Diagram Chasing in Abelian Categories

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In applications of the theory of homological algebra, results such as the Five Lemma are crucial. For abelian groups this result is proved by diagram chasing, a procedure not immediately available in a general abelian category. However, we can still prove the desired results by embedding our abelian category in the category of abelian groups. All of this material is taken from Mitchell's book on category theory [Mit65].

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1 Introduction

For our conventions regarding categories the reader is directed to our Abelian Categories (AC) notes. In particular recall that an *embedding* is a faithful functor which takes distinct objects to distinct objects.

Theorem 1. *Any small abelian category $\mathcal{A}$ has an exact embedding into the category of abelian groups.*

Proof. See [Mit65] Chapter 4, Theorem 2.6. □

Lemma 2. *Let $\mathcal{A}$ be an abelian category and $S \subseteq \mathcal{A}$ a nonempty set of objects. There is a full small abelian subcategory $\mathcal{B}$ of $\mathcal{A}$ containing S.*

Proof. See [Mit65] Chapter 4, Lemma 2.7. □

Combining results II 6.7 and II 7.1 of [Mit65] we have

Lemma 3. *Let $\mathcal{A}$ be an abelian category, $T : \mathcal{A} \longrightarrow \mathbf{Ab}$ an exact embedding. Then T preserves and reflects monomorphisms, epimorphisms, commutative diagrams, limits and colimits of finite diagrams, and exact sequences.*

1.1 Desired results

In the category of abelian groups, diagram chasing arguments are usually used either to establish a property (such as surjectivity) of a certain morphism, or to construct a new morphism between known objects. The first type of argument is easily lifted to a general abelian category using the above embeddings. Let us show how to prove the Five Lemma in any abelian category.

Lemma 4 (Five Lemma). *Suppose the following diagram is commutative and has exact rows in* **Ab***:*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ {\scriptstyle\gamma_1}\downarrow & & {\scriptstyle\gamma_2}\downarrow & & {\scriptstyle\gamma_3}\downarrow & & {\scriptstyle\gamma_4}\downarrow & & {\scriptstyle\gamma_5}\downarrow \\ B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \xrightarrow[\beta_3]{} & B_4 & \xrightarrow[\beta_4]{} & B_5 \end{array}$$

Then

(i) If γ_1 is an epimorphism and γ_2 and γ_4 are monomorphisms, then γ_3 is a monomorphism.

(ii) If γ_5 is a monomorphism and γ_2 and γ_4 are epimorphisms, then γ_3 is an epimorphism

(iii) If γ_1 is an epimorphism, γ_5 is a monomorphism, and γ_2 and γ_4 are isomorphisms, then γ_3 is an isomorphism.

We claim that if $\mathcal{A}$ is any abelian category, the above Lemma is true with **Ab** replaced by $\mathcal{A}$. Suppose we are given such a commutative diagram with exact rows, and let $\mathcal{B}$ be a small, full, abelian subcategory of $\mathcal{A}$ containing the $A_i, B_i, \alpha_i, \gamma_i, \beta_i$. Then the diagram is commutative and the rows are exact in $\mathcal{B}$. Let $T : \mathcal{B} \longrightarrow \mathbf{Ab}$ be an exact embedding.

(i) Suppose that γ_1 is an epimorphism, γ_2, γ_4 monomorphisms in $\mathcal{A}$. Then γ_1 is an epimorphism in $\mathcal{B}$, and γ_2, γ_4 are monomorphisms in $\mathcal{B}$. Hence if we map the diagram into **Ab** using T, we see that $T(\gamma_3)$ is a monomorphism. Since T reflects monomorphisms, this implies that γ_3 is a monomorphism in $\mathcal{B}$ and hence in $\mathcal{A}$.

(ii) Same as (i).

(iii) As above we find that $T(\gamma_3)$ is an isomorphism. Hence γ_3 is both a monomorphism and an epimorphism in $\mathcal{A}$, so since an abelian category is balanced, γ_3 is an isomorphism.

