Derived Functors

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In this note we give an exposition of some basic topics in homological algebra. Most of this material can be found in either [3] or [2], but for some topics the best reference is still [1].

Contents

1 Definitions **1**

2 Projective Resolutions **6**

3 Left Derived Functors **8**

4 Injective Resolutions **10**

5 Right Derived Functors **12**

6 The Long Exact (Co)Homology Sequence **14**

7 Two Long Exact Sequences **18**
- 7.1 Left Derived Functors . . . 18
- 7.2 Right Derived Functors . . . 23

8 Dimension Shifting **27**
- 8.1 Acyclic Resolutions . . . 31

9 Change of Base **33**

10 Homology and Colimits **37**

11 Cohomology and Limits **38**

12 Delta Functors **39**

1 Definitions

Throughout we work in an abelian category $\mathcal{A}$. We assume a strong axiom of choice that allows us to associate to every morphism $f : A \longrightarrow B$ a canonical kernel $Ker(f) \longrightarrow A$, cokernel $B \longrightarrow Coker(f)$, and image $Im(f) \longrightarrow B$. Similarly we pick a canonical zero object 0, and a canonical biproduct for any finite nonempty family of objects. We say $\mathcal{A}$ is *a category of modules* if it is $\mathbf{Mod}R$ or $R\mathbf{Mod}$ for some ring R (in particular $\mathbf{Ab}$ is a category of modules), and for a category of modules we choose the obvious canonical structures, with the subtle exception of choosing the identity $1_A : A \longrightarrow A$ to be the cokernel of any morphism $0 \longrightarrow A$ from a zero object. For a subobject $u : A \longrightarrow B$ we write $B \longrightarrow B/A$ for the cokernel. Given subobjects $u : A \longrightarrow B$ and $v : C \longrightarrow B$ we write $u \leq v$ to mean that u factors through v. The factorisation $A \longrightarrow C$ is monic, so we can talk about the quotient C/A.

Definition 1. A *chain complex* C in $\mathcal{A}$ is a collection of objects $\{C_n\}_{n\in\mathbb{Z}}$ together with morphisms $\partial_n : C_n \longrightarrow C_{n-1}$ satisfying $\partial_n\partial_{n+1} = 0$. We write a chain complex as a descending chain

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

The morphisms ∂_n are called the *differential* (or *boundary* operators). A *morphism* of chain complexes $\psi : C \longrightarrow D$ is a collection ψ of morphisms $\{\psi_n : C_n \longrightarrow D_n\}_{n\in\mathbb{Z}}$ satisfying $\psi_n\partial_{n+1} = \partial_{n+1}\psi_{n+1}$ for all $n \in \mathbb{Z}$. That is, they fit into a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi_{n+1}} & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_{n-1}} & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow[\partial_{n+1}]{} & D_n & \xrightarrow[\partial_n]{} & D_{n-1} & \longrightarrow & \cdots \end{array}$$

Composition is defined by $(\psi\varphi)_n = \psi_n\varphi_n$ and addition by $(\psi + \varphi)_n = \psi_n + \varphi_n$, which defines the preadditive category $\mathbf{Ch}\mathcal{A}$ of chain complexes in $\mathcal{A}$. Since $\partial_n\partial_{n+1} = 0$ we have $Im\partial_{n+1} \leq Ker\partial_n$ for all $n \in \mathbb{Z}$. The object $H_n(C) = Ker\partial_n/Im\partial_{n+1}$ is called the n-th *homology object* of the chain complex C.

Definition 2. A *cochain complex* C in $\mathcal{A}$ is a collection of objects $\{C^n\}_{n\in\mathbb{Z}}$ together with morphisms $\partial^n : C^n \longrightarrow C^{n+1}$ satisfying $\partial^{n+1}\partial^n = 0$. We write a cochain complex as an ascending chain

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \longrightarrow \cdots$$

The morphisms ∂^n are called the *coboundary* operators. A *morphism* of cochain complexes $\psi : C \longrightarrow D$ is a collection ψ of morphisms $\{\psi^n : C^n \longrightarrow D^n\}_{n\in\mathbb{Z}}$ satisfying $\psi^{n+1}\partial^n = \partial^n\psi^n$ for all $n \in \mathbb{Z}$. That is, they fit into a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{n-1} & \xrightarrow{\partial^{n-1}} & C^n & \xrightarrow{\partial^n} & C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi^{n-1}} & & \downarrow{\scriptstyle \psi^n} & & \downarrow{\scriptstyle \psi^{n+1}} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow[\partial^{n-1}]{} & D^n & \xrightarrow[\partial^n]{} & D^{n+1} & \longrightarrow & \cdots \end{array} \tag{1}$$

Composition is defined by $(\psi\varphi)^n = \psi^n\varphi^n$ and addition by $(\psi + \varphi)^n = \psi^n + \varphi^n$, which defines the preadditive category $\mathbf{coCh}\mathcal{A}$ of cochain complexes in $\mathcal{A}$. Since $\partial^{n+1}\partial^n = 0$ we have $Im\partial^{n-1} \leq Ker\partial^n$ for all $n \in \mathbb{Z}$. The object $H^n(C) = Ker\partial^n/Im\partial^{n-1}$ is called the n-th *cohomology object* of the cochain complex C.

Lemma 1. *There is an isomorphism of categories* $\mathbf{Ch}A \cong \mathbf{coCh}A$.

Proof. Given a chain complex $C = \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ define the cochain complex $F(C) = \{D^n, \delta^n\}_{n\in\mathbb{Z}}$ by $D^n = D_{-n}$ and $\delta^n = \partial_{-n}$. Given a morphism $\psi : C \longrightarrow C'$ define $F(\psi) : F(C) \longrightarrow F(C')$ by $F(\psi)^n = \psi_{-n}$. This is clearly an isomorphism. $\square$

Lemma 2. *Let* $\varphi : C \longrightarrow D$ *be a morphism of cochain complexes. Then*

(i) φ *is an isomorphism iff.* φ^n *is an isomorphism for all* $n \in \mathbb{Z}$*;*

(ii) φ *is an epimorphism iff.* φ^n *is an epimorphism for all* $n \in \mathbb{Z}$*;*

(iii) φ *is a monomorphism iff.* φ^n *is a monomorphism for all* $n \in \mathbb{Z}$*.*

The same result is true for chain complexes.

Proof. (i) is easily checked. Suppose $g : D^n \longrightarrow E$ is given with $g\varphi^n = 0$. We have to show that $g = 0$. We can fit g into a morphism of cochains with entries as depicted in the following diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C^{n-2} & \longrightarrow & C^{n-1} & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi^n} & & \downarrow & & \\
\cdots & \longrightarrow & D^{n-2} & \longrightarrow & D^{n-1} & \xrightarrow{\partial^{n-1}} & D^n & \longrightarrow & D^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle g\partial^{n-1}} & & \downarrow{\scriptstyle g} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & E & \xrightarrow[1]{} & E & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The composite of these two cochain morphisms is zero, so since φ is an epimorphism it follows that $g = 0$, as required. A similar construction proves (iii), and it is clear how to translate these results into statements about chain complexes. □

The categories $\mathbf{Ch}\mathcal{A}$ and $\mathbf{coCh}\mathcal{A}$ are abelian. First we give the definitions for $\mathbf{coCh}\mathcal{A}$.

Zero The 0 cochain is $\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$.

Finite products Given cochains $C_1, \ldots, C_r$ we define

$$(C_1 \oplus \cdots \oplus C_r)^n = C_1^n \oplus \cdots \oplus C_r^n$$
$$\partial^n = \partial_1^n \oplus \cdots \oplus \partial_r^n$$

It is not difficult to see that this is a cochain complex. The injection $u_i : C_i \longrightarrow C_1 \oplus \cdots \oplus C_r$ is pointwise the injection $C_i^n \longrightarrow C_1^n \oplus \cdots \oplus C_r^n$, and the projection p_i is also pointwise the projection $C_1^n \oplus \cdots \oplus C_r^n \longrightarrow C_i^n$. It is easy to check that $p_i u_j = \delta_{ij}$ and $\sum u_k p_k = 1$ so that these morphisms are indeed a biproduct.

Kernels and Cokernels Let $\varphi : C \longrightarrow D$ be a morphism of cochain complexes. For each $n \in \mathbb{Z}$ the commutative diagram (1) induces morphisms $Ker(\varphi^n) \longrightarrow Ker(\varphi^{n+1})$ for all n which are unique making the following diagram commute:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Ker(\varphi^n) & \longrightarrow & Ker(\varphi^{n+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & D^n & \longrightarrow & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

This cochain is a kernel for φ, where given any $\psi : E \longrightarrow C$ with $\varphi\psi = 0$ the unique factorisation through $Ker\varphi \longrightarrow C$ is pointwise the unique factorisation of ψ^n through $Ker(\varphi^n) \longrightarrow C^n$. A dual situation holds for cokernels.

Normal and Conormal Given our description of the kernel and cokernel, it is clear that since $\mathcal{A}$ is normal and conormal that every monomorphism in $\mathbf{coCh}\mathcal{A}$ is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Epi-Mono Factorisations Let $\varphi : C \longrightarrow D$ be a morphism of cochains. The morphisms $Im(\varphi^n) \longrightarrow D^n$ are kernels for $D^n \longrightarrow Coker(\varphi^n)$ so we induce morphisms $Im(\varphi^n) \longrightarrow Im(\varphi^{n+1})$ which form a cochain $Im(\varphi)$. The factorisations $C^n \longrightarrow Im(\varphi^n)$ give a morphism $C \longrightarrow Im(\varphi)$ which is a pointwise epimorphism and therefore an epimorphism of cochains. Similarly $Im(\varphi) \longrightarrow D$ is a pointwise monomorphism and therefore a monomorphism of cochains, so we have the desired factorisation. Note that $Im(\varphi) \longrightarrow D$ is the image of φ in $\mathbf{coCh}\mathcal{A}$.

This completes the proof that **coCh**$\mathcal{A}$ is abelian. The obvious translation of these structures to chains gives the definition of zero, biproducts, kernels, cokernels and images, which shows that **Ch**$\mathcal{A}$ is an abelian category. By picking a canonical zero object, canonical finite nonempty products and canonical kernels, cokernels and images for the category $\mathcal{A}$ we get canonical choices for all these structures in **coCh**$\mathcal{A}$ and **Ch**$\mathcal{A}$. Note that a sequence

$$C \xrightarrow{\varphi} D \xrightarrow{\psi} E$$

of chains is exact in **Ch**$\mathcal{A}$ if and only if for all $n \in \mathbb{Z}$ the following sequence is exact in $\mathcal{A}$

$$C_n \xrightarrow{\varphi_n} D_n \xrightarrow{\psi_n} E_n$$

The same is true of cochains.

Lemma 3. *Let D be a diagram in* **coCh**$\mathcal{A}$ *and suppose we have a cocone $\{\alpha_i : D_i \longrightarrow X\}_{i \in I}$ on this diagram with the property that $\{\alpha_i^n : D_i^n \longrightarrow X^n\}$ is a colimit for the diagram D^n in $\mathcal{A}$ for every $n \in \mathbb{Z}$. Then the α_i are a colimit for D in* **coCh**$\mathcal{A}$*.*

Lemma 4. *Let D be a diagram in* **coCh**$\mathcal{A}$ *and suppose we have a cone $\{\alpha_i : X \longrightarrow D_i\}_{i \in I}$ on this diagram with the property that $\{\alpha_i^n : X^n \longrightarrow D_i^n\}$ is a limit for the diagram D^n in $\mathcal{A}$ for every $n \in \mathbb{Z}$. Then the α_i are a limit for D in* **coCh**$\mathcal{A}$*.*

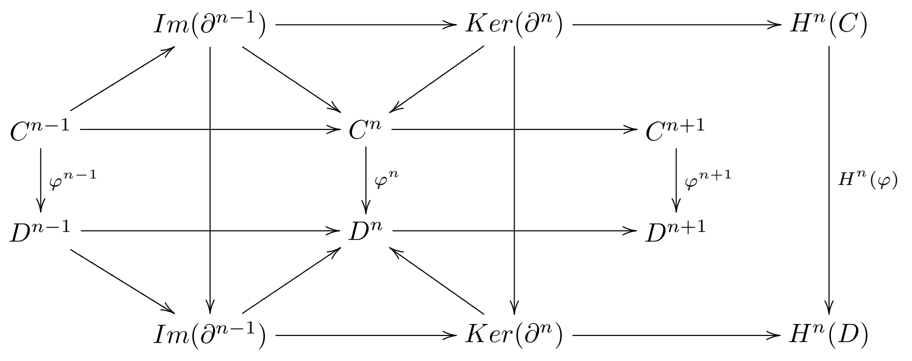
Suppose we have a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[g]{} & D \end{array}$$

Then there are unique morphisms $Ker(f) \longrightarrow Ker(g)$, $Coker(f) \longrightarrow Coker(g)$ and $Im(f) \longrightarrow Im(g)$ making the following diagrams commute

$$\begin{array}{ccc} Ker(f) & \longrightarrow & A \\ \downarrow & & \downarrow \\ Ker(g) & \longrightarrow & C \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & Coker(f) \\ \downarrow & & \downarrow \\ D & \longrightarrow & Coker(g) \end{array} \qquad \begin{array}{ccc} Im(f) & \longrightarrow & B \\ \downarrow & & \downarrow \\ Im(g) & \longrightarrow & D \end{array}$$

Let $\varphi : C \longrightarrow D$ be a morphism of cochains in **coCh**$\mathcal{A}$. For any $n \in \mathbb{Z}$ we induce a morphism $H^n(C) \longrightarrow H^n(D)$ which is unique making the following diagram commute



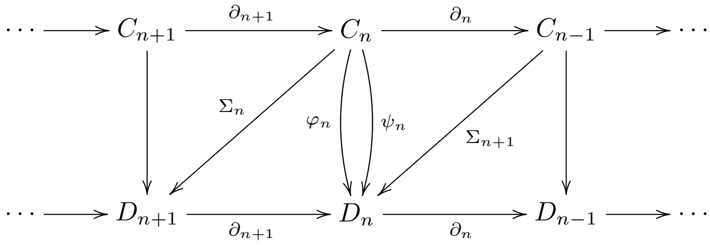
(2)

Hence $H^n(-)$ defines a covariant additive functor **coCh**$\mathcal{A} \longrightarrow \mathcal{A}$. A similar argument shows that $H_n(-)$ defines a covariant additive functor **Ch**$\mathcal{A} \longrightarrow \mathcal{A}$.

Definition 3. A *homotopy* $\Sigma : \varphi \longrightarrow \psi$ between two chain morphisms $\varphi, \psi : C \longrightarrow D$ is a collection of morphisms $\Sigma_n : C_n \longrightarrow D_{n+1}$ such that $\psi - \varphi = \partial\Sigma + \Sigma\partial$. That is,

$$\psi_n - \varphi_n = \partial_{n+1}\Sigma_n + \Sigma_{n-1}\partial_n \qquad \forall n \in \mathbb{Z}$$

as in the diagram



We say that φ, ψ are *homotopic*, and write $\varphi \simeq \psi$ if there exists a homotopy $\Sigma : \varphi \longrightarrow \psi$. If Σ is such a homotopy, then the morphisms $-\Sigma_n$ define a homotopy $-\Sigma : \psi \longrightarrow \varphi$, so there is a bijection between homotopies $\varphi \longrightarrow \psi$ and homotopies $\psi \longrightarrow \varphi$. We denote the set of all homotopies $\varphi \longrightarrow \psi$ by $Hom(\varphi, \psi)$.

Lemma 5. *The homotopy relation $\simeq$ is an equivalence relation, which is stable under composition. That is, if $\varphi, \psi : C \longrightarrow D$ are chain morphisms with $\varphi \simeq \psi$ then for any chain morphisms $\gamma : B \longrightarrow C$ and $\tau : D \longrightarrow E$ we have $\tau\varphi \simeq \tau\psi$ and $\varphi\gamma \simeq \psi\gamma$.*

Proof. The relation is clearly reflective and symmetric. To check transitivity, let $\psi - \varphi = \partial\Sigma + \Sigma\partial$ and $\xi - \psi = \partial T + T\partial$. An easy calculation shows that $\xi - \varphi = \partial(\Sigma + T) + (\Sigma + T)\partial$. For the results about composition, let us assume that $\psi - \varphi = \partial\Sigma + \Sigma\partial$. Then the morphisms $\tau_{n+1}\Sigma_n$ give rise to a homotopy $\tau\psi \simeq \tau\varphi$ and the morphisms $\Sigma_n\gamma_n$ give rise to a homotopy $\psi\gamma \simeq \varphi\gamma$. ☐

Proposition 6. *Let $\varphi, \psi : C \longrightarrow D$ be homotopic chain morphisms. Then $H_n(\varphi) = H_n(\psi)$ for every $n \in \mathbb{Z}$.*

Proof. It suffices to show that if $\varphi \simeq 0$ then $H_n(\varphi) = 0$ for all $n \in \mathbb{Z}$. Let $\Sigma : \varphi \longrightarrow 0$ be a homotopy, so $\varphi_n = \partial_{n+1}\Sigma_n + \Sigma_{n-1}\partial_n$ for all n. Let $n \in \mathbb{Z}$ be fixed, and let $k : Ker\partial_n \longrightarrow C_n$ be the kernel of ∂_n. Then

$$\varphi_n k = (\partial_{n+1}\Sigma_n + \Sigma_{n-1}\partial_n)k = \partial_{n+1}\Sigma_n k$$

So $\varphi_n k$ factors through $Im\partial_{n+1} \longrightarrow D_n$, from which we conclude that in the chain analogue of (2) there is a morphism $Ker\partial_n \longrightarrow Im\partial_{n+1}$ making the whole diagram commute. So the composite $Ker\partial_n \longrightarrow Ker\partial_n \longrightarrow H_n(D)$ must be zero, and therefore $H_n(\varphi) = 0$, as required. ☐

Lemma 7. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. Then F induces an additive functor $\mathbf{Ch}\mathcal{A} \longrightarrow \mathbf{Ch}\mathcal{B}$ (which we denote by the same symbol) defined by $F(C)_n = F(C_n)$ and $F(\varphi)_n = F(\varphi_n)$. This functor has the following properties*

(a) If $\Sigma : \varphi \longrightarrow \psi$ is a homotopy then so is $F(\Sigma) : F(\varphi) \longrightarrow F(\psi)$.

(b) If $\varphi \simeq \psi$ then $H_n(F(\varphi)) = H_n(F(\psi))$ for all $n \in \mathbb{Z}$.

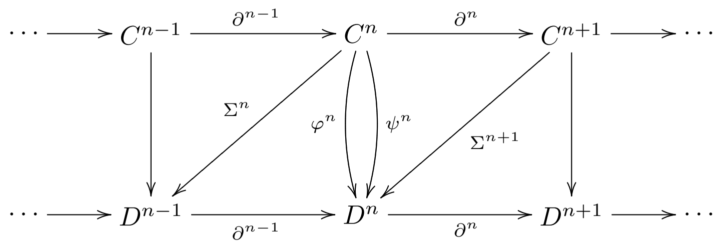
(c) If F is exact, so is the induced functor $\mathbf{Ch}\mathcal{A} \longrightarrow \mathbf{Ch}\mathcal{B}$.

Now for the cochain version of homotopy:

Definition 4. A *homotopy* $\Sigma : \varphi \longrightarrow \psi$ between two cochain morphisms $\varphi, \psi : C \longrightarrow D$ is a collection of morphisms $\Sigma^n : C^n \longrightarrow D^{n-1}$ such that $\psi - \varphi = \partial\Sigma + \Sigma\partial$. That is,

$$\psi^n - \varphi^n = \partial^{n-1}\Sigma^n + \Sigma^{n+1}\partial^n \qquad \forall n \in \mathbb{Z}$$

as in the diagram



We say that φ, ψ are *homotopic*, and write $\varphi \simeq \psi$ if there exists a homotopy $\Sigma : \varphi \longrightarrow \psi$. If Σ is such a homotopy, then the morphisms $-\Sigma_n$ define a homotopy $-\Sigma : \psi \longrightarrow \varphi$, so there is a bijection between homotopies $\varphi \longrightarrow \psi$ and homotopies $\psi \longrightarrow \varphi$. We denote the set of all homotopies $\varphi \longrightarrow \psi$ by $Hom(\varphi, \psi)$.

The following results are proved just as in the chain case:

Lemma 8. *The homotopy relation $\simeq$ is an equivalence relation, which is stable under composition. That is, if $\varphi, \psi : C \longrightarrow D$ are cochain morphisms with $\varphi \simeq \psi$ then for any cochain morphisms $\gamma : B \longrightarrow C$ and $\tau : D \longrightarrow E$ we have $\tau\varphi \simeq \tau\psi$ and $\varphi\gamma \simeq \psi\gamma$.*

Proposition 9. *Let $\varphi, \psi : C \longrightarrow D$ be homotopic cochain morphisms. Then $H^n(\varphi) = H^n(\psi)$ for every $n \in \mathbb{Z}$.*

Lemma 10. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. Then F induces an additive functor* $\mathbf{coCh}\mathcal{A} \longrightarrow \mathbf{coCh}\mathcal{B}$ *(which we denote by the same symbol) defined by $F(C)^n = F(C^n)$ and $F(\varphi)^n = F(\varphi^n)$. This functor has the following properties*

(a) If $\Sigma : \varphi \longrightarrow \psi$ is a homotopy then so is $F(\Sigma) : F(\varphi) \longrightarrow F(\psi)$.

