 and u_2, and written $Ker(u_1, u_2)$. We will see later (1.4.10) that while the fibred products exist in $\mathcal{C}$, so do kernels.

If $N = Ker(u_1, u_2)$, the canonical injection $Hom(T, N) \longrightarrow Hom(T, X)$ is thus of the form $Hom(1_T, j)$, where $j : N \longrightarrow X$ is a *monomorphism* of T; we say again therefore that the diagram of morphisms

$$N \xrightarrow[j]{} X \underset{u_2}{\overset{u_1}{\rightrightarrows}} Y \tag{44}$$

is *exact*, which is equivalent therefore to saying that, for *each* object T of $\mathcal{C}$, we have the exact diagram of maps

$$Hom(T, N) \xrightarrow[Hom(1_T, j)]{} Hom(T, X) \underset{Hom(1_T, u_2)}{\overset{Hom(1_T, u_1)}{\rightrightarrows}} Hom(T, Y)$$

thus (1.4.2.1) is in particular, commutative.

While $n = Ker(u_1, u_2)$ exists, in order that a morphism $f : T \longrightarrow X$ is such that $u_1 \circ f = u_2 \circ f$, it is necessary and sufficient, according to the above, that f is factorised in $f = j \circ g$, where $g : T \longrightarrow N$ is a morphism determined in a unique fashion.

1.5. Morphisms of representable functors.

1.5.1. *1.7.4.* The notation being the same as above, we define a *representable morphism* to be a functorial morphism

$$f : F \longrightarrow G$$

having the following property: for each object $X \in \mathcal{C}$ and each functorial morphism $u : h_X \longrightarrow G$, the functor $F \times_G h_X$ is representable.

PROPOSITION 22. (i) *Each isomorphism of functors is representable.*
(ii) *If $f : F \longrightarrow G$ and $g : G \longrightarrow H$ are two representable functorial morphisms, then so is $g \circ f : F \longrightarrow H$.*
(iii) *If $f : F \longrightarrow G$ is representable, then so is each functorial morphism $f_(H) : F \times_G H \longrightarrow H$ induced from f by the change of base $H \longrightarrow G$.*

PROOF. The assertion (i) results from the fact that an isomorphism is tranformed to an isomorphism by change of base (1.3.1), and the assertion (ii) is such that $F \times_H h_X \cong F \times_G (G \times_H h_X)$; as $G \times_H h_X$ is isomorphic with the functor h_Y, $F \times_G (G \times_H h_X)$, isomorphic with $F \times_G h_Y$, is representable. Similarly, to prove (iii) it suffices to note that $(F \times_G H) \times_H h_X$ is isomorphic with $F \times_G h_X$, which is representable. □

If $f : F \longrightarrow G$ is a representable functorial morphism, and if for each functorial morphism $h_X \longrightarrow G$, the functor $F \times_G h_X$ is represented by the object $Y \in \mathcal{C}$ (therefore is isomorphic with h_Y), therefore, for each morphism $Z \longrightarrow X$ of $\mathcal{C}$, the product $Y \times_X Z$ exists, since the functor $h_Y \times_{h_X} h_Z$ is representable by virtue of (1.7.5(iii)).

Proposition 23. *Let $G : \mathcal{C}^\star \longrightarrow$ **Sets** be a representable contravariant functor. In order that a functorial morphism $F \longrightarrow G$ be representable, it is necessary that the functor F be representable; this condition is sufficient if the category $\mathcal{C}$ has fibred products.*

Proof. The sufficiency of the condition results from (1.7.3). For the other part, if $G = h_S$, with an isomorphism given, and if the morphism $F \longrightarrow G$ is representable, the functor $F \times_G G = F \times_G h_S$ (where $G \longrightarrow G$ is the identity) is representable by definition; as it is identified with F, and this finishes the proof.