The second type of proof by diagram chasing (construction of a morphism) is more subtle. In the next section we will show how to lift such results to arbitrary abelian categories. The most important example is the Snake Lemma, and for convenience we give a proof in **Ab**.

Lemma 5 (Snake). *Suppose we have a commutative diagram with exact rows in* **Ab***:*

$$\begin{array}{ccccccc} & & A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' & & \end{array} \tag{1}$$

Then in the usual manner we obtain the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & Ker d' & \longrightarrow & Ker d & \longrightarrow & Ker d'' & & \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ & & A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' & & \\ & & \downarrow{\scriptstyle\varepsilon} & & \downarrow & & \downarrow & & \\ & & Coker d' & \longrightarrow & Coker d & \longrightarrow & Coker d'' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2}$$

There exists a canonical morphism $Kerd'' \longrightarrow Cokerd'$ making the following sequence exact:

$$Kerd' \longrightarrow Kerd \longrightarrow Kerd'' \longrightarrow Cokerd' \longrightarrow Cokerd \longrightarrow Cokerd''$$

Proof. By Theorem 12.3.4 of [Sch72] the top left square is a pullback and the bottom right square is a pushout. It follows that the kernel and cokernel sequences are exact.

Let α_2^{-1} and β_1^{-1} be right inverse *functions*. That is, α_2^{-1} and β_1^{-1} are defined on A'' and $Im\beta_1$ respectively in such a way that they are inverses on these domains for α_2, β_1 respectively (of course there may be many possible choices). Define $\omega : Kerd'' \longrightarrow Cokerd'$ by

$$\omega(x) = \varepsilon\beta_1^{-1}d\alpha_2^{-1}\gamma(x)$$

This makes sense, since $d''\gamma(x) = 0$ so if $\alpha_2(y) = \gamma(x)$ then $d(y) \in Ker\beta_2 = Im\beta_1$.

We show that ω is independent of the left inverses chosen. Since β_1 is injective there is only one choice for β_1^{-1}. Suppose $\alpha_2(y) = \alpha_2(y') = \gamma(x)$ for some $x \in Kerd''$. Then $y - y' \in Ker\alpha_2 = Im\alpha_1$, so $y = y' + \alpha_1(a)$ for some $a \in A'$. Hence $d(y) = d(y') + \beta_1 d'(a)$. So it is clear that we get the same $\omega(x)$ for both y and y'.

It is not difficult to check that ω is a morphism of abelian groups. The sequence

$$Kerd \longrightarrow Kerd'' \longrightarrow Cokerd' \longrightarrow Cokerd$$

is clearly of order two, and a little thought shows it is exact as well. This proves the result in **Ab**. Note that the morphism ω is canonical, since it is given by an explicit algorithm above. □

2 Walks in Abelian Categories

We want to use the embedding theorem to lift the Snake Lemma to an arbitrary abelian category. Suppose we are given a small abelian category $\mathcal{A}$ and an exact embedding $T : \mathcal{A} \longrightarrow \mathbf{Ab}$. Then we can identify T with an abelian subcategory of **Ab** . The construction of the connecting morphism above works by composing morphisms and their inverses and showing that the result is a morphism of groups with the required property. But since T is not in general full, it is not clear that we can reflect the connecting morphism back to $\mathcal{A}$ (one can avoid this by using a more powerful embedding, but one is still stuck with the problem of canonicity). However, we *can* reflect any morphism (such as the connecting morphism) constructed by a diagram chase. We make some general definitions:

Definition 1. A *relation* m from an abelian group A to an abelian group B is a subset of $A \times B$, written $m : A \longrightarrow B$. Given relations $m : A \longrightarrow B$ and $n : B \longrightarrow C$ we write $n \circ m$ for the following relation:

$$n \circ m = \{(a, c) \in A \times C \mid \text{There exists } b \in B \text{ with } (a, b) \in m, (b, c) \in n\}$$

If m, n are functions then $n \circ m$ is the usual composition. The *domain* of a relation $n : A \longrightarrow B$ is the set of all a for which there exists b with $(a, b) \in n$.