(b) If $\varphi \simeq \psi$ then $H^n(F(\varphi)) = H^n(F(\psi))$ for all $n \in \mathbb{Z}$.

(c) If F is exact, so is the induced functor $\mathbf{coCh}\mathcal{A} \longrightarrow \mathbf{coCh}\mathcal{B}$.

2 Projective Resolutions

Throughout we work with an abelian category $\mathcal{A}$ and chain complexes over $\mathcal{A}$.

Definition 5. A chain complex C is *positive* if $C_n = 0$ for all $n < 0$. In other words the complex looks like

$$\cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

A positive chain complex C is *projective* if C_n is projective for all $n \geq 0$. It is called *acyclic* if $H_n(C) = 0$ for $n \geq 1$. A positive chain complex C is acyclic if and only if the following sequence is exact

$$\cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow H_0(C) \longrightarrow 0$$

A *projective resolution* of an object A is a projective acyclic chain complex P together with a morphism $P_0 \longrightarrow A$ making the following sequence exact

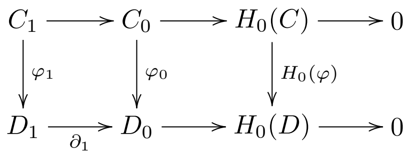
$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \qquad (3)$$

Equivalently, it is an exact sequence (3) with all the P_i projective.

Lemma 11. *The category* $\mathbf{Ch}^+\mathcal{A}$ *of positive chain complexes is an abelian subcategory of* $\mathbf{Ch}\mathcal{A}$.

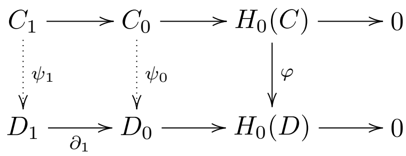
Proof. The canonical zero, products, kernels and cokernels for a morphism of positive chain complexes are themselves positive, so it is clear that $\mathbf{Ch}^+\mathcal{A}$ is an abelian subcategory. In particular this means that a sequence $C' \longrightarrow C \longrightarrow C''$ in $\mathbf{Ch}^+\mathcal{A}$ is exact iff. it is exact in $\mathbf{Ch}\mathcal{A}$, so iff. $C'_n \longrightarrow C_n \longrightarrow C''_n$ is exact in $\mathcal{A}$ for all $n \geq 0$. $\square$

If C is a positive chain complex then the construction of homology gives an epimorphism $C_0 \longrightarrow H_0(C)$ which is the cokernel of $Im(\partial_1)$ and therefore of $C_1 \longrightarrow C_0$. If $\varphi : C \longrightarrow D$ is a morphism of positive chain complexes then $H_0(\varphi) : H_0(C) \longrightarrow H_0(D)$ is just the morphism induced between the cokernels by the following commutative diagram



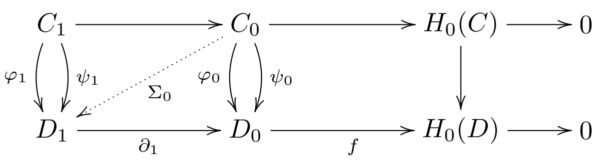
Theorem 12. *Let C, D be positive chain complexes with C projective and D acyclic. Then the map $\varphi \mapsto H_0(\varphi)$ gives a bijection between homotopy classes of chain morphisms $C \longrightarrow D$ and morphisms $H_0(C) \longrightarrow H_0(D)$.*

Proof. First we show that given any morphism $\varphi : H_0(C) \longrightarrow H_0(D)$ there is a chain morphism inducing φ on homology. The chain morphism $\psi : C \longrightarrow D$ is defined recursively. Consider the following diagram, whose bottom row is exact

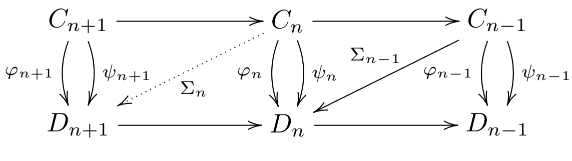


Since C_0 is projective we can induce $\psi_0 : C_0 \longrightarrow D_0$ making the right hand square commute. Since the bottom row is exact at D_0 and $\partial\psi_0\partial = 0$, we see that $\psi_0\partial$ factors through $Im(\partial_1) \longrightarrow D_0$. Since C_1 is projective and $D_1 \longrightarrow Im(\partial_1)$ is an epimorphism, we induce ψ_1 making the above diagram commute. For $n \geq 2$ we use the exactness of D at $n-1$ and the same argument to produce ψ_n. It is clear that ψ is a chain morphism which induces φ on the cokernels.

Now suppose φ, ψ are two chain morphisms inducing the same morphism $H_0(C) \longrightarrow H_0(D)$. Recursively we define a homotopy $\Sigma : \psi \longrightarrow \varphi$. First consider the diagram



Since φ_0, ψ_0 induce the same morphism on the cokernel, we must have $\varphi_0 - \psi_0$ factoring through $Kerf = Im\partial_1$. Since C_0 is projective and $D_1 \longrightarrow Im(\partial_1)$ is an epimorphism, there is $\Sigma_0 : C_0 \longrightarrow D_1$ with $\partial_1\Sigma_0 = \varphi_0 - \psi_0$. Suppose $n \geq 1$ and that $\Sigma_0, \ldots, \Sigma_{n-1}$ have been defined in such a way that $\varphi_r - \psi_r = \partial\Sigma_r + \Sigma_{r-1}\partial$ for $r \leq n-1$ (with $\Sigma_{-1}\partial$ being understood as 0), as in the diagram



We have

$$\partial(\varphi_n - \psi_n - \Sigma_{n-1}\partial) = (\varphi_{n-1} - \psi_{n-1} - \partial\Sigma_{n-1})\partial = \Sigma_{n-2}\partial\partial = 0$$

Hence $\varphi_n - \psi_n - \Sigma_{n-1}\partial$ factors through $Im(\partial_{n+1} : D_{n+1} \longrightarrow D_n)$, and using the fact that C_n is projective we obtain $\Sigma_n : C_n \longrightarrow D_{n+1}$ with the required property. Hence $\psi \simeq \varphi$, as claimed. Of course if $\psi \simeq \varphi$ then $H_0(\varphi) = H_0(\psi)$, which completes the proof. □

So this one situation where equality on homology at 0 always arises from a homotopy. An important special case is the following.

Corollary 13. *Suppose we are given projective resolutions of objects A, B in $\mathcal{A}$*

$$\begin{array}{ll} P: & \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \\ Q: & \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0 \end{array}$$

Then there is a bijection between homotopy classes of chain morphisms $P \longrightarrow Q$ and morphisms $A \longrightarrow B$. Given a chain morphism $\varphi : P \longrightarrow Q$ the corresponding morphism $\alpha : A \longrightarrow B$ is unique making the following diagram commute

$$\begin{array}{ccc} P_0 & \longrightarrow & A \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Q_0 & \longrightarrow & B \end{array}$$

3 Left Derived Functors

Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ *has enough projectives* if for every object A there is a projective object P_0 and an epimorphism $P_0 \longrightarrow A$. If $\mathcal{A}$ has enough projectives then it is clear that we can construct a projective resolution for any object A by taking the kernel $K \longrightarrow P_0$, finding another epimorphism $P_1 \longrightarrow K$ with P_1 projective, and repeating the process. Throughout this section $\mathcal{A}$ is an abelian category with enough projectives, and $\mathcal{B}$ is any abelian category.

Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive covariant functor. Suppose we have a projective resolution of A

$$P: \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

This gives rise to a chain complex of objects of $\mathcal{B}$

$$TP: \quad \cdots \longrightarrow T(P_n) \longrightarrow \cdots \longrightarrow T(P_1) \longrightarrow T(P_0) \longrightarrow 0$$

We define $L_n^P T(A) = H_n(TP)$ for $n \geq 0$. Let B be another object with projective resolution Q and let $\alpha : A \longrightarrow B$ be a morphism. By Corollary 13 there is a chain morphism $\varphi : P \longrightarrow Q$ making a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \alpha} & & \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

For $n \geq 0$ there is a morphism $H_n(T(\varphi)) : L_n^P T(A) \longrightarrow L_n^Q T(B)$, which depends only on α and not on the particular φ we use in the construction (since all candidate φ are homotopic).

Let $\mathcal{P}$ be an assignment of a projective resolution to every object of $\mathcal{A}$ (we need a strong axiom of choice to show that such an assignment exists) and for fixed $n \geq 0$ let $L_n T(A)$ denote the object calculated with the projective resolution chosen by $\mathcal{P}$. Let $L_n T(\alpha) : L_n T(A) \longrightarrow L_n T(B)$ denote the morphism induced by $\alpha : A \longrightarrow B$. Then it is easy to check that $L_n T : \mathcal{A} \longrightarrow \mathcal{B}$ is a covariant additive functor. If necessary we denote this functor by $L_n^{\mathcal{P}} T$ to indicate its dependence on $\mathcal{P}$.

Proposition 14. *Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive covariant functor. If $\mathcal{P}$ and $\mathcal{Q}$ are two assignments of projective resolutions to objects of $\mathcal{A}$, then for $n \geq 0$ there is a canonical natural equivalence $L_n^{\mathcal{P}} T \cong L_n^{\mathcal{Q}} T$.*

Proof. For an object A the morphism $1_A : A \longrightarrow A$ gives rise to chain morphisms $\xi : P \longrightarrow Q$ and $\eta : Q \rightarrow P$ of the corresponding projective resolutions. Moreover, $\xi\eta$ and $\eta\xi$ induce the identity on A and so by Corollary 13 they must be homotopic to the identity chain morphism. Applying T and passing to homology we obtain an isomorphism $L_n^{\mathcal{P}} T(A) \cong L_n^{\mathcal{Q}} T(A)$ which is easily checked to be natural in A. Since the isomorphism depends only on the homotopy class of ξ, η this isomorphism is canonical. $\square$

Definition 6. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant additive functor. For $n \geq 0$ the covariant additive functor $L_nT : \mathcal{A} \longrightarrow \mathcal{B}$ is called the n-th *left derived functor* of T. This functor depends on the projective resolutions chosen, but is independent of these choices up to canonical natural equivalence.

Proposition 15. *If $T : \mathcal{A} \longrightarrow \mathcal{B}$ is right exact, then L_0T and T are canonically naturally equivalent. If T is exact then $L_nT \cong 0$ for $n \neq 0$.*

Proof. Let P be a projective resolution of A. Then $P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ is exact, so $TP_1 \longrightarrow TP_0 \longrightarrow TA \longrightarrow 0$ is exact. Hence $H_0(TP) \cong TA$. This isomorphism is readily seen to be natural in A. It is clear that if T is exact all the higher left derived functors are zero. $\square$

Proposition 16. *For a projective object P, $L_nT(P) = 0$ for $n \geq 1$ and $L_0T(P) \cong T(P)$.*

Proof. This follows immediately from the fact that the chain complex P with $P_i = 0$ except for $P_0 = P$ is a projective resolution of P. $\square$

The next result shows how to calculate the values of left derived functors of functors with values in a category of modules from a "truncated" projective resolution.

Proposition 17. *Suppose we have an exact sequence in $\mathcal{A}$*

$$0 \longrightarrow K \xrightarrow{\mu} P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \qquad (4)$$

where $P_0, \ldots, P_n$ are all projective and $n \geq 0$. If $\mathcal{B}$ is a category of modules and $T : \mathcal{A} \longrightarrow \mathcal{B}$ is right exact then there is an exact sequence

$$0 \longrightarrow L_{n+1}T(A) \longrightarrow T(K) \xrightarrow{T(\mu)} T(P_n)$$

Proof. Find a projective P_{n+1} and an epimorphism $e : P_{n+1} \longrightarrow K$ and in the usual way continue this process to produce a projective resolution P whose differential $\partial_{n+1} : P_{n+1} \longrightarrow P_n$ is μe. Since T is right exact, the following diagram is commutative with exact rows

$$\begin{array}{ccccccc}
 & & T(P_{n+2}) & \xrightarrow{T(\partial_{n+2})} & T(P_{n+1}) & \xrightarrow{T(e)} & T(K) & \longrightarrow 0 \\
 & & \downarrow & & \downarrow{\scriptstyle T(\partial_{n+1})} & & \downarrow{\scriptstyle T(\mu)} & \\
0 & \longrightarrow & 0 & \longrightarrow & T(P_n) & = & T(P_n) &
\end{array}$$

The Snake Lemma provides us with an exact sequence

$$T(P_{n+2}) \longrightarrow KerT(\partial_{n+1}) \longrightarrow KerT(\mu) \longrightarrow 0$$

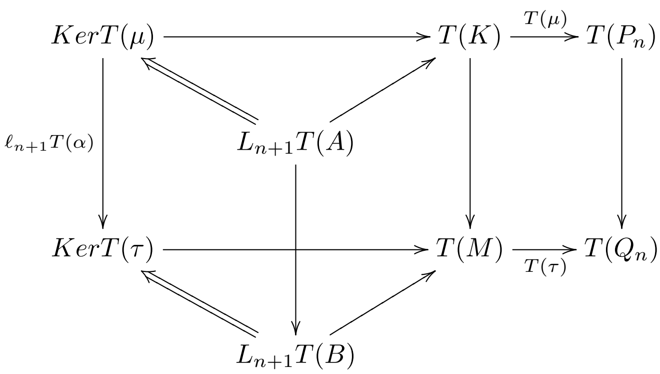
But of course $L_{n+1}T(A) = KerT(\partial_{n+1})/ImT(\partial_{n+2})$ so the modules $KerT(\mu)$ and $L_{n+1}T(A)$ are isomorphic. The morphism $L_{n+1}T(A) \longrightarrow T(K)$ fitting into the exact sequence is actually $x + ImT(\partial_{n+2}) \mapsto T(e)(x)$. $\square$

This idea can be pushed further. Suppose for some fixed $n \geq 0$ we have an exact sequence of the form (4) for every object A of $\mathcal{A}$, which we can extend to a projective resolution. Define the additive functor $\ell_{n+1}T : \mathcal{A} \longrightarrow \mathcal{B}$ as follows. The module $\ell_{n+1}T(A)$ is $KerT(\mu)$. A morphism $\alpha : A \longrightarrow B$ can be lifted to a chain morphism which induces $\alpha' : K \longrightarrow M$ in the following diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & K & \xrightarrow{\mu} & P_n & \longrightarrow & P_{n-1} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & M & \xrightarrow[\tau]{} & Q_n & \longrightarrow & Q_{n-1} & \longrightarrow \cdots \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

Let $\ell_{n+1}T(\alpha) : KerT(\mu) \longrightarrow KerT(\tau)$ be $x \mapsto T(\alpha')(x)$. As we will see in a moment, this is independent of the chain morphism lifting α and defines an additive functor $\ell_{n+1}T$.

If $L_{n+1}T$ is defined using projective resolutions obtained from these exact sequences, then it is not hard to see that there is a commutative diagram



which shows that $\ell_{n+1}T$ is well-defined and is naturally equivalent to $L_{n+1}T$. So provided $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a right exact functor into a category of modules we can calculate the left derived functors $L_1T, L_2T, \ldots$ using finite exact sequences instead of projective resolutions.

Let $\mathcal{A}$ be an abelian category with enough injectives and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a *contravariant* additive functor. Then $\mathcal{A}^{\text{op}}$ is an abelian category with enough projectives, so for $n \geq 0$ we can define the left derived functor L_nT of T^*, which is a contravariant additive functor $L_nT : \mathcal{A} \longrightarrow \mathcal{B}$. Let $\mathcal{I}$ be an assignment of injective resolutions to objects of $\mathcal{A}$. Then this is an assignment of projective resolutions to objects of $\mathcal{A}^{\text{op}}$ and we calculate L_nT as follows: let $I : 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ be the injective resolution of A, and consider the chain complex of objects of $\mathcal{B}$

$$TI : \cdots \longrightarrow T(I^2) \longrightarrow T(I^1) \longrightarrow T(I^0) \longrightarrow 0$$

Then $L_nT(A)$ is the homology object $H_n(TP)$. Given $\alpha : A \longrightarrow A'$ let $\varphi : I \longrightarrow I'$ be the chain morphism inducing α. Then $T(\varphi) : TI' \longrightarrow TI$ is a chain morphism and $L_nT(\alpha) : L_nT(A') \longrightarrow L_nT(A)$ is $H_nT(\varphi)$. We call L_nT the n-th *left derived functors* of T.

4 Injective Resolutions

Throughout we work with an abelian category $\mathcal{A}$ and cochain complexes over $\mathcal{A}$.

Definition 7. A cochain complex C is *positive* if $C^n = 0$ for all $n < 0$. In other words the complex looks like

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow \cdots$$

A positive cochain complex C is *injective* if C^n is injective for all $n \geq 0$. It is called *acyclic* if $H^n(C) = 0$ for $n \geq 1$. A positive cochain complex C is acyclic if and only if the following sequence is exact

$$0 \longrightarrow H^0(C) \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow \cdots$$

An *injective resolution* of an object A is an injective acyclic cochain complex I together with a morphism $A \longrightarrow I^0$ making the following sequence exact

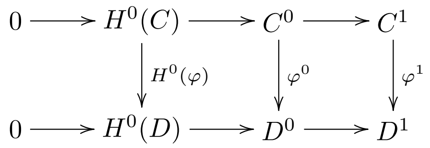
$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow \cdots \tag{5}$$

Equivalently, it is an exact sequence (5) with all the I^i injective.

Lemma 18. *The category* $\mathbf{coCh}^+\mathcal{A}$ *of positive cochain complexes is an abelian subcategory of* $\mathbf{coCh}\mathcal{A}$.

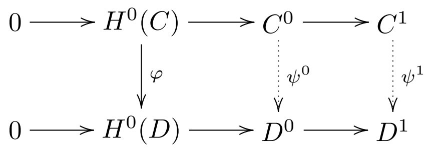
Proof. As before. Once again, we note that a sequence $C' \longrightarrow C \longrightarrow C''$ in $\mathbf{coCh}^+\mathcal{A}$ is exact iff. $C'_n \longrightarrow C_n \longrightarrow C''_n$ is exact in $\mathcal{A}$ for $n \geq 0$. $\square$

If C is a positive cochain complex then the construction of cohomology gives a monomorphism $H^0(C) \longrightarrow C^0$ which is kernel of $C^0 \longrightarrow C^1$. If $\varphi : C \longrightarrow D$ is a morphism of positive cochain complexes then $H^0(\varphi) : H^0(C) \longrightarrow H^0(D)$ is just the morphism induced between the kernels by the following commutative diagram



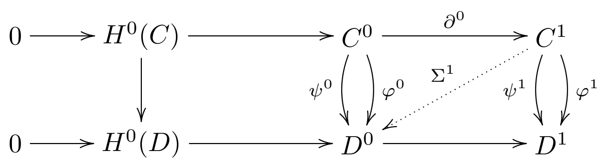
Theorem 19. *Let C, D be positive cochain complexes with C acyclic and D injective. Then the map $\varphi \mapsto H^0(\varphi)$ gives a bijection between homotopy classes of cochain morphisms $C \longrightarrow D$ and morphisms $H^0(C) \longrightarrow H^0(D)$.*

Proof. One could argue that since the dual of an abelian category is abelian, this follows from Theorem 12 by duality (note that now the morphisms go from the acyclic to the injective cochain, whereas before they went from the projective to the acyclic chain). Or one can just copy the proof of Theorem 12. First we show that given any morphism $\varphi : H^0(C) \longrightarrow H^0(D)$ there is a chain morphism inducing φ on cohomology. The chain morphism $\psi : C \longrightarrow D$ is defined recursively. Consider the following diagram, whose top row is exact:

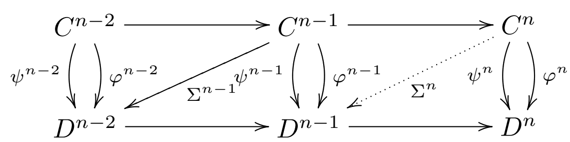


The fact that D^0 is injective means that we can find ψ^0 making the diagram commute. Since the top row is exact, $C^0 \longrightarrow Im(\partial^0)$ is the cokernel of $H^0(C) \longrightarrow C^0$. Since $\partial\psi^0\partial = 0$ we see that $C^0 \longrightarrow D^0 \longrightarrow Im(\partial^0)$ factors through $C^0 \longrightarrow Im(\partial^0)$ and using the fact that D^1 is injective we get ψ^1 making the diagram commute. Proceeding in this way we construct the cochain morphism ψ, which clearly induces φ on cohomology.