1.5.2. *1.7.7.* Let $\mathcal{P}$ be a collection of morphisms of the category $\mathcal{C}$; we suppose, that on composition of a morphism of $\mathcal{P}$ with an isomorphism (on the left or right) that the result is again in $\mathcal{P}$, and on the other hand that $\mathcal{P}$ satisfies the following condition: If $f : A \longrightarrow B$ belongs to $\mathcal{P}$, therefore, for each morphism $g : C \longrightarrow B$, the fibred product $A \times_B C$ exists and the projection $f_{(C)} = p_2 : A \times_B C \longrightarrow C$ belongs to $\mathcal{P}$ (i.e. $\mathcal{P}$ is *stable under change of base*).

This being so, we say that a representable functorial morphism $u : F \longrightarrow G$, where F and G are contravariant functors from $\mathcal{C}$ to **Sets**, is *represented by a morphism of* $\mathcal{P}$ if, for each object $X \in \mathcal{C}$ and for each functorial morphism $v : h_X \longrightarrow G$, the functorial morphism corresponding to $F \times_G h_X \longrightarrow h_X$, this is of the form h_w, where $w : Y \longrightarrow X$ is a morphism of $\mathcal{C}$ (Y being an object representing the functor $F \times_G h_X$), and such that w belongs to $\mathcal{P}$ (this condition does not depend on the object Y chosen, since if we compose a morphism of $\mathcal{P}$ with an isomorphism the result is still in $\mathcal{P}$). It is immediate that for each functorial morphism obtained from u by change of base (1.3.1) is also representable by the morphisms of $\mathcal{P}$.

□

1.6. Ringed Spaces.

1.6.1. *4.1.1.* A *ringed space* is a pair $(X, \mathcal{A})$ formed from a topological space X and a sheaf of rings (not necessarilly commutative) $\mathcal{A}$ on X; we say that X is the underlying topological space of the ringed space $(X, \mathcal{A})$, and $\mathcal{A}$ is the *structure sheaf.* This sheaf we sometimes also denote by $\mathcal{O}_X$, and the fibre at a point $x \in X$ is written $\mathcal{O}_{X,x}$ or simply $\mathcal{O}_x$ where there is no chance of confusion.

We denote by 1 or e the *unit section* of $\mathcal{O}_X$ (the unit element of $\Gamma(X, \mathcal{O}_X)$.

While $\mathcal{A}$ is a sheaf of commutative rings, we say that $(X, \mathcal{A})$ is a *commutative* ringed space. As in this treatise we mostly consider sheaves of commutative rings, it will be understood that whenever we talk about a ringed space $(X, \mathcal{A})$, that this space will be commutative.

The ringed spaces with not-necessarily commutative structure sheaves form a category, where we define a morphism $(X, \mathcal{A}) \longrightarrow (Y, \mathcal{B}$ as a pair $\Psi = (\psi, \theta)$ formed from a continuous map $\psi : X \longrightarrow Y$ and a ψ-morphism $\theta : \mathcal{B} \longrightarrow \mathcal{A}$ (3.5.1) of sheaves of rings. As the category of rings has inductive limits, for each $x \in X$, we have induced from θ a homomorphism of rings $\mathcal{B}_{\psi(x)} \longrightarrow \mathcal{A}_x$ (3.5.1); moreover, taking into account that $\psi_x \circ (\rho_{\mathcal{B}})_{\psi(x)}$ is an isomorphism (3.7.2), the preceeding homomorphism is identified with $\theta_x^\#$. **MORE**

1.6.2. *4.1.3.* We take the definition of a $\mathcal{A}$-module on a ringed space $(X, \mathcal{A})$ from $(G, II, 2.2)$; while $\mathcal{A}$ is a sheaf of not-necessarily commutative rings, by $\mathcal{A}$-module, we will mean a left $\mathcal{A}$-module, unless stated to the contrary. **MORE**

1.6.3. *4.1.4.* We say that a sheaf of rings $\mathcal{A}$ on a topological space X is *reduced* (resp. *integral*) if for each point x of X the fibre $\mathcal{A}_x$ is a reduced ring (resp. integral ring) ; we say that $\mathcal{A}$ is *reduced* if it is reduced at each point of X. **MORE**

