Derived Categories Part II

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October 5, 2006

In this second part of our notes on derived categories we define derived functors and establish their basic properties. The first major example is the derived Hom functor. Finally we prove that a grothendieck abelian category satisfies Brown representability.

Contents

1 Introduction

All notation and conventions are from our notes on Derived Functors. In particular we assume that every abelian category comes with canonical structures that allow us to define the cohomology of cochain complexes in an unambiguous way. If we write *complex* we mean *cochain complex*, and we write $\mathbf{C}(\mathcal{A})$ for the abelian category of all complexes in $\mathcal{A}$. As usual we write $A = 0$ to indicate that A is a zero object (not necessarily the canonical one). We use the terms *preadditive category* and *additive category* as defined in (AC,Section 2). The reader should be familiar with the contents of our notes on Derived Categories (DTC).

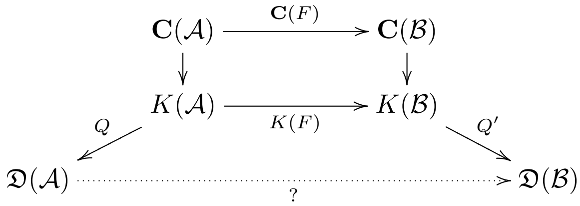
Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories, and suppose that $\mathcal{A}$ has enough injectives. In classical homological algebra one defines the right derived functors $R^iF : \mathcal{A} \longrightarrow \mathcal{B}$ (DF,Section 5) on an object $A \in \mathcal{A}$ by choosing an injective resolution $q : A \longrightarrow I$

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

and setting $R^iF(A) = H^iF(I)$ for $i \geq 0$. The more modern perspective is that you should work with the original complex $F(I)$ rather than take cohomology, and allow *complexes* in place of the object A by using the theory of resolutions of unbounded complexes developed in (DTC,Section 7). To realise this ambition one has to work in the derived category where we obtain a triangulated functor $\mathbb{R}F : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$. This packages all the classical derived functors $R^iF(-)$ into a single object, in a way similar to but more elegant than a delta functor (DF,Section 12).

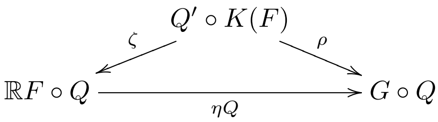
2 Derived Functors

For this section the reader should be familiar with the contents of (TRC,Section 5). If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor between abelian categories, then there is an induced triangulated functor $K(F) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ (DTC,Section 3.1) making the following diagram commute



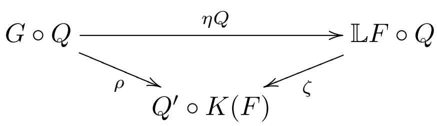
If F is exact then it lifts to a triangulated functor between the derived categories (DTC,Lemma 23), but in general this is not possible. More generally, the *right and left derived functors* of F, when they exist, are triangulated functors $\mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ with a certain universal property which says that they are "the best" such functors associated to F.

Definition 1. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. A *right derived functor of* F is a pair $(\mathbb{R}F, \zeta)$ consisting of a triangulated functor $\mathbb{R}F : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ and a trinatural transformation $\zeta : Q' \circ K(F) \longrightarrow \mathbb{R}F \circ Q$ with the following universal property: given any triangulated functor $G : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ and trinatural transformation $\rho : Q' \circ K(F) \longrightarrow G \circ Q$ there is *unique* trinatural transformation $\eta : \mathbb{R}F \longrightarrow G$ making the following diagram commute



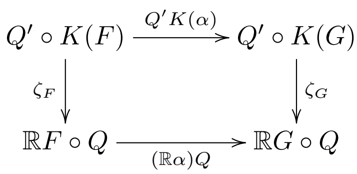
In the notation of (TRC,Definition 46) this says that the pair $(\mathbb{R}F, \zeta)$ is a right derived functor of the composite $Q' \circ K(F)$ with respect to the category $\mathcal{Z}$ of exact complexes in $K(\mathcal{A})$. By abuse of notation we often say that $\mathbb{R}F$ is a right derived functor of F, and drop ζ from the notation. Clearly if a right derived functor exists it is unique up to canonical trinatural equivalence.

Definition 2. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. A *left derived functor* of F is a pair $(\mathbb{L}F, \zeta)$ consisting of a triangulated functor $\mathbb{L}F : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ and a trinatural transformation $\zeta : \mathbb{L}F \circ Q \longrightarrow Q' \circ K(F)$ with the following universal property: given any triangulated functor $G : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ and trinatural transformation $\rho : G \circ Q \longrightarrow Q' \circ K(F)$ there is a *unique* trinatural transformation $\eta : G \longrightarrow \mathbb{L}F$ making the following diagram commute



In the notation of (TRC,Definition 50) this says that the pair $(\mathbb{L}F, \zeta)$ is a left derived functor of the composite $Q' \circ K(F)$ with respect to the category $\mathcal{Z}$ of exact complexes in $K(\mathcal{A})$. By abuse of notation we often say that $\mathbb{L}F$ is a left derived functor of F, and drop ζ from the notation. Clearly if a left derived functor exists it is unique up to canonical trinatural equivalence.

Definition 3. Let $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ be additive functors between abelian categories, and suppose that right derived functors $\mathbb{R}F, \mathbb{R}G$ exist. Given a natural transformation $\alpha : F \longrightarrow G$ there is a unique trinatural transformation $\mathbb{R}\alpha : \mathbb{R}F \longrightarrow \mathbb{R}G$ making the following diagram commute



Given another natural transformation $\beta : G \longrightarrow H$ we have $\mathbb{R}(\beta\alpha) = \mathbb{R}(\beta) \circ \mathbb{R}(\alpha)$ and similarly $\mathbb{R}(\alpha + \alpha') = \mathbb{R}\alpha + \mathbb{R}\alpha'$ and $\mathbb{R}1 = 1$.

Dually, suppose that left derived functors $\mathbb{L}F, \mathbb{L}G$ exist. There is a unique trinatural transformation $\mathbb{L}\alpha : \mathbb{L}F \longrightarrow \mathbb{L}G$ making the following diagram commute

$$\begin{array}{ccc} \mathbb{L}F \circ Q & \xrightarrow{(\mathbb{L}\alpha)Q} & \mathbb{L}G \circ Q \\ {\scriptstyle \zeta_F}\downarrow & & \downarrow{\scriptstyle \zeta_G} \\ Q' \circ K(F) & \xrightarrow[Q'K(\alpha)]{} & Q' \circ K(G) \end{array}$$

Given another natural transformation $\beta : G \longrightarrow H$ we have $\mathbb{L}(\beta\alpha) = \mathbb{L}(\beta) \circ \mathbb{L}(\alpha)$ and similarly $\mathbb{L}(\alpha + \alpha') = \mathbb{L}\alpha + \mathbb{L}\alpha'$ and $\mathbb{L}1 = 1$.

Remark 1. Let $U : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. If U is exact, then there is a unique triangulated functor $\mathfrak{D}(U) : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ lifting $K(U)$ (DTC,Lemma 23), which is actually a right and left derived functor of U. Since it is notationally very convenient, we will often simply write U in place of $\mathfrak{D}(U)$ when there is no chance of confusion.

Existence of derived functors follows from the existence of enough complexes satisfying an acyclicity condition. An example in classical homological algebra is the existence of "enough" injectives or projectives.

Definition 4. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor and X a complex in $\mathcal{A}$. We say that X is *right F-acyclic* if whenever there is a quasi-isomorphism $s : X \longrightarrow Y$ there exists a quasi-isomorphism $t : Y \longrightarrow Z$ such that $F(ts)$ is a quasi-isomorphism. In the notation of (TRC,Definition 48) this says that X is right $Q'K(F)$-acyclic with respect to the exact complexes $\mathcal{Z}$.

Dually we say that X is *left F-acyclic* if whenever there is a quasi-isomorphism $s : Y \longrightarrow X$ there exists a quasi-isomorphism $t : Z \longrightarrow Y$ such that $F(st)$ is a quasi-isomorphism. In the notation of (TRC,Definition 52) this says that X is left $Q'K(F)$-acyclic with respect to $\mathcal{Z}$. If X is left F-acyclic (resp. right F-acyclic) and exact, then $F(X)$ is exact (TRC,Proposition 115). On the other hand, if F is an exact functor then every complex X in $\mathcal{A}$ is both left and right F-acyclic.

We will show in Section 5 that there is no conflict with the old-fashioned notion of acyclicity as given for example in (DF,Definition 15) and (DF,Definition 14). Our first major result shows that the existence of sufficiently many right acyclic complexes guarantees the existence of a right derived functor. Hoinjective complexes are right acyclic for everything, so right derived functors usually exist.

Theorem 1. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. Suppose that for every complex X in $\mathcal{A}$ there exists a quasi-isomorphism $\eta_X : X \longrightarrow A_X$ with A_X right F-acyclic. Then F admits a right derived functor $(\mathbb{R}F, \zeta)$ with the following properties*

(i) For any complex X in $\mathcal{A}$ we have $\mathbb{R}F(X) = F(A_X)$ and $\zeta_X = F(\eta_X)$.

(ii) A complex X in $\mathcal{A}$ is right F-acyclic if and only if ζ_X is an isomorphism in $\mathfrak{D}(\mathcal{B})$.

Proof. This is a special case of (TRC,Theorem 116). Note that the triangulated functor F studied in (TRC,Theorem 116) is precisely the composite $Q' \circ K(F)$ in our current notation. □

Dually the existence of sufficiently many left acyclic complexes guarantees the existence of a left derived functor. Since hoprojectives are more rare than hoinjectives, there is usually some work involved in finding enough left acyclics (the hoflat complexes of our later notes being a good example).

Theorem 2. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. Suppose that for every complex X in $\mathcal{A}$ there exists a quasi-isomorphism $\eta_X : A_X \longrightarrow X$ with A_X left F-acyclic. Then F admits a left derived functor $(\mathbb{L}F, \zeta)$ with the following properties*

(i) *For any complex* X *in* $\mathcal{A}$ *we have* $\mathbb{L}F(X) = F(A_X)$ *and* $\zeta_X = F(\eta_X)$.

(ii) *A complex* X *in* $\mathcal{A}$ *is left* F*-acyclic if and only if* ζ_X *is an isomorphism in* $\mathfrak{D}(\mathcal{B})$.

Proof. This is a special case of (TRC,Theorem 125). □

Remark 2. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor satisfying the conditions of Theorem 1. Then since the right derived functor is unique, if $(\mathbb{R}F, \zeta)$ is *any* right derived functor of F we deduce that a complex X in $\mathcal{A}$ is right F-acyclic if and only if $\zeta_X : F(X) \longrightarrow \mathbb{R}F(X)$ is an isomorphism.

Dually if F satisfies the conditions of Theorem 2 and $(\mathbb{L}F, \zeta)$ is any left derived functor of F, then $\zeta_X : \mathbb{L}F(X) \longrightarrow F(X)$ is an isomorphism if and only if X is left F-acyclic.

Lemma 3. *Let* $\mathcal{A}$ *be an abelian category and* I *a complex in* $\mathcal{A}$. *Then* I *is hoinjective if and only if it is right* F*-acyclic with respect to* $\mathcal{Z}$ *for every triangulated functor* $F : K(\mathcal{A}) \longrightarrow \mathcal{T}$.

Proof. As usual $\mathcal{Z}$ denotes the triangulated subcategory of exact complexes in $K(\mathcal{A})$. The complex I is hoinjective if and only if every quasi-isomorphism of complexes $I \longrightarrow Y$ is a coretraction in $K(\mathcal{A})$ (DTC,Proposition 51). If this is the case then it is clear that I is right F-acyclic for any triangulated functor F. Conversely if I is right acyclic for the identity $1 : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$ then every quasi-isomorphism $I \longrightarrow Y$ admits a quasi-isomorphism $Y \longrightarrow Z$ such that the composite is an isomorphism in $K(\mathcal{A})$. It follows that $I \longrightarrow Y$ is a coretraction, as required. The second claim is (TRC,Proposition 115). □

Corollary 4. *Let* $\mathcal{A}$ *be an abelian category with enough hoinjectives,* $\mathcal{B}$ *an abelian category and* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *an additive functor. Then* F *has a right derived functor.*

Proof. This follows immediately from Theorem 2. In fact, in the more general sense of a right derived functor given in (TRC,Definition 46), any triangulated functor $T : K(\mathcal{A}) \longrightarrow \mathcal{T}$ has a right derived functor. □

Any grothendieck abelian category $\mathcal{A}$ has enough hoinjectives (DTC,Remark 49) so any additive functor $\mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories with $\mathcal{A}$ grothendieck abelian has a right derived functor. We introduce some notation to emphasise the analogy with the construction of ordinary derived functors in (DF,Section 5).

Definition 5. Let $\mathcal{A}$ be an abelian category. An *assignment of hoinjective resolutions for* $\mathcal{A}$ is an assignment to every complex X in $\mathcal{A}$ of a hoinjective complex I_X and quasi-isomorphism of complexes $\eta_X : X \longrightarrow I_X$. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories, where $\mathcal{A}$ has enough hoinjectives. Then Theorem 1 associates to any assignment $\mathcal{I}$ of hoinjective resolutions for $\mathcal{A}$ a canonical right derived functor $(\mathbb{R}_{\mathcal{I}}F, \zeta)$.

We will have occasion to use the following slightly more general version of the above, which once again is a special case of (TRC,Theorem 116).

Corollary 5. *Let* $\mathcal{A}$ *be an abelian category with enough hoinjectives,* $\mathcal{I}$ *an assignment of hoinjective resolutions and* $T : K(\mathcal{A}) \longrightarrow \mathcal{T}$ *a triangulated functor. Then there is a canonical right derived functor* $(\mathbb{R}_{\mathcal{I}}T, \zeta)$ *of* F *with the following properties*

(i) *For any complex* X *in* $\mathcal{A}$ *we have* $\mathbb{R}_{\mathcal{I}}T(X) = T(I_X)$ *and* $\zeta_X = T(\eta_X)$.

(ii) *A complex* X *in* $\mathcal{A}$ *is right* T*-acyclic if and only if* ζ_X *is an isomorphism in* $\mathcal{T}$.

Remark 3. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories where $\mathcal{A}$ has enough hoinjectives, with right derived functor $(\mathbb{R}F, \zeta)$. If X is a complex in $\mathcal{A}$ and $\alpha : X \longrightarrow I$ a hoinjective resolution then we have a commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\alpha)} & F(I) \\ {\scriptstyle \zeta_X}\downarrow & & \downarrow{\scriptstyle \zeta_I} \\ \mathbb{R}F(X) & \xrightarrow[\mathbb{R}F(\alpha)]{} & \mathbb{R}F(I) \end{array}$$

The right and bottom morphisms are isomorphisms, so we deduce a canonical isomorphism $\mathbb{R}F(X) \cong F(I)$ in $\mathfrak{D}(\mathcal{B})$, so in some sense you can use any resolution to calculate $\mathbb{R}F(X)$. If $f : X \longrightarrow Y$ is a morphism of $K(\mathcal{A})$ and $Y \longrightarrow J$ a hoinjective resolution then there is a unique morphism $g : I \longrightarrow J$ in $K(\mathcal{A})$ fitting into a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & J \end{array}$$

and the following diagram then commutes in $\mathfrak{D}(\mathcal{B})$

$$\begin{array}{ccc} \mathbb{R}F(X) & \Longrightarrow & F(I) \\ {\scriptstyle \mathbb{R}F(f)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \mathbb{R}F(Y) & \Longrightarrow & F(J) \end{array}$$

The following result tells us how to compose right derived functors.

Theorem 6. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough hoinjectives, $\mathcal{C}$ an abelian category and $F : \mathcal{A} \longrightarrow \mathcal{B}, G : \mathcal{B} \longrightarrow \mathcal{C}$ additive functors. If F sends hoinjectives to right G-acyclics, then there is a canonical trinatural equivalence*

$$\theta : \mathbb{R}(GF) \longrightarrow \mathbb{R}G \circ \mathbb{R}F$$

Proof. The conditions given for the theorem make the statement easy to read, but we will need something stronger in applications. Instead, let $F : \mathcal{A} \longrightarrow \mathcal{B}, G : \mathcal{B} \longrightarrow \mathcal{C}$ be additive functors between arbitrary abelian categories and assume that

(i) Every complex X in $\mathcal{A}$ admits a quasi-isomorphism $X \longrightarrow A_X$ with A_X right F-acyclic, such that $F(A_X)$ is right G-acyclic.