Now suppose φ, ψ are two chain morphisms inducing the same morphism $H^0(C) \longrightarrow H^0(D)$. Recursively we define a homotopy $\Sigma : \psi \longrightarrow \varphi$. First consider the diagram



Since $\varphi^0 - \psi^0$ gives zero on composition with $H^0(C) \longrightarrow C^0$ we see that $\varphi^0 - \psi^0$ factors through $C^0 \longrightarrow Im(\partial^0)$. Since D^0 is injective we obtain Σ^1 with $\varphi^0 - \psi^0 = \Sigma^1\partial^0$. Suppose $n \geq 1$ and that $\Sigma^0, \ldots, \Sigma^{n-1}$ have been defined in such a way that $\varphi^r - \psi^r = \Sigma^{r+1}\partial + \partial\Sigma^r$ for all $r \leq n-1$ (with Σ^0 being understood as 0). Consider the diagram



We have

$$(-\partial\Sigma^{n-1} + \varphi^{n-1} - \psi^{n-1})\partial = \partial(\varphi^{n-2} - \psi^{n-2} - \Sigma^{n-1}\partial) = \partial\partial\Sigma^{n-2} = 0$$

Once again we factor through the morphism $C^{n-1} \longrightarrow Im(\partial^{n-1})$ and use injectivity of D^{n-1} to obtain Σ^n with $\Sigma^n\partial + \partial\Sigma^{n-1} = \varphi^{n-1} - \psi^{n-1}$. Hence $\psi \simeq \varphi$, as claimed. $\square$

Corollary 20. *Suppose we are given injective resolutions of objects A, B in $\mathcal{A}$*

$$\begin{aligned} I: &\quad 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \\ J: &\quad 0 \longrightarrow B \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \end{aligned}$$

Then there is a bijection between homotopy classes of cochain morphisms $I \longrightarrow J$ and morphisms $A \longrightarrow B$. Given a cochain morphism $\varphi : I \longrightarrow J$ the corresponding morphism $\alpha : A \longrightarrow B$ is unique making the following diagram commute

$$\begin{array}{ccc} A & \longrightarrow & I_0 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\varphi_0} \\ B & \longrightarrow & J_0 \end{array}$$

5 Right Derived Functors

Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ *has enough injectives* if for every object A there is an injective object I_0 and a monomorphism $A \longrightarrow I_0$. If $\mathcal{A}$ has enough injectives then it is clear that we can construct an injective resolution for any object A by taking the cokernel $I_0 \longrightarrow C$, finding another monomorphism $C \longrightarrow I_1$ with I_1 injective, and repeating the process. Throughout this section $\mathcal{A}$ is an abelian category with enough injectives, and $\mathcal{B}$ is any abelian category.

Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive covariant functor. Suppose we have an injective resolution of A

$$I: \quad 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

This gives rise to a cochain complex of objects of $\mathcal{B}$

$$TI: \quad 0 \longrightarrow T(I^0) \longrightarrow T(I^1) \longrightarrow \cdots \longrightarrow T(I^n) \longrightarrow \cdots$$

We define $R^n_I T(A) = H^n(TI)$ for $n \geq 0$. Let B be another object with injective resolution J and let $\alpha : A \longrightarrow B$ be a morphism. By Corollary 20 there is a chain morphism $\varphi : I \longrightarrow J$ making a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi^0} & & \downarrow{\scriptstyle\varphi^1} & & \\ 0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \end{array}$$

For $n \geq 0$ there is a morphism $H^n(T(\varphi)) : R^n_I T(A) \longrightarrow R^n_J T(B)$, which depends only on α and not on the particular φ we use in the construction (since all candidate φ are homotopic).

Let $\mathcal{I}$ be an assignment of an injective resolution to every object of $\mathcal{A}$ (we need a strong axiom of choice to show that such an assignment exists) and for fixed $n \geq 0$ let $R^n T(A)$ denote the object calculated with the injective resolution chosen by $\mathcal{I}$. Let $R^n T(\alpha) : R^n T(A) \longrightarrow R^n T(B)$ denote the morphism induced by $\alpha : A \longrightarrow B$. Then it is easy to check that $R^n T : \mathcal{A} \longrightarrow \mathcal{B}$ is a covariant additive functor. If necessary we denote this functor by $R^n_{\mathcal{I}} T$ to indicate its dependence on $\mathcal{I}$.

Proposition 21. *Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant additive functor. If $\mathcal{I}$ and $\mathcal{J}$ are two assignments of injective resolutions to objects of $\mathcal{A}$, then for $n \geq 0$ there is a canonical natural equivalence $R^n_{\mathcal{I}} T \cong R^n_{\mathcal{J}} T$.*

Proof. For an object A the morphism $1_A : A \longrightarrow A$ gives rise to cochain morphisms $\xi : I \longrightarrow J$ and $\eta : J \longrightarrow I$ of the corresponding injective resolutions. Moreover, $\xi\eta$ and $\eta\xi$ induce the identity on A and so by Corollary 20 they must be homotopic to the identity cochain morphisms. Applying T and passing to cohomology we obtain an isomorphism $R^n_{\mathcal{I}} T(A) \cong R^n_{\mathcal{J}} T(A)$ which is easily checked to be natural in A. $\square$

Definition 8. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant additive functor. For $n \geq 0$ the covariant additive functor $R^nT : \mathcal{A} \longrightarrow \mathcal{B}$ is called the n-th *right derived functor* of T. This functor depends on the injective resolutions chosen, but is independent of these choices up to canonical natural equivalence.

Lemma 22. *If $T : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact, then R^0T and T are canonically naturally equivalent. If T is exact then $R^nT \cong 0$ for $n \neq 0$.*

Proof. Let I be an injective resolution of A. Then $0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1$ is exact, so $0 \longrightarrow TA \longrightarrow TI^0 \longrightarrow TI^1$ is exact. Hence $H^0(TA) \cong TA$. This isomorphism is readily seen to be natural in A. It is clear that if T is exact all the higher right derived functors are zero. □

Proposition 23. *For an injective object I, $R^nT(I) = 0$ for $n \geq 1$ and $R^0T(I) \cong T(I)$.*

Proof. This follows immediately from the fact that the cochain complex I with $I_i = 0$ except for $I_0 = I$ is an injective resolution of I. □

Proposition 24. *Suppose we have an exact sequence in $\mathcal{A}$*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow I^n \xrightarrow{\mu} C \longrightarrow 0 \tag{6}$$

where $I^0, \ldots, I^n$ are all injective and $n \geq 0$. If $\mathcal{B}$ is a category of modules and $T : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact then there is an exact sequence

$$T(I^n) \xrightarrow{T(\mu)} T(C) \longrightarrow R^{n+1}T(A) \longrightarrow 0$$

Proof. Find an injective I^{n+1} and a monomorphism $e : C \longrightarrow I^{n+1}$ and in the usual way continue this process to produce an injective resolution I whose differential $\partial^n : I^n \longrightarrow I^{n+1}$ is $e\mu$. Since T is left exact, the following diagram is commutative with exact rows

$$\begin{array}{ccccccc}
 & & T(I^n) & \Longrightarrow & T(I^n) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle T(\mu)}\downarrow & & {\scriptstyle T(\partial^n)}\downarrow & & \downarrow & & \\
0 & \longrightarrow & T(C) & \xrightarrow[T(e)]{} & T(I^{n+1}) & \xrightarrow[T(\partial^{n+1})]{} & T(I^{n+2}) & &
\end{array}$$

The Snake Lemma provides us with an exact sequence

$$0 \longrightarrow CokerT(\mu) \longrightarrow CokerT(\partial^n) \longrightarrow T(I^{n+2})$$

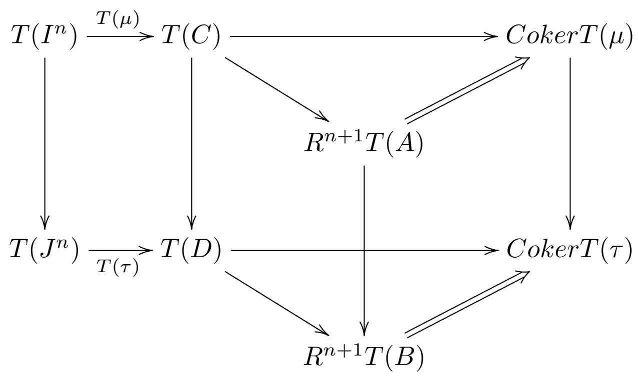
But of course $R^{n+1}T(A) = KerT(\partial^{n+1})/ImT(\partial^n)$ so the modules $CokerT(\mu)$ and $R^{n+1}T(A)$ are isomorphic. The morphism $T(C) \longrightarrow R^{n+1}T(A)$ fitting into the exact sequence is actually $x \mapsto T(e)(x) + ImT(\partial^n)$. □

Suppose for some fixed $n \geq 0$ we have an exact sequence of the form (6) for every object A of $\mathcal{A}$, which we can extend to an injective resolution. Define the additive functor $r^{n+1}T : \mathcal{A} \longrightarrow \mathcal{B}$ as follows. The module $r^{n+1}T(A)$ is $CokerT(\mu)$. A morphism $\alpha : A \longrightarrow B$ can be lifted to a cochain morphism which induces $\alpha' : C \longrightarrow D$ in the following diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & \cdots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \xrightarrow{\mu} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & {\scriptstyle\alpha'}\downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & \cdots & \longrightarrow & J^{n-1} & \longrightarrow & J^n & \xrightarrow[\tau]{} & D & \longrightarrow & 0
\end{array}$$

Let $r^{n+1}T(\alpha) : CokerT(\mu) \longrightarrow CokerT(\tau)$ be $x + ImT(\mu) \mapsto T(\alpha')(x) + ImT(\tau)$. As we will see in a moment, this is independent of the cochain morphism lifting α and defines an additive functor $r^{n+1}T$.

If $R^{n+1}T$ is defined using injective resolutions obtained from these exact sequences, then it is not hard to see that there is a commutative diagram



which shows that $r^{n+1}T$ is well-defined and is naturally equivalent to $R^{n+1}T$. So provided $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor into a category of modules we can calculate the right derived functors $R^1T, R^2T, \ldots$ using finite exact sequences instead of injective resolutions.

Let $\mathcal{A}$ be an abelian category with enough projectives and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a *contravariant* additive functor. Then $\mathcal{A}^{\text{op}}$ is an abelian category with enough injectives, so for $n \geq 0$ we can define the right derived functor R^nT of T^*, which is a contravariant additive functor $R^nT : \mathcal{A} \longrightarrow \mathcal{B}$. Let $\mathcal{P}$ be an assignment of projective resolutions to objects of $\mathcal{A}$. Then this is an assignment of injective resolutions to objects of $\mathcal{A}^{\text{op}}$ and we calculate R^nT as follows: let $P : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ be the projective resolution of A, and consider the cochain complex of objects of $\mathcal{B}$

$$TI : 0 \longrightarrow T(P_0) \longrightarrow T(P_1) \longrightarrow T(P_2) \longrightarrow \cdots$$

Then $R^nT(A)$ is the cohomology object $H^n(TI)$. Given $\alpha : A \longrightarrow A'$ let $\varphi : P \longrightarrow P'$ be the cochain morphism inducing α. Then $T(\varphi) : TP' \longrightarrow TP$ is a cochain morphism and $R^nT(\alpha) : R^nT(A') \longrightarrow R^nT(A)$ is $H^nT(\varphi)$. We call R^nT the n-th *right derived functor* of T.

6 The Long Exact (Co)Homology Sequence

Throughout this section $\mathcal{A}$ is an abelian category.

Lemma 25. *Let C be a chain complex in $\mathcal{A}$. For $n \in \mathbb{Z}$ there is a unique morphism $\widetilde{\partial_n} : Coker\partial_{n+1} \longrightarrow Ker\partial_{n-1}$ making the following diagram commute:*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \longrightarrow \cdots \\ & & & \downarrow & & \uparrow & & & \\ & & & Coker\partial_{n+1} & \longrightarrow & Ker\partial_{n-1} & & & \end{array}$$

Moreover this morphism fits into an exact sequence

$$0 \longrightarrow H_n(C) \longrightarrow Coker\partial_{n+1} \longrightarrow Ker\partial_{n-1} \longrightarrow H_{n-1}(C) \longrightarrow 0 \qquad (7)$$

Proof. The morphism $C_n \longrightarrow Im\partial_n$ is a quotient $C_n/Ker\partial_n$ and $Coker\partial_{n+1}$ is a quotient $C_n/Im\partial_{n+1}$ so the inclusion $Im\partial_{n+1} \leq Ker\partial_n$ gives rise to an epimorphism $Coker\partial_{n+1} \longrightarrow Im\partial_n$. Composed with $Im\partial_n \leq Ker\partial_{n-1}$ this gives us the desired morphism.

Consider the following commutative diagram, in which the rows and first two columns are exact

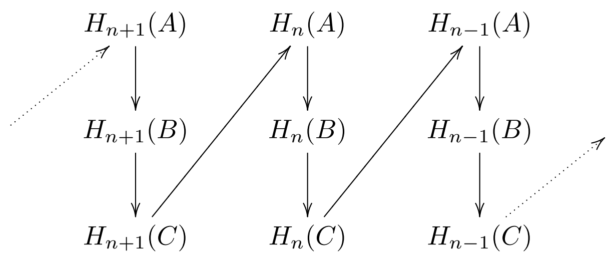
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Im\partial_{n+1} & \longrightarrow & Ker\partial_n & \longrightarrow & H_n(C) & \longrightarrow & 0 \\
 & & \Downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Im\partial_{n+1} & \longrightarrow & C_n & \longrightarrow & Coker\partial_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & Im\partial_n & \Longrightarrow & Im\partial_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The Nine Lemma shows that the third column is also exact. Composing with the monomorphism $Im\partial_n \longrightarrow Ker\partial_{n-1}$ we see that (7) is exact at $H_n(C)$ and $Coker\partial_{n+1}$. Since $Ker\partial_{n-1} \longrightarrow H_{n-1}(C)$ is a cokernel of $Im\partial_n \longrightarrow Ker\partial_{n-1}$ composing with the epimorphism $Coker\partial_{n+1} \longrightarrow Im\partial_n$ shows that the rest of the sequence is exact. □

Theorem 26. *Suppose we are given a short exact sequence of chain complexes in* $\mathcal{A}$

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

Then there is a canonical sequence of morphisms $\omega_n : H_n(C) \longrightarrow H_{n-1}(A)$ *for* $n \in \mathbb{Z}$ *called the* connecting morphisms *with the property that the following is a long exact sequence*



Proof. For $n \in \mathbb{Z}$ exactness of the sequences $0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0$ gives rise to exact sequences $0 \longrightarrow Ker\partial_n \longrightarrow Ker\partial_n \longrightarrow Ker\partial_n$ and $Coker\partial_n \longrightarrow Coker\partial_n \longrightarrow Coker\partial_n \longrightarrow 0$ (the first kernel being of $\partial_n : A_n \longrightarrow A_{n-1}$, the second of $\partial_n : B_n \longrightarrow B_{n-1}$ and so on), since taking kernels is left exact and taking cokernels is right exact. So for any $n \in \mathbb{Z}$ we get a commutative diagram where the middle two rows and all columns are exact

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & H_n(A) & \xrightarrow{H_n(\varphi)} & H_n(B) & \xrightarrow{H_n(\psi)} & H_n(C) & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & Coker\partial_{n+1} & \longrightarrow & Coker\partial_{n+1} & \longrightarrow & Coker\partial_{n+1} & \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \widetilde{\partial}_n} & & \downarrow{\scriptstyle \widetilde{\partial}_n} & & \downarrow{\scriptstyle \widetilde{\partial}_n} & \\
0 & \longrightarrow & Ker\partial_{n-1} & \longrightarrow & Ker\partial_{n-1} & \longrightarrow & Ker\partial_{n-1} & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & H_{n-1}(A) & \xrightarrow[H_{n-1}(\varphi)]{} & H_{n-1}(B) & \xrightarrow[H_{n-1}(\psi)]{} & H_{n-1}(C) & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}
\tag{8}
$$

Then by the Snake Lemma (which is valid in $\mathcal{A}$ by our Diagram Chasing notes) there is a canonical morphism $\omega_n : H_n(C) \longrightarrow H_{n-1}(A)$ with the property that the following sequence is exact

$$
H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\omega_n} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C)
$$

Piecing these exact sequences together gives the result. $\square$

The connecting morphism is canonical in the following sense: once we choose our canonical kernels, cokernels and images for all the morphisms in the category, then (8) is canonically constructed since it involves no choices (even the $\widetilde{\partial}$ morphisms are unique with a certain property) and there is a canonical choice for ω_n for each n (it has some unique property with respect to some of the morphisms in (8)). So the connecting morphisms ω_n depend only on the morphisms φ, ψ and the canonical structures on $\mathcal{A}$. If two people choose different canonical structures, then they may find different connecting morphisms (see Section 9).

Proposition 27. *The connecting morphisms are natural with respect to the exact sequence. Given a commutative diagram in* $\mathbf{Ch}\mathcal{A}$ *with exact rows*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow[\varphi']{} & B' & \xrightarrow[\psi']{} & C' & \longrightarrow & 0
\end{array}
$$

Then for every $n \in \mathbb{Z}$ *there is a commutative diagram*

$$
\begin{array}{ccccccccccc}
H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \xrightarrow{\omega} & H_{n-1}(A) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_n(A') & \longrightarrow & H_n(B') & \longrightarrow & H_n(C') & \xrightarrow[\omega']{} & H_{n-1}(A') & \longrightarrow & H_{n-1}(B') & \longrightarrow & H_{n-1}(C')
\end{array}
$$

where ω, ω' *are the respective connecting morphisms.*

Proof. We need only check commutativity of the middle square. Using α, β, γ we induce morphisms between the cokernels and kernels to obtain a commutative diagram with exact rows of the type given in our Diagram chasing result about naturality of the connecting morphism. This diagram is a “morphism” from the middle horizontal band of the diagrams (8) for A, B, C to the corresponding piece of the diagram for A', B', C'. Since the connecting morphism is natural with respect to such morphisms, it follows that the required diagram commutes (one checks that the morphisms $H_n(-)$ in our diagram above agree with the induced morphisms we get from the morphism of diagrams). □

As a Corollary we see that the connecting morphism is in a sense independent of the middle term of the sequence

Corollary 28. *Suppose we have a commutative diagram of chain sequences in $\mathcal{A}$ with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & {\scriptstyle 1_A}\Big\Downarrow & & {\scriptstyle \gamma}\Big\Downarrow & & {\scriptstyle 1_C}\Big\Downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow[\varphi']{} & B' & \xrightarrow[\psi']{} & C' & \longrightarrow & 0 \end{array} \tag{9}$$

Then both sequences have the same connecting morphisms $H_n(C) \longrightarrow H_{n-1}(A)$ for $n \in \mathbb{Z}$.

The same proofs (just replacing subscripts by superscripts and massaging indices, which don’t play any important role in the proof) work for cochains.

Theorem 29. *Suppose we are given a short exact sequence of cochain complexes in $\mathcal{A}$*

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

Then there is a canonical sequence of morphisms $\omega^n : H^n(C) \longrightarrow H^{n+1}(A)$ for $n \in \mathbb{Z}$ called the connecting morphisms *with the property that the following is a long exact sequence*

$$\begin{array}{ccccccc} & \nearrow & H^{n-1}(A) & & H^n(A) & & H^{n+1}(A) & \\ & & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \\ & & H^{n-1}(B) & & H^n(B) & & H^{n+1}(B) & \\ & & \downarrow & & \downarrow & & \downarrow & \nearrow \\ & & H^{n-1}(C) & & H^n(C) & & H^{n+1}(C) & \end{array}$$

Once again the connecting morphisms are canonical in the sense that they depend only on the morphisms φ, ψ and the canonical structures on $\mathcal{A}$.

Proposition 30. *The connecting morphisms are natural with respect to the exact sequence. Given a commutative diagram in* $\mathbf{coCh}\mathcal{A}$ *with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\Big\downarrow & & {\scriptstyle \beta}\Big\downarrow & & {\scriptstyle \gamma}\Big\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow[\varphi']{} & B' & \xrightarrow[\psi']{} & C' & \longrightarrow & 0 \end{array}$$

Then for every $n \in \mathbb{Z}$ there is a commutative diagram

$$\begin{array}{ccccccccccc} H^n(A) & \longrightarrow & H^n(B) & \longrightarrow & H^n(C) & \xrightarrow{\omega} & H^{n+1}(A) & \longrightarrow & H^{n+1}(B) & \longrightarrow & H^{n+1}(C) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ H^n(A') & \longrightarrow & H^n(B') & \longrightarrow & H^n(C') & \xrightarrow[\omega']{} & H^{n+1}(A') & \longrightarrow & H^{n+1}(B') & \longrightarrow & H^{n+1}(C') \end{array}$$

where ω, ω' are the respective connecting morphisms.

Corollary 31. *Suppose we have a commutative diagram of cochain sequences in $\mathcal{A}$ with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & {\scriptstyle 1_A}\Big\Downarrow & & {\scriptstyle \gamma}\Big\Downarrow & & {\scriptstyle 1_C}\Big\Downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow[\varphi']{} & B' & \xrightarrow[\psi']{} & C' & \longrightarrow & 0 \end{array} \tag{10}$$

Then both sequences have the same connecting morphisms $H^n(C) \longrightarrow H^{n+1}(A)$ for $n \in \mathbb{Z}$.

7 Two Long Exact Sequences

7.1 Left Derived Functors

Throughout this section we work in an abelian category $\mathcal{A}$. For an object A a *projective presentation* of A is just an epimorphism $P \longrightarrow A$ where P is projective. If we say $\varepsilon : P \longrightarrow A$ is a projective resolution of A we mean that the chain complex P together with $\varepsilon : P_0 \longrightarrow A$ is a projective resolution.