Any morphism $\alpha : A \longrightarrow B$ of abelian groups is a relation $A \longrightarrow B$. But α also gives rise to the *inverse* relation $\overline{\alpha} : B \longrightarrow A$ defined to be the set of all tuples (b, a) with $b = \alpha(a)$. Note that if $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ are morphisms then $\overline{\beta\alpha} = \overline{\alpha} \circ \overline{\beta}$.

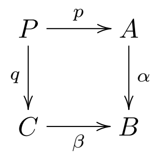
Definition 2. Let $\mathcal{A}$ be a subcategory of **Ab**, and let A, B be two objects of $\mathcal{A}$. An $\mathcal{A}$-*morphism* from A to B is a morphism $A \longrightarrow B$ belonging to $\mathcal{A}$, and an $\mathcal{A}$-*antimorphism* from A to B is the inverse relation of a morphism $B \longrightarrow A$ belonging to $\mathcal{A}$. Clearly $\mathcal{A}$-morphisms and $\mathcal{A}$-antimorphisms are closed under composition.

An $\mathcal{A}$-*relation* from A to B is a relation $A \longrightarrow B$ which can be written as the composition of relations $\alpha_n \circ \cdots \alpha_2 \circ \alpha_1$ where for each $1 \leq i \leq n$, α_i is an $\mathcal{A}$-morphism or an $\mathcal{A}$-antimorphism from A_i to A_{i+1} and $A_1 = A, A_{n+1} = B$. An $\mathcal{A}$-relation from A to B which is a function with domain A is called an $\mathcal{A}$-*function* from A to B. A *simple* $\mathcal{A}$-relation is a relation $A \longrightarrow B$ which can be written as the composite of an $\mathcal{A}$-antimorphism followed by an $\mathcal{A}$-morphism.

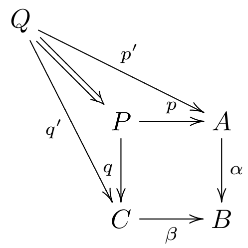
Proposition 6. *Let $\mathcal{A}$ be an abelian subcategory of* **Ab**. *Then all $\mathcal{A}$-relations are simple $\mathcal{A}$-relations. If a simple $\mathcal{A}$-relation is an $\mathcal{A}$-function, then it is actually a morphism in $\mathcal{A}$.*

Proof. To show that all $\mathcal{A}$-relations are simple, It would suffice to show that any composite of an $\mathcal{A}$-morphism followed by an $\mathcal{A}$-antimorphism was a simple relation, because then we could arrange the factors of any $\mathcal{A}$-relation to get it into the required form.

So suppose $\alpha : A \longrightarrow B, \beta : C \longrightarrow B$ are morphisms of $\mathcal{A}$ and we are dealing with the $\mathcal{A}$-relation $\overline{\beta}\alpha$. Form the pullback diagram in $\mathcal{A}$:

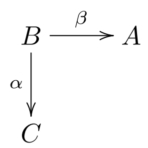


Since the inclusion is exact, this is a pullback diagram of abelian groups, so there is an isomorphism $P \cong Q$ and a commutative diagram



where $Q = \{(x, y) \in C \times A \,|\, \beta(x) = \alpha(y)\}$ is the canonical pullback of abelian groups. It is not hard to check that both $\overline{\beta}\alpha$ and $q'\overline{p'}$ are the relation $\{(a, c) \,|\, \alpha(a) = \beta(c)\}$ and one checks easily that $q\overline{p} = q'\overline{p'}$, so the proof is complete.