(ii) Every complex Y in $\mathcal{B}$ admits a quasi-isomorphism $Y \longrightarrow B_Y$ with B_Y right G-acyclic.

These conditions are satisfied under the conditions given in the statement of the theorem. Let $\mathcal{A}$ be the smallest triangulated subcategory of $K(\mathcal{A})$ containing A_X for every complex X in $\mathcal{A}$, and let $\mathcal{A}'$ be the class of all right G-acyclic complexes. By (TRC,Remark 77) the class $\mathcal{A}$ is right adapted for F, and by assumption $F(\mathcal{A}) \subseteq \mathcal{A}'$ so we are in the situation of of (TRC,Theorem 113), which in particular shows that the composite GF has a right derived functor. Let us elaborate a little on the consequences of this general result.

Let $(\mathbb{R}F, \zeta), (\mathbb{R}G, \omega)$ and $(\mathbb{R}(GF), \xi)$ be arbitrary right derived functors. We have a trinatural transformation

$$Q''K(GF) \xrightarrow{\omega K(F)} \mathbb{R}(G)Q'K(F) \xrightarrow{\mathbb{R}(G)\zeta} \mathbb{R}(G)\mathbb{R}(F)Q$$

which we denote by μ. This induces a trinatural transformation $\theta : \mathbb{R}(GF) \longrightarrow \mathbb{R}(G)\mathbb{R}(F)$ unique making the following diagram commute

$$\begin{array}{ccc} Q''K(GF) & \xrightarrow{\omega K(F)} & \mathbb{R}(G)Q'K(F) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \mathbb{R}(G)\zeta} \\ \mathbb{R}(GF)Q & \xrightarrow[\theta Q]{} & \mathbb{R}(G)\mathbb{R}(F)Q \end{array}$$

and it follows from (TRC,Theorem 113) that θ is a trinatural equivalence. In other words, the pair $(\mathbb{R}(G)\mathbb{R}(F), \mu)$ is a right derived functor of GF. $\square$

Corollary 7. *Let $\mathcal{A}$ be an abelian category with enough hoinjectives, $\mathcal{B}$ an abelian category and $F : \mathcal{A} \longrightarrow \mathcal{B}, U : \mathcal{B} \longrightarrow \mathcal{C}$ additive functors with U exact. If $(\mathbb{R}F, \zeta)$ is a right derived functor of F, then $(U \circ \mathbb{R}F, U\zeta)$ is a right derived functor of UF.*

Proof. This is a special case of (TRC,Lemma 117). That is, we simply observe that the lift of U to the derived category is its right derived functor, so we are in the situation of Theorem 6 and no longer require $\mathcal{B}$ to have enough hoinjectives. □

Theorem 8. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough hoprojectives, $\mathcal{C}$ an abelian category and $F : \mathcal{A} \longrightarrow \mathcal{B}, G : \mathcal{B} \longrightarrow \mathcal{C}$ additive functors. If F sends hoprojectives to left G-acyclics, then there is a canonical trinatural equivalence*

$$\theta : \mathbb{L}G \circ \mathbb{L}F \longrightarrow \mathbb{L}(GF)$$

Proof. Once again the conditions in the statement are unnecessarily restrictive. Instead, let $F : \mathcal{A} \longrightarrow \mathcal{B}, G : \mathcal{B} \longrightarrow \mathcal{C}$ be additive functors between arbitrary abelian categories and assume that

(i) Every complex X in $\mathcal{A}$ admits a quasi-isomorphism $A_X \longrightarrow A$ with A_X left F-acyclic, such that $F(A_X)$ is left G-acyclic.

(ii) Every complex Y in $\mathcal{B}$ admits a quasi-isomorphism $B_Y \longrightarrow Y$ with B_Y left G-acyclic.

These conditions are satisfied under the conditions given in the statement of the theorem. Let $\mathcal{A}$ be the smallest triangulated subcategory of $K(\mathcal{A})$ containing A_X for every complex X in $\mathcal{A}$, and let $\mathcal{A}'$ be the class of all left G-acyclic complexes. The class $\mathcal{A}$ is left adapted for F, and by assumption $F(\mathcal{A}) \subseteq \mathcal{A}'$ so we are in the situation of of (TRC,Theorem 121), which in particular shows that the composite GF has a left derived functor. Let us elaborate a little on the consequences of this general result.

Let $(\mathbb{L}F, \zeta), (\mathbb{L}G, \omega)$ and $(\mathbb{L}(GF), \xi)$ be arbitrary left derived functors. We have a trinatural transformation

$$\mathbb{L}(G)\mathbb{L}(F)Q \xrightarrow{\mathbb{L}(G)\zeta} \mathbb{L}(G)Q'K(F) \xrightarrow{\omega K(F)} Q''K(GF)$$

which we denote by μ. This induces a trinatural transformation $\theta : \mathbb{L}(G)\mathbb{L}(F) \longrightarrow \mathbb{L}(GF)$ unique making the following diagram commute

$$\begin{array}{ccc} \mathbb{L}(G)Q'K(F) & \xrightarrow{\omega K(F)} & Q''K(GF) \\ \uparrow{\scriptstyle \mathbb{L}(G)\zeta} & & \uparrow{\scriptstyle \xi} \\ \mathbb{L}(G)\mathbb{L}(F)Q & \xrightarrow[\theta Q]{} & \mathbb{L}(GF)Q \end{array}$$

and it follows from (TRC,Theorem 121) that θ is a trinatural equivalence. In other words, the pair $(\mathbb{L}(G)\mathbb{L}(F), \mu)$ is a left derived functor of GF. □

Let us establish some notation for the next result. Suppose we have two additive functors between abelian categories

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B} \qquad (1)$$

together with an adjunction $(\eta, \varepsilon) : G \dashv F$. Then the induced triangulated functors

$$K(\mathcal{A}) \underset{K(G)}{\overset{K(F)}{\rightleftarrows}} K(\mathcal{B})$$

are such that $K(G)$ is canonically left triadjoint to $K(F)$ (DTC,Lemma 25), with unit $K(\eta)$ and counit $K(\varepsilon)$. Suppose that $Q'K(F)$ is right adaptable and that $QK(G)$ is left adaptable, so that there exist derived functors $(\mathbb{R}F, \zeta)$ and $(\mathbb{L}G, \omega)$. With this notation,

Theorem 9. *There is a canonical triadjunction* $\mathbb{L}G \relbar\!\!\relbar\!\!\dashv \mathbb{R}F$ *represented by its unit and counit*

$$\eta^\diamondsuit : 1 \longrightarrow \mathbb{R}(F)\mathbb{L}(G)$$
$$\varepsilon^\diamondsuit : \mathbb{L}(G)\mathbb{R}(F) \longrightarrow 1$$

which are the unique trinatural transformations making the following diagrams commute

$$\begin{array}{ccc} Q' & \xrightarrow{\eta^\diamondsuit Q'} & \mathbb{R}(F)\mathbb{L}(G)Q' \\ {\scriptstyle Q'K(\eta)}\downarrow & & \downarrow{\scriptstyle \mathbb{R}(F)\omega} \\ Q'K(FG) & \xrightarrow[\zeta K(G)]{} & \mathbb{R}(F)QK(G) \end{array} \qquad \begin{array}{ccc} \mathbb{L}(G)Q'K(F) & \xrightarrow{\mathbb{L}(G)\zeta} & \mathbb{L}(G)\mathbb{R}(F)Q \\ {\scriptstyle \omega K(F)}\downarrow & & \downarrow{\scriptstyle \varepsilon^\diamondsuit Q} \\ QK(GF) & \xrightarrow[QK(\varepsilon)]{} & Q \end{array}$$

Proof. This is a special case of (TRC,Theorem 122) and (TRC,Remark 81). □

Remark 4. Typically we apply this result in the case where $\mathcal{A}$ has enough hoinjectives, so that the triangulated functor $Q'K(F)$ is trivially right adaptable. Existence of enough hoprojectives is more uncommon, so there is usually some work involved to show that $QK(G)$ is left adaptable. But the trivial case is already useful, as the next result shows.

Lemma 10. *Let* $\mathcal{A}$ *be an abelian category with enough hoinjectives,* $\mathcal{B}$ *an abelian category, and suppose we have an adjoint pair of additive functors as in* (1)*. If* G *is exact, then there is a triadjunction* $G \relbar\!\!\relbar\!\!\dashv \mathbb{R}F$.

Proof. This follows from Theorem 9. To be precise, given an adjunction $(\eta, \varepsilon) : G \relbar\!\!\relbar\!\!\dashv F$ and an arbitrary right derived functor $(\mathbb{R}F, \zeta)$ of F, there is a canonical triadjunction $\mathfrak{D}(G) \relbar\!\!\relbar\!\!\dashv \mathbb{R}F$ whose unit and counit are the unique trinatural transformations

$$\eta^\diamondsuit : 1 \longrightarrow \mathbb{R}(F)\mathfrak{D}(G)$$
$$\varepsilon^\diamondsuit : \mathfrak{D}(G)\mathbb{R}(F) \longrightarrow 1$$

making the following diagrams commute

$$\begin{array}{ccc} Q' & & \\ {\scriptstyle Q'K(\eta)}\downarrow & \searrow^{\eta^\diamondsuit Q'} & \\ & & \mathbb{R}(F)QK(G) \\ & \nearrow_{\zeta K(G)} & \\ Q'K(FG) & & \end{array} \qquad \begin{array}{ccc} & & \mathfrak{D}(G)\mathbb{R}(F)Q \\ & \nearrow^{\mathfrak{D}(G)\zeta} & \\ QK(GF) & & \downarrow{\scriptstyle \varepsilon^\diamondsuit Q} \\ & \searrow_{Q\varepsilon} & \\ & & Q \end{array}$$

□

2.1 Classical Derived Functors

We explained in the introduction how the triangulated functor $\mathbb{R}F : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ is meant to subsume the classical derived functors $R^iF(-) : \mathcal{A} \longrightarrow \mathcal{B}$. In this section we make the observation precise by showing that given an object $A \in \mathcal{A}$ there is an isomorphism

$$R^iF(A) \longrightarrow H^i\mathbb{R}F(A)$$

The proof proceeds by showing that the functors $H^i\mathbb{R}F : \mathcal{A} \longrightarrow \mathcal{B}$ form a universal δ-functor.

Remark 5. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and $T : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ a triangulated functor. Let $c = c_0 : \mathcal{A} \longrightarrow \mathfrak{D}(\mathcal{A})$ be the canonical additive, full embedding of $\mathcal{A}$ as complexes concentrated in degree zero (DTC,Definition 19) and for $n \in \mathbb{Z}$ let $H^n : \mathfrak{D}(\mathcal{B}) \longrightarrow \mathcal{B}$ be the cohomology functor (DTC,Definition 9). For $n \in \mathbb{Z}$ we denote by T^n the additive functor $H^n \circ T \circ c : \mathcal{A} \longrightarrow \mathcal{B}$. Given an exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{2}$$

The corresponding sequence of complexes is exact in $\mathbf{C}(\mathcal{A})$

$$0 \longrightarrow c(A) \longrightarrow c(B) \longrightarrow c(C) \longrightarrow 0$$

and therefore by (DTC,Proposition 20) there is a canonical morphism $-z : c(C) \longrightarrow \Sigma c(A)$ in $\mathfrak{D}(\mathcal{A})$ making a triangle $c(A) \longrightarrow c(B) \longrightarrow c(C) \longrightarrow \Sigma c(A)$. Applying T to this triangle we have a triangle $Tc(A) \longrightarrow Tc(B) \longrightarrow Tc(C) \longrightarrow \Sigma Tc(A)$ in $\mathfrak{D}(\mathcal{B})$. Taking cohomology we have for each $n \in \mathbb{Z}$ a canonical morphism $\delta^n : T^n(C) \longrightarrow T^{n+1}(A)$ and a long exact sequence in $\mathcal{B}$

$$\cdots \longrightarrow T^n(A) \longrightarrow T^n(B) \longrightarrow T^n(C) \longrightarrow T^{n+1}(A) \longrightarrow \cdots$$

Using (DTC,Lemma 21) one checks that this long exact sequence is natural in the short exact sequence (2) in the usual sense. Given another triangulated functor $S : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ and a trinatural transformation $\psi : T \longrightarrow S$ we have natural transformations $\psi^n = H^n\psi c : T^n \longrightarrow S^n$. Given a short exact sequence (2) we have a morphism of the long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & T^n(A) & \longrightarrow & T^n(B) & \longrightarrow & T^n(C) & \longrightarrow & T^{n+1}(A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi^n_A} & & \downarrow{\scriptstyle \psi^n_B} & & \downarrow{\scriptstyle \psi^n_C} & & \downarrow{\scriptstyle \psi^{n+1}_A} & & \\ \cdots & \longrightarrow & S^n(A) & \longrightarrow & S^n(B) & \longrightarrow & S^n(C) & \longrightarrow & S^{n+1}(A) & \longrightarrow & \cdots \end{array}$$

In particular, trinaturally equivalent triangulated functors $T, S : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ determine isomorphic sequences of functors.

Definition 6. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories and suppose that $(\mathbb{R}F, \zeta)$ is a right derived functor of F. For $n \in \mathbb{Z}$ we write $\mathbb{R}^n F$ for the additive functor $H^n \circ \mathbb{R}F \circ c$ of Remark 5.

Remark 6. Let $\mathcal{A}$ be an abelian category with enough injectives and hoinjectives, $\mathcal{B}$ an abelian category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Let A be an object of $\mathcal{A}$ and suppose we have an injective resolution

$$0 \longrightarrow A \overset{v}{\longrightarrow} I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots \tag{3}$$

Let I denote the complex $I^0 \longrightarrow I^1 \longrightarrow \cdots$ beginning in degree zero, so that v can be considered as a quasi-isomorphism of complexes $v : c(A) \longrightarrow I$. The complex I is certainly hoinjective, so every injective resolution of an object A gives rise to a hoinjective resolution of $c(A)$. Let $\mathscr{J}$ be an assignment of injective resolutions to the objects of $\mathcal{A}$. We define an assignment $\mathcal{I}$ of hoinjective resolutions for $\mathcal{A}$ as follows: for any $A \in \mathcal{A}$ the chosen resolution of $c(A)$ is obtained as above from the chosen resolution of A, and otherwise the resolution is arbitrary. We can use this assignment to define a right derived functor $(\mathbb{R}_{\mathcal{I}}F, \zeta)$ of F as in Definition 5. Explicitly

- Given an object $A \in \mathcal{A}$ with chosen resolution (3) the complex $\mathbb{R}_{\mathcal{I}}F(c(A))$ is

$$\cdots \longrightarrow 0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots$$

- Let A, B be objects of $\mathcal{A}$ with chosen resolutions I, J and let $f : A \longrightarrow B$ be a morphism. Lift this to a morphism of complexes $\varphi : I \longrightarrow J$. Then $\mathbb{R}_{\mathcal{I}}F(c(f))$ is the following morphism of complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & F(I^0) & \longrightarrow & F(I^1) & \longrightarrow & F(I^2) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle F(\varphi^0)} & & \downarrow{\scriptstyle F(\varphi^1)} & & \downarrow{\scriptstyle F(\varphi^2)} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & F(J^0) & \longrightarrow & F(J^1) & \longrightarrow & F(J^2) & \longrightarrow & \cdots \end{array}$$

Proposition 11. *Let $\mathcal{A}$ be an abelian category with enough injectives and hoinjectives, $\mathcal{B}$ an abelian category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor with right derived functor $(\mathbb{R}F, \zeta)$. Then $\mathbb{R}^n F = 0$ for $n < 0$ and the sequence $\{\mathbb{R}^n F\}_{n \geq 0}$ is a universal cohomological δ-functor.*

Proof. As in Remark 6 let $\mathcal{J}$ be an assignment of injective resolutions to the objects of $\mathcal{A}$, and let $\mathcal{I}$ be an induced assignment of hoinjective resolutions. Looking at the definition of $\mathbb{R}_{\mathcal{I}}F$ it is clear that $\mathbb{R}^n_{\mathcal{I}}F = 0$ for $n < 0$ and that there is an *equality* of functors $\mathbb{R}^n_{\mathcal{I}}F = R^n_{\mathcal{J}}F$ for $n \geq 0$, where R^n denotes the ordinary right derived functor of (DF,Section 5). In particular $\mathbb{R}^n_{\mathcal{I}}F(I) = 0$ for any injective object $I \in \mathcal{A}$ and $n > 0$.