Proposition 32. *Given a short exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

and projective presentations $\varepsilon' : P' \longrightarrow A'$ and $\varepsilon'' : P'' \longrightarrow A''$ there is a projective presentation $\varepsilon : P \longrightarrow A$ fitting into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \xrightarrow{\iota} & P & \xrightarrow{\pi} & P'' & \longrightarrow & 0 \\ & & {\scriptstyle \varepsilon'}\Big\downarrow & & {\scriptstyle \varepsilon}\Big\downarrow & & {\scriptstyle \varepsilon''}\Big\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow[\varphi]{} & A & \xrightarrow[\psi]{} & A'' & \longrightarrow & 0 \end{array}$$

Proof. Let $P = P' \oplus P''$ with ι, π the obvious morphisms. We define ε by components. The first component is $\varphi\varepsilon'$ and for the second we use the fact that P'' is projective to construct $\xi : P'' \longrightarrow A$ such that $\psi\xi = \varepsilon''$. Using the coproduct, it is clear that the diagram commutes. It follows from the 5-Lemma for abelian categories (see our Diagram Chasing notes) that ε is an epimorphism. □

Corollary 33. *Let $\mathcal{A}$ be an abelian category with enough projectives. Given a short exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

and projective resolutions $\varepsilon' : P' \longrightarrow A'$ and $\varepsilon'' : P'' \longrightarrow A''$ there is a projective resolution $\varepsilon : P \longrightarrow A$ and an exact sequence of chain complexes

$$0 \longrightarrow P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \longrightarrow 0$$

with the property that ι, π induce φ, ψ respectively.

Proof. By the previous result we can construct a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_0 & \longrightarrow & P_0 & \longrightarrow & P''_0 & \longrightarrow & 0 \\ & & {\scriptstyle \varepsilon'}\Big\downarrow & & {\scriptstyle \varepsilon}\Big\downarrow & & {\scriptstyle \varepsilon''}\Big\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow[\varphi]{} & A & \xrightarrow[\psi]{} & A'' & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

By the Snake Lemma the sequence of kernels $0 \longrightarrow Ker\varepsilon' \longrightarrow Ker\varepsilon \longrightarrow Ker\varepsilon'' \longrightarrow 0$ is exact. Repeating this procedure with the sequence of kernels and proceeding recursively, we construct an exact sequence of complexes $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$ where P', P, P'' are projective resolutions of A', A, A'' respectively. Note that $P_n = P'_n \oplus P''_n$ for $n \geq 0$ and ι_n, π_n are the injection and projection respectively, but the complex P is not necessarily the coproduct $P' \oplus P''$ since the differentials may not be the coproduct differentials. □

The resolution P is not unique, since its construction involved a lot of choices. But we can still say a few things about any resolution produced by the Corollary. There is a morphism $\sigma : P''_0 \longrightarrow A$ such that $\varepsilon = (\varphi\varepsilon', \sigma)$ and morphisms $\lambda_n : P''_n \longrightarrow P'_{n-1}$ for $n \geq 1$ such that the differential is

$$\partial_n = \begin{pmatrix} \partial'_n & \lambda_n \\ 0 & \partial''_n \end{pmatrix}$$

These morphisms satisfy the following relations:

$$\begin{aligned} \varepsilon'' &= \psi\sigma \\ \varphi\varepsilon'\lambda_1 + \sigma\partial''_1 &= 0 \\ \partial'_n\lambda_{n+1} + \lambda_n\partial''_{n+1} &= 0 \qquad n \geq 1 \end{aligned} \tag{11}$$

Theorem 34. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Suppose we have an exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0 \tag{12}$$

Then there exist canonical connecting morphisms $\omega_n : L_nT(A'') \longrightarrow L_{n-1}T(A')$ for $n \geq 1$ with the property that the following sequence is exact

$$\begin{aligned} \cdots \longrightarrow L_nT(A') \longrightarrow L_nT(A) \longrightarrow L_nT(A'') \xrightarrow{\omega_n} L_{n-1}T(A') \longrightarrow \cdots \\ \cdots \longrightarrow L_1T(A'') \xrightarrow{\omega_1} L_0T(A') \longrightarrow L_0T(A) \longrightarrow L_0T(A'') \longrightarrow 0 \end{aligned} \tag{13}$$

Moreover these connecting morphisms are natural in the exact sequence: if we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{\varphi} & A & \xrightarrow{\psi} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & B'' & \longrightarrow & 0 \end{array} \tag{14}$$

Then the following diagram commutes for all $n \geq 1$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & L_nT(A') & \longrightarrow & L_nT(A) & \longrightarrow & L_nT(A'') & \xrightarrow{\omega_n} & L_{n-1}T(A') & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & L_nT(B') & \longrightarrow & L_nT(B) & \longrightarrow & L_nT(B'') & \xrightarrow[\omega_n]{} & L_{n-1}T(B') & \longrightarrow \cdots \end{array}$$

Proof. Let $\mathcal{P}$ be an assignment of projective resolutions and assume the left derived functors are all calculated with respect to $\mathcal{P}$. Let $\varepsilon' : P' \longrightarrow A', \varepsilon'' : P'' \longrightarrow A''$ be arbitrary projective resolutions (not necessarily from $\mathcal{P}$) and let $\varepsilon : P \longrightarrow A$ be a resolution constructed by the technique of the Corollary. Since $P_n = P'_n \oplus P''_n$ for $n \in \mathbb{Z}$ and T is additive, it follows that the sequence

$$0 \longrightarrow TP' \longrightarrow TP \longrightarrow TP'' \longrightarrow 0 \tag{15}$$

is also short exact. Then Theorem 26 yields morphisms $\omega'_n : H_n(TP'') \longrightarrow H_{n-1}(TP')$ which fit into an exact sequence of the homology objects of (15). As in the proof of Proposition 14 there are canonical isomorphisms $L_nT(A'') \cong H_n(TP'')$ and $L_{n-1}T(A') \cong H_{n-1}(TP')$ for $n \geq 1$, and we define ω_n to be the unique morphism making the following diagram commute

$$\begin{array}{ccc} L_nT(A'') & \xrightarrow{\omega_n} & L_{n-1}T(A') \\ \Downarrow & & \Downarrow \\ H_n(TP'') & \xrightarrow[\omega_n']{} & H_{n-1}(TP') \end{array}$$

Then it is clear that the ω_n make the sequence (13) exact (it is not clear at this point that ω_n is independent of the choice of the resolutions P', P, P''). Next we prove naturality of these connecting morphisms. Suppose we are given projective resolutions $\varepsilon' : P' \longrightarrow A', \varepsilon'' : P'' \longrightarrow A'', \eta' : Q' \longrightarrow B'$ and $\eta'' : Q'' \longrightarrow B''$ and a commutative diagram with exact rows (14). Let $\varepsilon : P \longrightarrow A$ and $\eta : Q \longrightarrow B$ be projective resolutions produced by the Corollary. Let $F' : P' \longrightarrow Q'$ lift f' and $F'' : P'' \longrightarrow Q''$ lift f''. We claim there is a chain morphism $F : P \longrightarrow Q$ lifting f and giving a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F''} & & \\ 0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0 \end{array} \qquad (16)$$

In order to produce F, we will construct for $n \geq 0$ morphisms $\gamma_n : P_n'' \longrightarrow Q_n'$ (not a chain morphism) and then set

$$F_n = \begin{pmatrix} F_n' & \gamma_n \\ 0 & F_n'' \end{pmatrix} : P_n' \oplus P_n'' \longrightarrow Q_n' \oplus Q_n''$$

For $n \geq 0$ we let $u_n' : P_n' \longrightarrow P_n, p_n' : P_n \longrightarrow P_n'$ and $u_n'' : P_n'' \longrightarrow P_n, p_n'' : P_n \longrightarrow P_n''$ be the respective injections and projections, and similarly we use the notation v_n', q_n', v_n'', q_n'' for the product Q_n. We know there are morphisms $\sigma^P : P_0'' \longrightarrow A, \sigma^Q : Q_0'' \longrightarrow B$ and $\lambda_n : P_n'' \longrightarrow P_{n-1}', \mu_n : Q_n'' \longrightarrow Q_{n-1}'$ which satisfy the following relations for $n \geq 1$

$$\partial_n^P = \begin{pmatrix} \partial_n' & \lambda_n \\ 0 & \partial_n'' \end{pmatrix} \qquad \partial_n^Q = \begin{pmatrix} \partial_n' & \mu_n \\ 0 & \partial_n'' \end{pmatrix}$$

$$\varepsilon'' = \psi\sigma^P \qquad \eta'' = \beta\sigma^Q$$

$$\varphi\varepsilon'\lambda_1 + \sigma^P\partial_1'' = 0 \qquad \alpha\eta'\mu_1 + \sigma^Q\partial_1'' = 0$$

$$\partial_n'\lambda_{n+1} + \lambda_n\partial_{n+1}'' = 0 \qquad \partial_n'\mu_{n+1} + \mu_n\partial_{n+1}'' = 0$$

Also $\varepsilon = (\varphi\varepsilon', \sigma^P), \eta = (\alpha\eta', \sigma^Q)$ so $\sigma^P = \varepsilon u_0'', \sigma^Q = \eta v_0''$. Suppose that F is a lifting of f, so the map $\eta F_0 - f\varepsilon$ from $P_0 = P_0' \oplus P_0''$ is zero. Writing out the matrices we see that we must have $\alpha\eta'\gamma_0 = f\sigma^P - \sigma^Q F_0''$.

Assuming nothing about F we use the above to motivate the following definition of γ_0: one checks easily that $\beta(f\sigma^P - \sigma^Q F_0'') = 0$ so there is a unique morphism $\tau : P_0'' \longrightarrow B'$ with $\alpha\tau = f\sigma^P - \sigma^Q F_0''$. Use projectivity of P_0'' to lift τ to a morphism $\gamma_0 : P_0'' \longrightarrow Q_0'$ with the property that $\alpha\eta'\gamma_0 = f\sigma^P - \sigma^Q F_0''$. So we have constructed γ_0 with $\eta F_0 = f\varepsilon$.

For F to be a chain morphism we must have $d^Q F = F d^P$. Expanding this out shows that we have to construct morphisms $\gamma_n : P_n'' \longrightarrow Q_n'$ for $n \geq 1$ with the property that $\partial_n'\gamma_n = g_n$ where

$$g_n = \gamma_{n-1}\partial_n'' - \mu_n F_n'' + F_{n-1}'\lambda_n$$

The morphism $g_1 : P_1'' \longrightarrow Q_0'$ satisfies $\eta' g_1 = 0$, since

$$\begin{aligned} \alpha\eta' g_1 &= \alpha\eta'\gamma_0\partial_1'' + \alpha\eta' F_0'\lambda_1 - \alpha\eta'\mu_1 F_1'' \\ &= (f\sigma^P - \sigma^Q F_0'')\partial_1'' + \alpha\eta' F_0'\lambda_1 - \alpha\eta'\mu_1 F_1'' \\ &= f\sigma^P\partial_1'' - \sigma^Q F_0''\partial_1'' + \alpha\eta' F_0'\lambda_1 - \alpha\eta'\mu_1 F_1'' \end{aligned}$$

But $\sigma^Q F_0'' \partial_1'' = \sigma^Q \partial_1'' F_1'' = -\alpha\eta'\mu_1 F_1''$ and $f\sigma^P \partial_1'' = -f\varphi\varepsilon'\lambda_1 = -\alpha\eta' F_0'\lambda_1$ so it is clear that $\alpha\eta' g_1 = 0$ and therefore $\eta' g_1 = 0$. It follows that g_1 factors through $Im(Q_1'' \longrightarrow Q_0'')$ and we can use projectivity of P_1'' to lift to a morphism $\gamma_1 : P_1'' \longrightarrow Q_0''$ with $\partial_1'\gamma_1 = g_1$, as required. Given that we have constructed γ_n with $\partial_n'\gamma_n = g_n$ it is not hard to show that $\partial_{n+1}' g_{n+1} = 0$, so we can let γ_n be any lift of the factorisation of g_{n+1} through $Ker\partial_{n+1}' = Im\partial_{n+2}'$. We have constructed γ_n for all n, and it is clear that the morphism F thus defined has the required properties.

Using this fact and Proposition 27 in the case where both rows of (14) are the same and the vertical morphisms are identities, we see that the morphism $\omega_n : L_nT(A'') \longrightarrow L_{n-1}T(A')$ doesn't depend on the choices leading to the construction of the exact sequence $0 \longrightarrow TP' \longrightarrow TP \longrightarrow TP'' \longrightarrow 0$. That is, the connecting morphism depends only on the morphisms ψ, φ, the assignment of resolutions $\mathcal{P}$ and the canonical structures on $\mathcal{B}$. The naturality of the connecting morphisms ω_n for the left derived functors follows from the naturality of the connecting morphism for homology and the diagram (16). □

Corollary 35. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Then the functor $L_0T : \mathcal{A} \longrightarrow \mathcal{B}$ is right exact.*

The connecting morphisms are also independent of the choice of resolutions $\mathcal{P}$, in the following sense: if $\mathcal{Q}$ is another assignment of resolutions then, with the vertical isomorphisms canonical, there is a commutative diagram for all $n \geq 1$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & L_n^{\mathcal{P}}T(A') & \longrightarrow & L_n^{\mathcal{P}}T(A) & \longrightarrow & L_n^{\mathcal{P}}T(A'') & \xrightarrow{\omega_n^{\mathcal{P}}} & L_{n-1}^{\mathcal{P}}T(A') & \longrightarrow \cdots \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\ \cdots \longrightarrow & L_n^{\mathcal{Q}}T(A') & \longrightarrow & L_n^{\mathcal{Q}}T(A) & \longrightarrow & L_n^{\mathcal{Q}}T(A'') & \xrightarrow[\omega_n^{\mathcal{Q}}]{} & L_{n-1}^{\mathcal{Q}}T(A') & \longrightarrow \cdots \end{array}$$

We have proved naturality of the sequence (13) in the objects A. It is also natural in the functor.

Definition 9. Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $\tau : T \longrightarrow T'$ a natural transformation of additive covariant functors $T, T' : \mathcal{A} \longrightarrow \mathcal{B}$. Then for any chain complex X there is a chain morphism $\tau_X : TX \to T'X$ defined by $(\tau_X)_n = \tau_{X_n}$. In particular if $\mathcal{P}$ is an assignment of projective resolutions and if P is the resolution of A then we obtain a chain map $\tau_P : TP \longrightarrow T'P$. Taking homology gives a natural transformation of the left derived functors $L_n\tau : L_nT \longrightarrow L_nT'$ with $(L_n\tau)_A = H_n(\tau_P)$ for $n \geq 0$. Notice that $L_n(\tau\rho) = (L_n\tau)(L_n\rho)$, $L_n(\tau + \rho) = L_n(\tau) + L_n(\rho)$ and $L_n(1_T) = 1_{L_nT}$.

Lemma 36. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category, and $T, T' : \mathcal{A} \longrightarrow \mathcal{B}$ right exact functors. Then the following diagram commutes for any natural transformation $\tau : T \longrightarrow T'$*

$$\begin{array}{ccc} T & \xrightarrow{\tau} & T' \\ \Downarrow & & \Downarrow \\ L_0T & \xrightarrow[L_0\tau]{} & L_0T' \end{array}$$

Proposition 37. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category, and $\tau : T \longrightarrow T'$ a natural transformation of additive functors $T, T' : \mathcal{A} \longrightarrow \mathcal{B}$. Suppose we have an exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0 \tag{17}$$

Then the following diagram is commutative, where the connecting morphisms are canonical

$$\begin{array}{ccccccccc} \cdots \longrightarrow & L_nT(A') & \longrightarrow & L_nT(A) & \longrightarrow & L_nT(A'') & \xrightarrow{\omega^n} & L_{n-1}T(A') & \longrightarrow \cdots \\ & \downarrow{\scriptstyle (L_n\tau)_{A'}} & & \downarrow{\scriptstyle (L_n\tau)_A} & & \downarrow{\scriptstyle (L_n\tau)_{A''}} & & \downarrow{\scriptstyle (L_n\tau)_{A'}} & \\ \cdots \longrightarrow & L_nT'(A') & \longrightarrow & L_nT'(A) & \longrightarrow & L_nT'(A'') & \xrightarrow[\omega^n]{} & L_{n-1}T'(A') & \longrightarrow \cdots \end{array}$$

Proof. Let $\mathcal{P}$ be an assignment of projective resolutions with respect to which all derived functors are calculated. Let P', P'' be the assigned resolutions of A', A'' respectively and construct in the usual way a resolution P fitting into an exact sequence $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TP' & \longrightarrow & TP & \longrightarrow & TP'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_{P'}} & & \downarrow{\scriptstyle \tau_P} & & \downarrow{\scriptstyle \tau_{P''}} & & \\ 0 & \longrightarrow & T'P' & \longrightarrow & T'P & \longrightarrow & T'P'' & \longrightarrow & 0 \end{array}$$

Since $(L_n\tau)_{A''} = H_n(\tau_{P''})$ and $(L_n\tau)_{A'} = H_n(\tau_{P'})$ the commutativity of the required diagram follows immediately from the construction of the connecting morphisms in Theorem 34 and naturality of the connecting morphism of chain sequences in $\mathcal{B}$ with respect to diagrams like the one above. $\square$

Definition 10. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A sequence $T' \longrightarrow T \longrightarrow T''$ of additive functors $\mathcal{A} \longrightarrow \mathcal{B}$ and natural transformations is called *exact on projectives* if for every projective object P the sequence $0 \longrightarrow T'(P) \longrightarrow T(P) \longrightarrow T''(P) \longrightarrow 0$ is exact.

Proposition 38. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category, and suppose there is a sequence of additive functors $\mathcal{A} \longrightarrow \mathcal{B}$ which is exact on projectives*

$$T' \xrightarrow{\ \tau\ } T \xrightarrow{\ \rho\ } T''$$

Then for every object A there are canonical connecting morphisms $\omega_n : L_nT''(A) \longrightarrow L_{n-1}T'(A)$ for $n \geq 1$ with the property that the following sequence is exact

$$\begin{array}{c} \cdots \longrightarrow L_nT'(A) \xrightarrow{(L_n\tau)_A} L_nT(A) \xrightarrow{(L_n\rho)_A} L_nT''(A) \xrightarrow{\omega_n} L_{n-1}T'(A) \longrightarrow \cdots \\ \cdots \longrightarrow L_1T''(A) \xrightarrow{\omega_1} L_0T'(A) \xrightarrow{(L_0\tau)_A} L_0T(A) \xrightarrow{(L_0\rho)_A} L_0T''(A) \longrightarrow 0 \end{array} \tag{18}$$

This sequence is natural in both A and the exact sequence. For any morphism $\alpha : A \longrightarrow B$ the following diagram is commutative

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_nT'(A) & \longrightarrow & L_nT(A) & \longrightarrow & L_nT''(A) & \xrightarrow{\omega_n} & L_{n-1}T'(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & L_nT'(B) & \longrightarrow & L_nT(B) & \longrightarrow & L_nT''(B) & \xrightarrow[\omega_n]{} & L_{n-1}T'(B) & \longrightarrow & \cdots \end{array} \tag{19}$$

and for any commutative diagram of additive functors with rows exact on projectives

$$\begin{array}{ccccc} T' & \xrightarrow{\ \tau\ } & T & \xrightarrow{\ \rho\ } & T'' \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} \\ S' & \xrightarrow[\sigma]{} & S & \xrightarrow[\theta]{} & S'' \end{array} \tag{20}$$

the following diagram is commutative for any object A

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_nT'(A) & \longrightarrow & L_nT(A) & \longrightarrow & L_nT''(A) & \xrightarrow{\omega_n} & L_{n-1}T'(A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle (L_n\varphi')_A} & & \downarrow{\scriptstyle (L_n\varphi)_A} & & \downarrow{\scriptstyle (L_n\varphi'')_A} & & \downarrow{\scriptstyle (L_{n-1}\varphi')_A} & & \\ \cdots & \longrightarrow & L_nS'(A) & \longrightarrow & L_nS(A) & \longrightarrow & L_nS''(A) & \xrightarrow[\omega_n]{} & L_{n-1}S'(A) & \longrightarrow & \cdots \end{array} \tag{21}$$

Proof. Let $\mathcal{P}$ be an assignment of projective resolutions with respect to which all derived functors are calculated. Let P be the assigned resolution of A. Since the sequence of functors is exact on projectives the following sequence is exact

$$0 \longrightarrow T'P \longrightarrow TP \longrightarrow T''P \longrightarrow 0$$

The long exact homology sequence then yields the connecting morphisms ω_n and exactness of (18). Given a morphism $\alpha : A \longrightarrow B$ let Q be the resolution of B and lift α to a morphism of chain complexes $\varphi : P \longrightarrow Q$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T'P & \longrightarrow & TP & \longrightarrow & T''P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T'\varphi} & & \downarrow{\scriptstyle T\varphi} & & \downarrow{\scriptstyle T''\varphi} & & \\ 0 & \longrightarrow & T'Q & \longrightarrow & TQ & \longrightarrow & T''Q & \longrightarrow & 0 \end{array}$$

Naturality of the connecting morphism for homology with respect to such diagrams shows that (19) commutes. Commutativity of (21) follows similarly. □

The connecting morphisms depend only on τ, ρ, the canonical structures on $\mathcal{B}$ and the assignment of projective resolutions $\mathcal{P}$ used to calculate the left derived functors. In fact, they are also independent of the choice of resolutions $\mathcal{P}$, in the following sense: if $\mathcal{Q}$ is another assignment of resolutions then, with the vertical isomorphisms canonical, there is a commutative diagram for all $n \geq 1$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_n^{\mathcal{P}}T'(A) & \longrightarrow & L_n^{\mathcal{P}}T(A) & \longrightarrow & L_n^{\mathcal{P}}T''(A) & \xrightarrow{\omega_n^{\mathcal{P}}} & L_{n-1}T'(A) & \longrightarrow & \cdots \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & L_n^{\mathcal{Q}}T'(A) & \longrightarrow & L_n^{\mathcal{Q}}T(A) & \longrightarrow & L_n^{\mathcal{Q}}T''(A) & \xrightarrow[\omega_n^{\mathcal{Q}}]{} & L_{n-1}^{\mathcal{Q}}T'(A) & \longrightarrow & \cdots \end{array}$$

7.2 Right Derived Functors

For an object A an *injective presentation* of A is just a monomorphism $A \longrightarrow I$ where I is injective. If we say $\varepsilon : A \longrightarrow I$ is an injective resolution of A we mean that the cochain complex I together with $\varepsilon : A \longrightarrow I^0$ is an injective resolution.