1.6.4. *4.1.9.* A *locally ringed space* is a commutative ringed space $(X, \mathcal{O}_X)$ such that, for each $x \in X$, $\mathcal{O}_{X,x}$ is a local ring. We designate by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$, by $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ the residue field. For each $\mathcal{O}_X$-module $\mathcal{F}$, and each open U of X, and each point $x \in U$ and section $f \in \Gamma(U, \mathcal{F})$, we designate by $f(x) \in \kappa(x)$ the class of the germ $f_x \in \mathcal{F}_x$ modulo $\mathfrak{m}_x\mathcal{F}_x$, and we say that this is the *value* of f at the point x. The relation $f(x) = 0$ therefore signifies that $f_x \in \mathfrak{m}_x\mathcal{F}_x$; we will take care not to confuse this with the relation $f_x = 0_x$. We sometimes write U_f for the collection of $x \in U$ such that $f(x) \neq 0$ (in other words, $f_x \notin \mathfrak{m}_x \mathcal{F}_x$). We note that if $f \in \Gamma(X, \mathcal{F})$, the collection X_f is contained in $Supp(\mathcal{F})$.

PROPOSITION 24 (4.1.10). *Let $(X, \mathcal{O}_X)$ be a locally ringed space, $\mathcal{F}$, $\mathcal{G}$ two $\mathcal{O}_X$-modules. For each $x \in X$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x / \mathfrak{m}_x (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x$ is canonically identified with $(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x) \otimes_{\kappa(x)} (\mathcal{G}_x/\mathfrak{m}_x\mathcal{G}_x)$. If s (resp. t) is a section of $\mathcal{F}$ (resp. $\mathcal{G}$) over X, $(s \otimes t)(x)$ is identified with $s(x) \otimes t(x)$, and we have*

$$X_{s \otimes t} = X_s \cap X_t \tag{45}$$

PROOF. In effect, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x$ and $(\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x) \otimes_{\mathcal{O}_x} (\mathcal{O}_x/\mathfrak{m}_x)$ is canonically isomorphic with $(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x) \otimes_{\mathcal{O}_x} (\mathcal{G}_x/\mathfrak{m}_x\mathcal{G}_x)$, therefore with $(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x) \otimes_{\mathcal{O}_x/\mathfrak{m}_x} (\mathcal{G}_x/\mathfrak{m}_x\mathcal{G}_x)$; the relation (4.1.10.1) therefore results since in a tensor product of vector spaces, a product $a \otimes b$ of two vectors is nonzero if both a and b are. **CHECK**. □

1.7. Open immersions and morphisms representable by the open immersions.

1.7.1. *4.5.1.* Being given a ringed space $(X, \mathcal{O}_X)$, we define an *open immersion* of a ringed space $(Y, \mathcal{O}_Y)$ in $(X, \mathcal{O}_X)$ to be a morphism $f = (\psi, \theta) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ composed of a canonical injection $(U, \mathcal{O}_X|U) \longrightarrow (X, \mathcal{O}_X)$ of the ringed space induced on an open U in X, and an isomorphism $(Y, \mathcal{O}_Y) \cong (U, \mathcal{O}_X|U)$; this is a monomorphism. It is immediate that each isomorphism of ringed spaces is an open immersion, and that the composite of two open immersions is an open immersion.

