Dimensions

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1 Projective Dimension

If $\mathcal{A}$ is an abelian category with enough projectives, we can define for every object A a covariant additive functor $\underline{Ext}^n(A,-) : \mathcal{A} \longrightarrow \mathbf{Ab}$. This depends on the assignment of resolutions, but is independent of this choice up to a canonical natural equivalence. If $\mathcal{A}$ has enough injectives, we can define a covariant additive functor $Ext^n(A,-) : \mathcal{A} \longrightarrow \mathbf{Ab}$ which is also independent of the choice of resolutions, up to a canonical natural equivalence. If $\mathcal{A}$ has enough projectives and injectives, then for any choice of projective and injective resolutions there is a canonical natural equivalence $\underline{Ext}^n(A,-) \cong Ext^n(A,-)$.

Lemma 1. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then an object A is projective if and only if $\underline{Ext}(A,-) = 0$.*

Proof. Clearly if A is projective then $\underline{Ext}(A,-) = 0$. Suppose that $\underline{Ext}(A,-) = 0$ and find an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow 0$ with P projective. We obtain an exact sequence

$$0 \longrightarrow Hom(A,K) \longrightarrow Hom(A,P) \longrightarrow Hom(A,A) \longrightarrow \underline{Ext}(A,K)$$

Since $\underline{Ext}(A,K) = 0$ it follows that $Hom(A,P) \longrightarrow Hom(A,A)$ is surjective, so $P \longrightarrow A$ is a retraction, and therefore A is projective. □

The vanishing of the functors $Hom(A,-), \underline{Ext}(A,-), \underline{Ext}^2(A,-), \ldots$ measures "how projective" an object A is. In the next result we show that the vanishing of these functors is connected with the existence of a certain type of exact sequence.

Proposition 2. *Let $\mathcal{A}$ be an abelian category with enough projectives. The following statements are equivalent for $n \geq 0$ and any object A:*

(a) $\underline{Ext}^{n+1}(A,-) = 0$.

(b) $\underline{Ext}^n(A,-)$ *is right exact.*

(c) If there is an exact sequence with all P_i projective

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

then K is projective.

(d) There is an exact sequence with all P_i projective

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

Proof. For $n = 0$ condition (c) means that if we have an exact sequence $0 \longrightarrow K \longrightarrow A \longrightarrow 0$ then K is projective, which is equivalent to saying that A is projective.

$(a) \Rightarrow (b)$ Suppose we are given a short exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$. Then part of the long exact sequence of $\underline{Ext}(A, -)$ is

$$\underline{Ext}^n(A, B') \longrightarrow \underline{Ext}^n(A, B) \longrightarrow \underline{Ext}^n(A, B'') \longrightarrow \underline{Ext}^{n+1}(A, B')$$

since the last term is zero, it follows that $\underline{Ext}^n(A, -)$ is right exact.

$(b) \Rightarrow (c)$ If $n = 0$ then $\underline{Ext}(A, -) \cong Hom(A, -)$ which is right exact if and only if A is projective. So suppose $n \geq 1$ and that we are given an exact sequence of the stated form. Let $X \longrightarrow Y$ be an epimorphism. From our notes on dimension shifting of Ext we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} Hom(P_{n-1}, X) & \longrightarrow & Hom(K, X) & \longrightarrow & \underline{Ext}^n(A, X) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Hom(P_{n-1}, Y) & \longrightarrow & Hom(K, Y) & \longrightarrow & \underline{Ext}^n(A, Y) & \longrightarrow & 0 \end{array}$$

By assumption the first and third vertical morphisms are epimorphisms. It follows from the 5-Lemma that the middle morphism is also an epimorphism, which shows that K is projective.

$(c) \Rightarrow (d)$ This is trivial for $n = 0$. For $n \geq 1$ take a projective resolution of A and let $K \longrightarrow P_{n-1}$ be the image of $P_n \longrightarrow P_{n-1}$.

$(d) \Rightarrow (a)$ For $n = 0$ this follows from the fact that if A is projective then $\underline{Ext}(A, -) = 0$. For $n \geq 1$ we use dimension shifting to see that for any object B, $0 = \underline{Ext}(P_n, B) \cong \underline{Ext}^{n+1}(A, B)$. □

Corollary 3. *Let $\mathcal{A}$ be an abelian category with enough projectives, and let A be any object. Consider the sequence of additive functors $\mathcal{A} \longrightarrow$* **Ab**

$$Hom(A, -), \underline{Ext}(A, -), \underline{Ext}^2(A, -), \ldots, \underline{Ext}^n(A, -), \ldots$$

If $\underline{Ext}^n(A, -) = 0$ for an integer $n \geq 0$ then $\underline{Ext}^m(A, -) = 0$ for all $m \geq n$.

Proof. If $\underline{Ext}^n(A, -) = 0$ then it is trivially right exact, so $\underline{Ext}^{n+1}(A, -) = 0$. □

Definition 1. Let $\mathcal{A}$ be an abelian category with enough projectives. We define the *projective dimension* of an object A, denoted *proj.dim.A*, to be the smallest integer $n \geq -1$ with the property that $\underline{Ext}^{n+1}(A, -) = 0$. If no such integer exists, then we set $proj.dim.A = \infty$. So $proj.dim.A = -1$ if and only if A is a zero object, and for nonzero A, the projective dimension is the last nonzero position in the sequence

$$Hom(A, -), \underline{Ext}(A, -), \underline{Ext}^2(A, -), \ldots, \underline{Ext}^n(A, -), \ldots$$

Clearly $proj.dim.A = 0$ if and only if A is a nonzero projective. Proposition 2 shows that for nonzero A the projective dimension is the smallest integer $n \geq 0$ for which there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with all P_i projective, with the dimension being ∞ if no such sequence exists. So for $n \geq 0$ the conditions of Proposition 2 are all equivalent to $proj.dim.A \leq n$. Clearly isomorphic objects have the same projective dimension.

2 Injective Dimension

If $\mathcal{A}$ is an abelian category with enough injectives, we can define for every object A a contravariant additive functor $Ext^n(-, A) : \mathcal{A} \longrightarrow \mathbf{Ab}$. This depends on the assignment of resolutions, but is independent of this choice up to a canonical natural equivalence. If $\mathcal{A}$ has enough projectives, we can define a contravariant additive functor $\underline{Ext}^n(-, A) : \mathcal{A} \longrightarrow \mathbf{Ab}$ which is also independent of the choice of resolutions, up to a canonical natural equivalence. If $\mathcal{A}$ has enough projectives and injectives, then for any choice of projective and injective resolutions there is a canonical natural equivalence $Ext^n(-, A) \cong \underline{Ext}^n(-, A)$.