Since the right derived functor is unique up to trinatural equivalence, we deduce that $\mathbb{R}^nF = 0$ for $n < 0$. Remark 5 then shows how the sequence of additive functors $\{\mathbb{R}^nF\}_{n\geq 0}$ becomes a cohomological δ-functor. Since the functor $\mathbb{R}^nF$ for $n > 0$ must vanish on injectives, it follows from (DF,Theorem 74) that this δ-functor is universal. □

Corollary 12. *Let $\mathcal{A}$ be an abelian category with enough injectives and hoinjectives, $\mathcal{B}$ an abelian category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a left exact functor with right derived functor $(\mathbb{R}F, \zeta)$. There is a natural equivalence $\mathbb{R}^nF \cong R^nF$ for every $n \geq 0$.*

Proof. The equality $\mathbb{R}^0_{\mathcal{I}}F = R^0_{\mathcal{J}}F$ of the proof of Proposition 11 gives rise to an isomorphism of the universal cohomological δ-functors $\{\mathbb{R}^nF\}_{n\geq 0}$ and $\{R^n_{\mathcal{J}}F\}_{n\geq 0}$. □

2.2 Localisation

In this short section we want to elaborate a little on (DTC,Remark 48). Throughout this section $\mathcal{R}$ is a ringoid, J an additive topology on $\mathcal{R}$, and $\mathcal{D} = \mathbf{Mod}(\mathcal{R}, J)$ is the localisation with inclusion $i : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{R}$ and exact left adjoint a. Since a is exact and $\mathcal{D}$ grothendieck abelian, we have the following triangulated functors

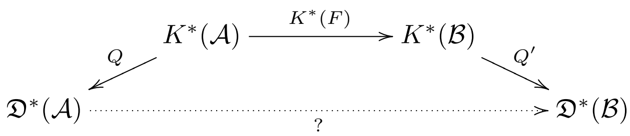
$$\mathfrak{D}(a) : \mathfrak{D}(\mathbf{Mod}\mathcal{R}) \longrightarrow \mathfrak{D}(\mathcal{D})$$
$$\mathbb{R}(i) : \mathfrak{D}(\mathcal{D}) \longrightarrow \mathfrak{D}(\mathbf{Mod}\mathcal{R})$$

It follows from Lemma 10 that $\mathfrak{D}(a)$ is actually left triadjoint to $\mathbb{R}(i)$.

2.3 Bounded Derived Functors

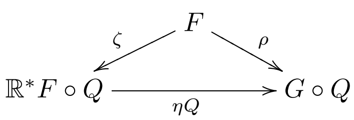
In this section whenever we write $K^*(-)$ or $\mathfrak{D}^*(-)$ we mean that the given statement holds with $*$ replaced by $+, -$ or b. See (DTC,Section 3.3) for the relevant definitions. One defines right and left derived functors on the bounded derived categories in exactly the same way as on the unbounded derived category, but we write down the definitions for convenience.

If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor between abelian categories, then there is an induced triangulated functor $K^*(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ as in the following diagram



If F is exact then it lifts to a triangulated functor between the derived categories but in general this is not possible.

Definition 7. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. A *right derived functor of F* is a pair $(\mathbb{R}^*F, \zeta)$ consisting of a triangulated functor $\mathbb{R}^*F : \mathfrak{D}^*(\mathcal{A}) \longrightarrow \mathfrak{D}^*(\mathcal{B})$ and a trinatural transformation $\zeta : Q' \circ K^*(F) \longrightarrow \mathbb{R}^*F \circ Q$ with the following universal property: given any triangulated functor $G : \mathfrak{D}^*(\mathcal{A}) \longrightarrow \mathfrak{D}^*(\mathcal{B})$ and trinatural transformation $\rho : Q' \circ K^*(F) \longrightarrow G \circ Q$ there is *unique* trinatural transformation $\eta : \mathbb{R}^*F \longrightarrow G$ making the following diagram commute



In the notation of (TRC,Definition 46) this says that the pair $(\mathbb{R}^*F, \zeta)$ is a right derived functor of the composite $Q' \circ K^*(F)$ with respect to the category $\mathcal{Z}^*$ of exact complexes in $K^*(\mathcal{A})$. By abuse of notation we often say that $\mathbb{R}^*F$ is a right derived functor of F, and drop ζ from the notation. Clearly if a right derived functor exists it is unique up to canonical trinatural equivalence.

Definition 8. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. A *left derived functor* of F is a pair $(\mathbb{L}^*F, \zeta)$ consisting of a triangulated functor $\mathbb{L}^*F : \mathfrak{D}^*(\mathcal{A}) \longrightarrow \mathfrak{D}^*(\mathcal{B})$ and a trinatural transformation $\zeta : \mathbb{L}^*F \circ Q \longrightarrow Q' \circ K^*(F)$ with the following universal property: given any triangulated functor $G : \mathfrak{D}^*(\mathcal{A}) \longrightarrow \mathfrak{D}^*(\mathcal{B})$ and trinatural transformation $\rho : G \circ Q \longrightarrow Q' \circ K^*(F)$ there is a *unique* trinatural transformation $\eta : G \longrightarrow \mathbb{L}^*F$ making the following diagram commute

$$\begin{array}{ccc} G \circ Q & \xrightarrow{\eta Q} & \mathbb{L}^*F \circ Q \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\zeta} & \\ & F & \end{array}$$

In the notation of (TRC,Definition 50) this says that the pair $(\mathbb{L}^*F, \zeta)$ is a left derived functor of the composite $Q' \circ K^*(F)$ with respect to the category $\mathcal{Z}^*$ of exact complexes in $K^*(\mathcal{A})$. By abuse of notation we often say that $\mathbb{L}^*F$ is a left derived functor of F, and drop ζ from the notation. Clearly if a left derived functor exists it is unique up to canonical trinatural equivalence.

Proposition 13. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. Then*

(i) If $\mathcal{A}$ has enough injectives then F has a right derived functor $\mathbb{R}^+F : \mathfrak{D}^+(\mathcal{A}) \longrightarrow \mathfrak{D}^+(\mathcal{B})$.

(ii) If $\mathcal{B}$ has enough projectives then F has a left derived functor $\mathbb{L}^-F : \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{B})$.

Proof. (i) It is clear that any bounded below hoinjective complex is right M-acyclic as an object of the triangulated category $K^+(\mathcal{A})$ with respect to $\mathcal{Z}^+$ for any triangulated functor $M : K^+(\mathcal{A}) \longrightarrow \mathcal{T}$. (TRC,Definition 48). Since any bounded below complex admits a quasi-isomorphism into a bounded below complex of injectives, it follows from (TRC,Theorem 116) that $\mathbb{R}^+F$ exists. (ii) A bounded above hoprojective complex is left M-acyclic as an object of $K^-(\mathcal{A})$ with respect to $\mathcal{Z}^-$ for any triangulated functor $M : K^-(\mathcal{A}) \longrightarrow \mathcal{T}$. So it follows from (TRC,Theorem 125) that $\mathbb{L}^-F$ exists. □

Lemma 14. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories where $\mathcal{A}$ has enough injectives and hoinjectives. The following diagram commutes up to canonical natural equivalence*

$$\begin{array}{ccc} \mathfrak{D}(\mathcal{A}) & \xrightarrow{\mathbb{R}F} & \mathfrak{D}(\mathcal{B}) \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle b} \\ \mathfrak{D}^+(\mathcal{A}) & \xrightarrow[\mathbb{R}^+F]{} & \mathfrak{D}^+(\mathcal{B}) \end{array}$$

Proof. Here a, b are the canonical full embeddings of (DTC,Lemma 37). Given a bounded below complex X in $\mathcal{A}$ we can find a bounded below complex of injectives I and a quasi-isomorphism of complexes $X \longrightarrow I$. This yields an isomorphism in $\mathfrak{D}(\mathcal{B})$

$$b\mathbb{R}^+F(X) \cong b\mathbb{R}^+F(I) \cong F(I) \cong (\mathbb{R}F)a(I) \cong (\mathbb{R}F)a(X)$$

which one checks is independent of the chosen resolution and natural in X. □

2.4 Linear Derived Categories

Recall that given a commutative ring k and a k-linear abelian category the triangulated categories $K(\mathcal{A})$ and $\mathfrak{D}(\mathcal{A})$ are canonically k-linear (DTC,Remark 11). If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a k-linear functor between k-linear abelian categories then it is clear that the triangulated functor $K(F) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ is also k-linear.

Lemma 15. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a k-linear functor between k-linear abelian categories. Then*

(i) If F satisfies the conditions of Theorem 1 then any right derived functor $\mathbb{R}F$ is k-linear.

(ii) If F satisfies the conditions of Theorem 2 then any left derived functor $\mathbb{L}F$ is k-linear.

Proof. We give the proof of (i), with (ii) being identical. It suffices to show that $\mathbb{R}F$ is k-linear where $(\mathbb{R}F, \zeta)$ is the right derived functor calculated using an assignment of resolutions as in Theorem 1. It is also enough to show that $\mathbb{R}F(\lambda \cdot f) = \lambda \cdot \mathbb{R}F(f)$ where $\lambda \in k$ and $f : X \longrightarrow Y$ is a morphism of *complexes* in $\mathcal{A}$. Using the explicit description of (TRC,Remark 75) this is straightforward. □

Lemma 16. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}, G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors between k-linear categories, and suppose that G is left adjoint to F with adjunction isomorphisms*

$$\theta_{A,B} : Hom_{\mathcal{A}}(GB, A) \longrightarrow Hom_{\mathcal{B}}(B, FA)$$

Then the following conditions are equivalent

(i) For every pair $A \in \mathcal{A}, B \in \mathcal{B}$ the map $\theta_{A,B}$ is k-linear.

(ii) F is a k-linear functor.

(iii) G is a k-linear functor.

In particular if one functor involved in an adjunction is k-linear, so is the other.

Proof. $(ii) \Rightarrow (i)$ Suppose that F is k-linear and let η be the unit of the adjunction. We have $\theta_{A,B}(\alpha) = F(\alpha)\eta_B$ so it is clear that $\theta_{A,B}$ is k-linear. Similarly one proves $(iii) \Rightarrow (i)$. For $(i) \Rightarrow (ii)$ suppose we are given a morphism $\alpha : A \longrightarrow A'$ in $\mathcal{A}$ and consider the following commutative diagram

$$\begin{array}{ccc} GFA & \xrightarrow{GF\alpha} & GFA' \\ {\scriptstyle \varepsilon_A}\downarrow & & \downarrow{\scriptstyle \varepsilon_{A'}} \\ A & \xrightarrow[\alpha]{} & A' \end{array}$$

From this diagram it is clear that $F(\alpha) = \theta_{A',FA}(\alpha\varepsilon_A)$, and therefore F is k-linear. The proof of $(i) \Rightarrow (iii)$ is identical, so we are done. □

3 Derived Hom

In this section we define for an abelian category $\mathcal{A}$ with enough hoinjectives the derived Hom

$$\mathbb{R}Hom^{\bullet}(-,-) : \mathfrak{D}(\mathcal{A})^{\text{op}} \times \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

The construction is similar to the construction of the Ext bifunctor in (EXT,Section 1). Later on we will want to consider the derived sheaf Hom, so it is convenient to develop the basic theory in as much generality as possible. Consequently, in the beginning of this section we work with arbitrary bifunctors additive in each variable and contravariant in the first.

Definition 9. Let $\mathcal{A}$ be a complete abelian category and C a bicomplex in $\mathcal{A}$. The *product totalisation complex* $Totp(C)$ of C is defined as follows. For $n \in \mathbb{Z}$ we have

$$Totp(C)^n = \prod_{i+j=n} C^{ij}$$

Let $p_{ij} : Totp(C)^{i+j} \longrightarrow C^{ij}$ be the projection out of the product. Then for $n \in \mathbb{Z}$ we define a morphism $\partial^n : Totp(C)^n \longrightarrow Totp(C)^{n+1}$ on components by

$$p_{ij}\partial^n = \partial_1^{(i-1)j} p_{(i-1)j} + (-1)^{n+1} \partial_2^{i(j-1)} p_{i(j-1)}$$

for any $i, j \in \mathbb{Z}$ with $i + j = n + 1$. One checks easily that $Totp(C)$ is indeed a complex in $\mathcal{A}$. Given a morphism of bicomplexes $\varphi : C \longrightarrow D$ we define a morphism of complexes $Totp(\varphi) : Totp(C) \longrightarrow Totp(D)$ by $Totp(\varphi)^n = \prod_{i+j=n} \varphi^{ij}$. This makes the product totalisation complex into an additive functor $Totp(-) : \mathbf{C}^2(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{A})$ from the category of bicomplexes in $\mathcal{A}$ to the category $\mathbf{C}(\mathcal{A})$.

Definition 10. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $H : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{B}$ a functor which is additive in each variable. For complexes X, Y in $\mathcal{A}$ we define a bicomplex $BH(X, Y)$ in $\mathcal{B}$ as follows. For $i, j \in \mathbb{Z}$ we have $BH(X, Y)^{ij} = H(X^{-i}, Y^j)$ and we define the differentials by

$$\partial_1^{ij} = H(\partial_X^{-i-1}, Y^j) : H(X^{-i}, Y^j) \longrightarrow H(X^{-i-1}, Y^j)$$
$$\partial_2^{ij} = H(X^{-i}, \partial_Y^j) : H(X^{-i}, Y^j) \longrightarrow H(X^{-i}, Y^{j+1})$$

Given morphisms of complexes $\varphi : X \longrightarrow X'$ and $\psi : Y \longrightarrow Y'$ there are morphisms of bicomplexes

$$BH(\varphi, Y) : BH(X', Y) \longrightarrow BH(X, Y), \quad BH(\varphi, Y)^{ij} = H(\varphi^{-i}, Y^j)$$
$$BH(X, \psi) : BH(X, Y) \longrightarrow BH(X, Y'), \quad BH(X, \psi)^{ij} = H(X^{-i}, \psi^j)$$

It is clear that $BH(X, \psi)BH(\varphi, Y) = BH(\varphi, Y')BH(X', \psi)$ so we have defined a functor additive in each variable

$$BH : \mathbf{C}(\mathcal{A})^{\mathrm{op}} \times \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}^2(\mathcal{B})$$
$$BH(\varphi, \psi)^{ij} = H(\varphi^{-i}, \psi^j)$$

Taking the product totalisation we have a functor $H^\bullet = Totp \circ BH$ additive in each variable

$$H^\bullet : \mathbf{C}(\mathcal{A})^{\mathrm{op}} \times \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$$
$$H^n(X, Y) = \prod_{i+j=n} H(X^{-i}, Y^j)$$

Definition 11. Let $\mathcal{A}$ be an abelian category and let X, Y be complexes in $\mathcal{A}$. We define a bicomplex $BHom(X, Y)$ in $\mathbf{Ab}$ as follows. For $i, j \in \mathbb{Z}$ we have

$$BHom(X, Y)^{ij} = Hom_{\mathcal{A}}(X^{-i}, Y^j)$$

The differentials are defined by $\partial_1^{ij}(f) = f\partial_X^{-i-1}$ and $\partial_2^{ij}(f) = \partial_Y^j f$. If $\mathcal{A}$ is R-linear for some ring R (EXT,Definition 3) then $BHom(X, Y)$ is a bicomplex in $R\mathbf{Mod}$. Totalising this bicomplex gives the complex $Hom^\bullet(X, Y)$ of abelian groups, defined by

$$Hom^n(X, Y) = \prod_{i+j=n} Hom_{\mathcal{A}}(X^{-i}, Y^j) = \prod_{q \in \mathbb{Z}} Hom_{\mathcal{A}}(X^q, Y^{q+n})$$
$$\partial^n((f_p)_{p \in \mathbb{Z}})_q = f_{q+1}\partial_X^q + (-1)^{n+1}\partial_Y^{q+n} f_q$$

and this defines a functor additive in each variable

$$Hom^\bullet(-, -) : \mathbf{C}(\mathcal{A})^{\mathrm{op}} \times \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathbf{Ab})$$
$$Hom^n(\varphi, \psi) = \prod_{q \in \mathbb{Z}} Hom_{\mathcal{A}}(\varphi^q, \psi^{q+n})$$

Example 1. Let $\mathcal{A}$ be an abelian category, A an object of $\mathcal{A}$ and Y a complex in $\mathcal{A}$. The complex $Hom^\bullet(A, Y)$ of abelian groups is canonically isomorphic to the complex $Hom_{\mathcal{A}}(A, Y)$ with alternating signs on the differentials

$$\cdots \longrightarrow Hom_{\mathcal{A}}(A, Y^0) \xrightarrow{-} Hom_{\mathcal{A}}(A, Y^1) \longrightarrow Hom_{\mathcal{A}}(A, Y^2) \xrightarrow{-} \cdots$$

which we denote by $(-1)^{\bullet+1}Hom_{\mathcal{A}}(A, Y)$. In particular $Hom^\bullet(A, Y)$ and $Hom_{\mathcal{A}}(A, Y)$ have the same cohomology.