Proposition 39. *Given a short exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

and injective presentations $\varepsilon' : A' \longrightarrow I''$ and $\varepsilon'' : A'' \longrightarrow I''$ there is an injective presentation $\varepsilon : A \longrightarrow I$ fitting into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I' & \xrightarrow{\iota} & I & \xrightarrow{\pi} & I'' & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varepsilon'} & & \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \varepsilon''} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\varphi]{} & A & \xrightarrow[\psi]{} & A'' & \longrightarrow & 0 \end{array}$$

Proof. Let $I = I' \oplus I''$ with ι, π the obvious morphisms. We define ε by components. The second component is $\varepsilon''\psi$ and for the first we use the fact that I' is injective to construct $\xi : A \longrightarrow I'$ such that $\xi\varphi = \varepsilon'$. Using the product, it is clear that the diagram commutes. It follows from the 5-Lemma for abelian categories (see our Diagram Chasing notes) that ε is a monomorphism. □

Corollary 40. *Let $\mathcal{A}$ be an abelian category with enough injectives. Given a short exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

and injective resolutions $\varepsilon' : A' \longrightarrow I'$ *and* $\varepsilon'' : I'' \longrightarrow A''$ *there is an injective resolution* $\varepsilon : A \longrightarrow I$ *and an exact sequence of cochain complexes*

$$0 \longrightarrow I' \xrightarrow{\iota} I \xrightarrow{\pi} I'' \longrightarrow 0$$

with the property that ι, π *induce* φ, ψ *respectively.*

Proof. The proof follows from the previous result just as in the chain case. The (nonunique) resolution produced can be described as follows (we use subscripts to avoid bad notation): for all $n \geq 0$, $I_n = I'_n \oplus I''_n$ and ι_n, π_n are part of this biproduct. There is a morphism $\sigma : A \longrightarrow I'_0$ and morphisms $\lambda_n : I''_n \longrightarrow I'_{n+1}$ for $n \geq 0$ such that

$$\varepsilon = \begin{pmatrix} \sigma \\ \varepsilon''\psi \end{pmatrix} \qquad \partial_n = \begin{pmatrix} \partial'_n & \lambda_n \\ 0 & \partial''_n \end{pmatrix}$$

These morphisms satisfy the following relations for $n \geq 0$

$$\begin{aligned} \varepsilon' &= \sigma\varphi \\ \partial'_0\sigma + \lambda_0\varepsilon''\psi &= 0 \\ \partial'_{n+1}\lambda_n + \lambda_{n+1}\partial''_n &= 0 \end{aligned}$$

□

Theorem 41. *Let* $\mathcal{A}$ *be an abelian category with enough injectives,* $\mathcal{B}$ *an abelian category, and let* $T : \mathcal{A} \longrightarrow \mathcal{B}$ *be an additive functor. Suppose we have an exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0 \tag{22}$$

Then there exist canonical connecting morphisms $\omega^n : R^nT(A'') \longrightarrow R^{n+1}T(A')$ *for* $n \geq 0$ *with the property that the following sequence is exact*

$$\begin{gathered} 0 \longrightarrow R^0T(A') \longrightarrow R^0T(A) \longrightarrow R^0T(A'') \xrightarrow{\omega^0} R^1T(A') \longrightarrow \cdots \\ \cdots \longrightarrow R^nT(A'') \xrightarrow{\omega^n} R^{n+1}T(A') \longrightarrow R^{n+1}T(A) \longrightarrow R^{n+1}T(A'') \longrightarrow \cdots \end{gathered} \tag{23}$$

Moreover these connecting morphisms are natural in the exact sequence: if we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{\varphi} & A & \xrightarrow{\psi} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & B'' & \longrightarrow & 0 \end{array} \tag{24}$$

Then the following diagram commutes for all $n \geq 0$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & R^nT(A') & \longrightarrow & R^nT(A) & \longrightarrow & R^nT(A'') & \xrightarrow{\omega^n} & R^{n+1}T(A') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & R^nT(B') & \longrightarrow & R^nT(B) & \longrightarrow & R^nT(B'') & \xrightarrow[\omega^n]{} & R^{n+1}T(B') & \longrightarrow & \cdots \end{array}$$

Proof. We just follow the proof of the chain case, making suitable modifications. We assume the right derived functors are all calculated using some assignment $\mathcal{I}$ of injective resolutions. To calculate the connecting morphisms you take any injective resolutions $\varepsilon' : A' \longrightarrow I', \varepsilon'' : A'' \longrightarrow I''$, construct $\varepsilon : A \longrightarrow I$ and use the connecting morphism of the long exact cohomology sequence arising from $0 \longrightarrow TI' \longrightarrow TI \longrightarrow TI'' \longrightarrow 0$, together with the isomorphisms of $R^nT(A')$ and $R^nT(A'')$ with the suitable cohomology objects in this sequence. This is independent of the resolutions I', I, I'' you use, and naturality follows easily. The connecting morphisms are canonical in the sense that they depend only on the morphisms φ, ψ, the assignment of resolutions $\mathscr{I}$ and the canonical structures on $\mathcal{B}$. $\square$

Corollary 42. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Then the functor $R^0T : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact.*

The connecting morphisms are also independent of the choice of resolutions $\mathscr{I}$, in the following sense: if $\mathscr{J}$ is another assignment of resolutions then, with the vertical isomorphisms canonical, there is a commutative diagram for all $n \geq 0$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & R^n_{\mathscr{I}}T(A') & \longrightarrow & R^n_{\mathscr{I}}T(A) & \longrightarrow & R^n_{\mathscr{I}}T(A'') & \xrightarrow{\omega^n_{\mathscr{I}}} & R^{n+1}_{\mathscr{I}}T(A') & \longrightarrow \cdots \\
& \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\
\cdots \longrightarrow & R^n_{\mathscr{I}}T(A') & \longrightarrow & R^n_{\mathscr{I}}T(A) & \longrightarrow & R^n_{\mathscr{J}}T(A'') & \xrightarrow[\omega^n_{\mathscr{J}}]{} & R^{n+1}_{\mathscr{J}}T(A') & \longrightarrow \cdots
\end{array}$$

We have proved naturality of the sequence (23) in the objects A. It is also natural in the functor.

Definition 11. Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category and $\tau : T \longrightarrow T'$ a natural transformation of additive covariant functors $T, T' : \mathcal{A} \longrightarrow \mathcal{B}$. Then for any cochain complex Y there is a cochain morphism $\tau_Y : TY \longrightarrow T'Y$ defined by $(\tau_Y)_n = \tau_{Y_n}$. In particular if $\mathscr{I}$ is an assignment of injective resolutions and if I is the resolution of A then we obtain a cochain map $\tau_I : TI \longrightarrow T'I$. Taking cohomology gives a natural transformation of the right derived functors $R^n\tau : R^nT \longrightarrow R^nT'$ with $(R^n\tau)_A = H^n(\tau_I)$ for $n \geq 0$. Notice that $R^n(\tau\rho) = (R^n\tau)(R^n\rho)$, $R^n(\tau + \rho) = R^n(\tau) + R^n(\rho)$ and $R^n(1_T) = 1_{R^nT}$.

Lemma 43. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category, and $T, T' : \mathcal{A} \longrightarrow \mathcal{B}$ left exact functors. Then the following diagram commutes for any natural transformation $\tau : T \longrightarrow T'$*

$$\begin{array}{ccc}
T & \xrightarrow{\tau} & T' \\
\Downarrow & & \Downarrow \\
R^0T & \xrightarrow[R^0\tau]{} & R^0T'
\end{array}$$

Proposition 44. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category and $\tau : T \longrightarrow T'$ a natural transformation of additive functors $T, T' : \mathcal{A} \longrightarrow \mathcal{B}$. Suppose we have an exact sequence*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0 \qquad (25)$$

Then the following diagram is commutative, where the connecting morphisms are canonical

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & R^nT(A') & \longrightarrow & R^nT(A) & \longrightarrow & R^nT(A'') & \xrightarrow{\omega^n} & R^{n+1}T(A') & \longrightarrow \cdots \\
& \downarrow {\scriptstyle (R^n\tau)_{A'}} & & \downarrow {\scriptstyle (R^n\tau)_{A}} & & \downarrow {\scriptstyle (R^n\tau)_{A''}} & & \downarrow {\scriptstyle (R^n\tau)_{A'}} & \\
\cdots \longrightarrow & R^nT'(A') & \longrightarrow & R^nT'(A) & \longrightarrow & R^nT'(A'') & \xrightarrow[\omega^n]{} & R^{n+1}T'(A') & \longrightarrow \cdots
\end{array}$$

Definition 12. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A sequence $T' \longrightarrow T \longrightarrow T''$ of additive functors $\mathcal{A} \longrightarrow \mathcal{B}$ and natural transformations is called *exact on injectives* if for every injective object I the sequence $0 \longrightarrow T'(I) \longrightarrow T(I) \longrightarrow T''(I) \longrightarrow 0$ is exact.

Proposition 45. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category and suppose there is a sequence of additive functors $\mathcal{A} \longrightarrow \mathcal{B}$ which is exact on injectives*

$$T' \xrightarrow{\tau} T \xrightarrow{\rho} T''$$

Then for every object A there are canonical connecting morphisms $\omega^n : R^nT''(A) \longrightarrow R^{n+1}T'(A)$ for $n \geq 0$ with the property that the following sequence is exact

$$0 \longrightarrow R^0T'(A) \xrightarrow{(R^0\tau)_A} R^0T(A) \xrightarrow{(R^0\rho)_A} R^0T''(A) \xrightarrow{\omega^0} R^1T'(A) \longrightarrow \cdots$$
$$\cdots \longrightarrow R^nT'(A) \xrightarrow{(R^n\tau)_A} R^nT(A) \xrightarrow{(R^n\rho)_A} R^nT''(A) \xrightarrow{\omega^n} R^{n+1}T'(A) \longrightarrow \cdots$$

This sequence is natural in both A and the exact sequence. For any morphism $\alpha : A \longrightarrow B$ the following diagram is commutative

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & R^nT'(A) & \longrightarrow & R^nT(A) & \longrightarrow & R^nT''(A) & \xrightarrow{\omega^n} & R^{n+1}T'(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & R^nT'(B) & \longrightarrow & R^nT(B) & \longrightarrow & R^nT''(B) & \xrightarrow[\omega^n]{} & R^{n+1}T'(B) & \longrightarrow & \cdots \end{array} \tag{26}$$

and for any commutative diagram of additive functors with rows exact on injectives

$$\begin{array}{ccccc} T' & \xrightarrow{\tau} & T & \xrightarrow{\rho} & T'' \\ {\scriptstyle\varphi'}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi''}\downarrow \\ S' & \xrightarrow[\sigma]{} & S & \xrightarrow[\theta]{} & S'' \end{array} \tag{27}$$

the following diagram is commutative for any object A

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & R^nT'(A) & \longrightarrow & R^nT(A) & \longrightarrow & R^nT''(A) & \xrightarrow{\omega^n} & R^{n+1}T'(A) & \longrightarrow & \cdots \\ & & {\scriptstyle (R^n\varphi')_A}\downarrow & & {\scriptstyle (R^n\varphi)_A}\downarrow & & {\scriptstyle (R^n\varphi'')_A}\downarrow & & {\scriptstyle (R^{n+1}\varphi')_A}\downarrow & & \\ \cdots & \longrightarrow & R^nS'(A) & \longrightarrow & R^nS(A) & \longrightarrow & R^nS''(A) & \xrightarrow[\omega^n]{} & R^{n+1}S'(A) & \longrightarrow & \cdots \end{array} \tag{28}$$

Proof. Let $\mathcal{I}$ be an assignment of projective resolutions with respect to which all derived functors are calculated. Let I be the assigned resolution of A. Since the sequence of functors is exact on injectives the following sequence is exact

$$0 \longrightarrow T'I \longrightarrow TI \longrightarrow T''I \longrightarrow 0$$

The long exact cohomology sequence then yields the connecting morphisms ω^n and the various other claims follow in the same way as for left derived functors. $\square$

The connecting morphisms depend only on τ, ρ, the canonical structures on $\mathcal{B}$ and the assignment of injective resolutions $\mathcal{I}$ used to calculate the right derived functors. They are also independent of the choice of resolutions $\mathcal{I}$, in the following sense: if $\mathcal{J}$ is another assignment of resolutions then, with the vertical isomorphisms canonical, there is a commutative diagram for all $n \geq 0$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & R^n_{\mathcal{I}}T'(A) & \longrightarrow & R^n_{\mathcal{I}}T(A) & \longrightarrow & R^n_{\mathcal{I}}T''(A) & \xrightarrow{\omega^n_{\mathcal{I}}} & R^{n+1}T'(A) & \longrightarrow & \cdots \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & R^n_{\mathcal{J}}T'(A) & \longrightarrow & R^n_{\mathcal{J}}T(A) & \longrightarrow & R^n_{\mathcal{J}}T''(A) & \xrightarrow[\omega^n_{\mathcal{J}}]{} & R^{n+1}_{\mathcal{J}}T'(A) & \longrightarrow & \cdots \end{array}$$

8 Dimension Shifting

Definition 13. Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. An object Q is called *left T-acyclic* if $L_i T(Q) = 0$ for all $i \geq 1$. If there is no chance of confusion we say that Q is *left acyclic* or even just *acyclic*. It is clear that projective objects are acyclic.

Proposition 46. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Suppose we have an exact sequence in $\mathcal{A}$ with P projective*

$$0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0$$

Then the canonical connecting morphism $\omega_n : L_n T(A) \longrightarrow L_{n-1}T(M)$ is an isomorphism for all $n \geq 2$. If T is right exact there is an exact sequence

$$0 \longrightarrow L_1 T(A) \longrightarrow T(M) \longrightarrow T(P)$$

Proof. More generally assume that P is left T-acyclic. The first claim follows immediately from exactness of the long exact sequence of derived functors. If T is right exact, let $L_1T(A) \longrightarrow T(M)$ be the canonical connecting morphism $L_1T(A) \longrightarrow L_0T(M)$ followed by the canonical isomorphism $L_0T(M) \cong T(M)$. Then it is clear that the sequence given above is exact. $\square$

More generally, we have

Proposition 47 (Dimension Shifting). *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Suppose we have an exact sequence in $\mathcal{A}$ with all P_i projective and $m \geq 0$*

$$0 \longrightarrow M \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \tag{29}$$

Then there are canonical isomorphisms $\rho_n : L_nT(A) \longrightarrow L_{n-m-1}T(M)$ for $n \geq m+2$, and if T is right exact there is an exact sequence

$$0 \longrightarrow L_{m+1}T(A) \longrightarrow T(M) \longrightarrow T(P_m) \tag{30}$$

Both the isomorphisms ρ_n and the exact sequence (30) are natural in T, in the sense that for a natural transformation $\tau : T \longrightarrow T'$, $n \geq m+2$ and $m \geq 0$ the following two diagrams commute

$$\begin{array}{ccc} L_nT(A) & \xRightarrow{\rho_n} & L_{n-m-1}T(M) \\ \downarrow & & \downarrow \\ L_nT'(A) & \xRightarrow[\rho_n]{} & L_{n-m-1}T'(M) \end{array} \qquad \begin{array}{ccccccc} 0 & \longrightarrow & L_{m+1}T(A) & \longrightarrow & T(M) & \longrightarrow & T(P_m) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L_{m+1}T'(A) & \longrightarrow & T'(M) & \longrightarrow & T'(P_m) \end{array}$$

Proof. More generally we can assume that the P_i are left T-acyclic. If $m = 0$ then we are in the situation of the previous Proposition, and we let ρ_n be the connecting morphism. If $m \geq 1$ then for $0 \leq i \leq m-1$ let $K_i \longrightarrow P_i$ be an image of $P_{i+1} \longrightarrow P_i$. We have exact sequences

$$\begin{gathered} 0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \\ 0 \longrightarrow K_1 \longrightarrow P_1 \longrightarrow K_0 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow K_{m-1} \longrightarrow P_{m-1} \longrightarrow K_{m-2} \longrightarrow 0 \\ 0 \longrightarrow M \longrightarrow P_m \longrightarrow K_{m-1} \longrightarrow 0 \end{gathered}$$

Putting together the connecting morphisms for all these sequences gives an isomorphism $\rho_n : L_nT(A) \longrightarrow L_{n-m-1}T(M)$ for $n \geq m+2$, which depends only on the canonical structures on $\mathcal{A}$ and $\mathcal{B}$. If T is right exact and $m \geq 1$ let $L_{m+1}T(A) \longrightarrow T(M)$ be the composite

$L_{m+1}T(A) \cong L_mT(K_0) \cong \cdots \cong L_1T(K_{m-1}) \longrightarrow L_0T(M) \cong T(M)$. This clearly fits into the required exact sequence. Naturality of the morphisms ρ_n and the sequence (30) follows immediately from Proposition 37. It is not difficult to check that the isomorphisms ρ_n and morphism $L_{m+1}T(A) \longrightarrow T(M)$ do not depend on the images K_i chosen, so they depend only on the assignment of resolutions used to calculate the derived functors, the canonical structures on $\mathcal{B}$, and the exact sequence (29). □

The morphisms ρ_n and $L_{m+1}T(A) \longrightarrow T(M)$ are also independent of the assignment of resolutions used to calculate the left derived functors, in the following sense: if $\mathcal{P}, \mathcal{Q}$ are two assignments of projective resolutions, and we are in the situation of the Proposition, then it is not hard to check that the following diagrams commute for $n \geq m+2$ (vertical isomorphisms canonical)

$$\begin{array}{ccc} L_n^{\mathcal{P}}T(A) & \xRightarrow{\rho_n} & L_{n-m-1}^{\mathcal{P}}T(M) \\ \Downarrow & & \Downarrow \\ L_n^{\mathcal{Q}}T(A) & \xRightarrow[\rho_n]{} & L_{n-m-1}^{\mathcal{Q}}T(M) \end{array} \qquad \begin{array}{ccc} L_{m+1}^{\mathcal{P}}T(A) & \longrightarrow & T(M) \\ \Downarrow & & \| \\ L_{m+1}^{\mathcal{Q}}T(A) & \longrightarrow & T(M) \end{array}$$

The next result shows that Proposition 47 is natural in the exact sequence.

Proposition 48. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Suppose we have a commutative diagram in $\mathcal{A}$ with exact rows, all P_i, Q_i left T-acyclic and $m \geq 0$*

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_m & \longrightarrow & P_{m-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \psi_m} & & \downarrow{\scriptstyle \psi_{m-1}} & & & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & N & \longrightarrow & Q_m & \longrightarrow & Q_{m-1} & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Then we claim that the morphisms of Proposition 47 fit into the following commutative diagrams for $n \geq m+2$

$$\begin{array}{ccc} L_nT(A) & \Longrightarrow & L_{n-m-1}T(M) \\ {\scriptstyle L_nT(\alpha)}\downarrow & & \downarrow{\scriptstyle L_{n-m-1}T(\beta)} \\ L_nT(B) & \Longrightarrow & L_{n-m-1}T(N) \end{array} \qquad \begin{array}{ccccccc} 0 & \longrightarrow & L_{m+1}T(A) & \longrightarrow & T(M) & \longrightarrow & T(P_m) \\ & & \downarrow{\scriptstyle L_{m+1}T(\alpha)} & & \downarrow{\scriptstyle T(\beta)} & & \downarrow{\scriptstyle T(\psi_m)} \\ 0 & \longrightarrow & L_{m+1}T(B) & \longrightarrow & T(N) & \longrightarrow & T(Q_m) \end{array}$$

Proof. Both statements follow easily from the naturality of the connecting morphism with respect to morphisms of exact sequences (note that the right hand diagram only makes sense for T right exact). □

Let $\mathcal{A}$ be an abelian category with enough projectives, and let $\mathcal{B}$ be a category of modules. If $T : \mathcal{A} \longrightarrow \mathcal{B}$ is right exact then Propositions 47 and 17 both produce exact sequences

$$0 \longrightarrow L_{n+1}T(A) \longrightarrow T(M) \longrightarrow T(P)$$

from an exact sequence

$$0 \longrightarrow M \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with all P_i projective. In Proposition 47 the morphism $L_{n+1}T(A) \longrightarrow T(M)$ was defined as the composite of connecting morphisms, whereas in Proposition 17 we gave the map explicitly. It would be nice to know that these two maps are the same, which is what we now prove. We begin with a technical result that does most of the work.