1.7.2. *4.5.2.* We now show that if $j : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ is an open immersion and $g : (Z, \mathcal{O}_Z) \longrightarrow (X, \mathcal{O}_X)$ is an arbitrary morphism of ringed spaces, therefore, in the category of ringed spaces, the fibred product $Y \times_X Z$ existes (for the morphisms j and g), and the projection $j' = p_2 : Y \times_X Z \longrightarrow Z$ is an open immersion. With an isomorphism near, we can suppose that Y is an open of X, $\mathcal{O}_Y = \mathcal{O}_X|Y$, and j the canonical injection. If $g = (\psi, \theta)$, let P be the open $\psi^{-1}(Y)$ of Z; we proceed to see that the ringed space $(P, \mathcal{O}_P)$ induced by Z upon the open P of Z is the fibre product we seek, when one defines the projection $p_2 : P \longrightarrow Z$ as the canonical injection, and the projection $p_1 : P \longrightarrow Y$ as equal to the morphism (ψ_1, θ_1), where ψ_1 is the restriction of ψ to P, considered as a map to $Y \supseteq \psi(P)$, and $\theta_1^\#$ the restriction of $\theta^\# : \psi^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Z$ to the open P, $\psi_1^{-1}(\mathcal{O}_X)$ being identified with $\psi^{-1}(\mathcal{O}_X)|P$ (3.7.1). It is clear that we have $g \circ p_2 = j \circ p_1$. For to show that in this way we have well obtained a fibre product, we consider a ringed space T and two morphisms $u = (\rho, \alpha) : T \longrightarrow Y$, $v = (\sigma, \beta) : T \longrightarrow Z$, such that $j \circ u = g \circ v$. This last relation shows firstly that $\psi(\sigma(T)) \subseteq U$, therefore $\sigma(T) \subseteq \psi^{-1}(U) = P$; if we denote by τ the map σ considered as a map from T to P, we therefore have $\tau^{-1}(\mathcal{O}_P) = \sigma^{-1}(\mathcal{O}_Z)$; we define therefore a morphism $w = (\tau, \gamma) : T \longrightarrow P$ in absorbing $\gamma^\# : \tau^{-1}(\mathcal{O}_P) \longrightarrow \mathcal{O}_T$ equally with $\beta^\#$, and it is clear that the morphism is the only one such that $p_2 \circ w = v$. It remains to see that $p_1 \circ w = u$; but it is clear that $\psi_1 \circ \tau$ is equal with $\psi \circ \sigma$, therefore with ρ; we therefore have $\tau^{-1}(\psi_1^{-1}(\mathcal{O}_Y)) = \sigma^{-1}(\psi^{-1}(\mathcal{O}_X))$ and $\gamma^\# \circ \theta_1^\#$ is equal with $\beta^\# \circ \theta^\#$, therefore with $\alpha^\#$, and this finishes the proof.

1.7.3. *4.5.3.* The open immersions are therefore the morphisms from the category *Esp.ann* of ringed spaces, which satisfy the conditions of (1.7.7) and we can therefore talk of functorial morphisms $F \longrightarrow G$ (where F and G are contravariant functors from *Esp.ann* to *Ens*), which are *representable by the open immersions*. It is the same when we consider the category *Esp.ann/S* in place of *Esp.ann*, where S is a ringed space.

PROPOSITION 25. *Let S be a ringed space, $F : (Esp.ann/S)^o \longrightarrow Ens$ a contravariant functor, $(F_i)_{i \in I}$ a family of subfunctors of F (1.7.1). We suppose that the following hypothesis hold:*

(i) *Each of the canonical functorial morphisms $u_i : F_i \longrightarrow F$ is representable by an open immersion.*
(ii) *For each ringed S-space X, the map $U \mapsto F(U)$, where U ranges over the collection of ringed S-spaces induced on the opens of X, is a sheaf of sets.*

(iii) *For each ringed S-space Z and each functorial morphism $h_Z \longrightarrow F$, if Z_i is a ringed S-space representing the functor $F_i \times_F h_Z$ (1.7.4) and U_i the image in Z of the space underlying Z_i (image which is an open of Z, since the morphism $Z_i \longrightarrow Z$ is an open immersion by (i)), therefore the U_i form a cover of Z.*

(iv) *Every functor F_i is representable by a ringed S-space X_i.*

To these conditions, F is representable by a ringed S-space X, and the image of the underlying spaces X_i by the morphisms $X_i \longrightarrow X$ (which are the open immersions by (i)), form an open cover of X.