Now suppose a simple $\mathcal{A}$-relation $\alpha\overline{\beta}$ for morphisms $\beta : B \longrightarrow A$ and $\alpha : B \longrightarrow C$ is actually a function $A \longrightarrow C$. So we have a diagram



This relation being defined on all of A implies that β is an epimorphism. Let $K \longrightarrow B$ be the kernel of β in $\mathcal{A}$ and therefore in **Ab**. Then since $\alpha\overline{\beta}$ is a function the composite $K \longrightarrow B \longrightarrow C$ is zero. Since $B \longrightarrow A$ is the cokernel of $K \longrightarrow B$ this means that there is a morphism $\theta : A \longrightarrow C$ in $\mathcal{A}$ such that $B \longrightarrow A \longrightarrow C = \alpha$. It is not hard to see that $\alpha\overline{\beta} = \theta$, completing the proof. □

Corollary 7. *Let $\mathcal{A}$ be an abelian subcategory of* **Ab**. *Then all $\mathcal{A}$-functions are $\mathcal{A}$-morphisms.*

How do you show that an $\mathcal{A}$-relation is an $\mathcal{A}$-function?

Lemma 8. *Let $\mathcal{A}$ be a subcategory of* **Ab**. *An $\mathcal{A}$-relation $\alpha_n \circ \cdots \circ \alpha_1$ from A to B is an $\mathcal{A}$-function if and only if for every $a \in A$ there is a sequence $a_1, \ldots, a_{n+1}$ with $a_i \in A_i$, $a = a_1$ and $(a_i, a_{i+1}) \in \alpha_i$ for $1 \le i \le n$, and moreover any such sequence terminates with the same element of B.*

Proof. Note that by definition $\alpha_n \circ \cdots \circ \alpha_1$ is the set of all pairs (a, b) for which there exists a sequence $a_1, \cdots, a_{n+1}$ with $a_i \in A_i$, $a_1 = a, a_{n+1} = b$ and $(a_i, a_{i+1}) \in \alpha_i$ for $1 \le i \le n$, so this is obvious. □

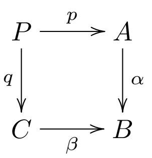
So given an abelian subcategory $\mathcal{A}$ of **Ab** and two objects $A, B \in \mathcal{A}$, suppose we can get from A to B by going along some morphisms of $\mathcal{A}$ and "backwards" along other morphisms. Suppose that this path makes sense, in that any $a \in A$ can be carried to B by mapping along the morphisms and taking some inverse image along the backwards morphisms (we don't require that arbitrary choices of inverse images will lead to such a sequence, only that at each antimorphism we can make some choice which does lead to a sequence), and moreover that this end result *does not depend on which inverse image we choose at any backward step.* Then this process defines a morphism of abelian groups $A \longrightarrow B$ and moreover this morphism belongs to $\mathcal{A}$. We say the morphism $A \longrightarrow B$ was constructed *by diagram chasing.*

The way is this is usually applied is by taking an abelian category $\mathcal{A}$, finding a small, abelian subcategory $\mathcal{C}$ containing an interesting diagram, and an embedding $T : \mathcal{C} \longrightarrow \mathbf{Ab}$. If we construct morphisms between objects in $\mathcal{C}$ using a diagram chase then we lift the result up to $\mathcal{C}$. It is not immediately obvious that this result will be independent of the embedding T.

Definition 3. Let $\mathcal{A}$ be an abelian category. A *walk* $W : A \longrightarrow B$ in $\mathcal{A}$ is a sequence of morphisms $\alpha_1, \dots, \alpha_n$ where for each $1 \le i \le n$, $\alpha_i : A_i \longrightarrow A_{i+1}$ or $\alpha_i : A_{i+1} \longrightarrow A_i$ (called a *forward* and *backward* step respectively) and $A_1 = A, A_{n+1} = B$. The *length* of a walk is the number of occurring morphisms. A walk is *simple* if all the backward steps occur before any forward step.