Proposition 49. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ a category of modules, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a right exact functor. Let A be an object of $\mathcal{A}$ with projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and let $e : P_1 \longrightarrow K, \mu : K \longrightarrow P_0$ be an epi-mono factorisation of $\partial_1 = \mu e$ with $A \neq K$, so we have an exact sequence*

$$0 \longrightarrow K \xrightarrow{\mu} P_0 \longrightarrow A \longrightarrow 0$$

If we choose the projective resolution $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow K \longrightarrow 0$ then the connecting morphism $\omega_1 : L_1T(A) \longrightarrow L_0T(K)$ is the canonical injection $KerT(\partial_1)/ImT(\partial_2) \longrightarrow T(P_1)/ImT(\partial_2)$ given by $x + ImT(\partial_2) \mapsto x + ImT(\partial_2)$ and for $n > 1$ the connecting morphism ω_n is the identity.

Proof. All derived functors are calculated relative to the assignment $\mathcal{P}$ of projective resolutions which chooses the given resolution for A and the resolution $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow K \longrightarrow 0$ for K, with $P_1 \longrightarrow K$ being e. As in Corollary 33 we produce a resolution T of P_0 with $T_n = P_{n+1} \oplus P_n$ for $n \geq 0$. The differentials $\partial_n : T_n \longrightarrow T_{n-1}$ are constructed as follows. The epimorphism $T_0 \longrightarrow P_0$ is $(\partial_1, 1)$. Let K be a pullback of $\partial_1 : P_1 \longrightarrow P_0$ and $1 : P_0 \longrightarrow P_0$. Then the induced morphism $K \longrightarrow T_0$ is a kernel for $T_0 \longrightarrow P_0$. Similar arguments for higher n show that we can arrange for the differentials for $n \geq 1$ to be of the form

$$\partial_n = \begin{pmatrix} \partial_n^K & \lambda_n \\ 0 & \partial_n^A \end{pmatrix} \tag{31}$$

where $\lambda_n : P_n \longrightarrow P_n$ is the identity. The morphism ω_1 is the connecting morphism of the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & CokerT(\partial_2^K) & \longrightarrow & CokerT(\partial_2) & \longrightarrow & CokerT(\partial_2^A) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T(P_1) & \longrightarrow & T(P_1 \oplus P_0) & \longrightarrow & T(P_0) \end{array}$$

We can identify the module $T(P_1 \oplus P_0)$ with $T(P_1) \oplus T(P_0)$ and similarly for $T(P_2 \oplus P_1)$. Given $x + ImT(\partial_2^A) \in L_1T(A)$ the image in $CokerT(\partial_2^A)$ is just $x + ImT(\partial_2)$. We can choose the preimage in $CokerT(\partial_2)$ to be $(0, x) + ImT(\partial_2)$, which maps to $(x, T(\partial_1^A)(x)) = (x, 0) \in T(P_1 \oplus P_0)$ since by assumption $x \in KerT(\partial_1^A)$. Then we can choose the preimage $x \in T(P_1)$, which shows that ω_1 is the desired map. A similar argument shows that the higher connecting morphisms are identities. □

Corollary 50. *Let $\mathcal{A}$ be an infinite abelian category with enough projectives, $\mathcal{B}$ a category of modules, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a right exact functor. Suppose we have an exact sequence in $\mathcal{A}$ with all P_i projective and $m \geq 0$*

$$0 \longrightarrow M \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

Assume that the resolution P chosen for A ends with $P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and let $e : P_{m+1} \longrightarrow M$ be the unique factorisation of $P_{m+1} \longrightarrow P_m$ through $M \longrightarrow P_m$. Then

- *(i) The morphism $L_{m+1}T(A) \longrightarrow T(M)$ is given by $x + ImT(\partial_{m+2}) \mapsto T(e)(x)$.*
- *(ii) If the resolution chosen for M is the one ending in $e : P_{m+1} \longrightarrow M$ obtained from P, then for $n \geq m + 2$ the isomorphism $\rho_n : L_nT(A) \longrightarrow L_{n-m-1}T(M)$ is the identity map.*

Proof. (*i*) By induction on m. If $m = 0$ then $L_1T(A) \longrightarrow T(M)$ is the connecting morphism $L_1T(A) \longrightarrow L_0T(M)$ followed by the canonical isomorphism $L_0T(M) \cong T(M)$. Using the previous Lemma it is easy to check this has the required form (we can assume $M \neq A$ by replacing M by an isomorphic copy and using naturality of the connecting morphism in the exact sequence). For $m \geq 1$ choose an image $K_0 \longrightarrow P_0$ of $P_1 \longrightarrow P_0$ with $K_0 \neq A$. So we have exact sequences

$$0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow K_0 \longrightarrow 0$$

Let $\mathcal{P}$ be an assignment of resolutions which chooses a resolution for A ending in $P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and chooses the resolution for K_0 ending in $P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow K_0 \longrightarrow 0$ obtained from the resolution of A. Let e be as in the statement of the Corollary. The canonical morphism $L_{m+1}T(A) \longrightarrow T(M)$ is the composite of the connecting isomorphism $L_{m+1}T(A) \cong L_mT(K_0)$ with the morphism $L_mT(K_0) \longrightarrow T(M)$ defined for the second exact sequence above. So by the inductive hypothesis and Proposition 49 the morphism $L_{m+1}T(A) \longrightarrow T(M)$ has the desired form. Since this morphism is independent of the choice of K_0, the result holds for *any* assignment of resolutions $\mathcal{P}$ which chooses a resolution for A ending in $P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ (it need not choose a special resolution for any particular K_0). The proof of *(ii)* is similar. $\square$

Definition 14. Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. An object Q is called *right T-acyclic* if $R^iT(Q) = 0$ for all $i \geq 1$. If there is no chance of confusion we say that Q is *right acyclic* or even just *acyclic*. It is clear that injective objects are right acyclic.

The dual version of Proposition 47 is the following

Proposition 51. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category, and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Suppose we have an exact sequence in $\mathcal{A}$ with all I^i injective and $m \geq 0$*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{m-1} \longrightarrow I^m \longrightarrow M \longrightarrow 0$$

Then there are canonical isomorphisms $\rho^n : R^nT(M) \longrightarrow R^{n+m+1}T(A)$ for $n \geq 1$, and if T is left exact there is an exact sequence

$$T(I^m) \longrightarrow T(M) \longrightarrow R^{m+1}T(A) \longrightarrow 0 \tag{32}$$

Both the isomorphisms ρ^n and the exact sequence (32) are natural in T, in the sense that for a natural transformation $\tau : T \longrightarrow T'$, $n \geq 1$ and $m \geq 0$ the following two diagrams commute

$$\begin{array}{ccc} R^nT(M) & \xrightarrow{\rho^n} & R^{n+m+1}T(A) \\ \downarrow & & \downarrow \\ R^nT'(M) & \xrightarrow[\rho^n]{} & R^{n+m+1}T'(A) \end{array} \qquad \begin{array}{ccccccc} T(I^m) & \longrightarrow & T(M) & \longrightarrow & R^{m+1}T(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ T'(I^m) & \longrightarrow & T'(M) & \longrightarrow & R^{m+1}T'(A) & \longrightarrow & 0 \end{array}$$

Proof. We work more generally with right T-acyclic I^i and proceed in the same way as for left derived functors. Take canonical images and form a list of exact sequences

$$\begin{gathered} 0 \longrightarrow K_m \longrightarrow I^m \longrightarrow M \longrightarrow 0 \\ 0 \longrightarrow K_{m-1} \longrightarrow I^{m-1} \longrightarrow K_m \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow K_1 \longrightarrow I^1 \longrightarrow K_2 \longrightarrow 0 \\ 0 \longrightarrow A \longrightarrow I^0 \longrightarrow K_1 \longrightarrow 0 \end{gathered}$$

For $n \geq 1$ the connecting morphisms for all these sequences are isomorphisms, so we can take the composite $R^nT(M) \cong R^{n+1}T(K_m) \cong \cdots \cong R^{n+m+1}T(A)$. For $m = 0$, ρ^n is just the connecting morphism of $0 \longrightarrow A \longrightarrow I^0 \longrightarrow M \longrightarrow 0$, and the exact sequence (32) involves the morphism $T(M) \longrightarrow R^1T(A)$ given by the canonical isomorphism $T(M) \cong R^0T(M)$ followed by the connecting morphism. For $m \geq 1$ we take $T(M) \cong R^0T(M)$ followed by $R^0T(M) \longrightarrow R^1T(K_m)$ and then a sequence of connecting isomorphisms $R^1T(K_m) \cong \cdots \cong R^{m+1}T(A)$. Naturality is easy to check. $\square$

Proposition 52. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Suppose we have a commutative diagram in $\mathcal{A}$ with exact rows, all I^i, J^i right T-acyclic and $m \geq 0$*

$$\begin{CD} 0 @>>> A @>>> I^0 @>>> \cdots @>>> I^{m-1} @>>> I^m @>>> M @>>> 0\\ @. @V{\alpha}VV @V{\psi^0}VV @. @V{\psi^{m-1}}VV @V{\psi^m}VV @V{\beta}VV @.\\ 0 @>>> B @>>> J^0 @>>> \cdots @>>> J^{m-1} @>>> J^m @>>> N @>>> 0 \end{CD}$$

Then we claim that the morphisms of Proposition 51 fit into the following commutative diagrams for $n \geq 1$

$$\begin{CD} R^nT(M) @>>> R^{n+m+1}T(A)\\ @V{R^nT(\beta)}VV @VV{R^{n+m+1}T(\alpha)}V\\ R^nT(N) @>>> R^{n+m+1}T(B) \end{CD} \qquad \begin{CD} T(I^m) @>>> T(M) @>>> R^{m+1}T(A) @>>> 0\\ @V{T(\psi^m)}VV @V{T(\beta)}VV @V{R^{m+1}T(\alpha)}VV @.\\ T(J^m) @>>> T(N) @>>> R^{m+1}T(B) @>>> 0 \end{CD}$$

Proof. Both statements follow easily from the naturality of the connecting morphism with respect to morphisms of exact sequences (note that the right hand diagram only makes sense for T left exact). ☐

8.1 Acyclic Resolutions

Definition 15. Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. If A is an object of $\mathcal{A}$ then a *left T-acyclic resolution* of A is an exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

with all F_i left T-acyclic. Alternatively if $\mathcal{A}$ has enough injectives then a *right T-acyclic resolution* of A is an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with all I^i right T-acyclic.

The next result says that we can calculate left derived functors using acyclic resolutions. An important application of this result is the calculation of Tor groups from flat resolutions.

Proposition 53. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a right exact functor. Suppose that we have a left T-acyclic resolution of an object A of $\mathcal{A}$*

$$F : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

Then there are canonical isomorphisms $\tau_n : L_nT(A) \longrightarrow H_n(TF)$ for $n \geq 0$. These isomorphisms are natural in A, in the following sense: given a commutative diagram in $\mathcal{A}$ with exact rows and all F_i, G_i left T-acyclic

$$\begin{CD} \cdots @>>> F_1 @>>> F_0 @>>> A @>>> 0\\ @. @V{\psi_1}VV @V{\psi_0}VV @V{\alpha}VV @.\\ \cdots @>>> G_1 @>>> G_0 @>>> B @>>> 0 \end{CD}$$

the following diagram commutes for $n \geq 0$

$$\begin{CD} L_nT(A) @>>> H_n(TF)\\ @V{L_nT(\alpha)}VV @VV{H_n(T\psi)}V\\ L_nT(B) @>>> H_n(TG) \end{CD}$$

Proof. Here TF denotes the chain complex $\cdots \longrightarrow T(F_1) \longrightarrow T(F_0) \longrightarrow 0$ in $\mathcal{B}$. Fix an assignment of projective resolutions $\mathcal{P}$ with respect to which all left derived functors are calculated. Since T is right exact there is a canonical isomorphism $\tau_0 : L_0T(A) \cong T(A) \cong H_0(TF)$.

For $n = 1$ let $\mu : K \longrightarrow F_{n-1}$ be the kernel of $F_0 \longrightarrow A$, and for $n \geq 1$ let $\mu : K \longrightarrow F_{n-1}$ be the kernel of $F_{n-1} \longrightarrow F_{n-2}$. So for $n \geq 1$ we have an exact sequence

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

Let $e : F_n \longrightarrow K$ be the unique factorisation of $\partial : F_n \longrightarrow F_{n-1}$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & T(F_{n+1}) & \longrightarrow & T(F_n) & \xrightarrow{T(e)} & T(K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle T(\partial)} & & \downarrow{\scriptstyle T(\mu)} & & \\
0 & \longrightarrow & 0 & \longrightarrow & T(F_{n-1}) & \Longrightarrow & T(F_{n-1}) & &
\end{array}$$

It follows from the Snake Lemma that the following sequence is exact

$$T(F_{n+1}) \longrightarrow KerT(\partial) \longrightarrow KerT(\mu) \longrightarrow 0$$

and therefore we have a canonical isomorphism $H_n(TF) \cong KerT(\mu)$. Using the exact sequence (30) of Proposition 47 (with $m = n-1$) we have a canonical isomorphism $KerT(\mu) \cong L_nT(A)$, and therefore by composition we have the required canonical isomorphism $\tau_n : L_nT(A) \cong H_n(TF)$. Naturality in A follows from Proposition 48. □

Proposition 54. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a left exact functor. Suppose that we have a right T-acyclic resolution of an object A of $\mathcal{A}$*

$$I : 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

Then there are canonical isomorphisms $\sigma^n : R^nT(A) \longrightarrow H^n(TI)$ for $n \geq 0$. These isomorphisms are natural in A, in the following sense: given a commutative diagram in $\mathcal{A}$ with exact rows and I^i, J^i right T-acyclic

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \psi^0} & & \downarrow{\scriptstyle \psi^1} & & \\
0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots
\end{array}$$

the following diagram commutes for $n \geq 0$

$$\begin{array}{ccc}
R^nT(A) & \Longrightarrow & H^n(TI) \\
{\scriptstyle R^nT(\alpha)}\downarrow & & \downarrow{\scriptstyle H^n(T\psi)} \\
R^nT(B) & \Longrightarrow & H^n(TJ)
\end{array}$$

Proof. Here TI denotes the cochain complex $0 \longrightarrow T(I^0) \longrightarrow T(I^1) \longrightarrow \cdots$ in $\mathcal{B}$. Fix an assignment of injective resolutions $\mathcal{I}$ with respect to which all right derived functors are calculated. Since T is left exact there is a canonical isomorphism $\sigma^0 : R^0T(A) \cong T(A) \cong H^0(TI)$.

For $n = 1$ let $\mu : I^{n-1} \longrightarrow C$ be the cokernel of $A \longrightarrow I^0$, and for $n \geq 1$ let $\mu : I^{n-1} \longrightarrow C$ be the cokernel of $I^{n-2} \longrightarrow I^{n-1}$. So for $n \geq 1$ we have an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow C \longrightarrow 0$$

Let $e : C \longrightarrow I^n$ be the unique factorisation of $\partial : I^{n-1} \longrightarrow I^n$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & T(I^{n-1}) & \Longrightarrow & T(I^{n-1}) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle T(\mu)}\downarrow & & \downarrow{\scriptstyle T(\partial)} & & \downarrow & & \\
0 & \longrightarrow & T(C) & \xrightarrow[T(e)]{} & T(I^n) & \longrightarrow & T(I^{n+1}) & &
\end{array}$$

It follows from the Snake Lemma that the following sequence is exact

$$0 \longrightarrow CokerT(\mu) \longrightarrow CokerT(\partial) \longrightarrow T(I^{n+1})$$

But by Lemma 7 the canonical morphism $H^n(TI) \longrightarrow CokerT(\partial)$ is also a kernel of the morphism $CokerT(\partial) \longrightarrow T(I^{n+1})$, so we have a canonical isomorphism $H^n(TI) \cong CokerT(\mu)$. Using the exact sequence (32) of Proposition 51 we have a canonical isomorphism $CokerT(\mu) \cong R^nT(A)$, and therefore by composition we have the required canonical isomorphism $\sigma^n : R^nT(A) \cong H^n(TI)$. Naturality in A follows from Proposition 52. □

Remark 1. The reader may object that the definition of the isomorphism $R^nT(A) \longrightarrow H^n(TI)$ is incredibly opaque, and they'd be right. Here is the problem. Given a right T-acyclic resolution

$$Q : 0 \longrightarrow A \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \cdots$$

and an injective resolution

$$I : 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

There is a morphism of complexes $s : Q \longrightarrow I$ lifting the identity, which yields morphisms $H^nT(s) : H^n(TQ) \longrightarrow H^n(TI) = R^nT(A)$. But it is not clear to me how to show directly that this is an isomorphism, which is why we resort to the construction above. It seems very likely that σ^n is just $H^nT(s)^{-1}$ up to a sign, but checking this might require some patience (one can immediately reduce to the case $I = Q$ which should not be too difficult).

In most cases where we actually use acyclic resolutions to calculate, one manages somehow to avoid this issue and show $H^nT(s)$ is an isomorphism by other means. See for example Remark 2. It is possible to show that $H^nT(s)$ is always an isomorphism, but the easy proof (DTC2,Lemma 44) requires some more sophisticated category theory (namely, derived categories).

9 Change of Base

We have defined the derived functors of additive functors $T : \mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories. Given a functor $U : \mathcal{B} \longrightarrow \mathcal{C}$ (a "change of coefficients") what is the relationship between the functors $U \circ L_nT$ and $L_n(UT)$? In the case where U is faithful and exact we would expect this relationship to be an isomorphism, and that is what we shall prove in this section. First we need some technical results.

Proposition 55. *Let $\mathcal{A}$ be an abelian category, and suppose there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} & & A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' & & \end{array} \tag{33}$$

The connecting morphism $\omega : Kerd'' \longrightarrow Cokerd'$ is independent of the kernels and cokernels chosen for d', d, d'' in the sense that if $\underline{Ker}, \underline{Coker}$ denote another set of choices, the following diagram commutes

$$\begin{array}{ccc} Kerd'' & \xrightarrow{\omega} & Cokerd' \\ \Downarrow & & \Downarrow \\ \underline{Ker}d'' & \xrightarrow[\underline{\omega}]{} & \underline{Coker}d' \end{array}$$

where $\underline{\omega}$ is the connecting morphism and the vertical isomorphisms are canonical.

Proof. This follows immediately from naturality of the connecting morphism with respect to morphisms of the diagram (33). In this case the morphism is the identity, but we choose different sets of kernels and cokernels for each copy. □

Corollary 56. *Suppose $U : \mathcal{B} \longrightarrow \mathcal{C}$ is a faithful exact functor between abelian categories, and suppose there is a commutative diagram D with exact rows in $\mathcal{B}$*

$$\begin{array}{ccccccc} & & A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' & & \end{array} \tag{34}$$

Let $\omega : Kerd'' \longrightarrow Cokerd'$ be the canonical connecting morphism and $\rho : KerU(d'') \longrightarrow CokerU(d')$ the canonical connecting morphism of the diagram UD. Then the following diagram commutes

$$\begin{array}{ccc} U(Kerd'') & \xrightarrow{U(\omega)} & U(Cokerd') \\ \Downarrow & & \Downarrow \\ KerU(d'') & \xrightarrow[\rho]{} & CokerU(d') \end{array}$$

where the vertical isomorphisms are canonical.

Proof. When we refer to the "canonical" connecting morphisms we mean the morphisms constructed using the canonical kernels and cokernels in $\mathcal{B}, \mathcal{C}$. Using the previous Proposition it suffices to show that $U(\omega)$ is the connecting morphism τ for the diagram UD in $\mathcal{C}$ where the chosen kernels and cokernels are the images under U of the canonical ones in $\mathcal{B}$. Let $\gamma : Kerd'' \longrightarrow A''$ and $\varepsilon : B' \longrightarrow Cokerd'$ be canonical in $\mathcal{B}$ and let W be the walk $\gamma, \alpha_2, d, \beta_1, \varepsilon$. Denote by UW the walk in $\mathcal{C}$ given by taking the images of the morphisms in W. We use the unique properties of the morphisms ω, τ given in our Diagram Chasing notes.