Proof. □

(4.5.6) An arbitrary contravariant functor

$$F : (Esp.ann/S)^o \longrightarrow Ens \tag{46}$$

defines, for each ringed S-space a presheaf of sets $U \mapsto F(U)$ on X (U varying over the set of ringed S-spaces inducted on the opens of X). Where for each ringed S-space X, this presheaf is a sheaf (condition (ii) of (4.5.4)), we say again that F is a sheaf on the category $Esp.ann/S$ (expression which appears here as an abuse of language, but will be justified by the general theory of sheaves on categories); these functors form therefore a full subcategory of the category $Hom((Esp.ann/S)^o, \mathbf{Sets})$, denoted $Fais/(Esp.ann/S)$. It results from (1.1.3) and (3.2.6) that a projective limit of functors of $Hom((Esp.ann/S)^o, \mathbf{Sets})$ which are sheaves, is again a sheaf.

A functor (4.5.6.1) which is representable is always a sheaf: it is clear in effect that the presheaf $U \mapsto Hom_S(U, Z)$ on a ringed space X is a sheaf, because for open collection (U_α) of an open U of X, the data of an S-morphism from U to Z is equivalent to giving a family of S-morphisms $f_\alpha : U_\alpha \longrightarrow Z$ such that for each pair of indices α, β the restrictions of f_α and of f_β to $U_\alpha \cap U_\beta$ coincide.

We can therefore say, taking into account (1.1.8), that we have a fully faithful functor $Z \mapsto h_Z$ from the category $Esp.ann/S$ to the category $Fais/(Esp.ann/S)$ permitting the identification of the first with a full subcategory of the second.

(4.5.7) All the results of this section are valuable without change where in place of the category of ringed spaces we consider the subcategory (non-full) of locally ringed spaces (4.1.12): in effect, a ringed space X obtained by collection of locally ringed spaces X_i is locally ringed, and if a morphism of ringed spaces $X \longrightarrow S$ (where S is locally ringed) is such that its restriction to each X_i is a morphism of locally ringed spaces, this is a morphism of locally ringed spaces.

CHAPTER 2

The Language of Schemes

(2.3.6) We note firstly that the remarks of (0,4.5.6) are valuable without modification where we replace it throughout the category $Esp.ann/S$ with the category Sch/S of S-schemes (where S is an arbitrary scheme), and in particular by the category Sch of all schemes. Consider on the other hand the full subcategory Aff of Sch, formed of the affine schemes; if $G : Aff^o \longrightarrow \mathbf{Sets}$ is a contravariant functor from Aff to **Sets**, for each affine scheme X, we can again consider the map $U \mapsto G(U)$, where U covers the collection of open affines of X. As the open affines form a base $\mathfrak{B}_X$ of the topology of X (1.1.2), the map $U \mapsto G(U)$ is a presheaf on $\mathfrak{B}_X$ with values in **Sets**(0,3.2.1); where, for each affine scheme X, $U \mapsto G(U)$ is a sheaf on $\mathfrak{B}_X$ in the sense of (0,3.2.2) we say again that G is a sheaf on the category Aff; the functors G having this property form again a full subcategory of the category $Hom(Aff^o, \mathbf{Sets})$ that we will denote $Fais/Aff$. This being so, for each contravariant functor $F : Sch^o \longrightarrow \mathbf{Sets}$, we can consider the restriction $F|_{Aff^o} : Aff^o \longrightarrow \mathbf{Sets}$ and it is clear that

$$F \mapsto F|Aff^o \tag{47}$$

is a functor from $Hom(Sch^o, \mathbf{Sets})$ to $Hom(Aff^o, \mathbf{Sets})$. We will see that the restriction of this functor to the full subcategory $Fais/Sch$ (0,4.5.6) of $Hom(Sch^o, \mathbf{Sets})$ defines in fact an equivalence of categories