Let $T : \mathcal{A} \longrightarrow \mathbf{Ab}$ be an exact embedding. Any walk in $\mathcal{A}$ determines an $\mathcal{A}$-relation $T(W)$ in $\mathbf{Ab}$. The concatenation of two walks determines the composed relation, and a simple walk determines a simple $\mathcal{A}$-relation. We say two walks W, W' between the same objects are T-*equivalent* if $T(W) = T(W')$ and *equivalent* if they are T-equivalent for *every* exact embedding $T : \mathcal{A} \longrightarrow \mathbf{Ab}$.

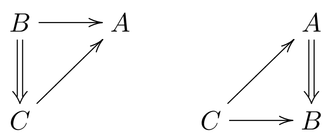
Proposition 9. *Let $\mathcal{A}$ be an abelian category and suppose the following diagram is a pullback:*



Then the walks α, β and p, q are equivalent.

Proof. We already proved the first claim in Proposition 6. □

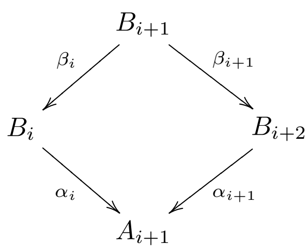
Corollary 10. *Let $\mathcal{A}$ be an abelian category and suppose the following diagrams are commutative, with isomorphisms as indicated:*



Then the two ways from A to C in both diagrams give equivalent walks.

This motivates the following definition:

Definition 4. Let $\mathcal{A}$ be an abelian category and let $W, V : A \longrightarrow B$ be two walks. We write $V \preceq W$ if either V, W are the same walk, or they both have the same length $n \ge 1$ and there exists $1 < i < n$ such that $\alpha_j = \beta_j$ for $1 \le j \le n$ with $j \ne i, j \ne i+1$, and there is a pullback diagram:



So either the walks are the same, or the walk W is obtained from V by forming a pullback diagram at some pair of morphisms in V consisting of a forward step followed by a backward step. We say W is *obtained from V by pullback* if there is a finite sequence of walks $V_1, \ldots, V_{m+1}$ with $V_1 = V, V_{m+1} = W$ and $V_i \preceq V_{i+1}$ for $1 \leq i \leq m$. Clearly W is equivalent to V.

Lemma 11. *Let V be a walk in an abelian category $\mathcal{A}$. Then there is a simple walk W obtained from V by pullback. Hence any walk is equivalent to a simple walk β, α consisting of one backward and one forward step.*

Proof. Simply use the pullback technique to shift all the backward steps to the beginning of the walk, which gives a simple walk obtained from V by pullback. Then compose all the backward steps and all the forward steps, to get the required simple walk β, α. □

2.1 Diagram chasing

We say a walk W in an abelian category $\mathcal{A}$ is a *function walk* if $T(W)$ is a function for every exact embedding $T : \mathcal{A} \longrightarrow \mathbf{Ab}$.

Proposition 12. *Let $\mathcal{A}$ be a small abelian category, and W a walk in $\mathcal{A}$. Then W is a function walk if and only if whenever β, α is a simple walk consisting of one backward and one forward step which is equivalent to W, β is an epimorphism and α factors through β.*

Proof. Suppose W is a function walk, and let $T : \mathcal{A} \longrightarrow \mathbf{Ab}$ be an exact embedding. Then $T(W) = T(\alpha)\overline{T(\beta)}$ is a function, from which it follows by the argument in the proof of Proposition 6 we see that β is an epimorphism and α factors through β. Conversely, suppose β is an epimorphism and $\alpha = \gamma\beta$ and let $T : \mathcal{A} \longrightarrow \mathbf{Ab}$ be an exact embedding. Then clearly $T(\alpha)\overline{T(\beta)} = T(\gamma)$, which is a function. □

So being a function walk in a small abelian category is actually independent of any particular embedding, it is a condition which is intrinsic to the category. We have shown that the following conditions on a walk W in a small abelian category are equivalent:

(i) For every exact embedding $T : \mathcal{A} \longrightarrow \mathbf{Ab}$, $T(W)$ is a function.