Let $\mathcal{A}$ be a small, full, abelian subcategory of $\mathcal{B}$ containing the diagram (34) and our canonical kernels and cokernels for d', d, d'' and let $\mathcal{E}$ be a small, full, abelian subcategory of $\mathcal{C}$ containing all the objects $U(A), A \in \mathcal{A}$. We know that W is a function walk in $\mathcal{A}$ and UW is a function walk in $\mathcal{E}$ (this is what we proved in our Diagram Chasing notes on the Snake Lemma). Let $S : \mathcal{E} \longrightarrow \mathbf{Ab}$ be an exact imbedding. The composite $SU : \mathcal{A} \longrightarrow \mathbf{Ab}$ is exact and faithful but not necessarily distinct on objects, but we can find a naturally equivalent functor $Q : \mathcal{A} \longrightarrow \mathbf{Ab}$ which is an exact imbedding (Mitchell $II, 10.4$). We have a commutative diagram of relations in $\mathbf{Ab}$

$$\begin{array}{ccc} SU(Kerd'') & \xrightarrow{S(UW)} & SU(Cokerd') \\ \Downarrow & & \Downarrow \\ Q(Kerd'') & \xrightarrow[Q(W)]{} & Q(Cokerd') \end{array}$$

But $Q(W) = Q(\omega)$ and $S(UW) = S(\tau)$ and by naturality of the isomorphism $Q \cong SU$ it follows that $SU(\omega) = S(\tau)$ so $U(\omega) = \tau$, as required. □

Proposition 57. *Let $U : \mathcal{B} \longrightarrow \mathcal{C}$ be an exact functor between abelian categories. Then for every chain complex X and $n \in \mathbb{Z}$ there is a canonical isomorphism $U(H_n(X)) \cong H_n(UX)$ natural in X. The following diagram of functors therefore commutes up to natural equivalence*

$$\begin{array}{ccc} \mathbf{Ch}\mathcal{B} & \xrightarrow{U} & \mathbf{Ch}\mathcal{C} \\ {\scriptstyle H_n}\downarrow & & \downarrow{\scriptstyle H_n} \\ \mathcal{B} & \xrightarrow[U]{} & \mathcal{C} \end{array}$$

Proof. We choose canonical structures for $\mathcal{B}, \mathcal{C}$ with respect to which all homology objects are defined. Let X be a chain complex in $\mathcal{B}$ and $n \in \mathbb{Z}$. Exactness of U means that there are canonical isomorphisms $U(Im\partial_{n+1}) \cong ImU(\partial_{n+1})$ and $U(Ker\partial_n) \cong KerU(\partial_n)$ which induce an isomorphism $U(H_n(X)) \longrightarrow H_n(UX)$ for fitting into a commutative diagram

$$\begin{array}{ccccc} U(Im\partial_{n+1}) & \longrightarrow & U(Ker\partial_n) & \longrightarrow & U(H_n(X)) \\ \Downarrow & & \Downarrow & & \Downarrow \\ ImU(\partial_{n+1}) & \longrightarrow & KerU(\partial_n) & \longrightarrow & H_n(UX) \end{array}$$

These isomorphisms are easily checked to be natural in the chain complex X. $\square$

Proposition 58. *Let $U : \mathcal{B} \longrightarrow \mathcal{C}$ be a faithful exact functor between abelian categories. Suppose we have an exact sequence of chain complexes in $\mathcal{B}$*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

Let $\omega_n : H_n(X'') \longrightarrow H_{n-1}(X')$ be the canonical connecting morphisms for this sequence, and $\rho_n : H_n(UX'') \longrightarrow H_{n-1}(UX')$ the canonical connecting morphisms for its image in $\mathcal{C}$. Then the following diagram commutes for $n \in \mathbb{Z}$

$$\begin{array}{ccc} U(H_n(X'')) & \xrightarrow{U(\omega_n)} & U(H_{n-1}(X')) \\ \Downarrow & & \Downarrow \\ H_n(UX'') & \xrightarrow[\rho_n]{} & H_{n-1}(UX') \end{array}$$

Proof. Form the diagram (8) of Theorem 26 for the sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in $\mathcal{B}$ and map it to $\mathcal{C}$ using U. Then the middle two squares give a diagram isomorphic to the analogous diagram constucted for $0 \longrightarrow UX' \longrightarrow UX \longrightarrow UX'' \longrightarrow 0$ in $\mathcal{C}$ (the isomorphisms being of the form $U(Coker\partial_{n+1}) \cong CokerU(\partial_{n+1})$ and $U(Ker\partial_{n-1}) \cong KerU(\partial_{n-1})$). This morphism of diagrams induces morphisms of the kernels and cokernels of the vertical maps, and it is easily checked that these are the isomorphisms are the ones of the form $U(H_n(Y)) \longrightarrow H_n(UY)$ defined above. The result now follows from naturality of the connecting morphism with respect to morphisms of diagrams of this sort, and the proof of Corollary 56 which shows that $U(\omega_n)$ is in fact the connecting morphism for one of these diagrams. $\square$

Proposition 59. *Let $\mathcal{A}$ be an abelian category with enough projectives and $U : \mathcal{B} \longrightarrow \mathcal{C}$ an exact functor between abelian categories. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Then for $n \geq 0$ there is a canonical natural equivalence $L_n(UT) \cong U \circ L_nT$. If U is faithful and there is an exact sequence in $\mathcal{A}$*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

Then the following diagram commutes for all $n \geq 1$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & U(L_nTA') & \longrightarrow & U(L_nTA) & \longrightarrow & U(L_nTA'') & \xrightarrow{U(\omega_n)} & U(L_{n-1}TA') & \longrightarrow & \cdots \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & L_n(UT)(A') & \longrightarrow & L_n(UT)(A) & \longrightarrow & L_n(UT)(A'') & \xrightarrow[\omega_n]{} & L_{n-1}(UT)(A') & \longrightarrow & \cdots \end{array}$$

Proof. Let $\mathcal{P}$ be an assignment of projective resolutions with respect to which the derived functors are calculated. Given an object A with resolution P let $\varphi_A : U(H_n(TP)) \longrightarrow H_n(UTP)$ be the canonical isomorphism defined in Proposition 57. Since $U(H_n(TP)) = U(L_nT(A))$ and $H_n(UTP) = L_n(UT)(A)$ the first part of the proof follows easily from the fact that this isomorphism is natural in the chain complex.

If U is faithful then the second claim follows from the previous Proposition and the construction of the connecting morphisms in Theorem 34. $\square$

Proposition 60. *Let $\mathcal{A}$ be an abelian category with enough projectives and $U : \mathcal{B} \longrightarrow \mathcal{C}$ a faithful exact functor between abelian categories. Let $T', T, T'' : \mathcal{A} \longrightarrow \mathcal{B}$ be additive functors and suppose the following sequence is exact on projectives*

$$T' \longrightarrow T \longrightarrow T''$$

Then for every object A the following diagram commutes for all $n \geq 1$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & U(L_nT'A) & \longrightarrow & U(L_nTA) & \longrightarrow & U(L_nT''A) & \xrightarrow{U(\omega_n)} & U(L_{n-1}T'A) & \longrightarrow & \cdots \\
 & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
\cdots & \longrightarrow & L_n(UT')(A) & \longrightarrow & L_n(UT)(A) & \longrightarrow & L_n(UT'')(A) & \xrightarrow[\omega_n]{} & L_{n-1}(UT')(A) & \longrightarrow & \cdots
\end{array}$$

Proof. Let an object A be given, and let P be the projective resolution of A. Then the sequence of chain complexes $0 \longrightarrow T'P \longrightarrow TP \longrightarrow T''P \longrightarrow 0$ is exact and the ω_n are the connecting morphisms of the corresponding long exact sequence. So the result follows immediately from Proposition 58. $\square$

There are dual results for right derived functors

Proposition 61. *Let $U : \mathcal{B} \longrightarrow \mathcal{C}$ be an exact functor between abelian categories. Then for every cochain complex X and $n \in \mathbb{Z}$ there is a canonical isomorphism $U(H^n(X)) \cong H^n(UX)$ natural in X. The following diagram of functors therefore commutes up to natural equivalence*

$$\begin{array}{ccc}
\mathbf{coCh}\mathcal{B} & \xrightarrow{U} & \mathbf{coCh}\mathcal{C} \\
{\scriptstyle H^n}\downarrow & & \downarrow{\scriptstyle H^n} \\
\mathcal{B} & \xrightarrow[U]{} & \mathcal{C}
\end{array}$$

Proposition 62. *Let $U : \mathcal{B} \longrightarrow \mathcal{C}$ be a faithful exact functor between abelian categories. Suppose we have an exact sequence of cochain complexes in $\mathcal{B}$*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

Let $\omega^n : H^n(X'') \longrightarrow H^{n+1}(X')$ be the canonical connecting morphisms for this sequence, and $\rho^n : H^n(UX'') \longrightarrow H^{n+1}(UX')$ the canonical connecting morphisms for its image in $\mathcal{C}$. Then the following diagram commutes for $n \in \mathbb{Z}$

$$\begin{array}{ccc}
U(H^n(X'')) & \xrightarrow{U(\omega^n)} & U(H^{n+1}(X')) \\
\Downarrow & & \Downarrow \\
H^n(UX'') & \xrightarrow[\rho^n]{} & H^{n+1}(UX')
\end{array}$$

Proposition 63. *Let $\mathcal{A}$ be an abelian category with enough injectives and $U : \mathcal{B} \longrightarrow \mathcal{C}$ an exact functor between abelian categories. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Then for $n \geq 0$ there is a canonical natural equivalence $R^n(UT) \cong U \circ R^nT$. If U is faithful and there is an exact sequence in $\mathcal{A}$*

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

Then the following diagram commutes for all $n \geq 0$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & U(R^nTA') & \longrightarrow & U(R^nTA) & \longrightarrow & U(R^nTA'') & \xrightarrow{U(\omega^n)} & U(R^{n+1}TA') & \longrightarrow & \cdots \\
 & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
\cdots & \longrightarrow & R^n(UT)(A') & \longrightarrow & R^n(UT)(A) & \longrightarrow & R^n(UT)(A'') & \xrightarrow[\omega^n]{} & R^{n+1}(UT)(A') & \longrightarrow & \cdots
\end{array}$$

Proposition 64. *Let $\mathcal{A}$ be an abelian category with enough injectives and $U : \mathcal{B} \longrightarrow \mathcal{C}$ a faithful exact functor between abelian categories. Let $T', T, T'' : \mathcal{A} \longrightarrow \mathcal{B}$ be additive functors and suppose the following sequence is exact on injectives*

$$T' \longrightarrow T \longrightarrow T''$$

Then for every object A the following diagram commutes for all $n \geq 0$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & U(R^nT'A) & \longrightarrow & U(R^nTA) & \longrightarrow & U(R^nT''A) & \xrightarrow{U(\omega^n)} & U(R^{n+1}T'A) & \longrightarrow \cdots \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\ \cdots \longrightarrow & R^n(UT')(A) & \longrightarrow & R^n(UT)(A) & \longrightarrow & R^n(UT'')(A) & \xrightarrow[\omega^n]{} & R^{n+1}(UT')(A) & \longrightarrow \cdots \end{array}$$

10 Homology and Colimits

For some necessary background to this section, the reader should consult (AC,Section 2.2).

Lemma 65. *If $\mathcal{A}$ is a cocomplete abelian category then so are* **Ch**$\mathcal{A}$ *and* **coCh**$\mathcal{A}$. *A cocone is a colimit if and only if it is a colimit pointwise. If $\mathcal{A}, \mathcal{B}$ are abelian categories with $\mathcal{A}$ cocomplete, and if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor, then*

(i) If F preserves coproducts, then so do **Ch**$\mathcal{A} \longrightarrow$ **Ch**$\mathcal{B}$ *and* **coCh**$\mathcal{A} \longrightarrow$ **coCh**$\mathcal{B}$.

(ii) If F preserves direct limits, then so do **Ch**$\mathcal{A} \longrightarrow$ **Ch**$\mathcal{B}$ *and* **coCh**$\mathcal{A} \longrightarrow$ **coCh**$\mathcal{B}$.

(iii) If F preserves all colimits, then so do **Ch**$\mathcal{A} \longrightarrow$ **Ch**$\mathcal{B}$ *and* **coCh**$\mathcal{A} \longrightarrow$ **coCh**$\mathcal{B}$.

Proof. It suffices to show that both categories have arbitrary coproducts. Let I be a nonempty index set, and suppose we are given chain complexes A_i for $i \in I$. For each $n \in \mathbb{Z}$ take a coproduct $A_{i,n} \longrightarrow C_n$ and let the differential $C_n \longrightarrow C_{n-1}$ be induced by the $A_{i,n} \longrightarrow A_{i,n-1}$. It is not hard to check that this is a coproduct of chain complexes, and a similar construction works for cochains. The other claims are easily checked. □

Lemma 66. *Let $\mathcal{A}$ be a cocomplete abelian category with exact coproducts. Then for $n \in \mathbb{Z}$ the functors $H_n :$* **Ch**$\mathcal{A} \longrightarrow \mathcal{A}$ *and $H^n :$* **coCh**$\mathcal{A} \longrightarrow \mathcal{A}$ *preserve coproducts.*

Proof. The functors H_n, H^n are additive, so they trivially preserves initial objects. We prove that H_n preserves nonempty coproducts, with the proof for H^n being similar. Let $\{A_i\}_{i \in I}$ be a nonempty family of chain complexes and $A_i \longrightarrow \bigoplus_i A_i$ a coproduct. Then for each $n \in \mathbb{Z}$ the morphisms $A_{i,n} \longrightarrow (\bigoplus_i A_i)_n$ are a coproduct in $\mathcal{A}$. Since taking coproducts is exact, the coproduct of the images and kernels for the sequences A_i give images and kernels for $\bigoplus A_i$ (although probably not the canonical ones used to calculate $H_n(\bigoplus A_i)$). Taking some coproduct of the exact sequences $Im(\partial_{n+1}) \longrightarrow Ker(\partial_n) \longrightarrow H_n(A_i)$ therefore provides an isomorphism $\bigoplus H_n(A_i) \cong H_n(\bigoplus A_i)$, and it is not hard to show that $H_n(A_i) \longrightarrow \bigoplus H_n(A_i) \cong H_n(\bigoplus A_i)$ is $H_n(A_i \longrightarrow \bigoplus A_i)$, which completes the proof. □

Definition 16. An abelian category $\mathcal{A}$ is *infinite* if for every object A, there is an infinite number of objects of $\mathcal{A}$ isomorphic to A. This avoids technical complications when defining derived functors.

Proposition 67. *Let $\mathcal{A}, \mathcal{B}$ be cocomplete abelian categories with exact coproducts, and suppose $\mathcal{A}$ is infinite and has enough projectives. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ preserves coproducts, then so does L_nF for $n \geq 0$.*

Proof. The functors L_nF are additive, and therefore preserve zero objects, so we can restrict our attention to nonempty coproducts $\oplus_i A_i$. If the chosen resolution for A_i is $P^i : \cdots \longrightarrow P_1^i \longrightarrow$

$P^i_0 \longrightarrow A_i \longrightarrow 0$ then since $\mathcal{A}$ has exact coproducts the following sequence is a projective resolution for $\oplus A_i$

$$\oplus P^i : \cdots \longrightarrow \oplus P^i_1 \longrightarrow \oplus P^i_0 \longrightarrow \oplus A_i \longrightarrow 0$$

and we can assume this is the chosen resolution of $\oplus A_i$. The functor $\mathbf{Ch}\mathcal{A} \longrightarrow \mathbf{Ch}\mathcal{B}$ induced by F preserves coproducts, as does $H_n : \mathbf{Ch}\mathcal{B} \longrightarrow \mathcal{B}$. Since the resolution $\oplus P^i$ is a coproduct in $\mathbf{Ch}\mathcal{A}$ for the resolutions P^i it follows immediately that the morphisms $L_nF(A_i) \longrightarrow L_nF(\oplus A_i)$ are a coproduct for $n \geq 0$. □

Lemma 68. *Let $\mathcal{A}$ be a cocomplete abelian category with exact direct limits. Then for $n \in \mathbb{Z}$ the functors $H_n : \mathbf{Ch}\mathcal{A} \longrightarrow \mathcal{A}$ and $H^n : \mathbf{coCh}\mathcal{A} \longrightarrow \mathcal{A}$ preserve direct limits.*

Let $\{A_i, \pi_{ij}\}_{i\in I}$ be a direct system of modules (right or left) and let $u_i : A_i \longrightarrow A$ be *any* colimit of this family. Since the canonical direct limit is a candidate for this role, we see that $u_i(a) = 0$ iff. $\pi_{ij}(a) = 0$ for some $i \leq j$ and every element of A is $\rho_i(a)$ for some $a \in A_i$ and some $i \in I$.

Lemma 69. *Let $\mathcal{A}$ be a category of modules. Suppose $A = \varinjlim A_i$ is a direct limit. Then there exist projective resolutions P_i of A_i forming a direct system such that $P = \varinjlim P_i$ is a projective resolution of A.*

Proof. We assume that $\mathcal{A}$ is $R\mathbf{Mod}$ or $\mathbf{Mod}R$ for some ring R. Let $u_i : A_i \longrightarrow A$ be any direct limit. For each i let F_i be the free module on the elements of A_i, and F the free module on the elements of A, and define epimorphisms $F_i \longrightarrow A_i$ and $F \longrightarrow A$ in the obvious way. Let the morphisms π_{ij} and u_i induce canonical morphisms $\mu_{ij} : F_i \longrightarrow F_j$ and $F_i \longrightarrow F$ respectively. Then the $\{F_i, \mu_{ij}\}_{i\in I}$ are a direct system, and we claim the morphisms $F_i \longrightarrow F$ are a colimit. Let $v_i : A_i \longrightarrow C$ be the canonical direct limit and $\tau : C \longrightarrow F$ induced by the morphisms $F_i \longrightarrow F$. It suffices to show that τ is an isomorphism.

It is clear that τ is surjective since every element of A is in the image of some $A_i \longrightarrow A$. Suppose that for some $i \in I$ there are elements $b_1, \ldots, b_n \in A_i$ such that a formal sum $r_1 \cdot b_1 + \cdots + r_n \cdot b_n$ in F_i has its image in C mapped by τ to zero in F (we assume all $b_i \neq 0$). We can partition the b_i up into sets $G_1, \ldots, G_r$, with each element of G_i being mapped to the same element under $A_i \longrightarrow A$, and the coefficients for the elements of G_i all adding to zero. So we can reduce to the case where every b_i is mapped to the same element of A and $r_1 + \cdots + r_n = 0$. For each pair of indices $1 \leq i, j \leq n$ there is k with the property that $\pi_{ik}(b_i) = \pi_{jk}(b_j)$. So we can find a single index q with $\pi_{1q}(b_1) = \cdots = \pi_{nq}(b_n)$. It is clear that $r_1 \cdot b_1 + \cdots + r_n \cdot b_n \in F_i$ is mapped to zero in F_q, and hence is also zero in C, as required.

The kernels $Ker(F_i \longrightarrow A_i)$ and $Ker(F \longrightarrow A)$ form another direct limit, and we can repeat this process to produce the desired projective resolutions P_i of A_i and P of A, together with morphisms $P_i \longrightarrow P_j$ for $i \leq j$ giving a direct system in $\mathbf{Ch}\mathcal{A}$, and morphisms $P_i \longrightarrow P$ which are a colimit in $\mathbf{Ch}\mathcal{A}$. □

Proposition 70. *Let $\mathcal{A}$ be a category of modules, and let $\mathcal{B}$ be a cocomplete abelian category with exact direct limits. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ preserves direct limits, then so does L_nF for $n \geq 0$.*

Proof. We assume that $\mathcal{A}$ is $R\mathbf{Mod}$ or $\mathbf{Mod}R$ for some ring R. Let $\{A_i, \pi_{ij}\}_{i\in I}$ be a direct system with colimit $u_i : A_i \longrightarrow A$. Find projective resolutions P of the A_i and a projective resolution P of A so that $P = \varinjlim P_i$ as above. Calculate the L_nF relative to these choices of resolution. Since the functors $\mathbf{Ch}\mathcal{A} \longrightarrow \mathbf{Ch}\mathcal{B}$ and $H_n : \mathbf{Ch}\mathcal{B} \longrightarrow \mathcal{B}$ preserve direct limits, the morphisms $L_nT(A_i) \longrightarrow L_nT(A)$ are a direct limit for all $n \geq 0$, as required. □

11 Cohomology and Limits

For some necessary background to this section, the reader should consult (AC,Section 2.2).

Lemma 71. *If $\mathcal{A}$ is a complete abelian category then so are $\mathbf{Ch}\mathcal{A}$ and $\mathbf{coCh}\mathcal{A}$. A cone is a limit if and only if it is a limit pointwise. If $\mathcal{A}, \mathcal{B}$ are abelian categories with $\mathcal{A}$ complete, and if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor, then* *(i) If F preserves products, then so do $\mathbf{Ch}\mathcal{A} \longrightarrow \mathbf{Ch}\mathcal{B}$ and $\mathbf{coCh}\mathcal{A} \longrightarrow \mathbf{coCh}\mathcal{B}$.*

(iii) If F preserves all limits, then so do $\mathbf{Ch}\mathcal{A} \longrightarrow \mathbf{Ch}\mathcal{B}$ and $\mathbf{coCh}\mathcal{A} \longrightarrow \mathbf{coCh}\mathcal{B}$.