Lemma 4. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then an object A is injective if and only if $Ext(-, A) = 0$.*

In an analogous way the vanishing of the functors $Hom(-, A), Ext(-, A), Ext^2(-, A), \ldots$ measures "how injective" an object A is. As we would expect

Proposition 5. *Let $\mathcal{A}$ be an abelian category with enough injectives. The following statements are equivalent for $n \geq 0$ and any object A:*

(a) $Ext^{n+1}(-, A) = 0$.

(b) $Ext^n(-, A)$ *is right exact.*

(c) If there is an exact sequence with all I^i injective

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow C \longrightarrow 0$$

then C is injective.

(d) There is an exact sequence with all I^i injective

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow I^n \longrightarrow 0$$

Proof. For $n = 0$ the condition (c) means that if we have an exact sequence $0 \longrightarrow A \longrightarrow C \longrightarrow 0$, then C is injective, which is equivalent to saying that A is injective. Condition (b) means that $Ext^n(-, A) : \mathcal{A}^{\text{op}} \longrightarrow \mathbf{Ab}$ is right exact as a functor $\mathcal{A}^{\text{op}} \longrightarrow \mathbf{Ab}$. In other words it sends kernels of $\mathcal{A}$ to cokernels. The proof is straightforward. $\square$

Corollary 6. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let A be any object. Consider the sequence of contravariant additive functors $\mathcal{A} \longrightarrow \mathbf{Ab}$*

$$Hom(-, A), Ext(-, A), Ext^2(-, A), \ldots, Ext^n(-, A), \ldots$$

If $Ext^n(-, A) = 0$ for an integer $n \geq 0$ then $Ext^m(-, A) = 0$ for all $m \geq n$.

Definition 2. Let $\mathcal{A}$ be an abelian category with enough injectives. We define the *injective dimension* of an object A, denoted $inj.dim.A$, to be the smallest integer $n \geq -1$ with the property that $Ext^{n+1}(-, A) = 0$. If no such integer exists, then we set $inj.dim.A = \infty$. So $inj.dim.A = -1$ if and only if A is a zero object, and for nonzero A, the injective dimension is the last nonzero position in the sequence

$$Hom(-, A), Ext(-, A), Ext^2(-, A), \ldots, Ext^n(-, A), \ldots$$

Clearly $inj.dim.A = 0$ if and only if A is a nonzero injective. For nonzero A the injective dimension is the smallest integer $n \geq 0$ for which there exists an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow I^n \longrightarrow 0$$

with all I^i injective, with the dimension being ∞ if no such sequence exists. So for $n \geq 0$ the conditions of Proposition 5 are all equivalent to $inj.dim.A \leq n$. Clearly isomorphic objects have the same injective dimension.

Lemma 7. *Let $\mathcal{A}$ be an abelian category with enough injectives (or projectives). Consider the sequence of bifunctors $\mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathbf{Ab}$*

$$Hom(-,-), Ext(-,-), Ext^2(-,-), \ldots, Ext^n(-,-), \ldots$$

If $Ext^n(-,-) = 0$ for an integer $n \geq 0$ then $Ext^m(-,-) = 0$ for all $m \geq n$.

Proof. Suppose $\mathcal{A}$ has enough projectives. If $Ext^n(-,-) = 0$ then the functor $\underline{Ext}^n(A,-) : \mathcal{A} \longrightarrow \mathbf{Ab}$ is zero for every object A, which means that $\underline{Ext}^{n+1}(A,-) = 0$ for every object A, and therefore $Ext^{n+1}(-,-) = 0$. Similarly if $\mathcal{A}$ has enough injectives. $\square$

3 Global Dimension

We say an abelian category $\mathcal{A}$ is *zero* if all the morphism sets $Hom_{\mathcal{A}}(A, B)$ are zero abelian groups, which is equivalent to the functor $Hom(-,-)$ being zero.

Definition 3. Let $\mathcal{A}$ be an abelian category with enough injectives (or projectives). The *global dimension* of $\mathcal{A}$, denoted $gl.dim.\mathcal{A}$, is the least integer $n \geq -1$ such that the bifunctor $Ext^{n+1}(-,-)$ is zero, or ∞ if no such integer exists. So $gl.dim.\mathcal{A} = -1$ if and only if $\mathcal{A}$ is zero, and for a nonzero abelian category $\mathcal{A}$, the global dimension is the largest integer $n \geq 0$ for which there exists objects A, B with $Ext^n(A, B) \neq 0$.

Proposition 8. *Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. Then*

$$gl.dim.\mathcal{A} = sup\{proj.dim.A \mid A \in \mathcal{A}\} = sup\{inj.dim.A \mid A \in \mathcal{A}\}$$

Remark 1. Let $\mathcal{A}$ be an abelian category with enough projectives. Then we can define the functors $\underline{Ext}^n(-, A)$ for $n \geq 0$ and use them to define the injective dimension (in terms of the vanishing of these functors). Similarly if $\mathcal{A}$ has enough injectives we could define the projective dimension. But these dimensions would not be equal to the dimensions defined by looking at the shortest finite injective (resp. projective) resolutions. So we only consider projective dimensions for categories with enough projectives, and injective dimensions for categories with enough injectives.

Remark 2. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an equivalence of abelian categories. Then F is exact and preserves injectives and projectives, so $\mathcal{A}$ has enough injectives (resp. projectives) if and only if $\mathcal{B}$ does.

- If $\mathcal{A}, \mathcal{B}$ have enough projectives, then $proj.dim.A = proj.dim.F(A)$.
- If $\mathcal{A}, \mathcal{B}$ have enough injectives, then $inj.dim.A = inj.dim.F(A)$.

So equivalent abelian categories have the same global dimension (recall global dimension is only defined for categories with either enough injectives or enough projectives). In particular, if R is a commutative ring, then the projective and injective dimensions of a module M as a right and left R-module agree. Also, if $\phi : R \longrightarrow S$ is an isomorphism of rings, restricting scalars gives isomorphisms $\mathbf{Mod}S \cong \mathbf{Mod}R$ and $S\mathbf{Mod} \cong R\mathbf{Mod}$. So if M is an S-module then $proj.dim_S M = proj.dim_R M$ and $inj.dim_S M = inj.dim_R M$ for both left and right modules.