Since we will encounter it several times, it is convenient for us to introduce a notation for complexes obtained in the above way by alternating the signs on the differentials.

Definition 12. Let $\mathcal{A}$ be an abelian category. We can define an additive automorphism Λ of $\mathbf{C}(\mathcal{A})$ which sends a complex X to the complex $\Lambda(X) = (-1)^{\bullet+1}X$ with the same objects but differentials $\partial^n_{\Lambda(X)} = (-1)^{n+1}\partial^n_X$, and sends a morphism of complexes $\psi : X \longrightarrow Y$ to the same morphism $\Lambda(\psi)^n = \psi^n$ between the modified complexes. This extends to an additive automorphism $\Lambda : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$. If we define $\phi : \Lambda\Sigma X \longrightarrow \Sigma\Lambda X$ by $\phi^n = (-1)^{n+1}1_{X^{n+1}}$ then the pair (Λ, ϕ) is a triangulated functor $K(\mathcal{A}) \longrightarrow K(\mathcal{A})$ with $\Lambda^2 = 1$. Since Λ preserves exactness, there is a unique triangulated functor $\Lambda : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ making the following diagram commute

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{\Lambda} & K(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathfrak{D}(\mathcal{A}) & \xrightarrow[\Lambda]{} & \mathfrak{D}(\mathcal{A}) \end{array}$$

And we have $\Lambda^2 = 1$ for the bottom triangulated functor as well. For complexes X, Y in $\mathcal{A}$ there is a canonical isomorphism of complexes

$$\alpha : Hom^\bullet(X, (-1)^{\bullet+1}Y) \longrightarrow (-1)^{\bullet+1}Hom^\bullet(X, Y)$$

defined by $p_q\alpha^n = (-1)^{qn+\frac{q(q+1)}{2}}p_q$. This sign factor is necessary for α to be a morphism of complexes. One checks that this is natural in both variables.

Remark 7. Let $\mathcal{A}$ be an abelian category and X a complex in $\mathcal{A}$. There is a canonical isomorphism of complexes $\omega : X \longrightarrow (-1)^{\bullet+1}X$ defined by

$$\omega^n = (-1)^{\frac{n(n+1)}{2}} = \begin{cases} 1 & n \equiv 0 \text{ or } 3 \bmod 4 \\ -1 & n \equiv 1 \text{ or } 2 \bmod 4 \end{cases}$$

Graphically, the signs on ω alternate in pairs

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & X^{-3} & \longrightarrow & X^{-2} & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \\ & & \downarrow - & & \downarrow - & & \downarrow + & & \downarrow + & & \downarrow - & & \downarrow - & & \\ \cdots & \longrightarrow & X^{-3} & \xrightarrow{+} & X^{-2} & \xrightarrow{-} & X^{-1} & \xrightarrow{+} & X^0 & \xrightarrow{-} & X^1 & \xrightarrow{+} & X^2 & \xrightarrow{-} & \cdots \end{array}$$

It is clear that ω is natural, so in the notation of Definition 12 it is a trinatural equivalence $1 \longrightarrow \Lambda$. This means that we can safely ignore the distinction between X and $(-1)^{\bullet+1}X$. In particular if A is an object of $\mathcal{A}$ and Y a complex in $\mathcal{A}$ the complex $Hom^\bullet(A, Y)$ is canonically isomorphic in $\mathbf{C}(\mathbf{Ab})$ to the complex $Hom_{\mathcal{A}}(A, Y)$ described in Remark 1.

The Hom complex has the expected limit and colimit preserving properties.

Lemma 17. *Let $\mathcal{A}$ be a complete, cocomplete abelian category and X a complex in $\mathcal{A}$. Then the additive functors*

$$Hom^\bullet(X, -) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathbf{Ab})$$
$$Hom^\bullet(-, X) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathbf{Ab})$$

send products to products and coproducts to products respectively, and $Hom^\bullet(X, -)$ is left exact.

Proof. To be precise, for $Hom^\bullet(X, -)$ to preserve products we only need $\mathcal{A}$ to be complete, and for $Hom^\bullet(-, X)$ to send coproducts to products we only need $\mathcal{A}$ to be cocomplete.

Suppose that $\mathcal{A}$ is complete and let $\{Y_\lambda\}_{\lambda\in\Lambda}$ be a nonempty family of complexes. Let $\prod_\lambda Y_\lambda$ be the any product and observe that we have an isomorphism

$$Hom^n(X, \prod_\lambda Y_\lambda) = \prod_q Hom_{\mathcal{A}}(X^q, \prod_\lambda Y_\lambda^{q+n}) \cong \prod_{q,\lambda} Hom_{\mathcal{A}}(X^q, Y_\lambda^{q+n}) \cong \prod_\lambda Hom^n(X, Y_\lambda)$$

which when composed with the projection to $Hom^n(X, Y_\lambda)$ is just $Hom^n(X, -)$ of the projection $\prod Y_\lambda \longrightarrow Y_\lambda$. This shows that $Hom^\bullet(X, -)$ preserves products. If $\mathcal{A}$ is instead cocomplete then one shows $Hom^\bullet(-, X)$ sends coproducts to products in the same way.

It is straightforward to check that if $\psi : Y \longrightarrow Y'$ is a monomorphism of complexes in $\mathcal{A}$, then the same is true of $Hom^\bullet(X, \psi)$. If instead ψ is a retraction in each degree, then the same is also true of $Hom^\bullet(X, \psi)$. □

Proposition 18. *Let $\mathcal{A}$ be an abelian category and let X, Y be complexes in $\mathcal{A}$. For $n \in \mathbb{Z}$ there is a canonical isomorphism of abelian groups natural in both variables*

$$\omega : H^n(Hom^\bullet(X, Y)) \longrightarrow Hom_{K(\mathcal{A})}(X, \Sigma^n Y)$$
$$\omega((f_p)_{p\in\mathbb{Z}} + Im\partial^{n-1}) = [F] \text{ where } F^p = f_p$$

Proof. We want to calculate the cohomology of the following complex of abelian groups

$$\cdots \longrightarrow Hom^{n-1}(X, Y) \longrightarrow Hom^n(X, Y) \longrightarrow Hom^{n+1}(X, Y) \longrightarrow \cdots$$

which we can expand as

$$\cdots \longrightarrow \prod_{q\in\mathbb{Z}} Hom_{\mathcal{A}}(X^q, Y^{q+n-1}) \longrightarrow \prod_{q\in\mathbb{Z}} Hom_{\mathcal{A}}(X^q, Y^{q+n}) \longrightarrow \prod_{q\in\mathbb{Z}} Hom_{\mathcal{A}}(X^q, Y^{q+n+1}) \longrightarrow \cdots$$

It is easy to check that a sequence $(f_p)_{p\in\mathbb{Z}}$ belongs to $Ker(\partial^n)$ if and only if it defines a morphism of complexes $f : X \longrightarrow \Sigma^n Y$. So there is a canonical isomorphism of abelian groups $Ker(\partial^n) \longrightarrow Hom_{\mathbf{C}(\mathcal{A})}(X, \Sigma^n Y)$. One checks that this isomorphism identifies sequences in $Im(\partial^{n-1})$ with morphisms $X \longrightarrow \Sigma^n Y$ which are null-homotopic, so taking quotients we have the desired canonical isomorphism ω. Naturality in both variables is easily checked. □

Corollary 19. *Let $\mathcal{A}$ be an abelian category and I a complex in $\mathcal{A}$. Then I is hoinjective if and only if for every exact complex Z the complex $Hom^\bullet(Z, I)$ is exact.*

Proof. This follows from Proposition 18 and the definition of a hoinjective complex. □

Corollary 20. *Let $\mathcal{A}$ be an abelian category and P a complex in $\mathcal{A}$. Then P is hoprojective if and only if for every exact complex Z the complex $Hom^\bullet(P, Z)$ is exact.*

It is this property of Corollary 19 that allows us to lift $Hom^\bullet(-, -)$ to a bifunctor on the derived category. To allow us to treat the normal Hom and the sheaf Hom at the same time, we introduce the following condition.

Definition 13. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $H : \mathcal{A}^{\text{op}} \times \mathcal{A} \longrightarrow \mathcal{B}$ a functor which is additive in each variable. We say that H is *homlike* if it has the property that the complex $H^\bullet(Z, I)$ is exact in $\mathcal{B}$ whenever Z is exact and I hoinjective.

Throughout the rest of this section $\mathcal{A}, \mathcal{B}$ are abelian categories and $H : \mathcal{A}^{\text{op}} \times \mathcal{A} \longrightarrow \mathcal{B}$ is a functor additive in each variable.

Lemma 21. *Suppose we are given homotopic morphisms of complexes $\varphi \simeq \varphi' : X \longrightarrow X'$ and $\psi \simeq \psi' : Y \longrightarrow Y'$ in $\mathcal{A}$. Then $H^\bullet(\varphi, \psi) \simeq H^\bullet(\varphi', \psi')$.*

Proof. It suffices to show that $H^\bullet(Z, \psi) \simeq H^\bullet(Z, \psi')$ and $H^\bullet(\varphi, Z) \simeq H^\bullet(\varphi', Z)$ for any complex Z and morphisms as in the statement of the Lemma. Let $\Sigma : \psi \longrightarrow \psi'$ be a homotopy and define a morphism $\Lambda^n : H^n(Z, Y) \longrightarrow H^n(Z, Y')$ by

$$\Lambda^n : \prod_{i+j=n} H(Z^{-i}, Y^j) \longrightarrow \prod_{i+j=n-1} H(Z^{-i}, (Y')^j)$$
$$p_{ij}\Lambda^n = (-1)^n H(Z^{-i}, \Sigma^{n-i}) p_{i(j+1)}$$

One checks that Λ is a homotopy $H^\bullet(Z,\psi) \longrightarrow H^\bullet(Z,\psi')$. On the other hand if $\Sigma : \varphi \longrightarrow \varphi'$ is a homotopy then we define a morphism $\Lambda^n : H^n(X',Z) \longrightarrow H^{n-1}(X,Z)$ by

$$\Lambda^n : \prod_{i+j=n} H((X')^{-i}, Z^j) \longrightarrow \prod_{i+j=n-1} H(X^{-i}, Z^j)$$
$$p_{ij}\Lambda^n = H(\Sigma^{-i}, Z^j)p_{(i+1)j}$$

One checks that Λ is a homotopy $H^\bullet(\varphi,Z) \longrightarrow H^\bullet(\varphi',Z)$ as required. □

Definition 14. The functor $H^\bullet : \mathbf{C}(\mathcal{A})^{\mathrm{op}} \times \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$ extends to a functor additive in each variable $H^\bullet : K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ which makes the following diagram commute

$$\begin{array}{ccc} \mathbf{C}(\mathcal{A})^{\mathrm{op}} \times \mathbf{C}(\mathcal{A}) & \longrightarrow & \mathbf{C}(\mathcal{B}) \\ \downarrow & & \downarrow \\ K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) & \longrightarrow & K(\mathcal{B}) \end{array}$$

Proof. We define the new functor $H^\bullet$ on objects as before, and on morphisms by $H^\bullet([\varphi],[\psi]) = [H^\bullet(\varphi,\psi)]$ which is well-defined by Lemma 21. It is clear that this is a functor additive in each variable. □

For $X \in \mathbf{C}(\mathcal{A})$ we have the additive partial functor $H^\bullet(X,-) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$. For any complex Y we have an equality of objects $H^n(X,\Sigma Y) = \Sigma H^n(X,Y)$ in $\mathcal{B}$ and a natural equivalence

$$\rho : H^\bullet(X,-)\Sigma \longrightarrow \Sigma H^\bullet(X,-), \quad \rho^n_Y = (-1)^{n+1}$$

We have a contravariant additive functor $H^\bullet(-,X) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$ and a natural equivalence

$$\tau : H^\bullet(-,X)\Sigma^{-1} \longrightarrow \Sigma H^\bullet(-,X), \quad p_{ij}\tau^n_Y = p_{(i-1)j}$$

Lemma 22. *Let $u : X \longrightarrow Y$ be a morphism of complexes and Z any complex. Then*

(i) There is a canonical isomorphism of complexes $H^\bullet(Z,C_u) \cong C_{H^\bullet(Z,u)}$ in $\mathcal{B}$.

(ii) There is a canonical isomorphism of complexes $H^\bullet(C_u,Z) \cong C_{H^\bullet(\Sigma u,Z)}$ in $\mathcal{B}$.

Proof. (*i*) For $n \in \mathbb{Z}$ we have a canonical isomorphism $\alpha^n : H^n(Z,C_u) \longrightarrow C^n_{H^\bullet(Z,u)}$ in $\mathcal{B}$

$$\begin{aligned} H^n(Z,C_u) &= \prod_{i+j=n} H(Z^{-i}, C_u^j) \\ &= \prod_{i+j=n} H(Z^{-i}, X^{j+1} \oplus Y^j) \\ &\cong \prod_{i+j=n} H(Z^{-i}, X^{j+1}) \oplus H(Z^{-i}, Y^j) \\ &\cong \left(\prod_{i+j=n} H(Z^{-i}, X^{j+1}) \right) \oplus \left(\prod_{i+j=n} H(Z^{-i}, Y^j) \right) \\ &= H^{n+1}(Z,X) \oplus H^n(Z,Y) = C^n_{H^\bullet(Z,u)} \end{aligned}$$

Write α^n as a matrix $\alpha^n = \begin{pmatrix} a^n \\ b^n \end{pmatrix}$ and set $\theta^n = \begin{pmatrix} (-1)^{n+1}a^n \\ b^n \end{pmatrix}$. This is still an isomorphism and $\theta : H^\bullet(Z,C_u) \longrightarrow C_{H^\bullet(Z,u)}$ is the desired isomorphism of complexes.

(*ii*) For $n \in \mathbb{Z}$ we have a canonical isomorphism $\beta^n : H^n(C_u, Z) \longrightarrow C^n_{H^\bullet(\Sigma u, Z)}$ in $\mathcal{B}$

$$\begin{aligned} H^n(C_u, Z) &= \prod_{i+j=n} H(C_u^{-i}, Z^j) \\ &= \prod_{i+j=n} H(X^{-i+1} \oplus Y^{-i}, Z^j) \\ &\cong \left(\prod_{i+j=n} H(X^{-i+1}, Z^j) \right) \oplus \left(\prod_{i+j=n} H(Y^{-i}, Z^j) \right) \\ &\cong H^n(\Sigma X, Z) \oplus H^{n+1}(\Sigma Y, Z) \cong C^n_{H^\bullet(\Sigma u, Z)} \end{aligned}$$

One checks that β is in fact an isomorphism of complexes. □

Lemma 23. *For any complex Z the pair $(H^\bullet(Z,-), \rho)$ is a triangulated functor $K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ and the pair $(H^\bullet(-,Z), \tau)$ is a triangulated functor $K(\mathcal{A})^{op} \longrightarrow K(\mathcal{B})$.*

Proof. It suffices to show that given a morphism of complexes $u : X \longrightarrow Y$ and the induced triangle $X \longrightarrow Y \longrightarrow C_u \longrightarrow \Sigma X$ in $K(\mathcal{A})$ that the following candidate triangle in $K(\mathcal{B})$ is a triangle

$$H^\bullet(Z, X) \longrightarrow H^\bullet(Z, Y) \longrightarrow H^\bullet(Z, C_u) \longrightarrow H^\bullet(Z, \Sigma X) \cong \Sigma H^\bullet(Z, X) \tag{4}$$

Using the isomorphism of Lemma 22(*i*) we have a commutative diagram in $K(\mathcal{B})$

$$\begin{array}{ccccccc} H^\bullet(Z,X) & \longrightarrow & H^\bullet(Z,Y) & \longrightarrow & H^\bullet(Z,C_u) & \longrightarrow & \Sigma H^\bullet(Z,X) \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle \theta}\downarrow & & {\scriptstyle 1}\downarrow \\ H^\bullet(Z,X) & \longrightarrow & H^\bullet(Z,Y) & \longrightarrow & C_{H^\bullet(Z,u)} & \longrightarrow & \Sigma H^\bullet(Z,X) \end{array}$$

From which it follows that (4) is a triangle, as required. The second claim follows similarly from Lemma 22(*ii*). □

Definition 15. Let $\mathcal{A}$ be an abelian category with enough hoinjectives, $\mathcal{B}$ an abelian category and $H : \mathcal{A}^{\text{op}} \times \mathcal{A} \longrightarrow \mathcal{B}$ a functor which is additive in each variable. For each complex Z in $\mathcal{A}$ it follows from Corollary 5 that the triangulated functor $Q' \circ H^\bullet(Z,-) : K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ has a right derived functor

$$\mathbb{R}H^\bullet(Z,-) : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$$

To be precise, for each assignment $\mathcal{I}$ of hoinjective resolutions for $\mathcal{A}$ we have a canonical right derived functor $\mathbb{R}_\mathcal{I} H^\bullet(Z,-)$ of $Q' \circ H^\bullet(Z,-)$. In particular $\mathbb{R}_\mathcal{I} H^\bullet(Z,X) = H^\bullet(Z, I_X)$.