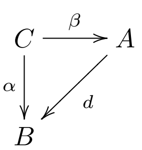
(ii) For some exact embedding $T : \mathcal{A} \longrightarrow \mathbf{Ab}$, $T(W)$ is a function.

(iii) Whenever β, α is a simple walk consisting of one backward and one forward step which is equivalent to W, β is an epimorphism and α factors through β.

Definition 5. Let $\mathcal{A}$ be an abelian category. A walk $W : A \longrightarrow B$ is *amenable to a diagram chase* if there is a small, full, abelian subcategory $\mathcal{C}$ of $\mathcal{A}$ containing W in which W is a function walk. A morphism $d : A \longrightarrow B$ is said to be *constructed from W by a diagram chase* if for every small, full, abelian subcategory $\mathcal{C}$ of $\mathcal{A}$ in which W is a function walk we have $T(d) = T(W)$ for any exact embedding $T : \mathcal{C} \longrightarrow \mathbf{Ab}$.

Theorem 13. *Let $\mathcal{A}$ be an abelian category with an amenable walk $W : A \longrightarrow B$. Then there is precisely one morphism $d : A \longrightarrow B$ constructed from W by a diagram chase.*

Proof. Let $\mathcal{C}$ be a small, full, abelian subcategory of $\mathcal{A}$ containing W in which W is a function walk (by assumption such a subcategory exists). Let β, α be an equivalent simple walk consisting of one backward and one forward step. Then by Proposition 12, β is an epimorphism and α factors through β. Let $d : A \longrightarrow B$ be the unique morphism making the following diagram commute:



For any exact embedding $T : \mathcal{C} \longrightarrow \mathbf{Ab}$ we have $T(d) = T(W)$, so this morphism d does not depend on the simple walk β, α. We have produced a morphism $d_{\mathcal{C}} : A \longrightarrow B$ with the property that $T(d_{\mathcal{C}}) = T(W)$ for any exact embedding $T : \mathcal{C} \longrightarrow \mathbf{Ab}$.

To complete the proof it would be enough to show that $d_{\mathcal{C}} = d_{\mathcal{C}'}$ for any other small, full, abelian subcategory $\mathcal{C}'$ in which W is a function walk. Let $\mathcal{B}$ be a small, full, abelian subcategory of $\mathcal{A}$ containing $\mathcal{C}, \mathcal{C}'$ and let $S : \mathcal{B} \longrightarrow \mathbf{Ab}$ be an exact embedding. Then $S|_{\mathcal{C}}$ and $S|_{\mathcal{C}'}$ are exact embeddings of the respective subcategories in $\mathbf{Ab}$ and therefore by construction $S(d_{\mathcal{C}}) = S(W) = S(d_{\mathcal{C}'})$. Since S is faithful we see that $d_{\mathcal{C}} = d_{\mathcal{C}'}$, as required.

So given an amenable walk W pick any small, full, abelian subcategory $\mathcal{C}$ in which W is a function walk and let $d = d_{\mathcal{C}}$. Then d is constructed from W by a diagram chase, in the precise sense above. ☐

Lemma 14. *The Snake Lemma holds in any abelian category $\mathcal{A}$.*

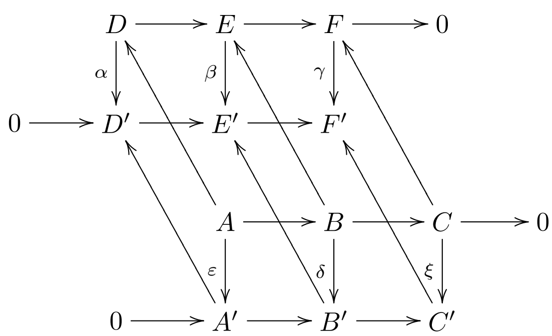
Proof. That is, if we replace $\mathbf{Ab}$ by $\mathcal{A}$ in the statement of the Snake Lemma, the result is still true. Note that given the first diagram, choose any kernels and cokernels for the vertical morphisms, and induce morphisms between them in the canonical way. Then the sequences of kernels and cokernels are exact (the argument given in the proof of the Lemma in $\mathbf{Ab}$ uses categorical arguments valid in $\mathcal{A}$). So we have to prove the existence of $\omega : Kerd'' \longrightarrow Cokerd'$ and show the extended sequence is exact. We do this by showing that the walk $W : \gamma, \alpha_2, d, \beta_1, \varepsilon$ is amenable in $\mathcal{A}$.