Definition 4. Let R be a ring. The *right global dimension* of R, denoted $r.gl.dim(R)$ is the global dimension of the abelian category $\mathbf{Mod}R$. Similarly the *left global dimension* of R, denoted $l.gl.dim(R)$, is the global dimension of $R\mathbf{Mod}$. It follows from the previous remark that $l.gl.dim(R^{\text{op}}) = r.gl.dim(R)$. In particular if R is commutative, $l.gl.dim(R) = r.gl.dim(R)$, and we denote this common dimension by $gl.dim(R)$.

Lemma 9. *A right R-module B is injective if and only if $Ext(R/I, B) = 0$ for all right ideals I, and similarly for left modules and left ideals.*

Proof. The long exact Ext sequence for $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ includes

$$Hom(R,B) \longrightarrow Hom(I,B) \longrightarrow Ext(R/I,B) \longrightarrow 0$$

By Baer's criterion, B is injective if and only if the first map is surjective for all right ideals I, that is, if $Ext(R/I,B) = 0$. ☐

Proposition 10. *The following numbers are the same for any ring R:*

(a) $sup\{inj.dim.B \,|\, B \in \mathbf{Mod}R\}$

(b) $sup\{proj.dim.B \,|\, B \in \mathbf{Mod}R\}$

(c) $sup\{proj.dim.R/I \,|\, I \text{ is a right ideal of } R\}$

(d) $sup\{d \,|\, Ext^d_R(A,B) \neq 0 \text{ for some right modules } A,B\}$

(e) $r.gl.dim(R)$

Proof. The equivalence of the numbers $(a),(b),(d),(e)$ has already been shown. In the case where R is the zero ring it is clear that (c) agrees with the other numbers (all of which are -1). Since $(a) \geq (c)$ we can reduce to the case where R is nonzero and $d = sup\{proj.dim.R/I\}$ is such that $0 \leq d < \infty$. So $Ext^{d+1}(R/I,B) = 0$ for any right ideal I and right R-module B.

If $d = 0$ this means that every right R-module is injective, so $sup\{inj.dim.B\} = 0$ and we are done. If $d \geq 1$ suppose that B is a right R-module and find an exact sequence with all J^i injective

$$0 \longrightarrow B \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \longrightarrow J^{d-1} \longrightarrow M \longrightarrow 0$$

Then by dimension shifting we have $0 = Ext^{d+1}(R/I,B) \cong Ext(R/I,M)$ for any right ideal I, so M is injective, and hence $inj.dim.B \leq d$. Hence $(a) = (c)$ and the proof is complete. ☐

Proposition 11. *The following numbers are the same for any ring R:*

(a) $sup\{inj.dim.B \,|\, B \in R\mathbf{Mod}\}$

(b) $sup\{proj.dim.B \,|\, B \in R\mathbf{Mod}\}$

(c) $sup\{proj.dim.R/I \,|\, I \text{ is a left ideal of } R\}$

(d) $sup\{d \,|\, Ext^d_R(A,B) \neq 0 \text{ for some left modules } A,B\}$

(e) $l.gl.dim(R)$

Lemma 12. *Let $\mathcal{A}$ be an abelian category, and suppose $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence. Then*

(i) If $\mathcal{A}$ has enough projectives, then $proj.dim.B \leq max\{proj.dim.A, proj.dim.C\}$ with equality unless $proj.dim.C = proj.dim.A + 1$.

(ii) If $\mathcal{A}$ has enough injectives, then $inj.dim.B \leq max\{inj.dim.A, inj.dim.C\}$ with equality unless $inj.dim.A = inj.dim.C + 1$.

Proof. (a) For any object X we have a long exact sequence

$$\begin{aligned}0 \longrightarrow \underline{Ext}(C,X) \longrightarrow \underline{Ext}(B,X) \longrightarrow \underline{Ext}(A,X) \longrightarrow \\ \longrightarrow \underline{Ext}^2(C,X) \longrightarrow \underline{Ext}^2(B,X) \longrightarrow \underline{Ext}^2(A,X) \longrightarrow \\ \longrightarrow \underline{Ext}^3(C,X) \longrightarrow \underline{Ext}^3(B,X) \longrightarrow \underline{Ext}^3(A,X) \longrightarrow \cdots\end{aligned}$$

The claim is trivial unless both $proj.dim.A, proj.dim.C$ are finite. In that case, the first and third terms in every row of the above sequence vanish past $e = max\{proj.dim.A, proj.dim.C\}$ and so the $\underline{Ext}^n(B,X)$ terms must also vanish. Hence $proj.dim.B \leq e$. If $proj.dim.C < proj.dim.A$

then for all $n > proj.dim.C$ and all objects X we have $\underline{Ext}^n(B, X) \cong \underline{Ext}^n(A, X)$, so it is clear that $proj.dim.B = e$. A similar argument applies if $proj.dim.C > proj.dim.A + 1$. If $e = proj.dim.C = proj.dim.A$ then the result is trivial for $e = -1$, so assume $e \geq 0$ and find objects X, Y with $\underline{Ext}^e(C, X) \neq 0, \underline{Ext}^e(A, Y) \neq 0$. Then both functors are nonzero on $X \oplus Y$, so $\underline{Ext}^e(B, X \oplus Y) \neq 0$ and hence $proj.dim.B = e$. This takes care of every case other than $proj.dim.C = proj.dim.A + 1$. One proves (b) in much the same way using the long exact sequence in the first variable. □

Proposition 13. *Let $\mathcal{A}$ be an infinite cocomplete abelian category with exact coproducts and enough projectives. Then for any nonempty collection of objects $\{A_i\}_{i \in I}$ we have $proj.dim.(\oplus_i A_i) = sup\{proj.dim.A_i \,|\, i \in I\}$.*

Proof. A nonempty coproduct $\oplus_i A_i$ is zero if and only if every A_i is zero. For every object B we have an isomorphism of groups

$$\underline{Ext}^n(\oplus_i A_i, B) \cong \prod_i \underline{Ext}^n(A_i, B)$$

So the functor $\underline{Ext}^n(\oplus_i A_i, -)$ is nonzero if and only if some $\underline{Ext}^n(A_i, -)$ is nonzero, which completes the proof. □

Proposition 14. *Let $\mathcal{A}$ be an infinite complete abelian category with exact products and enough injectives. Then for any nonempty collection of objects $\{A_i\}_{i \in I}$ we have $inj.dim(\prod_i A_i) = sup\{inj.dim.A_i \,|\, i \in I\}$.*

Both results apply to the abelian categories **Ab**, R**Mod** and **Mod**R for a ring R. There are a couple of immediate consequences of these results:

- Let R be a ring. For any R-module M and nonempty set I we have $proj.dim.(\oplus_i M) = proj.dim.M$ and $inj.dim.(\prod_i M) = inj.dim.M$.
- Let $\phi : R \longrightarrow S$ be a morphism of rings and let P be a projective S-module. Then for some nonempty set I there is a S-module Q with $\oplus_i S = P \oplus Q$. This expression still holds as modules over R, so we have

  $$proj.dim_R S = proj.dim_R(\oplus_i S) = proj.dim_R(P \oplus Q) = sup\{proj.dim_R P, proj.dim_R Q\}$$

  It follows that $proj.dim_R P \leq proj.dim_R S$.
- Let R be a ring with $r.gl.dim(R) = \infty$. Then there actually exists a right R-module M with $proj.dim.M = inj.dim.M = \infty$ (and similarly for left global dimension and left modules). Since $r.gl.dim(R) = sup\{proj.dim.B\} = sup\{inj.dim.B\}$ we simply take a sequence of modules $M_1, M_2, \ldots$ with strictly increasing projective dimensions. The coproduct $M_1 = \oplus_i M_i$ must have infinite projective dimension, and a similar argument applies to the construction of a module M_2 with infinite injective dimension. Then the module $M = M_1 \oplus M_2$ has both dimensions infinite.

Proposition 15. *Let R be a ring and $x \in R$ a central regular element which is not a unit. Then $proj.dim_R R/x = 1$.*

Proof. To explain the notation, since x is central we have $Rx = xR$ and we denote the module $R/xR = R/Rx$ simply by R/x. We claim that its projective dimension as a right and left module are both equal to 1. The map $R \longrightarrow R$ defined by multiplication with x gives a projective resolution of the module R/x

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/x \longrightarrow 0 \tag{1}$$

So for a right R-module B the Ext groups $Ext^n(R/x, B)$ are the cohomology of the sequence

$$0 \longrightarrow Hom(R, B) \longrightarrow Hom(R, B) \longrightarrow 0 \longrightarrow \cdots$$

So we have group isomorphisms $Ext(R/x, B) \cong B/Bx$ and $Ext^n(R/x, B) = 0$ for $n > 1$. This shows that $proj.dim_R R/x \leq 1$. The fact that x is not a unit on either side means that the module R/x is nonzero and the group $R/Rx = Ext(R/x, R)$ is nonzero, so $proj.dim_R R/x = 1$. A similar argument applies when we consider R/x as a left R-module. □

Let R be a commutative integral domain. Then $proj.dim_R R/(x) \leq 1$ for every $x \in R$. From the previous results we know that for nonzero x, $proj.dim_R R/(x) = 1$ if and only if x is not a unit. It follows from Proposition 11 that the global dimension of any commutative principal ideal domain which is not a field is 1. In particular $gl.dim(\mathbb{Z}) = 1$. We will see later that this also follows from the fact that such a ring is Dedekind.

4 Flat dimension

In this section R is a fixed ring. There is a bifunctor $Tor_n(-,-) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ which depends on an assignment of resolutions but is independent of this choice up to a canonical natural equivalence. We obtain covariant additive functors $Tor_n(A, -), Tor_n(-, A) : \mathcal{A} \longrightarrow \mathbf{Ab}$ for any object A which are determined up to natural equivalence.

Lemma 16. *A right R-module A is flat if and only if $Tor(A, -) = 0$ and a left R-module B is flat if and only if $Tor(-, B) = 0$.*

For a right R-module A the vanishing of the functors $A \otimes -, Tor(A, -), Tor_2(A, -), \ldots$ measures "how flat" the module A is. Similarly for a left R-module B the vanishing of the functors $- \otimes B, Tor(-, B), Tor_2(-, B), \ldots$ measures how flat the module B is.

Proposition 17. *The following statements are equivalent for $n \geq 0$ and any right R-module A:*

(a) $Tor_{n+1}(A, -) = 0$.

(b) $Tor_n(A, -)$ *is left exact.*

(c) If there is an exact sequence of right R-modules with all F_i flat

$$0 \longrightarrow M \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

then M is flat.

(d) There is an exact sequence of right R-modules with all F_i flat

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

Proof. For $n = 0$ condition (c) means that if we have an exact sequence $0 \longrightarrow M \longrightarrow A \longrightarrow 0$ then M is flat, which is equivalent to saying that A is flat.

$(a) \Rightarrow (b)$ Suppose we are given a short exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$. Part of the long exact sequence for Tor is

$$Tor_{n+1}(A, B'') \longrightarrow Tor_n(A, B') \longrightarrow Tor_n(A, B) \longrightarrow Tor_n(A, B'')$$

Since $Tor_{n+1}(A, -)$ it follows that $Tor_n(A-)$ is left exact.

$(b) \Rightarrow (c)$ For $n = 0$ this is trivial. So suppose $n \geq 1$ and that we are given an exact sequence of the stated form. Let $N \longrightarrow N'$ be a monomorphism of left R-modules. From our notes on dimension shifting of Tor we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & Tor_n(A, N) & \longrightarrow & M \otimes_R N & \longrightarrow & F_{n-1} \otimes_R N \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Tor_n(A, N') & \longrightarrow & M \otimes_R N' & \longrightarrow & F_{n-1} \otimes_R N' \end{array}$$

Using the 5-Lemma we see that $M \otimes_R N \longrightarrow M \otimes_R N'$ is a monomorphism, so M is flat.

$(c) \Rightarrow (d)$ This is trivial for $n = 0$. For $n \geq 1$ take a free resolution and let $M \longrightarrow F_{n-1}$ be the image of $F_n \longrightarrow F_{n-1}$.

$(d) \Rightarrow (a)$ For $n = 0$ this follows from the fact that if A is flat, $Tor(A, -) = 0$. For $n \geq 1$ we use dimension shifting to see that for any object B, $0 = Tor(F_n, B) \cong Tor_{n+1}(A, B)$. □

The same proof gives the result for Tor on the right:

Proposition 18. *The following statements are equivalent for $n \geq 0$ and any left R-module B:*

(a) $Tor_{n+1}(-, B) = 0$.

(b) $Tor_n(-, B)$ *is left exact.*

(c) If there is an exact sequence of left R-modules with all F_i flat

$$0 \longrightarrow M \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

then M is flat.