We use the notation of Definition 15 and fix an assignment $\mathcal{I}$ of hoinjective resolutions. Given a morphism $\psi : Z \longrightarrow Z'$ in $K(\mathcal{A})$ we can define a trinatural transformation

$$H^\bullet(\psi,-) : H^\bullet(Z',-) \longrightarrow H^\bullet(Z,-)$$
$$H^\bullet(\psi,-)_X = H^\bullet(\psi, X)$$

This gives rise to a trinatural transformation $Q'H^\bullet(\psi,-) : Q' \circ H^\bullet(Z',-) \longrightarrow Q' \circ H^\bullet(Z,-)$ which by (TRC,Definition 49) induces a canonical trinatural transformation

$$\mathbb{R}_\mathcal{I} H^\bullet(\psi,-) : \mathbb{R}_\mathcal{I} H^\bullet(Z',-) \longrightarrow \mathbb{R}_\mathcal{I} H^\bullet(Z,-)$$

which by (TRC,Lemma 118) must have the form $\mathbb{R}_\mathcal{I} H^\bullet(\psi, X) = Q'H^\bullet(\psi, I_X)$, where $i_X : X \longrightarrow I_X$ is the hoinjective resolution chosen by $\mathcal{I}$. Moreover we have

$$\mathbb{R}_\mathcal{I} H^\bullet(\psi,-) \circ \mathbb{R}_\mathcal{I} H^\bullet(\psi',-) = \mathbb{R}_\mathcal{I} H^\bullet(\psi'\psi,-)$$
$$\mathbb{R}_\mathcal{I} H^\bullet(\psi+\psi',-) = \mathbb{R}_\mathcal{I} H^\bullet(\psi,-) + \mathbb{R}_\mathcal{I} H^\bullet(\psi',-)$$
$$\mathbb{R}_\mathcal{I} H^\bullet(1,-) = 1$$

For any complex X in $\mathcal{A}$ we write L_X for the additive functor $K(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathfrak{D}(\mathcal{B})$ defined on objects by $L_X(Z) = \mathbb{R}_{\mathcal{I}} H^\bullet(Z, X)$ and on a morphism $\psi : Z \longrightarrow Z'$ by $L_X(\psi) = \mathbb{R}_{\mathcal{I}} H^\bullet(\psi, X)$. In fact this is equal as an additive functor to the composite $Q'H^\bullet(-, I_X) : K(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathfrak{D}(\mathcal{B})$, so L_X becomes by Lemma 23 a triangulated functor in a canonical way. If we make the further assumption that H is *homlike*, then L_X contains the exact complexes of $K(\mathcal{A})^{\mathrm{op}}$ in its kernel, and therefore induces a triangulated functor

$$\mathbb{R}_{\mathcal{I}} H^\bullet(-, X) : \mathfrak{D}(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathfrak{D}(\mathcal{B})$$

Lemma 24. *For morphisms $\varphi : X \longrightarrow X'$ and $\psi : Z \longrightarrow Z'$ in $\mathfrak{D}(\mathcal{A})$ we have*

$$\mathbb{R}_{\mathcal{I}} H^\bullet(Z, \varphi)\mathbb{R}_{\mathcal{I}} H^\bullet(\psi, X) = \mathbb{R}_{\mathcal{I}} H^\bullet(\psi, X')\mathbb{R}_{\mathcal{I}} H^\bullet(Z', \varphi)$$

Proof. First write $\varphi = Q(g)Q(f)^{-1}$ and $\psi = Q(t)^{-1}Q(s)$ for suitable morphisms of complexes. Using (TRC,Remark 75) we can write both sides of the equation as a sequence of morphisms of the form $Q'H^\bullet(-,-)$ or $Q'H^\bullet(-,-)^{-1}$. Then use the fact that $H^\bullet(-,-)$ is a bifunctor to commute the terms past each other and check that they are equal. □

Definition 16. Let $\mathcal{A}$ be an abelian category with enough hoinjectives, $\mathcal{B}$ an abelian category, $H : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{B}$ a homlike functor and $\mathcal{I}$ an assignment of hoinjective resolutions for $\mathcal{A}$. Then there is a canonical functor additive in each variable

$$\mathbb{R}_{\mathcal{I}} H^\bullet(-,-) : \mathfrak{D}(\mathcal{A})^{\mathrm{op}} \times \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$$

with $\mathbb{R}_{\mathcal{I}} H^\bullet(\psi, \varphi)$ defined to be the equal composites of Lemma 24. For complexes Z, X we have $\mathbb{R}_{\mathcal{I}} H^\bullet(Z, X) = H^\bullet(Z, I_X)$ where $X \longrightarrow I_X$ is the chosen resolution. As part of the data we have a morphism in $\mathfrak{D}(\mathcal{B})$ trinatural in both variables

$$\zeta : H^\bullet(Z, Q) \longrightarrow \mathbb{R}H^\bullet(Z, Q)$$

In particular we have the derived Hom functor

$$\mathbb{R}_{\mathcal{I}} Hom^\bullet(-,-) : \mathfrak{D}(\mathcal{A})^{\mathrm{op}} \times \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

defined by $\mathbb{R}_{\mathcal{I}} Hom^\bullet(X, Y) = Hom^\bullet(X, I_Y)$.

Remark 8. With the notation of Definition 16 the additive functor $\mathbb{R}_{\mathcal{I}} H^\bullet(-,-)$ has partial functors in each variable which are triangulated functors

$$\mathbb{R}_{\mathcal{I}} H^\bullet(X, -) : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$$
$$\mathbb{R}_{\mathcal{I}} H^\bullet(-, Y) : \mathfrak{D}(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathfrak{D}(\mathcal{B})$$

and moreover these triangulated structures are compatible. That is, one checks that for a morphism $\alpha : Y \longrightarrow Y'$ in $\mathfrak{D}(\mathcal{A})$ the following diagram commutes in $\mathfrak{D}(\mathcal{B})$

$$\begin{array}{ccc}
\mathbb{R}_{\mathcal{I}} H^\bullet(\Sigma^{-1}X, Y) & \xrightarrow{\mathbb{R}_{\mathcal{I}} H^\bullet(\Sigma^{-1}X, \alpha)} & \mathbb{R}_{\mathcal{I}} H^\bullet(\Sigma^{-1}X, Y') \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
\Sigma\mathbb{R}_{\mathcal{I}} H^\bullet(X, Y) & \xrightarrow[\Sigma\mathbb{R}_{\mathcal{I}} H^\bullet(X, \alpha)]{} & \Sigma\mathbb{R}_{\mathcal{I}} H^\bullet(X, Y')
\end{array}$$

and for a morphism $\alpha : X \longrightarrow X'$ in $\mathfrak{D}(\mathcal{A})$ the following diagram commutes in $\mathfrak{D}(\mathcal{B})$

$$\begin{array}{ccc}
\mathbb{R}_{\mathcal{I}} H^\bullet(X', \Sigma Y) & \xrightarrow{\phi} & \Sigma\mathbb{R}_{\mathcal{I}} H^\bullet(X', Y) \\
{\scriptstyle \mathbb{R}_{\mathcal{I}} H^\bullet(\psi, \Sigma Y)}\downarrow & & \downarrow{\scriptstyle \Sigma\mathbb{R}_{\mathcal{I}} H^\bullet(\psi, Y)} \\
\mathbb{R}_{\mathcal{I}} H^\bullet(X, \Sigma Y) & \xrightarrow[\phi]{} & \Sigma\mathbb{R}_{\mathcal{I}} H^\bullet(X, Y)
\end{array}$$

Lemma 25. *With the notation of Definition 16 suppose we have assignments $\mathcal{I}, \mathcal{J}$ of hoinjective resolutions. For complexes X, Y in $\mathcal{A}$ there is a canonical isomorphism in $\mathfrak{D}(\mathcal{B})$ natural in both variables*

$$\mathbb{R}_{\mathcal{I}} H^\bullet(X,Y) \longrightarrow \mathbb{R}_{\mathcal{J}} H^\bullet(X,Y)$$

which on the partial functors gives trinatural equivalences

$$\begin{aligned} \mathbb{R}_{\mathcal{I}} H^\bullet(X,-) &\longrightarrow \mathbb{R}_{\mathcal{J}} H^\bullet(X,-) \\ \mathbb{R}_{\mathcal{I}} H^\bullet(-,Y) &\longrightarrow \mathbb{R}_{\mathcal{J}} H^\bullet(-,Y) \end{aligned} \tag{5}$$

Proof. By definition of a right derived functor we have trinatural transformations

$$\begin{aligned} \zeta : \mathbb{R}_{\mathcal{I}} H^\bullet(X,-)Q &\longrightarrow QH^\bullet(X,-) \\ \zeta' : \mathbb{R}_{\mathcal{J}} H^\bullet(X,-)Q &\longrightarrow QH^\bullet(X,-) \end{aligned}$$

and therefore a trinatural equivalence $\mu : \mathbb{R}_{\mathcal{I}} H^\bullet(X,-) \longrightarrow \mathbb{R}_{\mathcal{J}} H^\bullet(Y,-)$ which is the unique trinatural transformation making the following diagram commute

$$\begin{array}{ccccc} & & QH^\bullet(X,-) & & \\ & \overset{\zeta}{\swarrow} & & \overset{\zeta'}{\searrow} & \\ \mathbb{R}_{\mathcal{I}} H^\bullet(X,-)Q & & \xrightarrow[\mu Q]{} & & \mathbb{R}_{\mathcal{J}} H^\bullet(X,-)Q \end{array}$$

This yields the desired isomorphism $\mathbb{R}_{\mathcal{I}} H^\bullet(X,Y) \longrightarrow \mathbb{R}_{\mathcal{J}} H^\bullet(X,Y)$ which one checks is also natural in X. It is also clear that the partial functors are trinatural equivalences. □

Lemma 26. *Let $\mathcal{A}$ be an abelian category with enough hoinjectives and X, Y complexes in $\mathcal{A}$. Then there is a canonical isomorphism of abelian groups natural in both variables*

$$H^n(\mathbb{R}Hom^\bullet(X,Y)) \longrightarrow Hom_{\mathfrak{D}(\mathcal{A})}(X, \Sigma^n Y)$$

Proof. To be precise, we mean that once you fix an assignment of hoinjective resolutions $\mathcal{I}$ there is a canonical isomorphism $H^n(\mathbb{R}_{\mathcal{I}} Hom^\bullet(X,Y)) \longrightarrow Hom_{\mathfrak{D}(\mathcal{A})}(X, \Sigma^n Y)$. This follows from Proposition 18 since

$$\begin{aligned} H^n(\mathbb{R}_{\mathcal{I}} Hom^\bullet(X,Y)) &= H^n(Hom^\bullet(X, I_Y)) \\ &\cong Hom_{K(A)}(X, \Sigma^n I_Y) \\ &\cong Hom_{\mathfrak{D}(A)}(X, \Sigma^n I_Y) \\ &\cong Hom_{\mathfrak{D}(A)}(X, \Sigma^n Y) \end{aligned}$$

naturality in both variables with respect to morphisms in $\mathfrak{D}(X)$ is easily checked. □

Combining this with (DTC2,Proposition 18) we see that the cohomology of $Hom^\bullet(-,-)$ calculates morphisms in $K(\mathcal{A})$ and the cohomology of $\mathbb{R}Hom^\bullet(-,-)$ calculates morphisms in $\mathfrak{D}(\mathcal{A})$. Next we check that these isomorphisms are compatible in the obvious way.

Lemma 27. *Let $\mathcal{A}$ be an abelian category with enough hoinjectives and X, Y complexes in $\mathcal{A}$. Then the following diagram commutes*

$$\begin{array}{ccc} H^n(Hom^\bullet(X,Y)) & \longrightarrow & Hom_{K(\mathcal{A})}(X, \Sigma^n Y) \\ \downarrow & & \downarrow \\ H^n(\mathbb{R}Hom^\bullet(X,Y)) & \longrightarrow & Hom_{\mathfrak{D}(\mathcal{A})}(X, \Sigma^n Y) \end{array}$$

Proof. Fix an assignment of hoinjectives $\mathcal{I}$ to calculate $\mathbb{R}_{\mathcal{I}} Hom(-,-)$. As part of the definition we have a natural transformation $\zeta : Q \circ Hom^\bullet(-,-) \longrightarrow \mathbb{R}_{\mathcal{I}} Hom(-,-)Q$ which is what gives rise upon taking cohomology to the morphism on the left. The top and bottom morphisms are from (DTC2,Proposition 18) and Lemma 26 respectively.

If $\alpha : Y \longrightarrow I_Y$ is the chosen resolution of Y then $\zeta : Hom^\bullet(X,Y) \longrightarrow \mathbb{R}Hom^\bullet(X,Y) = Hom^\bullet(X, I_Y)$ is simply $Hom^\bullet(X, \alpha)$. It is then clear from the construction of the bottom isomorphism that the diagram commutes. □

3.1 The New Ext

Definition 17. Let $\mathcal{A}$ be an abelian category. Given objects $X, Y \in \mathcal{A}$ and $i \in \mathbb{Z}$ we define

$$\underline{E}xt^i_{\mathcal{A}}(X,Y) = Hom_{\mathfrak{D}(\mathcal{A})}(X, \Sigma^i Y)$$

where we consider X, Y as complexes in degree zero. This is a (large) abelian group which is clearly functorial in each variable. It is additive in each variable and contravariant in the first. From (DTC,Lemma 32) we deduce that $\underline{E}xt^i_{\mathcal{A}}(X,Y) = 0$ whenever $i < 0$. From (DTC,Proposition 28) we have a canonical isomorphism of abelian groups natural in both variables

$$Hom_{\mathcal{A}}(X,Y) \longrightarrow \underline{E}xt^0(X,Y)$$

Given a short exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ in $\mathcal{A}$ there is by (DTC,Proposition 20) a canonical morphism $z : Y'' \longrightarrow \Sigma Y'$ in $\mathfrak{D}(\mathcal{A})$ fitting into a triangle

$$Y' \longrightarrow Y \longrightarrow Y'' \xrightarrow{-z} \Sigma Y'$$

For $i \geq 0$ we have a canonical morphism of abelian groups $\omega^i : \underline{E}xt^i_{\mathcal{A}}(X, Y'') \longrightarrow \underline{E}xt^{i+1}_{\mathcal{A}}(X, Y')$ defined to be $Hom_{\mathfrak{D}(\mathcal{A})}(X, -\Sigma^i z)$. These connecting morphisms fit into a long exact sequence

$$\begin{aligned} 0 \longrightarrow Hom_{\mathcal{A}}(X,Y') \longrightarrow Hom_{\mathcal{A}}(X,Y) \longrightarrow Hom_{\mathcal{A}}(X,Y'') \longrightarrow \\ \underline{E}xt^1_{\mathcal{A}}(X,Y') \longrightarrow \underline{E}xt^1_{\mathcal{A}}(X,Y) \longrightarrow \underline{E}xt^1_{\mathcal{A}}(X,Y'') \longrightarrow \\ \underline{E}xt^2_{\mathcal{A}}(X,Y') \longrightarrow \underline{E}xt^2_{\mathcal{A}}(X,Y) \longrightarrow \underline{E}xt^2_{\mathcal{A}}(X,Y'') \longrightarrow \cdots \end{aligned}$$

Similarly, given a short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in $\mathcal{A}$ we have a canonical morphism $z : X'' \longrightarrow \Sigma X'$ in $\mathfrak{D}(\mathcal{A})$ and a triangle

$$X' \longrightarrow X \longrightarrow X'' \xrightarrow{-z} \Sigma X'$$

For $i \geq 0$ we have a canonical morphism of abelian groups $\delta^i : \underline{E}xt^i_{\mathcal{A}}(X', Y) \longrightarrow \underline{E}xt^{i+1}_{\mathcal{A}}(X'', Y)$ defined to be the following composite

$$Hom_{\mathfrak{D}(\mathcal{A})}(X', \Sigma^i Y) \xrightarrow{Hom(-\Sigma^{-1} z, \Sigma^i Y)} Hom_{\mathfrak{D}(\mathcal{A})}(\Sigma^{-1} X'', \Sigma^i Y) \Longrightarrow Hom_{\mathfrak{D}(\mathcal{A})}(X'', \Sigma^{i+1} Y)$$

These connecting morphisms fit into a long exact sequence

$$\begin{aligned} 0 \longrightarrow Hom_{\mathcal{A}}(X'',Y) \longrightarrow Hom_{\mathcal{A}}(X,Y) \longrightarrow Hom_{\mathcal{A}}(X',Y) \longrightarrow \\ \underline{E}xt^1_{\mathcal{A}}(X'',Y) \longrightarrow \underline{E}xt^1_{\mathcal{A}}(X,Y) \longrightarrow \underline{E}xt^1_{\mathcal{A}}(X',Y) \longrightarrow \\ \underline{E}xt^2_{\mathcal{A}}(X'',Y) \longrightarrow \underline{E}xt^2_{\mathcal{A}}(X,Y) \longrightarrow \underline{E}xt^2_{\mathcal{A}}(X',Y) \longrightarrow \cdots \end{aligned}$$

Remark 9. This new definition of the group $\underline{E}xt^i_{\mathcal{A}}(X,Y)$ is remarkable for several reasons. Firstly, we do not require $\mathcal{A}$ to have enough injectives, and secondly the definition is canonical: one need not choose resolutions in either variable in order to calculate the group. We also obtain the long exact sequences and functoriality in both variables very cheaply.