Let $\mathcal{C}$ be a small, full, abelian subcategory of $\mathcal{A}$ containing all the objects in the larger second diagram in the statement of the Snake Lemma. In $\mathcal{C}$ the rows and columns of the diagram are still exact, since they were exact in $\mathcal{A}$. Let $T : \mathcal{C} \longrightarrow \mathbf{Ab}$ be an exact embedding, which identifies $\mathcal{C}$ with an abelian subcategory of $\mathbf{Ab}$. Apply the Snake Lemma in $\mathbf{Ab}$ to see that $T(W)$ is a function, which shows that W is amenable to a diagram chase. Let $\omega : Kerd'' \longrightarrow Cokerd'$ be the unique morphism constructed from W by a diagram chase, which exists and is unique by Theorem 13. Then $T(\omega) = T(W)$, and since T reflects exact sequences, we see that the required sequence of kernels in $\mathcal{A}$ followed by ω, followed by cokernels in $\mathcal{A}$ is exact. ☐

We call ω the *connecting morphism*, and it is the unique morphism $\omega : Kerd'' \longrightarrow Cokerd'$ in $\mathcal{A}$ with the property that for any small, full, abelian subcategory $\mathcal{C}$ in which W is a function walk, and for any exact embedding $T : \mathcal{C} \longrightarrow \mathbf{Ab}$ we have $T(\omega) = T(W)$, which means that $T(\omega)$ is the canonical connecting morphism for the image under T of the diagram in $\mathbf{Ab}$ (provided $\mathcal{C}$ contains the whole diagram, and not just the walk). Note that ω depends only on the morphisms in the walk W, so if we choose alternative kernels for d', d or cokernels for d, d'' we will get the same connecting morphism.

We claim that the connecting morphism is natural with respect to the diagram in the Snake Lemma. More precisely:

Lemma 15. *Suppose there is a morphism of diagrams of the form (1) in $\mathcal{A}$. That is, a commutative diagram with exact rows:*



We claim that the induced morphisms between the kernels and cokernels fit into a commutative *diagram*

$$
\begin{array}{ccccccccccc}
Ker\alpha & \longrightarrow & Ker\beta & \longrightarrow & Ker\gamma & \xrightarrow{\omega} & Coker\alpha & \longrightarrow & Coker\beta & \longrightarrow & Coker\gamma \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Ker\varepsilon & \longrightarrow & Ker\delta & \longrightarrow & Ker\xi & \xrightarrow[\omega']{} & Coker\varepsilon & \longrightarrow & Coker\delta & \longrightarrow & Coker\xi
\end{array}
$$

where ω and ω' are the canonical connecting morphisms.

Proof. Pick kernels and cokernels for the morphisms α, β, γ and ε, δ, ξ and let $\mathcal{C}$ be a small, full, abelian subcategory of $\mathcal{A}$ containing all these objects and let $T : \mathcal{C} \longrightarrow \mathbf{Ab}$ be an exact embedding. Then $T(\omega), T(\omega')$ are the connecting morphisms for the respective diagrams, so we reduce to proving the result in **Ab**. But using the explicit algorithm for the connecting morphisms, this is straightforward to check. $\square$

References

[Mit65] Barry Mitchell, *Theory of categories*, Pure and Applied Mathematics, Vol. XVII, Academic Press, New York, 1965. MR MR0202787 (34 #2647)

[Sch72] Horst Schubert, *Categories*, Springer-Verlag, New York, 1972, Translated from the German by Eva Gray. MR MR0349793 (50 #2286)