(d) There is an exact sequence of left R-modules with all F_i flat

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

Corollary 19. *Let A be a right R-module and consider the following sequence of additive functors* $R\mathbf{Mod} \longrightarrow \mathbf{Ab}$

$$A \otimes_R -, Tor(A, -), Tor_2(A, -), \ldots, Tor_n(A, -), \ldots$$

If $Tor_n(A, -) = 0$ for an integer $n \geq 0$ then $Tor_m(A, -) = 0$ for all $m \geq n$. Similarly if B is a left R-module there is a sequence of additive functors $\mathbf{Mod}R \longrightarrow \mathbf{Ab}$

$$- \otimes_R B, Tor(-, B), Tor_2(-, B), \ldots, Tor_n(-, B), \ldots$$

If $Tor_n(-, B) = 0$ for an integer $n \geq 0$ then $Tor_m(-, B) = 0$ for all $m \geq n$.

Definition 5. Let R be a ring and A a right R-module. We define the *flat dimension* of A, denoted $flat.dim.A$, to be the smallest integer $n \geq -1$ with the property that $Tor_{n+1}(A, -) = 0$. If no such integer exists, then we set $flat.dim.A = \infty$. So $flat.dim.A = -1$ if and only if A is a zero module, and for nonzero A, the flat dimension is the last nonzero position in the sequence

$$A \otimes_R -, Tor(A, -), Tor_2(A, -), \ldots, Tor_n(A, -), \ldots$$

Clearly $flat.dim.A = 0$ if and only if A is a nonzero flat module. For nonzero A the flat dimension is the smallest integer $n \geq 0$ for which there exists an exact sequence of right R-modules

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

with all F_i flat, with the dimension being ∞ if no such sequence exists. Clearly isomorphic modules have the same flat dimension.

Definition 6. Let R be a ring and B a left R-module. We define the *flat dimension* of B, denoted $flat.dim.B$, to be the smallest integer $n \geq -1$ with the property that $Tor_{n+1}(-, B) = 0$. If no such integer exists, then we set $flat.dim.B = \infty$. So $flat.dim.B = -1$ if and only if B is a zero module, and for nonzero B, the flat dimension is the last nonzero position in the sequence

$$- \otimes_R B, Tor(-, B), Tor_2(-, B), \ldots, Tor_n(-, B), \ldots$$

Clearly $flat.dim.B = 0$ if and only if B is a nonzero flat module. For nonzero B the flat dimension is the smallest integer $n \geq 0$ for which there exists an exact sequence of left R-modules

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

with all F_i flat, with the dimension being ∞ if no such sequence exists. Clearly isomorphic modules have the same flat dimension.

Lemma 20. *Consider the sequence of bifunctors $\mathcal{A} \times \mathcal{A} \longrightarrow \mathbf{Ab}$*

$$- \otimes_R -, Tor(-,-), Tor_2(-,-), \ldots, Tor_n(-,-), \ldots$$

If $Tor_n(-,-) = 0$ for an integer $n \geq 0$ then $Tor_m(-,-) = 0$ for all $m \geq n$.

Clearly R is the zero ring if and only if the functor $- \otimes_R -$ is zero.

Definition 7. The *Tor dimension* of a ring R, denoted $tor.dim(R)$, is the least integer $n \geq -1$ such that the bifunctor $Tor_{n+1}(-,-)$ is zero, or ∞ if no such integer exists. So $tor.dim(R) = -1$ if and only if R is zero, and for a nonzero ring R, the Tor dimension is the largest integer $n \geq 0$ for which there exists a right module A and a left module B with $Tor_n(A,B) \neq 0$.

Lemma 21. *A left R-module B is flat if and only if $Tor(R/I, B) = 0$ for all right ideals I, and a right R-module A is flat if and only if $Tor(A, R/J) = 0$ for all left ideals J.*

Proof. The long exact Tor sequence for $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ includes

$$Tor(R/I, B) \longrightarrow I \otimes_R B \longrightarrow R \otimes_R B \longrightarrow R/I \otimes_R B$$

From our Strenstrom notes, we know that B is flat if and only if $I \otimes_R B \longrightarrow R \otimes_R B$ is injective for every right ideal I, which is if and only if $Tor(R/I, B) = 0$. In fact, we can even restrict to *finitely generated* right ideals I in the statement of the Lemma. The dual version follows similarly. □

Proposition 22. *The following numbers are the same for any ring R:*

(a) $sup\{flat.dim.A \mid A \in \mathbf{Mod}R\}$

(b) $sup\{flat.dim.R/J \mid J \text{ is a right ideal of } R\}$

(c) $sup\{flat.dim.B \mid B \in R\mathbf{Mod}\}$

(d) $sup\{flat.dim.R/I \mid I \text{ is a left ideal of } R\}$

(e) $sup\{d \mid Tor_d(A,B) \neq 0 \text{ for some modules } A, B\}$

(f) $tor.dim(R)$

Proof. If R is the zero ring, these are all trivially -1, so assume R is nonzero. It is clear that $(a), (c), (e), (f)$ are all equal. We show that $(b) = (c)$ (then $(a) = (d)$ follows from the same argument applied to R^{op}). Since $(c) \geq (b)$ we can reduce to the case where $d = sup\{flat.dim.R/J\}$ is such that $0 \leq d < \infty$. So $Tor_{d+1}(R/J, B) = 0$ for any right ideal J and left R-module B.

If $d = 0$ this means that every left R-module B is flat, so $sup\{flat.dim.B\} = 0$ and we are done. If $d \geq 1$ suppose that B is a left R-module and find an exact sequence with all F_i flat

$$0 \longrightarrow K \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

Then by dimension shifting, $0 = Tor_{d+1}(R/J, B) \cong Tor(R/J, K)$ for any right ideal J, so K is flat and $flat.dim.B \leq d$. Hence $(b) = (c)$ and the proof is complete. □

Lemma 23. *Let R be a ring and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ an exact sequence of left or right R-modules. Then $flat.dim.B \leq max\{flat.dim.A, flat.dim.C\}$ with equality unless $flat.dim.C = flat.dim.A + 1$.*

Next we make some observations concering the dimensions defined so far for rings:

- Let R be a ring and let A be a right R-module. Then the projective, injective and flat dimension of A as a right module are equal to the respective dimension for A viewed as a left module over R^{op}. In particular, if R is commutative the right and left dimensions agree.

- Let $\phi : R \longrightarrow S$ be an isomorphism of rings, $\mathbf{Mod}R \cong \mathbf{Mod}S$ and $R\mathbf{Mod} \cong S\mathbf{Mod}$ the corresponding isomorphisms. These isomorphisms identify projective, injective, free and flat modules. So if M is an S-module then $proj.dim_R M = proj.dim_S M$, $inj.dim_R M = inj.dim_S M$ and $tor.dim_R M = tor.dim_S M$ for right and left modules.