Proof. It suffices to show that both categories have arbitrary products. As before we simply take the arbitrary pointwise products and induce morphisms making them into a (co)chain. This is a product, which shows that *any* product must be a product pointwise. The other claims are straightforward to check. □

Lemma 72. *Let $\mathcal{A}$ be a complete abelian category with exact products. Then for $n \in \mathbb{Z}$ the functors $H_n : \mathbf{Ch}\mathcal{A} \longrightarrow \mathcal{A}$ and $H^n : \mathbf{coCh}\mathcal{A} \longrightarrow \mathcal{A}$ preserve products.*

Proposition 73. *Let $\mathcal{A}, \mathcal{B}$ be complete abelian categories with exact products, and suppose $\mathcal{A}$ is infinite and has enough injectives. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ preserves products, then so does R^nF for $n \geq 0$.*

Proof. The functors R^nF are additive, and therefore preserve zero objects, so we can restrict our attention to nonempty products $\prod_i A_i$. If the chosen resolution for A_i is $I_i : 0 \longrightarrow A_i \longrightarrow I_i^0 \longrightarrow I_i^1 \longrightarrow \cdots$ then since $\mathcal{A}$ has exact products the following sequence is an injective resolution for $\prod A_i$

$$\prod I^i : 0 \longrightarrow \prod A_i \longrightarrow \prod I_i^0 \longrightarrow \prod I_i^1 \longrightarrow \cdots$$

and we can assume this is the chosen resolution of $\prod A_i$. The functor $\mathbf{coCh}\mathcal{A} \longrightarrow \mathbf{coCh}\mathcal{B}$ induced by F preserves products, as does $H^n : \mathbf{coCh}\mathcal{B} \longrightarrow \mathcal{B}$. Since the resolution $\prod I_i$ is a product in $\mathbf{coCh}\mathcal{A}$ for the resolutions I_i it follows immediately that the morphisms $R^nF(\oplus A_i) \longrightarrow R^nF(A_i)$ are a product for $n \geq 0$. □

12 Delta Functors

Let $\mathcal{A}$ be an abelian category with enough projectives. We have associated to an additive functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ a sequence of additive functors $L_0T, L_1T, \ldots$ together with connecting morphisms $\omega_n : L_nT(A'') \longrightarrow L_{n-1}T(A')$ for any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$. The proper way to "package" the sequence of functors and the associated connecting morphisms is the concept of a *δ-functor*.

Definition 17. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *homological δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a sequence $\{T_n\}_{n \geq 0}$ of additive functors $T_n : \mathcal{A} \longrightarrow \mathcal{B}$ together with an assignment of a morphism

$$\delta_n : T_n(A'') \longrightarrow T_{n-1}(A')$$

to every $n \geq 1$ and short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of $\mathcal{A}$, satisfying the following properties

1. For each such short exact sequence in $\mathcal{A}$, we have a long exact sequence

$$\begin{gathered} \cdots \longrightarrow T_n(A') \longrightarrow T_n(A) \longrightarrow T_n(A'') \xrightarrow{\delta_n} T_{n-1}(A') \longrightarrow \cdots \\ \cdots \longrightarrow T_1(A'') \xrightarrow{\delta_1} T_0(A') \longrightarrow T_0(A) \longrightarrow T_0(A'') \longrightarrow 0 \end{gathered} \tag{35}$$

   In particular the functor T_0 is right exact.

2. For every commutative diagram in $\mathcal{A}$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array} \tag{36}$$

the following diagram commutes for all $n \geq 1$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & T_n(A') & \longrightarrow & T_n(A) & \longrightarrow & T_n(A'') & \xrightarrow{\delta_n} & T_{n-1}(A') & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & T_n(B') & \longrightarrow & T_n(B) & \longrightarrow & T_n(B'') & \xrightarrow[\delta_n]{} & T_{n-1}(B') & \longrightarrow \cdots \end{array}$$

Definition 18. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *cohomological δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a sequence $\{T^n\}_{n\geq 0}$ of additive functors $T^n : \mathcal{A} \longrightarrow \mathcal{B}$ together with an assignment of a morphism

$$\delta^n : T^n(A'') \longrightarrow T^{n+1}(A')$$

to every $n \geq 0$ and short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of $\mathcal{A}$, satisfying the following properties

1. For each such short exact sequence in $\mathcal{A}$, we have a long exact sequence

$$\begin{array}{c} 0 \longrightarrow T^0(A') \longrightarrow T^0(A) \longrightarrow T^0(A'') \xrightarrow{\delta^0} T^1(A') \longrightarrow \cdots \\ \cdots \longrightarrow T^n(A'') \xrightarrow{\delta^n} T^{n+1}(A') \longrightarrow T^{n+1}(A) \longrightarrow T^{n+1}(A'') \longrightarrow \cdots \end{array} \tag{37}$$

   In particular the functor T_0 is left exact.

2. For every commutative diagram in $\mathcal{A}$ with exact rows of the form (36) the following diagram commutes for all $n \geq 0$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & T^n(A') & \longrightarrow & T^n(A) & \longrightarrow & T^n(A'') & \xrightarrow{\delta^n} & T^{n+1}(A') & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & T^n(B') & \longrightarrow & T^n(B) & \longrightarrow & T^n(B'') & \xrightarrow[\delta^n]{} & T^{n+1}(B') & \longrightarrow \cdots \end{array}$$

Definition 19. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *contravariant homological δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a homological δ-functor between $\mathcal{A}^{\text{op}}$ and $\mathcal{B}$. Similarly a *contravariant cohomological δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a cohomological δ-functor between $\mathcal{A}^{\text{op}}$ and $\mathcal{B}$.

Example 1. *(i)* Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. For an assignment of projective resolutions $\mathcal{P}$ the left derived functors $\{L_nT\}_{n\geq 0}$ together with the connecting morphisms ω_n defined in Theorem 34 are a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$.

(ii) Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category, and let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. For every assignment of injective resolutions $\mathcal{I}$ the right derived functors $\{R^nT\}_{n\geq 0}$ together with the connecting morphisms ω^n defined in Theorem 41 are a cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$.

Definition 20. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *morphism* $\psi : S \longrightarrow T$ of homological δ-functors is a family of natural transformations $\{\psi_n : S_n \longrightarrow T_n\}_{n\geq 0}$ with the property that for every short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of $\mathcal{A}$ the following diagram commutes for $n \geq 1$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & S_n(A') & \longrightarrow & S_n(A) & \longrightarrow & S_n(A'') & \xrightarrow{\delta_n} & S_{n-1}(A') & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & T_n(A') & \longrightarrow & T_n(A) & \longrightarrow & T_n(A'') & \xrightarrow[\delta_n]{} & T_{n-1}(A') & \longrightarrow \cdots \end{array}$$

Morphisms of homological δ-functors can be composed in the obvious way, with the identity morphism given pointwise by the identity natural transformation. A homological δ-functor T is *universal* if given any homological δ-functor S and natural transformation $f : S_0 \longrightarrow T_0$, there exists a *unique* morphism of homological δ-functors $\psi : S \longrightarrow T$ with $\psi_0 = f$.

A *morphism* $\psi : S \longrightarrow T$ of cohomological δ-functors is a family of natural transformations $\{\psi^n : S^n \longrightarrow T^n\}_{n\geq 0}$ with the property that for every short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of $\mathcal{A}$ the following diagram commutes for $n \geq 0$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & S^n(A') & \longrightarrow & S^n(A) & \longrightarrow & S^n(A'') & \xrightarrow{\delta^n} & S^{n+1}(A') & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & T^n(A') & \longrightarrow & T^n(A) & \longrightarrow & T^n(A'') & \xrightarrow[\delta^n]{} & T^{n+1}(A') & \longrightarrow & \cdots
\end{array}$$

Morphisms cohomological δ-functors can be composed in the obvious way. A cohomological δ-functor T is *universal* if given any cohomological δ-functor S and natural transformation $f : T^0 \longrightarrow S^0$, there exists a *unique* morphism of cohomological δ-functors $\psi : T \longrightarrow S$ with $\psi^0 = f$.

Example 2. Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Suppose we have two assignments of projective resolutions $\mathcal{P}, \mathcal{Q}$ with associated homological δ-functors $L^{\mathcal{P}}, L^{\mathcal{Q}}$ of left derived functors. Then the canonical natural equivalences $\psi_n : L^{\mathcal{P}}_n \longrightarrow L^{\mathcal{Q}}_n$ form an isomorphism of homological δ-functors $\psi : L^{\mathcal{P}} \longrightarrow L^{\mathcal{Q}}$. A similar observation holds if $\mathcal{A}$ has enough injectives and we have two assignments of injective resolutions.

Definition 21. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. A universal homological δ-functor T between $\mathcal{A}$ and $\mathcal{B}$ together with a natural equivalence $u : T_0 \cong F$ is called a *left satellite* of F and the functors $T_n : \mathcal{A} \longrightarrow \mathcal{B}$ are called *left satellite functors* of F. Left satellites are unique up to canonical isomorphism if they exist, in the sense that if (S, v) is another left satellite of F there is a unique isomorphism $\psi : T \longrightarrow S$ of homological δ-functors with $v\psi^0 = u$.

A universal cohomological δ-functor T between $\mathcal{A}$ and $\mathcal{B}$ together with a natural equivalence $T^0 \cong F$ is called a *right satellite* of F and the functors $T^n : \mathcal{A} \longrightarrow \mathcal{B}$ are called *right satellite functors* of F. Right satellites are unique up to canonical isomorphism if they exist, since if (S, v) is another right satellite of F there is a unique isomorphism $\psi : T \longrightarrow S$ of cohomological δ-functors with $v\psi_0 = u$.

Definition 22. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. An additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called *effaceable* if for each object A of $\mathcal{A}$ there is a monomorphism $u : A \longrightarrow I$ such that $F(u) = 0$. We call F *coeffaceable* if for every A there is an epimorphism $u : P \longrightarrow A$ such that $F(u) = 0$.

Example 3. Suppose that $\mathcal{A}$ has enough projectives and let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Then the left derived functors L_nT of T are coeffaceable for $n \geq 1$ since every object A admits an epimorphism $P \longrightarrow A$ with P projective, and $L_nT(P) = 0$ for $n \geq 1$. Similarly if $\mathcal{A}$ has enough injectives then the right derived functors R^nT are effaceable for $n \geq 1$.

Theorem 74. *Let T be a homological δ-functor between abelian categories $\mathcal{A}, \mathcal{B}$. If the additive functors $T_n : \mathcal{A} \longrightarrow \mathcal{B}$ are coeffaceable for $n \geq 1$ then T is universal. If T is a cohomological δ-functor between abelian categories with T^n effaceable for $n \geq 1$ then T is universal.*

Corollary 75. *Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a right exact functor. Then the homological δ-functor of left derived functors $\{L_nT\}_{n\geq 0}$ together with the canonical natural equivalence $L_0T \cong T$ form a left satellite of T.*

Proof. Choose an assignment of projective resolutions $\mathcal{P}$ with respect to which all left derived functors are calculated and let $u : L_0T \longrightarrow T$ be the canonical natural equivalence. Then by Theorem 74 the homological δ-functor $\{L_nT\}_{n\geq 0}$ together with u is a left satellite of T, since the functors L_nT are coeffaceable for $n \geq 1$. □

Corollary 76. *Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category and $T : \mathcal{A} \longrightarrow \mathcal{B}$ a left exact functor. Then the cohomological δ-functor of right derived functors $\{R^nT\}_{n\geq 0}$ together with the canonical natural equivalence $R^0T \cong T$ form a right satellite of T.*

Definition 23. Let $T = \{T_n\}_{n\geq 0}$ be a homological δ-functor between abelian categories $\mathcal{A}, \mathcal{B}$ and let $U : \mathcal{B} \longrightarrow \mathcal{C}$ be an exact functor, where $\mathcal{C}$ is another abelian category. The composites $UT_n : \mathcal{A} \longrightarrow \mathcal{C}$ are additive functors. Given an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $\mathcal{A}$ let $\delta_n : T_n(A'') \longrightarrow T_{n-1}(A')$ be the connecting morphism and $U\delta_n$ its image under U. Then the functors $\{UT_n\}_{n\geq 0}$ together with the $U\delta_n$ form a homological δ-functor between $\mathcal{A}$ and $\mathcal{C}$ which we denote by UT. If $\psi : S \longrightarrow T$ is a morphism of homological δ-functors between $\mathcal{A}, \mathcal{B}$ then $U\psi : US \longrightarrow UT$ is a morphism of homological δ-functors

Similarly if $U : \mathcal{C} \longrightarrow \mathcal{A}$ is an exact functor then we have additive functors $T_nU : \mathcal{C} \longrightarrow \mathcal{B}$ and for an exact sequence $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ in $\mathcal{A}$ we have the connecting morphism $\delta_n : T_nU(C'') \longrightarrow T_{n-1}U(C')$ of the exact sequence $0 \longrightarrow UC' \longrightarrow UC \longrightarrow UC'' \longrightarrow 0$ in $\mathcal{A}$. The functors $\{T_nU\}_{n\geq 0}$ together with these connecting morphisms form a homological δ-functor between $\mathcal{C}$ and $\mathcal{B}$ which we denote by TU. If $\psi : S \longrightarrow T$ is a morphism of homological δ-functors between $\mathcal{A}, \mathcal{B}$ then $\psi U : SU \longrightarrow TU$ is a morphism of homological δ-functors.

Definition 24. Let $T = \{T^n\}_{n\geq 0}$ be a cohomological δ-functor between abelian categories $\mathcal{A}, \mathcal{B}$ and let $U : \mathcal{B} \longrightarrow \mathcal{C}$ be an exact functor, where $\mathcal{C}$ is another abelian category. The composites $UT^n : \mathcal{A} \longrightarrow \mathcal{C}$ are additive functors. Given an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $\mathcal{A}$ let $\delta^n : T^n(A'') \longrightarrow T^{n+1}(A')$ be the connecting morphism and $U\delta^n$ its image under U. Then the functors $\{UT^n\}_{n\geq 0}$ together with the $U\delta^n$ form a cohomological δ-functor between $\mathcal{A}$ and $\mathcal{C}$ which we denote by UT. If $\psi : S \longrightarrow T$ is a morphism of cohomological δ-functors between $\mathcal{A}, \mathcal{B}$ then $U\psi : US \longrightarrow UT$ is a morphism of cohomological δ-functors.

Similarly if $U : \mathcal{C} \longrightarrow \mathcal{A}$ is an exact functor then we have additive functors $T^nU : \mathcal{C} \longrightarrow \mathcal{B}$ and for an exact sequence $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ in $\mathcal{A}$ we have the connecting morphism $\delta^n : T^nU(C'') \longrightarrow T^{n+1}U(C')$ of the exact sequence $0 \longrightarrow UC' \longrightarrow UC \longrightarrow UC'' \longrightarrow 0$ in $\mathcal{A}$. The functors $\{T^nU\}_{n\geq 0}$ together with these connecting morphisms form a cohomological δ-functor between $\mathcal{C}$ and $\mathcal{B}$ which we denote by TU. If $\psi : S \longrightarrow T$ is a morphism of cohomological δ-functors between $\mathcal{A}, \mathcal{B}$ then $\psi U : SU \longrightarrow TU$ is a morphism of cohomological δ-functors.

We can now give an alternative proof (and slight generalisation) of the results on base change.

Proposition 77. *Suppose we have a commutative diagram of abelian categories and additive functors where $\mathcal{A}, \mathcal{A}'$ have enough injectives, U, u are exact, T, T' are left exact and the functor U sends injective objects of $\mathcal{A}$ into right T'-acyclic objects of $\mathcal{A}'$*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{B} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{A}' & \xrightarrow[T']{} & \mathcal{B}' \end{array}$$

Then there are canonical natural equivalences $u \circ R^n(T) \cong R^n(u \circ T) \cong R^n(T') \circ U$ for $n \geq 0$. Given an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

the following diagram commutes for $n \geq 0$

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & uR^nT(A') & \longrightarrow & uR^nT(A) & \longrightarrow & uR^nT(A'') & \xrightarrow{\delta^n} & uR^{n+1}T(A') & \longrightarrow \cdots \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\ \cdots \longrightarrow & R^n(uT)(A') & \longrightarrow & R^n(uT)(A) & \longrightarrow & R^n(uT)(A'') & \xrightarrow{\delta^n} & R^{n+1}(uT)(A') & \longrightarrow \cdots \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\ \cdots \longrightarrow & R^n(T')(UA') & \longrightarrow & R^n(T')(UA) & \longrightarrow & R^n(T')(UA'') & \xrightarrow[\delta^n]{} & R^{n+1}(T')(UA') & \longrightarrow \cdots \end{array}$$

Proof. Choose assignments of injective resolutions $\mathcal{I}, \mathcal{I}'$ for $\mathcal{A}, \mathcal{A}'$ respectively, with respect to which all right derived functors are calculated. Let $G : \mathcal{A} \longrightarrow \mathcal{B}'$ be the composite $u \circ T = T' \circ U$. Then using Definition 24 we have *three* cohomological δ-functors between $\mathcal{A}$ and $\mathcal{B}'$

$$\{u \circ R^n T\}_{n \geq 0}, \{R^n G\}_{n \geq 0}, \{R^n(T') \circ U\}_{n \geq 0}$$

The assumption that U sends injective objects of $\mathcal{A}$ to right T'-acyclic objects of $\mathcal{A}'$ means that all three of these cohomological δ-functors are universal by Theorem 74. But all three functors $u \circ R^0 T, R^0 G, R^0(T') \circ U$ are canonically naturally equivalent to G. So by uniqueness of right satellites there are canonical isomorphisms of cohomological δ-functors $\psi : \{u \circ R^n T\}_{n \geq 0} \cong \{R^n G\}_{n \geq 0}, \varphi : \{R^n G\}_{n \geq 0} \cong \{R^n(T') \circ U\}_{n \geq 0}$ and $\theta : \{u \circ R^n T\}_{n \geq 0} \cong \{R^n(T') \circ U\}_{n \geq 0}$ and moreover $\theta = \varphi \circ \psi$. Commutativity of the large diagram is part of the definition of a morphism of cohomological δ-functors. □

Remark 2. With the notation of Proposition 77 make the further assumption that u is *faithful* and fix assignments of injective resolutions $\mathcal{I}, \mathcal{I}'$ to the objects of $\mathcal{A}, \mathcal{A}'$ respectively. Let A be an object of $\mathcal{A}$ with injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

Then by assumption UI is a right T-acyclic resolution of UA. Let J be the chosen injective resolution of UA. Then by Theorem 19 we can lift the identity to a morphism of cochain complexes $UI \longrightarrow J$ and we therefore have a canonical morphism on cohomology $H^n(T'UI) \longrightarrow R^n(T')(UA)$. But $T'UI = uTI$ and by Proposition 61 there is a canonical isomorphism $H^n(uTI) \cong u(H^n(TI)) = uR^nT(A)$. For $n \geq 0$ the composite gives a canonical morphism

$$\mu^n_A : uR^nT(A) \longrightarrow R^n(T')(UA)$$

This morphism is natural in A, so we have a natural transformation $\mu^n : u \circ R^n T \longrightarrow R^n(T') \circ U$ for $n \geq 0$. The natural transformation μ^0 is the composite of the canonical natural equivalences $u \circ R^0 T \cong G \cong R^0(T') \circ U$. Therefore to show that μ^n is the isomorphism θ^n defined in the proof of Proposition 77 for $n \geq 0$, we need only show that μ is a morphism of cohomological δ-functors $\{u \circ R^n T\}_{n \geq 0} \longrightarrow \{R^n(T') \circ U\}_{n \geq 0}$.

Suppose we are given a short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $\mathcal{A}$. Let $\varepsilon' : A' \longrightarrow I', \varepsilon'' : A'' \longrightarrow I''$ and $\eta' : UA' \longrightarrow J', \eta'' : UA'' \longrightarrow J''$ be the chosen injective resolutions and use Corollary 40 to construct injective resolutions $\varepsilon : A \longrightarrow I$ and $\eta : UA \longrightarrow J$ fitting into short exact sequences of cochain complexes

$$0 \longrightarrow I' \longrightarrow I \longrightarrow I'' \longrightarrow 0$$
$$0 \longrightarrow J' \longrightarrow J \longrightarrow J'' \longrightarrow 0$$

with each morphism lifting the appropriate morphism of $\mathcal{A}$ or $\mathcal{A}'$. By Theorem 19 we can lift the identities to morphisms of cochain complexes $g : UI' \longrightarrow J$, $f : UI \longrightarrow J, e : UI'' \longrightarrow J''$ so that we have a diagram of cochain complexes in $\mathcal{B}'$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & uTI' & \longrightarrow & uTI & \longrightarrow & uTI'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T'(g)} & & \downarrow{\scriptstyle T'(f)} & & \downarrow{\scriptstyle T'(e)} & & \\ 0 & \longrightarrow & T'J' & \longrightarrow & T'J & \longrightarrow & T'J'' & \longrightarrow & 0 \end{array}$$

There is no reason to expect this diagram to commute, but nonetheless by (DTC,Theorem 9) and (DTC,Proposition 10) we have a commutative diagram on cohomology

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^n(uTI') & \longrightarrow & H^n(uTI) & \longrightarrow & H^n(uTI'') & \xrightarrow{\delta^n} & H^{n+1}(uTI') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^n(T'J') & \longrightarrow & H^n(T'J) & \longrightarrow & H^n(T'J'') & \xrightarrow[\delta^n]{} & H^{n+1}(T'J') & \longrightarrow & \cdots \end{array}$$

Examining the construction of the connecting morphisms in Theorem 41 and using Proposition 62 we see that the following diagram commutes for $n \geq 0$

$$\begin{array}{ccc} uR^nT(A'') & \xrightarrow{u(\delta^n)} & uR^{n+1}T(A') \\ \downarrow{\scriptstyle \mu^n_{A''}} & & \downarrow{\scriptstyle \mu^{n+1}_{A'}} \\ R^n(T')(UA'') & \xrightarrow[\delta^n]{} & R^n(T')(UA') \end{array}$$

Therefore $\mu : \{u \circ R^n T\}_{n \geq 0} \longrightarrow \{R^n(T') \circ U\}_{n \geq 0}$ is a morphism of cohomological δ-functors. Since $\mu^0 = \theta^0$ it follows by definition of a universal cohomological δ-functor that $\mu = \theta$. Therefore the natural transformations μ^n are all natural equivalences.

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