Lemma 28. *Let $\mathcal{A}$ be an abelian category with enough injectives. Given objects $X, Y \in \mathcal{A}$ and $i \geq 0$ there is a canonical isomorphism of abelian groups natural in both variables*

$$Ext^i_{\mathcal{A}}(X,Y) \longrightarrow \underline{E}xt^i_{\mathcal{A}}(X,Y)$$

where the first abelian group is the usual one.

Proof. Fix an assignment of injective resolutions for $\mathcal{A}$ and use it to define the bifunctor $Ext^i_{\mathcal{A}}(-,-)$ (EXT,Proposition 1). Before we can proceed with the proof we need to make a small observation. Let Q be any complex in $\mathcal{A}$ and consider X as a complex in degree zero as usual. Then the complex $Hom^\bullet(X,Q)$ of abelian groups is canonically isomorphic to the complex $Hom_{\mathcal{A}}(X,Q)$ with alternating signs on the differentials

$$\cdots \longrightarrow Hom_{\mathcal{A}}(X,Q^0) \xrightarrow{-} Hom_{\mathcal{A}}(X,Q^1) \longrightarrow Hom_{\mathcal{A}}(X,Q^2) \xrightarrow{-} \cdots$$

In any case the sign changes do not affect cohomology, so we have a canonical isomorphism of abelian groups $H^n(Hom^\bullet(X,Q)) \longrightarrow H^n(Hom_{\mathcal{A}}(X,Q))$. Now suppose we are given objects $X,Y \in \mathcal{A}$ and $i \geq 0$ and let I be the chosen injective resolution of Y. That is, we are given a quasi-isomorphism of complexes $v : Y \longrightarrow I$. This becomes an isomorphism in the derived category, so using Proposition 18 we have a canonical isomorphism of abelian groups

$$\begin{aligned} Ext^i_{\mathcal{A}}(X,Y) &= H^i(Hom_{\mathcal{A}}(X,I)) \\ &\cong H^i(Hom^\bullet(X,I)) \\ &\cong Hom_{K(\mathcal{A})}(X,\Sigma^i I) \\ &\cong Hom_{\mathfrak{D}(\mathcal{A})}(X,\Sigma^i I) \\ &\cong Hom_{\mathfrak{D}(\mathcal{A})}(X,\Sigma^i Y) \\ &= \underline{Ext}^i_{\mathcal{A}}(X,Y) \end{aligned}$$

naturality in both variables is easily checked. □

Remark 10. Let k be a commutative ring and $\mathcal{A}$ a k-linear abelian category, so that the morphism sets in $\mathfrak{D}(A)$ are canonically k-modules (DTC,Remark 11). In particular $\underline{Ext}^i_{\mathcal{A}}(X,Y)$ is a k-module for $X,Y \in \mathcal{A}$ and $i \in \mathbb{Z}$. If $\mathcal{A}$ has enough injectives then the usual group $Ext^i_{\mathcal{A}}(X,Y)$ is also a k-module (EXT,Section 4.1), and it is easy to check that the isomorphism of Lemma 28 is of k-modules.

Lemma 29. *Let $\mathcal{A}$ be an abelian category. An object $X \in \mathcal{A}$ is projective in $\mathcal{A}$ if and only if $Hom_{\mathfrak{D}(\mathcal{A})}(X,\mathfrak{D}(\mathcal{A})^{\leq -1}) = 0$. That is, if and only if $Hom_{\mathfrak{D}(\mathcal{A})}(X,Q) = 0$ for every $Q \in \mathfrak{D}(\mathcal{A})^{\leq -1}$.*

Proof. If X is projective then it is hoprojective as a complex, so for any $Q \in \mathfrak{D}(\mathcal{A})^{\leq -1}$ we have

$$Hom_{\mathfrak{D}(\mathcal{A})}(X,Q) \cong Hom_{K(\mathcal{A})}(X,Q)$$

We may as well assume that as a complex Q is zero above -1, since we can replace Q by $Q_{\leq -1}$ in $\mathfrak{D}(\mathcal{A})$. Then a morphism of complexes $X \longrightarrow Q$ must be zero, from which we deduce that $Hom_{\mathfrak{D}(\mathcal{A})}(X,Q) = 0$. Conversely if $Hom_{\mathfrak{D}(\mathcal{A})}(X,\mathfrak{D}(\mathcal{A})^{\leq -1}) = 0$ then in particular for any $Y \in \mathcal{A}$ we have

$$\underline{Ext}^1_{\mathcal{A}}(X,Y) = Hom_{\mathfrak{D}(\mathcal{A})}(X,\Sigma^1 Y) = 0$$

Using the long exact sequences of $\underline{Ext}$'s it is now clear that X is projective. Observe that we did not need $\mathcal{A}$ to have enough injectives. □

Remark 11. If $\mathcal{A}$ is an abelian category with enough hoinjectives then for $X,Y \in \mathcal{A}$ and $n \in \mathbb{Z}$ we have by Lemma 26 a canonical isomorphism of abelian groups natural in both variables

$$H^n(\mathbb{R}Hom^\bullet(X,Y)) \longrightarrow \underline{Ext}^n_{\mathcal{A}}(X,Y)$$

which is quite interesting. In particular $\mathbb{R}Hom^\bullet(X,Y)$ belongs to $\mathfrak{D}(\mathcal{A})^{\geq 0}$ (that is, it is exact at negative positions).

3.2 Hoinjectives and Inverse Limits

In this section we give an argument due to Spaltenstein [Spa88] that hoinjective complexes are closed under split inverse limits. Our proof in (DTC,Section 5.1) is more general, but the technique here might be useful elsewhere. We begin with a useful technical lemma from [Spa88].

Lemma 30. *Suppose we have four inverse systems and morphisms between them in* **Ab**, *as in the following diagram*

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & D_2 \\
\downarrow{\scriptstyle \partial_A^2} & & \downarrow{\scriptstyle \partial_B^2} & & \downarrow{\scriptstyle \partial_C^2} & & \downarrow{\scriptstyle \partial_D^2} \\
A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\
\downarrow{\scriptstyle \partial_A^1} & & \downarrow{\scriptstyle \partial_B^1} & & \downarrow{\scriptstyle \partial_C^1} & & \downarrow{\scriptstyle \partial_D^1} \\
A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 & \xrightarrow{h_0} & D_0
\end{array}
$$

in which the vertical morphisms in the first two columns are surjective and $g_i f_i = 0, h_i g_i = 0$ *for* $i \geq 0$. *Taking inverse limits we have a sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

with $gf = 0, hg = 0$. *Suppose* $j \geq 0$ *is such that for every* $i > j$ *the sequence*

$$Ker\partial_A^i \longrightarrow Ker\partial_B^i \longrightarrow Ker\partial_C^i \longrightarrow Ker\partial_D^i$$

is exact. Then there is a canonical isomorphism $Kerg/Imf \longrightarrow Kerg_j/Imf_j$.

Proof. The following commutative diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_j & \xrightarrow{f_j} & B_j & \xrightarrow{g_j} & C_j & \xrightarrow{h_j} & D_j
\end{array}
$$

induces a morphism of abelian groups $\theta : Kerg/Imf \longrightarrow Kerg_j/Imf_j$, which we claim is an isomorphism. To see that it is injective, let $(m_k)_{k\geq 0}$ be a sequence belonging to $Kerg$ with $m_j \in Imf_j$. We have to define a sequence $(t_k)_{k\geq 0}$ in A with $f_k(t_k) = m_k$. By assumption we can find $t_j \in A_j$ with $f_j(t_j) = m_j$. If we define $t_0, t_1, \ldots, t_{j-1}$ to be the images of this t_j under the morphisms ∂ of the direct system then it is clear that $f_k(t_k) = m_k$ for $0 \leq k \leq j$.

Suppose we are given a partial sequence $t_0, \ldots, t_i$ for some $i \geq j$ with $f_k(t_k) = m_k$ for $0 \leq k \leq i$ and $\partial(t_{k+1}) = t_k$ for $k \geq 0$. Using exactness of the kernel sequence and a diagram chase, we deduce that there is an element $t_{i+1} \in A_{i+1}$ with $f_{i+1}(t_{i+1}) = m_{i+1}$ and $\partial(t_{i+1}) = t_i$. By a simple Zorn's Lemma argument we can produce the desired sequence $(t_k)_{k\geq 0}$ in A mapping to $(m_k)_{k\geq 0}$, so θ is injective. Surjectivity of θ is checked in much the same way. $\square$

Definition 18. Let $\mathcal{A}$ be a complete abelian category. An inverse system of complexes in $\mathcal{A}$

$$\cdots \longrightarrow G_3 \xrightarrow{\mu_3} G_2 \xrightarrow{\mu_2} G_1 \xrightarrow{\mu_1} G_0 \qquad (6)$$

is a *split inverse system* if every μ_n is a retraction in each degree (in particular an epimorphism). If $\mathcal{I}$ is a nonempty class of complexes in $\mathcal{A}$ containing the zero objects and closed under isomorphism, then we say that $\mathcal{I}$ is *closed under split inverse limits* if the limit of every split inverse system whose objects belong to $\mathcal{I}$, also belongs to $\mathcal{I}$.

Lemma 31. *The class of all exact complexes in* $\mathbf{C}(\mathbf{Ab})$ *is closed under split inverse limits.*

Proof. Suppose we are given a direct system (6) of exact complexes of abelian groups in which every μ_n is a retraction in each degree. Then applying Lemma 30 (where going "up the page" in the lemma corresponds to increasing n in our inverse system) we deduce that $\varprojlim_{n\geq 0} G_n$ is exact. □

Proposition 32. *Let* $\mathcal{A}$ *be a complete abelian category. The class of all hoinjective complexes in* $\mathbf{C}(\mathcal{A})$ *is closed under split inverse limits.*

Proof. Let (6) be a split inverse system with every G_n hoinjective, and let G be the inverse limit. We have to show that G is hoinjective. Let Z be an exact complex in $\mathcal{A}$. From (DTC,Lemma 67) we have a short exact sequence of complexes

$$0 \longrightarrow G \longrightarrow \prod_n G_n \xrightarrow{1-\nu} \prod_n G_n \longrightarrow 0$$

which is actually split exact in every degree. Therefore we have a short exact sequence of complexes of abelian groups, split exact in each degree

$$0 \longrightarrow Hom^\bullet(Z,G) \longrightarrow \prod_n Hom^\bullet(Z,G_n) \xrightarrow{1-\nu} \prod_n Hom^\bullet(Z,G_n) \longrightarrow 0$$

where we use Lemma 17. But this means that $Hom^\bullet(Z,G)$ is the inverse limit of the following inverse system of exact complexes

$$\cdots \longrightarrow Hom^\bullet(Z,G_2) \longrightarrow Hom^\bullet(Z,G_1) \longrightarrow Hom^\bullet(Z,G_0)$$

Each morphism in this system is a retraction in each degree, so we deduce from Lemma 31 that $Hom^\bullet(Z,G)$ is exact. Therefore G is hoinjective and the proof is complete. □

4 Dimension of Functors

Let $\mathcal{A}$ be an abelian category. Recall from (DTC,Definition 20) that for each $n \in \mathbb{Z}$ we have full replete additive subcategories $\mathfrak{D}(\mathcal{A})^{\geq n}$ and $\mathfrak{D}(\mathcal{A})^{\leq n}$ of $\mathfrak{D}(\mathcal{A})$ consisting of complexes whose cohomology is bounded below and above by n respectively. Given a triangulated functor $T : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ one can ask "how badly" a complex can grow when you apply T. The measure of this badness is the *dimension* of T.

Definition 19. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $\mathcal{S}$ a fragile triangulated subcategory of $\mathfrak{D}(\mathcal{A})$. Given a triangulated functor $T : \mathcal{S} \longrightarrow \mathfrak{D}(\mathcal{B})$ we define the *upper dimension* dim^+T and *lower dimension* dim^-T by

$$dim^+T = inf\{d \geq 0 \,|\, T(X) \in \mathfrak{D}(\mathcal{B})^{\leq(n+d)} \text{ for all } n \in \mathbb{Z}, X \in \mathfrak{D}(\mathcal{A})^{\leq n}\}$$
$$dim^-T = inf\{d \geq 0 \,|\, T(X) \in \mathfrak{D}(\mathcal{B})^{\geq(n-d)} \text{ for all } n \in \mathbb{Z}, X \in \mathfrak{D}(\mathcal{A})^{\geq n}\}$$

with either infimum equal to ∞ if the sets are empty. So the dimensions are elements of the set $\{0, 1, 2, \ldots, \infty\}$. We say that T is *bounded above* if $dim^+T < \infty$ and *bounded below* if $dim^-T < \infty$. If T is bounded above and below, we say it is *bounded*. Clearly naturally equivalent triangulated functors have the same upper and lower dimensions.

Remark 12. With the notation of Definition 19 let $d \geq 0$ be an integer for which there exists *some* $n \in \mathbb{Z}$ with $T(X) \in \mathfrak{D}(\mathcal{B})^{\leq(n+d)}$ for every $X \in \mathfrak{D}(\mathcal{A})^{\leq n}$. Then this is true for *every* $n \in \mathbb{Z}$ since $\mathcal{S}$ is closed under translation and T is triangulated. Similarly for the lower dimension.

It is clear that the sets in Definition 19 over which the infimums dim^+T, dim^-T are taken are upwards closed. That is, if an integer $d \geq 0$ has the proper of the previous paragraph then so does any integer $e \geq d$.

Remark 13. To be clear, with the notation of Definition 19 the triangulated functor T is

- *Bounded above* if for some $d \geq 0, n \in \mathbb{Z}$ we have $T(\mathcal{S} \cap \mathfrak{D}(\mathcal{A})^{\leq n}) \subseteq \mathfrak{D}(\mathcal{B})^{\leq (n+d)}$.
- *Bounded below* if for some $d \geq 0, n \in \mathbb{Z}$ we have $T(\mathcal{S} \cap \mathfrak{D}(\mathcal{A})^{\geq n}) \subseteq \mathfrak{D}(\mathcal{B})^{\geq (n-d)}$.