- Isomorphic rings have the same left global dimension, right global dimension, and Tor dimension.

- The zero ring has $gl.dim(0) = tor.dim(0) = -1$.

- For every right or left R-module M we have $flat.dim.M \leq proj.dim.M$. Hence for any ring R we have $tor.dim(R) \leq l.gl.dim(R)$ and $tor.dim(R) \leq r.gl.dim(R)$.

- If k is a field, then every module over k is projective, injective and flat, so $gl.dim(k) = tor.dim(k) = 0$.

Lemma 24. *Let R be a commutative domain. Then the following statements are equivalent*

(i) R is a field.

(ii) $gl.dim(R) = 0$.

(iii) $tor.dim(R) = 0$.

Proof. $(i) \Rightarrow (ii)$ If R is a field, then every module over R is projective, so $gl.dim(R) = 0$. Since $tor.dim(R) \leq gl.dim(R)$ the assertion $(ii) \Rightarrow (iii)$ is trivial. $(iii) \Rightarrow (i)$ Suppose that $tor.dim(R) = 0$ but $0 \neq x \in R$ is not a unit. Then $R/(x)$ is a nonzero module, which must be flat. By tensoring with the injection $(x) \longrightarrow R$ we see that $(x)/(x^2) \longrightarrow R/(x)$ is a monomorphism, which implies that $(x) = (x^2)$ and hence that x is a unit, which is a contradiction. $\square$

Proposition 25. *If R is right noetherian, then*

(i) $flat.dim.A = proj.dim.A$ for every finitely generated right R-module A.

(ii) $tor.dim(R) = r.gl.dim(R)$.

Similarly if R is left noetherian, flat dimension agrees with projective dimension for finitely generated left modules, and $tor.dim(R) = l.gl.dim(R)$.

Proof. Since we can compute $tor.dim(R)$ and $r.gl.dim(R)$ using the modules R/I, it suffices to prove (i). Since $flat.dim.A \leq proj.dim.A$ it suffices to suppose that $flat.dim.A = n$ with $0 < n < \infty$ and prove that $proj.dim.A \leq n$. As R is right noetherian, there is an exact sequence

$$0 \longrightarrow M \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

in which the P_i are finitely generated free modules and M is finitely presented. Since $flat.dim.A = n$ it follows that M is flat, and therefore projective, since any finitely presented flat module is projective. This implies that $proj.dim.A \leq n$, as required. $\square$

Corollary 26. *Let R be a ring which is right and left noetherian. Then*

$$r.gl.dim(R) = l.gl.dim(R) = tor.dim(R)$$

In particular this is the case for a commutative noetherian ring.

Proposition 27. *Let R be a commutative Dedekind domain. Then $gl.dim(R) = tor.dim(R) = 1$.*

Proof. We know from our Ext notes that $Ext^n_R(A, B) = 0$ for $n \geq 2$ and R-modules A, B, so $0 \leq gl.dim(R) \leq 1$. But $gl.dim(R) = 0$ if and only if R is a field, and a Dedekind domain is not a field. The global dimension agrees with the Tor dimension since R is noetherian. $\square$

Corollary 28. *Let R be a commutative principal ideal domain which is not a field. Then $tor.dim(R) = gl.dim(R) = 1$. In particular both dimensions are 1 for $\mathbb{Z}$ and $k[x]$ for any field k.*

In what follows we distinguish between the bifunctors Tor and $\underline{Tor}$. Let R be a ring and $x \in R$ a central regular element. Then R/x is a right R-module with projective resolution (1). So for a left R-module B the groups $Tor_n^R(R/x, B)$ are the homology of the top complex of abelian groups in the following diagram, which has homology isomorphic to the bottom complex

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & R \otimes_R B & \xrightarrow{x \otimes 1} & R \otimes_R B & \longrightarrow & 0 \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & B & \xrightarrow[x]{} & B & \longrightarrow & 0 \end{array} \tag{2}$$

There is a canonical left R/x-module structure on the group $Tor_n^R(R/x, B)$ (see our Tor notes on bimodules). The action of $a + x \in R/x$ is calculated by lifting left multiplication $R/x \longrightarrow R/x$ by $a + x$ to a morphism of resolutions

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a+x} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow[x]{} & R & \longrightarrow & R/x & \longrightarrow & 0 \end{array}$$

Tensoring with B and taking homology gives the action $Tor_n^R(R/x, B) \longrightarrow Tor_n^R(R/x, B)$ of $a+x$. Reading off the homology from the bottom row of (2) we see that there are isomorphisms of left R/x-modules

$$\underline{Tor}_n^R(R/x, B) \cong Tor_n^R(R/x, B) \cong \begin{cases} B/xB & n = 0 \\ \{b \in B \,|\, xb = 0\} & n = 1 \\ 0 & n > 1 \end{cases}$$

where all three modules on the right have the canonical R/x-module structure.

Now let A be a right R-module. The complex (1) is also a resolution of R/x as a left R-module. The groups $\underline{Tor}_n^R(A, R/x)$ are the homology of the top complex of abelian groups in the following diagram, which has homology isomorphic to the bottom complex

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A \otimes_R R & \xrightarrow{1 \otimes x} & A \otimes_R R & \longrightarrow & 0 \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow[x]{} & A & \longrightarrow & 0 \end{array} \tag{3}$$

There is a canonical right R/x-module structure on the group $\underline{Tor}_n^R(A, R/x)$ calculated in the same way as above, this time lifting right multiplication $R/x \longrightarrow R/x$ by $a + x$ to a morphism of the resolutions, and taking homology. We conclude that there are isomorphisms of right R/x-modules

$$Tor_n^R(A, R/x) \cong \underline{Tor}_n^R(A, R/x) \cong \begin{cases} A/Ax & n = 0 \\ \{a \in A \,|\, ax = 0\} & n = 1 \\ 0 & n > 1 \end{cases}$$

where all three modules on the right have the canonical R/x-module structure.

Proposition 29. *Let R be a ring and $x \in R$ a central regular element which is not a unit. Then $tor.dim_R R/x = 1$.*

Proof. We claim that the Tor dimension of R/x is 1 as both a left and right R-module. It is clear from the above comments that in both cases $tor.dim_R R/x \leq 1$. Since $Tor_1^R(R/x, R/x) \cong R/x$ it follows from the fact that x is not a unit that $tor.dim_R R/x = 1$. $\square$

5 Change of Rings

Recall that in a ring R an element x is *central* if $xy = yx$ for all $y \in R$.