$$Fais/Sch \cong Fais/Aff \tag{48}$$

It is clear in effect that the image in $Fais/Sch$ of (2.3.6.1) is a subcategory $Fais/Aff$ in virtue of (0,3.2.2); we show firstly that the restriction of (2.3.6.1) to $Fais/Sch$ is fully faithful. This results from a part of that which, if $F \in Fais/Sch$ and if X is a scheme, the data of $F(U)$ for the U open affines of X entirely determine the $F(U)$s for all open U of X by the formula $F(U) = \varinjlim F(U_\alpha)$, where (U_α) is the set ordered of open affines contained in U (0,3.2.1); on the other hand if $\varphi : F \longrightarrow G$ is a functorial morphism (for F, G in $Fais/Sch$), with the same notations the data of the maps $\varphi(U_\alpha) : F(U_\alpha) \longrightarrow G(U_\alpha)$ entirely determines $\varphi(U) : F(U) \longrightarrow G(U)$, this map being the projective limit of the projective system of maps $\varphi(U_\alpha)$ by definition of a functorial morphism. It remains to be seen that each functor $G \in Fais/Aff$ is of the form $F|Aff^p$ for a $F \in Fais/Sch$. For this, we define, four each scheme X, $F(X)$ as the projective limit of the $G(U_\alpha)$, where $U_\alpha)$ is the projective system of open affines in X; in order to obtain a functor, it is again necessary to define $F(u)$ for each morphism of schemes $u : X \longrightarrow Y$; we will take $F(u)$ to be the projective limit of the maps $G(u_{\alpha\beta}) : G(V_\beta) \longrightarrow G(U_\alpha)$, where (U_α) is defined as above , (V_β) is the projective system of open affines in Y, the pairs (α, β) are only those such that $u(U_\alpha) \subseteq V_\beta$, and $u_{\alpha\beta} : U_\alpha \longrightarrow V_\beta$ is the restriction of u. The verification of the fact that F is a functor is immediate. Finally, the fact tha tF is a sheaf results from the fact that the open affines of X form a base of the topology of X (2.1.3) and of (0,3.2.2).

We saw (1.6.5) that the category Aff^o is canonically equivalent to the category Ann of rings; this equivalence also defines therefore the equivalence of categories $Hom(Aff^o, \mathbf{Sets}$ and $Hom(Ann, \mathbf{Sets})$. We say by abuse of language that a functor (covariant) $Ann \longrightarrow \mathbf{Sets}$ is a sheaf if its image by the preceding equivalence is a sheaf on Aff; denote by $Fais/Ann$ the category of these functors. Taking into account that, for each ring A, the collection of open affines $D(f) = Spec(A_f)$ form a base of the topology of $Spec(A)$, we can again, by virtue of (0,3.2.2), express that the functor $F : Ann \longrightarrow \mathbf{Sets}$

belongs to $Fais/Ann$ in saying that, for each family $(f_i)_{i \in I}$ of elements of A such that $\sum_i Af_i = A$, the diagram of sets

$$FIXME$$

is exact (0,1.4.1); we elsewhere in this exposition limit ourselves to families (f_i) finite generating A. We have in this way characterised the category $Fais/Ann$ without utilising the notion of schemes.

Recall now that the data of a scheme X canonically defines a contravariant functor $h_X : Y \mapsto Hom(Y, X)$ from Sch to **Sets**(0,1.1.1) and we have remarked (0,4.5.6) that the functor is a sheaf on Sch; we say moreover (0,1.1.8) that $h : X \mapsto h_X$ is a fully faithful functor from Sch to $Fais/Sch$. By virtue of the equivalence (2.3.6.2) and of the definition of $Fais/Ann$, we say that we can canonically identify the category Sch with a full subcategory of the category $Fais/Ann$, a scheme X being identified with the sheaf $A \mapsto Hom(Spec(A), X)$ on Ann. We will take care to note that there are the functors in $Fais/Ann$ which are not isomorphic to a sheaf coming from a scheme.

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