Lemma 33. *Let $\mathcal{A}$ be an abelian category with enough injectives and hoinjectives, $\mathcal{B}$ an abelian category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Then $dim^{-}\mathbb{R}F = 0$.*

Proof. In other words, for any $n \in \mathbb{Z}$ and $X \in \mathfrak{D}(A)^{\geq n}$ we have $\mathbb{R}F(X) \in \mathfrak{D}(B)^{\geq n}$. Replacing X by its truncation we can assume that $X^i = 0$ for $i < n$. Then by (DTC,Corollary 74) we can replace X by a complex of injectives I with $I^i = 0$ for $i < n$. Since $\mathbb{R}F(I) \cong F(I)$ the proof is complete. □

Lemma 34. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $T : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ a triangulated functor. If $U : \mathcal{A}' \longrightarrow \mathcal{A}$ is an exact functor then*

$$dim^{+}T \geq dim^{+}(T \circ U), \quad dim^{-}T \geq dim^{-}(T \circ U)$$

Similarly if $u : \mathcal{B} \longrightarrow \mathcal{B}'$ is exact then

$$dim^{+}T \geq dim^{+}(u \circ T), \quad dim^{-}T \geq dim^{-}(u \circ T)$$

Just for convenience in the next few results, we say that a fragile triangulated subcategory $\mathcal{S} \subseteq \mathfrak{D}(\mathcal{A})$ is *upper truncation closed* if whenever $X \in \mathcal{S}$ we have $X_{\geq n} \in \mathcal{S}$ for $n \in \mathbb{Z}$. It is *lower truncation closed* if whenever $X \in \mathcal{S}$ we have $X_{\leq n} \in \mathcal{S}$ for $n \in \mathbb{Z}$. For example if $\mathcal{C}$ is a plump subcategory of $\mathcal{A}$ then $\mathfrak{D}_{\mathcal{C}}(\mathcal{A})$ has both these properties as a subcategory of $\mathfrak{D}(\mathcal{A})$.

Lemma 35. *With the notation of Definition 19 suppose that $\mathcal{S}$ is upper and lower truncation closed. Then for $d \geq 0$ the following conditions are equivalent*

(a) $dim^{+}T \leq d$.

(b) If $X \in \mathcal{S}$ and $n \in \mathbb{Z}$ then $X \in \mathfrak{D}(\mathcal{A})^{\leq n}$ implies $T(X) \in \mathfrak{D}(\mathcal{B})^{\leq (n+d)}$.

(c) If $X \in \mathcal{S}$ and $n \in \mathbb{Z}$ then $H^iT(q) : H^i(TX) \longrightarrow H^i(TX_{\geq n})$ is an isomorphism for $i \geq n+d$, where $q : X \longrightarrow X_{\geq n}$ is canonical.

Proof. $(a) \Rightarrow (b)$ is trivial. $(b) \Rightarrow (c)$ follows from the long exact sequence in $\mathcal{B}$ we obtain from applying $H^i(-)$ to the following triangle in $\mathcal{S}$ (this is why we need $\mathcal{S}$ to be closed under truncation)

$$X_{\leq (n-1)} \longrightarrow X \longrightarrow X_{\geq n} \longrightarrow \Sigma X_{\leq (n-1)}$$

$(c) \Rightarrow (a)$ If $X \in \mathfrak{D}(\mathcal{A})^{\leq n} \cap \mathcal{S}$ then $q : X \longrightarrow X_{\geq (n+1)}$ is zero. But by assumption $H^iT(q)$ is an isomorphism for $i \geq n+d+1$, from which we deduce that $H^i(TX) = 0$ for $i \geq n+d+1$ and therefore $T(X) \in \mathfrak{D}(\mathcal{B})^{\leq (n+d)}$ as required. □

Lemma 36. *With the notation of Definition 19 suppose that $\mathcal{S}$ is upper and lower truncation closed. Then for $d \geq 0$ the following conditions are equivalent*

(a) $dim^{-}T \leq d$.

(b) If $X \in \mathcal{S}$ and $n \in \mathbb{Z}$ then $X \in \mathfrak{D}(\mathcal{A})^{\geq n}$ implies $T(X) \in \mathfrak{D}(\mathcal{B})^{\geq (n-d)}$.

(c) If $X \in \mathcal{S}$ and $n \in \mathbb{Z}$ then $H^iT(v) : H^i(TX_{\leq n}) \longrightarrow H^i(TX)$ is an isomorphism for $i \leq n-d$ where $v : X_{\leq n} \longrightarrow X$ is canonical.

The condition (c) of Lemma 35 appears technical, but it is probably the characterisation of upper dimension that is most used in practice. It simply says that to calculate the cohomology of the complex $T(X)$ you can reduce to calculating the cohomology of T applied to a bounded below complex. In particular

Lemma 37. *With the notation of Definition 19 suppose that $\mathcal{S}$ is upper and lower truncation closed. Then for a complex $X \in \mathcal{S}$ we have*

(i) If T is bounded above and $T(X_{\geq n}) = 0$ for every $n \in \mathbb{Z}$, then $T(X) = 0$.

(ii) If T is bounded below and $T(X_{\leq n}) = 0$ for every $n \in \mathbb{Z}$, then $T(X) = 0$.

Proposition 38. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $S, T : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ triangulated functors and $\psi : S \longrightarrow T$ a trinatural transformation. Suppose that $\psi_A : S(A) \longrightarrow T(A)$ is an isomorphism for every $A \in \mathcal{A}$. Then for any $X \in \mathfrak{D}(\mathcal{A})$*

(i) If X is bounded then ψ_X is an isomorphism.

(ii) If S, T are bounded above and X bounded above then ψ_X is an isomorphism.

(iii) If S, T are bounded below and X bounded below then ψ_X is an isomorphism.

(iv) If S, T are bounded then ψ_X is an isomorphism.

Proof. Let $\mathcal{T}$ be the full subcategory of $\mathfrak{D}(\mathcal{A})$ consisting of the complexes X such that ψ_X is an isomorphism. This is a triangulated subcategory of $\mathfrak{D}(\mathcal{A})$ (TRC,Remark 30). By assumption it contains all the objects of $\mathcal{A}$ and therefore by (DTC,Lemma 79) it contains any bounded complex, which proves (i). For (ii) let d be an integer such that $dim^+T \leq d, dim^+S \leq d$. It would suffice to show that $H^i(\psi_X) : H^i(SX) \longrightarrow H^i(TX)$ is an isomorphism for $i \in \mathbb{Z}$. We want to replace X by something which is bounded but which appears the same to $H^i(\psi_X)$. Fix $i \in \mathbb{Z}$ and let $q : X \longrightarrow X_{\geq n}$ be canonical with $n = i - d$. Then by Lemma 35(c) we have a commutative diagram

$$\begin{array}{ccc} H^i(SX) & \xrightarrow{H^i(\psi_X)} & H^i(TX) \\ {\scriptstyle H^iS(q)}\downarrow & & \downarrow{\scriptstyle H^iT(q)} \\ H^i(SX_{\geq n}) & \xrightarrow[H^i(\psi_{X_{\geq n}})]{} & H^i(TX_{\geq n}) \end{array}$$

in which the vertical morphisms are isomorphisms. Since $X_{\geq n}$ is bounded $\psi_{X_{\geq n}}$ is an isomorphism, from which it follows that $H^i(\psi_X)$ is an isomorphism, completing the proof of (ii). The proof of (iii) proceeds similarly. To prove (iv) let X be any complex in $\mathcal{A}$ and consider the following triangle in $\mathfrak{D}(\mathcal{A})$

$$X_{\leq 0} \longrightarrow X \longrightarrow X_{\geq 1} \longrightarrow \Sigma X_{\leq 0} \qquad (7)$$

It gives rise to a morphism of triangles in $\mathfrak{D}(\mathcal{B})$

$$\begin{array}{ccccccc} S(X_{\leq 0}) & \longrightarrow & S(X) & \longrightarrow & S(X_{\geq 1}) & \longrightarrow & \Sigma S(X_{\leq 0}) \\ {\scriptstyle \psi_{X_{\leq 0}}}\downarrow & & {\scriptstyle \psi_X}\downarrow & & {\scriptstyle \psi_{X_{\geq 1}}}\downarrow & & \downarrow \\ T(X_{\leq 0}) & \longrightarrow & T(X) & \longrightarrow & T(X_{\geq 1}) & \longrightarrow & \Sigma T(X_{\leq 0}) \end{array}$$

in which the first and third vertical morphisms are isomorphisms by $(ii), (iii)$. We conclude that ψ_X is an isomorphism, as claimed. □

The same proof applies slightly more generally to yield the following.

Proposition 39. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{C}$ a plump subcategory of $\mathcal{A}$, $S, T : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ triangulated functors and $\psi : S \longrightarrow T$ a trinatural transformation. Suppose that $\psi_A : S(A) \longrightarrow T(A)$ is an isomorphism for every $A \in \mathcal{C}$. Then for any $X \in \mathfrak{D}_{\mathcal{C}}(\mathcal{A})$*

(i) If X is bounded then ψ_X is an isomorphism.

(ii) If S, T are bounded above and X bounded above then ψ_X is an isomorphism.

(iii) If S, T are bounded below and X bounded below then ψ_X is an isomorphism.

(iv) If S, T are bounded then ψ_X is an isomorphism.

Proof. Let $\mathcal{T}$ be as in the proof of Proposition 38. By assumption it contains all of the objects of $\mathcal{C}$ and therefore by (DTC,Lemma 81) it contains any bounded complex in $\mathfrak{D}_{\mathcal{C}}(\mathcal{A})$, which proves (i). The rest of the proof proceeds as before. □

Proposition 40. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{D} \subseteq \mathcal{B}$ plump subcategories, and $T : \mathfrak{D}_{\mathcal{C}}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ a triangulated functor. Suppose that $T(A) \in \mathfrak{D}_{\mathcal{D}}(\mathcal{B})$ for all $A \in \mathcal{C}$. Then for any $X \in \mathfrak{D}_{\mathcal{C}}(\mathcal{A})$*

(i) If X is bounded then $T(X) \in \mathfrak{D}_{\mathcal{D}}(\mathcal{B})$.

(ii) If T is bounded above and X bounded above then $T(X) \in \mathfrak{D}_{\mathcal{D}}(\mathcal{B})$.

(iii) If T is bounded below and X bounded below then $T(X) \in \mathfrak{D}_{\mathcal{D}}(\mathcal{B})$.

(iv) If T is bounded then $T(X) \in \mathfrak{D}_{\mathcal{D}}(\mathcal{B})$.

Proof. Let $\mathcal{T}$ be the full subcategory of $\mathfrak{D}_{\mathcal{C}}(\mathcal{A})$ consisting of the complexes X such that $T(X) \in \mathfrak{D}_{\mathcal{D}}(\mathcal{B})$. One checks that this is a triangulated subcategory of $\mathfrak{D}_{\mathcal{C}}(\mathcal{A})$. By assumption it contains all the objects of $\mathcal{C}$ and therefore by the argument of (DTC,Lemma 81) it contains any bounded complex, which proves (i). For (ii) suppose that $dim^+ T = d$. Fix $i \in \mathbb{Z}$ and set $n = i - d$. From Lemma 35(c) we deduce an isomorphism $H^i(TX) \cong H^i(TX_{\geq n})$ in $\mathcal{B}$. Since $X_{\geq n}$ is bounded, this later cohomology object belongs to $\mathcal{D}$, and therefore $H^i(TX) \in \mathcal{D}$ as well. (iii) is proved in the same way, and we deduce (iv) by applying T to the triangle (7). □

5 Acyclic Complexes

The following simple observation allows us to bridge the gap between resolutions of objects in classical homological algebra, and the resolutions of complexes used to define derived functors on unbounded complexes.

Proposition 41. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories and $\mathcal{P} \subseteq \mathcal{A}$ a class of objects which is closed under isomorphism and contains all the zero objects, such that*

(i) Every object X in $\mathcal{A}$ admits an epimorphism $P \longrightarrow X$ for some $P \in \mathcal{P}$.

(ii) If $P, Q \in \mathcal{P}$ then $P \oplus Q \in \mathcal{P}$.

(iii) For every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{A}$, if $B, C \in \mathcal{P}$ then also $A \in \mathcal{P}$ and the following sequence is exact

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

Then every bounded above complex in $\mathcal{P}$ is left F-acyclic.

Proof. When we say that X is a complex in $\mathcal{P}$, we mean it is a complex in $\mathcal{A}$ such that $X^i \in \mathcal{P}$ for every $i \in \mathbb{Z}$. In particular our class $\mathcal{P}$ is a *smothering class* in the sense of (DTC,Definition 30). Therefore by (DTC,Proposition 69) any bounded above complex X in $\mathcal{A}$ admits a quasi-isomorphism $P \longrightarrow X$ where P is a bounded above complex of objects in $\mathcal{P}$.

Let X be a bounded above complex in $\mathcal{P}$, and let $Y \longrightarrow X$ be a quasi-isomorphism. Let $n \in \mathbb{Z}$ be such that $X^i = 0$ for $i > n$. Then the canonical morphism $Y_{\leq n} \longrightarrow Y$ is a quasi-isomorphism. By the above we can find a quasi-isomorphism $X' \longrightarrow Y_{\leq n}$ with X' a bounded above complex in $\mathcal{P}$. So to show that X is left F-acyclic we need to show that any quasi-isomorphism between bounded above complexes in $\mathcal{P}$ is sent to a quasi-isomorphism by F. For this it suffices to show that $F(Z)$ is exact whenever Z is an exact, bounded above complex in $\mathcal{P}$ (here we use (ii) to see that mapping cones stay inside $\mathcal{P}$). Suppose Z is of the form

$$\cdots \longrightarrow Z^{n-3} \longrightarrow Z^{n-2} \longrightarrow Z^{n-1} \longrightarrow Z^n \longrightarrow 0 \longrightarrow \cdots$$

which we can decompose into a series of exact sequences

$$\begin{gathered} 0 \longrightarrow K^1 \longrightarrow Z^{n-1} \longrightarrow Z^n \longrightarrow 0 \\ 0 \longrightarrow K^2 \longrightarrow Z^{n-2} \longrightarrow K^1 \longrightarrow 0 \\ 0 \longrightarrow K^3 \longrightarrow Z^{n-3} \longrightarrow K^2 \longrightarrow 0 \\ \vdots \end{gathered}$$

From (iii) we deduce that the objects $K_1, K_2, \ldots$ belong to $\mathcal{P}$, and therefore under F each short exact sequence is carried to a short exact sequence in $\mathcal{B}$. Piecing the series back together, we deduce that $F(Z)$ is exact in $\mathcal{B}$ as required. □

Dually, we have

Proposition 42. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories and $\mathcal{I} \subseteq \mathcal{A}$ a class of objects which is closed under isomorphism and contains all the zero objects, such that*

(i) Every object X in $\mathcal{A}$ admits a monomorphism $X \longrightarrow I$ for some $I \in \mathcal{I}$.

(ii) If $I, J \in \mathcal{I}$ then $I \oplus J \in \mathcal{I}$.

(iii) For every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{A}$, if $A, B \in \mathcal{I}$ then also $C \in \mathcal{I}$ and the following sequence is exact

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

Then every bounded below complex in $\mathcal{I}$ is right F-acyclic.

The next result shows that acyclicity for complexes agrees with acyclicity for objects, in the sense of our Derived Functors notes. This avoids some potential ambiguity in applications.

Corollary 43. *Let $\mathcal{A}$ be an abelian category with enough injectives and hoinjectives, $\mathcal{B}$ an abelian category, and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Then*

(a) Let X be a bounded below complex in $\mathcal{A}$ such that every X^i is right F-acyclic in the sense of (DF,Definition 14). Then X is right F-acyclic in the sense of Definition 4.

(b) An object $A \in \mathcal{A}$ is right F-acyclic in the sense of (DF,Definition 14) if and only if it is right F-acyclic as a complex in the sense of Definition 4.

Proof. (a) Take $\mathcal{I}$ equal to the class of all objects of $\mathcal{A}$ which are right F-acyclic in the sense of (DF,Definition 14). Since this includes the injective objects, the condition (i) of Proposition 42 is satisfied. Condition (ii) follows from the fact that the derived functors R^iF are additive, and (iii) from a long exact sequence argument. The conclusion of Proposition 42 is what we wanted to show.

(b) It follows from (a) that an object $A \in \mathcal{A}$ which is right F-acyclic in the sense of (DF,Definition 15) is right F-acyclic as a complex. For the converse, let $(\mathbb{R}F, \zeta)$ be a right derived functor of F. By (TRC,Theorem 116)(ii) we have an isomorphism $\zeta_A : F(A) \longrightarrow \mathbb{R}F(A)$ in $\mathfrak{D}(\mathcal{B})$. Since the complex $F(A)$ only has cohomology in degree zero, we deduce from Corollary 12 that $0 = H^i(\mathbb{R}F(A)) \cong R^iF(A)$ for $i > 0$. Therefore A is right F-acyclic in the sense of (DF,Definition 15) and the proof is complete. □

The following result should have an elementary proof using double complexes and spectral sequences, but we give it here as an example of how amazing the machinery really is.