Theorem 30 (First Change of Rings). *Let x be a regular central element of a ring R. If A is a nonzero R/x-module with $proj.dim_{R/x}A$ finite, then*

$$proj.dim_R A = 1 + proj.dim_{R/x} A$$

Proof. We prove the result for left modules, with the proof for right modules being similar. Since $A \neq 0$ we must also have $R/x \neq 0$, so x is not a right or a left unit. It follows from Proposition 15 that $proj.dim_R(R/x) = 1$. As x is a regular element and $xA = 0$ (and projective modules are regular-torsion-free) it follows that A cannot be a projective R-module, so $proj.dim_R A \geq 1$. On the other hand, if A is a projective R/x-module, then by the comments following Proposition 14 we see that $proj.dim_R A = proj.dim_R(R/x) = 1$.

We proceed by induction on $proj.dim_{R/x}A$. If $proj.dim_{R/x}A \geq 1$ find an exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0 \tag{4}$$

with P a projective R/x-module, so that $proj.dim_{R/x}A = proj.dim_{R/x}M + 1$ by Lemma 12. Since A is not projective we have $M \neq 0$ and so by the inductive hypothesis $proj.dim_{R/x}A = proj.dim_{R/x}M + 1 = proj.dim_R M$. The sequence (4) is also an exact sequence of R-modules, so

$$proj.dim_R P \leq max\{proj.dim_R M, proj.dim_R A\}$$

with equality unless $proj.dim_R A = proj.dim_R M + 1$. Since P is a nonzero projective R/x-module, we have already shown that $proj.dim_R P = 1$. So to complete the proof it suffices to eliminate the possibility that $proj.dim_R A = 1 = proj.dim_{R/x}A$.

Find an exact sequence of R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with F free. If $proj.dim_R A = 1$ then K is a projective R-module, so $K/xK \cong R/x \otimes_R K$ is a projective R/x-module. The module R/x is an R/x-R-bimodule, so we get an exact sequence of left R/x-modules

$$0 \longrightarrow \underline{Tor}_1^R(R/x, A) \longrightarrow K/xK \longrightarrow F/xF \longrightarrow A \longrightarrow 0$$

If $proj.dim_{R/x}A \leq 2$ then $\underline{Tor}_1^R(R/x, A)$ is a projective left R/x-module. But there is an isomorphism of left R/x-modules

$$\underline{Tor}_1^R(R/x, A) \cong \{a \in A \,|\, xa = 0\} = A$$

So $proj.dim_{R/x}A = 0$ and it is impossible to have $proj.dim_R A = 1 = proj.dim_{R/x}A$, which is what we wanted to show. ☐

Theorem 31 (Second Change of Rings). *Let x be a regular central element of a ring R. If A is a left R-module and x is regular on A, then*

$$proj.dim_R A \geq proj.dim_{R/x}(A/xA)$$

Similarly if A is a right R-module and x is regular on A, then $proj.dim_R A \geq proj.dim_{R/x}(A/Ax)$.

Proof. We prove the result for left modules, with the proof for right modules being similar. If $proj.dim_R A$ is ∞ or -1 there is nothing to prove, so we assume $proj.dim_R A = n$ with $0 \leq n < \infty$ and proceed by induction on n. If A is a projective R-module, then A/xA is a projective R/x-module, so the result is true in the case $n = 0$. If $proj.dim_R A \geq 1$, find an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with F free. Since K is a submodule of F, and free modules are regular-torsion-free, it follows that x is regular on K. Then by Lemma 12 we see that $proj.dim_R K = n-1$, sso by the inductive hypothesis $proj.dim_{R/x}(K/xK) \leq n-1$. Tensoring with R/x yields the exact sequence of left R/x-modules

$$0 \longrightarrow \underline{Tor}_1^R(R/x, A) \longrightarrow K/xK \longrightarrow F/xF \longrightarrow A/xA \longrightarrow 0$$

As x is regular on A, $\underline{Tor}_1^R(R/x, A) \cong \{a \in A \,|\, xa = 0\} = 0$. Again using Lemma 12 and the fact that F/xF is projective, we see that either A/xA is projective or $proj.dim_{R/x}(A/xA) = 1 + proj.dim_{R/x}(K/xK) \leq 1 + (n-1) = proj.dim_R A$. ☐

See our notes on polynomial rings for the definition of the left $R[x_1, \ldots, x_n]$-module $A[x_1, \ldots, x_n]$ for any left R-module A (and similarly for right modules), and the proof that $R[x_1, \ldots, x_n]$ is a flat right and left module over R for $n \geq 1$. It is clear that x is a central regular element of $R[x]$.

Corollary 32. *For any ring R we have $proj.dim_{R[x]}(A[x]) = proj.dim_R A$ for every R-module A.*

Proof. We prove the result for left modules, with the proof for right modules being similar. The isomorphism of rings $R \cong R[x]/x$ identifies the $R[x]/x$-module $A[x]/xA$ with the R-module A. Hence $proj.dim_{R[x]/x}(A[x]/xA) = proj.dim_R A$. But since $x \in R[x]$ is regular on $A[x]$ it follows from the second Change of Rings Theorem that $proj.dim_{R[x]} A[x] \geq proj.dim_R A$. On the other hand, if there is an exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with all the P_i projective then tensoring with the flat left R-module $R[x]$ gives an exact sequence of left $R[x]$-modules

$$0 \longrightarrow P_n[x] \longrightarrow P_{n-1}[x] \longrightarrow \cdots \longrightarrow P_0[x] \longrightarrow A[x] \longrightarrow 0$$

where the modules $P_i[x] \cong R[x] \otimes_R P_i$ are projective since $R[x] \otimes_R - : R\mathbf{Mod} \longrightarrow R[x]\mathbf{Mod}$ has an exact right adjoint. Hence $proj.dim_{R[x]} A[x] \leq proj.dim_R A$ and so the two are equal. □

In particular, an R-module A is projective if and only if $A[x]$ is a projective $R[x]$-module. If A is an R-module, then A becomes an $R[x]$-module by setting the action of x to be zero. In other words, use the morphism $R[x] \longrightarrow R[x]/x \cong R$. It is a consequence of the first Change of Rings Theorem that $proj.dim_{R[x]} A = 1 + proj.dim_R A$, provided A is nonzero and has finite projective dimension.

Theorem 33. *If R is a nonzero ring and $R[x_1, \ldots, x_n]$ denotes a polynomial ring in n variables, then $l.gl.dim(R[x_1, \ldots, x_n]) = n + l.gl.dim(R)$ and $r.gl.dim(R[x_1, \ldots, x_n]) = n + r.gl.dim(R)$.*

Proof. We give the proof for left global dimension. It is sufficient to treat the case of $n = 1$. If $l.gl.dim(R) = \infty$ then by the above Corollary $l.gl.dim(R[x]) = \infty$, so we may assume $l.gl.dim(R) = n$ with $0 \leq n < \infty$. If A is a nonzero R-module with $proj.dim_R A$ finite, then A becomes an $R[x]$-module and $proj.dim_{R[x]} A = 1 + proj.dim_R A$, so $l.gl.dim(R[x]) \geq 1 + n$. Given an $R[x]$-module M consider the sequence of $R[x]$-modules

$$0 \longrightarrow R[x] \otimes_R M \xrightarrow{\beta} R[x] \otimes_R M \xrightarrow{\mu} M \longrightarrow 0 \qquad (5)$$

where μ is multiplication and β is defined by the bilinear map

$$\beta(t \otimes m) = t \cdot (x \otimes m - 1 \otimes (x \cdot m))$$

for $t \in R[x]$ and $m \in M$. We know from the previous Corollary that

$$proj.dim_{R[x]}(R[x] \otimes_R M) = proj.dim_R M \leq n$$

Suppose for a moment that (5) is exact. Then by Lemma 12

$$proj.dim_{R[x]} M \leq proj.dim_{R[x]}(R[x] \otimes_R M) + 1 \leq n + 1$$

Taking the supremum over all M shows that $l.gl.dim(R[x]) \leq 1 + n$. So to finish the proof, we must establish the claim that (5) is exact.

It is clear that $\mu\beta = 0$. To see that β is injective, note that $R[x] \otimes_R M$ is isomorphic as an $R[x]$-module to $M[x]$, so every nonzero element f can be written uniquely as a polynomial with coefficients in M, with the leading coefficient m_k nonzero

$$f = x^k \otimes m_k + \cdots + x^2 \otimes m_2 + x \otimes m_1 + 1 \otimes m_0$$

The leading term of $\beta(f)$ is $x^{k+1} \otimes m_k$, so if $\beta(f) = 0$ we have $m_k = 0$ and consequently $f = 0$. Finally, we show by induction on k that if $f \in Ker\mu$ then $f \in Im\beta$. Since $\mu(1 \otimes m) = m$ the case $k = 0$ is trivial. If $k \geq 1$ then $\mu(f) = \mu(g)$ for the polynomial $g = f - \beta(x^{k-1} \otimes m_k)$ of lower degree. By induction if $f \in Ker\mu$ then $g = \beta(h)$ for some h, and hence $f = \beta(h + x^{k-1} \otimes m_k)$. □

Corollary 34 (Hilbert's Theorem on Syzygies). *If k is a field, then the polynomial ring $k[x_1, \ldots, x_n]$ has global dimension n. Thus for any exact sequence of modules*

$$0 \longrightarrow M \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

If the X_i are all projective, so is M. If the X_i are all injective, so is A.

Lemma 35. *Let R be a commutative local ring and M a finitely generated R-module. Then M is free $\Leftrightarrow$ M is projective $\Leftrightarrow$ M is flat.*

Proof. See our Matsumura notes. □

Lemma 36. *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$, and let A be a finitely generated R-module. If $x \in \mathfrak{m}$ is a regular element which is regular on A, then A/xA is a free R/x-module if and only if A is a free R-module.*

Proof. The functor $R/x \otimes_R - : R\mathbf{Mod} \longrightarrow R/x\mathbf{Mod}$ preserves all colimits, so if A is free then so is A/xA. Suppose that A/xA is a free R/x-module. If $A/xA = 0$ then $A = 0$ by Nakayama. So assume $A/xA \neq 0$ and pick elements $u_1, \ldots, u_n$ mapping onto a basis of A/xA. We claim they form a basis of A. Since $(u_1, \ldots, u_r)A + xA = A$ it follows from Nakayama's Lemma that $(u_1, \ldots, u_n)A = A$, so the u_i at least span A. To show that u_i are linearly independent, suppose $\sum r_i u_i = 0$ for $r_i \in R$. In A/xA, the images of the u_i are linearly independent over R/x, so $r_i \in xR$ for all i. As x is regular on R and A we can divide to get a well-defined quotient $r_i/x \in R$ such that $\sum (r_i/x)u_i = 0$ in A. Continuing this process, we get a sequence of elements $r_i, r_i/x, r_i/x^2, \ldots$. Suppose for a moment that $r_i \neq 0$. If $r_i/x \in (r_i)$ we would have $r_i = xr_ib$ for some nonzero $b \in R$, which implies that $r_i(1 - xb) = 0$, contradicting the fact that since $x \in \mathfrak{m}$, the element $1 - xb$ is a unit. In this way we show that $(r_i), (r_i/x), (r_i/x^2), \ldots$ is a strictly increasing chain of ideals in R, which is impossible since R is noetherian. Hence all the $r_i = 0$, as required. □

Theorem 37 (Third Change of Rings). *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$, and let A be a finitely generated R-module. If $x \in \mathfrak{m}$ is a regular element which is regular on A, then*

$$proj.dim_R A = proj.dim_{R/x}(A/xA)$$

Proof. We know $\geq$ holds by the second Change of Rings Theorem, and we shall prove equality by induction on $n = proj.dim_{R/x}(A/xA)$. If $A/xA = 0$ then $xA = A$ so $A = 0$ by Nakayama. If $n = 0$ then A/xA is projective, hence a free R/x-module since R/x is local. It follows from the previous Lemma that A is a free R-module, so $proj.dim_R A = 0$. Now assume $n > 0$ and find an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with F finite free. Since $\underline{Tor}_1^R(R/x, A) \cong \{a \in A \,|\, xa = 0\} = 0$ we have an exact sequence of R/x-modules

$$0 \longrightarrow K/xK \longrightarrow F/xF \longrightarrow A/xA \longrightarrow 0$$

As F/xF is free, $proj.dim_{R/x}(K/xK) = n - 1$. Free modules are regular-torsion-free, so x is regular on K, which is also finitely generated since R is noetherian. By the inductive hypothesis $proj.dim_R K = n - 1$, which implies that $proj.dim_R A = n$ and completes the proof. □

Corollary 38. *Let R be a commutative noetherian local ring, and let A be a nonzero finitely generated R-module with $proj.dim_R A < \infty$. If $x \in \mathfrak{m}$ is regular and regular on A, then*

$$proj.dim_R(A/xA) = 1 + proj.dim_R A$$

Proof. Combine the first and third Change of Rings Theorems. □