Lemma 44. *Let $\mathcal{A}$ be an abelian category with enough injectives and hoinjectives, $\mathcal{B}$ an abelian category, and $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Suppose an object $A \in \mathcal{A}$ has two right F-acyclic resolutions*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$
$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots$$

If $\psi : I \longrightarrow J$ is a morphism of complexes lifting the identity on A then $F(\psi) : FI \longrightarrow FJ$ is a quasi-isomorphism. In particular we have a canonical isomorphism $H^i(FI) \longrightarrow H^i(FJ)$ for every $i \geq 0$.

Proof. By a right F-acyclic resolution we mean that these sequences are exact, and all the I^i, J^i are right F-acyclic. Write $p : A \longrightarrow I^0$ and $q : A \longrightarrow J^0$ for the first two morphisms. Then the morphism of coomplexes $p : A \longrightarrow I$ is a quasi-isomorphism of A with a right F-acyclic complex, and the same is true of $q : A \longrightarrow J$. In $\mathfrak{D}(\mathcal{A})$ the morphism $Q(\psi) : I \longrightarrow J$ is an isomorphism. Let $(\mathbb{R}F, \zeta)$ be a right derived functor of F. Then we have a commutative diagram in $\mathfrak{D}(B)$

$$\begin{array}{ccc} FI & \xrightarrow{\zeta_I} & \mathbb{R}F(I) \\ {\scriptstyle F(\psi)}\downarrow & & \downarrow{\scriptstyle \mathbb{R}F(Q(\psi))} \\ FJ & \xrightarrow[\zeta_J]{} & \mathbb{R}F(J) \end{array}$$

In which the horizontals morphisms are isomorphisms by (TRC,Theorem 116)(ii). We deduce that $F(\psi)$ is an isomorphism in $\mathfrak{D}(\mathcal{B})$, which means that $F(\psi)$ is a quasi-isomorphism as claimed. □

Remark 14. With the notation of Lemma 44 if J is an injective resolution of A then the morphism ψ always exists, and we deduce a canonical isomorphism $H^i(FI) \longrightarrow R^iF(A)$. This isomorphism is natural in A in the following sense: given a commutative diagram in $\mathcal{A}$ with exact rows and I^i, J^i right F-acyclic

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \\ & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \psi^0}\downarrow & & {\scriptstyle \psi^1}\downarrow & & {\scriptstyle \psi^2}\downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \end{array}$$

the following diagram commutes for $i \geq 0$

$$\begin{array}{ccc} H^i(FI) & \longrightarrow & R^iF(A) \\ {\scriptstyle H^i(F\psi)}\downarrow & & \downarrow{\scriptstyle R^iF(\alpha)} \\ H^i(FJ) & \longrightarrow & R^iF(B) \end{array}$$

This gives a much more elegant proof of (DF,Proposition 54). Note we are not claiming the isomorphism here is equal to the one given there, but it seems likely that they agree up to some sign.

Remark 15. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. Let $\mathcal{A}_F$ denote the full subcategory of $K(\mathcal{A})$ consisting of the right F-acyclic complexes, and similarly let ${}_F\mathcal{A}$ be the full subcategory of left F-acyclic complexes. By (TRC,Proposition 115) and (TRC,Proposition 124) these are both triangulated subcategories of $K(\mathcal{A})$, and F sends exact complexes in $\mathcal{A}_F$ or ${}_F\mathcal{A}$ to exact complexes in $\mathcal{B}$.

Lemma 45. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a coproduct preserving additive functor between grothendieck abelian categories, whose right derived functor $\mathbb{R}F$ also preserves coproducts. Then $\mathcal{A}_F$ is a localising subcategory of $K(\mathcal{A})$.*

Proof. Let $(\mathbb{R}F, \zeta)$ be a right derived functor of F. A complex X in $\mathcal{A}$ is right F-acyclic if and only if ζ_X is an isomorphism in $\mathfrak{D}(\mathcal{B})$. The hypotheses mean that $\zeta : Q'K(F) \longrightarrow \mathbb{R}(F)Q$ is a trinatural transformation of coproduct preserving triangulated functors, so the desired conclusion follows from (TRC,Remark 30). □

6 Brown Representability

Recall that a *ringoid* is a small preadditive category. See our Rings with Several Objects notes for the definition of modules over a ringoid. Given a ringoid $\mathcal{R}$ and $A \in \mathcal{R}$, we denote by H_A the representable functor $Hom_{\mathcal{R}}(-, A)$, and write $\mathfrak{D}(\mathcal{R})$ for the derived category of the abelian category $\mathbf{Mod}\mathcal{R}$. Our rings are not necessarily commutative. Given an abelian category $\mathcal{A}$ and $n \in \mathbb{Z}$, we have the full embedding $c_n : \mathcal{A} \longrightarrow \mathfrak{D}(\mathcal{A})$ defined in (DTC,Section 3.2). The reader may substitute "ring" for "ringoid" everywhere without any loss.

Lemma 46. *Let $\mathcal{A}$ be a cocomplete abelian category with exact coproducts. If A is a compact and projective object of $\mathcal{A}$, then $c_i(A)$ is compact in $K(\mathcal{A})$ and $\mathfrak{D}(\mathcal{A})$.*

Proof. Given a nonempty family of complexes $\{X_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{A}$ we have, using compactness of A (DF,Lemma 66) and (DTC,Proposition 57), the following isomorphism

$$\begin{aligned}
Hom_{\mathfrak{D}(\mathcal{A})}(c_i(A), \bigoplus_{\lambda\in\Lambda} X_\lambda) &\cong Hom_{\mathcal{A}}(A, H^i(\bigoplus_{\lambda\in\Lambda} X_\lambda)) \\
&\cong Hom_{\mathcal{A}}(A, \bigoplus_{\lambda\in\Lambda} H^i(X_\lambda)) \\
&\cong \bigoplus_{\lambda\in\Lambda} Hom_{\mathcal{A}}(A, H^i(X_\lambda)) \\
&\cong \bigoplus_{\lambda\in\Lambda} Hom_{\mathfrak{D}(\mathcal{A})}(c_i(A), X_\lambda)
\end{aligned}$$

which shows that $c_i(A)$ is compact in $\mathfrak{D}(\mathcal{A})$ (AC,Proposition 87). A similar calculation shows that $c_i(A)$ is also compact in $K(\mathcal{A})$. □

Proposition 47. *If $\mathcal{R}$ is a ringoid then the complexes $\{c_i(H_A)\}_{i\in\mathbb{Z}, A\in\mathcal{R}}$ form a compact generating set for $\mathfrak{D}(\mathcal{R})$.*

Proof. See (TRC3,Definition 9) for the definition of a compact generating set. It follows from Lemma 46 that the complexes $c_i(H_A)$ are all compact. For $i \in \mathbb{Z}$ and $A \in \mathcal{R}$ and any complex X in $\mathbf{Mod}\mathcal{R}$ we have by (DTC,Proposition 57) and the Yoneda lemma an isomorphism of abelian groups natural in X

$$Hom_{\mathfrak{D}(\mathcal{R})}(c_i(H_A), X) \cong Hom_{\mathcal{R}}(H_A, H^i(X)) \cong H^i(X)(A)$$

If the first of these sets is zero for every $i \in \mathbb{Z}, A \in \mathcal{R}$ then X is exact, and therefore zero in $\mathfrak{D}(\mathcal{R})$. This proves that $\{c_i(H_A)\}_{i\in\mathbb{Z}, A\in\mathcal{R}}$ is a compact generating set for $\mathfrak{D}(\mathcal{R})$. □

Corollary 48. *If $\mathcal{R}$ is a ringoid then an additive functor $\mathfrak{D}(\mathcal{R})^{op} \longrightarrow$ **Ab** or $\mathfrak{D}(\mathcal{R}) \longrightarrow$ **Ab** is representable if and only if it is homological and product preserving.*

Proof. We know from (DTC,Corollary 114) that the portly triangulated category $\mathfrak{D}(\mathcal{R})$ is only *mildly portly* (TRC3,Definition 11). We also need to be careful about what we mean by a *representable* functor defined on $\mathfrak{D}(\mathcal{R})^{\mathrm{op}}$ or $\mathfrak{D}(\mathcal{R})$ (TRC3,Definition 12). The result now follows from the fact that $\mathfrak{D}(\mathcal{R})$ is compactly generated and therefore satisfies the representability theorem and the dual representability theorem (TRC3,Corollary 32). □

Although our convention is that $\mathfrak{D}(R)$ stands for the derived category of *right* modules over R, it is obvious that by looking at the opposite ring we get the next few results for $\mathfrak{D}(R) = \mathfrak{D}(R\mathbf{Mod})$ as well.

Corollary 49. *If R is a ring then $\mathfrak{D}(R)$ is a compactly generated portly triangulated category. It is compactly generated by the complexes $\{c_i(R)\}_{i\in\mathbb{Z}}$. Moreover for any complex X of R-modules there is a canonical isomorphism of abelian groups natural in X*

$$Hom_{\mathfrak{D}(R)}(c_i(R), X) \longrightarrow H^i(X)$$

Remark 16. This result says that for modules over a ring cohomology is a representable functor. It is represented in degree i by the complex $c_i(R)$. But one should not expect cohomology functors $H^i(-)$ on $\mathfrak{D}(\mathcal{A})$ for arbitrary abelian categories to be represented in this way, because in general cohomology does not commute with products.

Now applying the major result of (TRC2,Section 6) we can classify the compact objects of $\mathfrak{D}(R)$ for any ring R. First we need a useful result on splitting idempotents, taken from [BN93].

Proposition 50. *Let R be a ring and $\mathcal{P}$ the full subcategory of $\mathfrak{D}(R)$ consisting of complexes isomorphic in $\mathfrak{D}(R)$ to a bounded complex of finitely generated projective modules. Then $\mathcal{P}$ is a thick triangulated subcategory of $\mathfrak{D}(R)$.*

Proof. The full subcategory $\mathcal{P}$ is certainly replete and closed under suspension, so by (TRC,Lemma 33) to show it is a triangulated subcategory it suffices to show closure under mapping cones. Suppose we are given $X, Y \in \mathcal{P}$ and a morphism $u : X \longrightarrow Y$ in $\mathfrak{D}(R)$. Since X is hoprojective, u is actually given by a morphism of complexes (DTC,Corollary 50). In the canonical triangle

$$X \longrightarrow Y \longrightarrow C_u \longrightarrow \Sigma X$$

it is clear that C_u is a bounded complex of finitely generated projectives, which shows that $\mathcal{P}$ is a triangulated subcategory. The subtle part is to show that this subcategory is thick. By (TRC,Corollary 84) it suffices to show that idempotents split in $\mathcal{P}$. From the homotopy theoretic point of view [BN93] the standard way to show that idempotents split is using totalisation arguments.

Let $e : X \longrightarrow X$ be an idempotent morphism in $\mathcal{P}$. We may as well assume X is a bounded complex of finitely generated projectives. Since such a complex is hoprojective, we can also assume that e arises from a morphism of complexes. To split e, we proceed as in (TRC,Proposition 89). In the construction of the sequence $Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots$ for the first sequence of (TRC,Proposition 89) we always extend triangles by taking the mapping cone on the level of complexes. It is then not difficult to see that each Y_i is a bounded complex of finitely generated projectives, and that the sequence

$$Y_1 \longrightarrow Y_2 \to Y_3 \longrightarrow \cdots \tag{8}$$

on the level of morphisms of *complexes*, stabilises in each degree. That is, for each $n \in \mathbb{Z}$ the sequence of modules $Y_1^n \longrightarrow Y_2^n \longrightarrow Y_3^n \longrightarrow \cdots$ eventually becomes a long chain of identities. In each degree the direct limit is this stable value, so $\varinjlim_i Y_i$ is a bounded above complex of finitely generated projectives. By (DTC,Proposition 65) this direct limit is also the homotopy colimit in $\mathfrak{D}(R)$. That is, $Y = \varinjlim_i Y_i$ is a totalisation of the first sequence of (TRC,Proposition 89). Similarly one constructs a totalisation Z of the second sequence which is a bounded above complex of finitely generated projectives.

So finally we have written X as a coproduct (in $\mathfrak{D}(R)$) of two bounded above complexes Y, Z of finitely generated projectives, such that $X \longrightarrow Y \longrightarrow X$ is e. The next step is to reduce Y, Z to complexes which are bounded.

Since X is bounded below, there is $n \in \mathbb{Z}$ such that $X^i = 0$ for $i < n$. In other words, the canonical morphism of complexes $X \longrightarrow X_{\geq n}$ is an isomorphism in $\mathfrak{D}(R)$. The functor $(-)_{\geq n} : \mathfrak{D}(R) \longrightarrow \mathfrak{D}(R)$ is additive (HRT,Definition 3) so we have $X = Y_{\geq n} \oplus Z_{\geq n}$ in $\mathfrak{D}(R)$ and moreover the composite $X_{\geq n} \longrightarrow Y_{\geq n} \longrightarrow X_{\geq n}$ is still e. So to complete the proof that e splits in $\mathcal{P}$ we need only show that $Y_{\geq n}$ is still a complex of finitely generated projectives. Explicitly, $Y_{\geq n}$ is

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Coker\partial_Y^{n-1} \longrightarrow Y^{n+1} \longrightarrow Y^{n+2} \longrightarrow \cdots$$

so it is clear that the only problem is in showing that $Coker\partial_Y^{n-1}$ is projective. Firstly we observe that since X is hoprojective, we have

$$\begin{aligned} 0 &= Hom_{\mathfrak{D}(R)}(X, \mathfrak{D}(R)^{\leq(n-1)}) \\ &= Hom_{\mathfrak{D}(R)}(Y_{\geq n}, \mathfrak{D}(R)^{\leq(n-1)}) \oplus Hom_{\mathfrak{D}(R)}(Z_{\geq n}, \mathfrak{D}(R)^{\leq(n-1)}) \end{aligned}$$

from which we deduce $Hom_{\mathfrak{D}(R)}(Y_{\geq n}, \mathfrak{D}(R)^{\leq(n-1)}) = 0$. The complex S formed from $Y_{\geq n}$ by replacing $Coker\partial_Y^{n-1}$ with zero is certainly hoprojective, so if we form a triangle like the one in (DTC,Remark 32) and apply $Hom_{\mathfrak{D}(R)}(-, Q)$ for some $Q \in \mathfrak{D}(R)^{\leq(n-1)}$ we deduce

$$Hom_{\mathfrak{D}(R)}(c_n(Coker\partial_Y^{n-1}), Q) = 0$$

It follows that $Hom_{\mathfrak{D}(R)}(Coker\partial_Y^{n-1}, \mathfrak{D}(R)^{\leq -1}) = 0$ and this is enough to show that $Coker\partial_Y^{n-1}$ is projective by Lemma 29. Therefore $Y_{\geq n}$ is a bounded complex of finitely generated projectives; that is, $Y \in \mathcal{P}$, and the proof is complete. □

Proposition 51. *Let R be a ring. In $\mathfrak{D}(R)$ the compact objects are precisely those isomorphic in $\mathfrak{D}(R)$ to a bounded complex of finitely generated projective modules.*

Proof. Set $\mathcal{T} = \mathfrak{D}(R)$. It follows from Corollary 49 and (TRC3,Lemma 17) that $\mathcal{T}^c$ is the smallest thick triangulated subcategory of $\mathcal{T}$ containing the complexes $\{c_i(R)\}_{i\in\mathbb{Z}}$. Let $\mathcal{P}$ be the full subcategory of $\mathcal{T}$ consisting of complexes isomorphic (in $\mathcal{T}$) to a bounded complexes of finitely generated projective modules.

Let P be a finitely generated projective R-module. Then P is compact in $\mathbf{Mod}R$ so the complex $c_i(P)$ is compact in $\mathcal{T}$ for any $i \in \mathbb{Z}$ by Lemma 46. It follows from (DTC,Lemma 79) that every bounded complex of finitely generated projectives is compact in $\mathcal{T}$. That is, we have an inclusion $\mathcal{P} \subseteq \mathcal{T}^c$. But by Lemma 50 the triangulated subcategory $\mathcal{P}$ is thick, and since it contains the complexes $c_i(R)$ we must have equality $\mathcal{P} = \mathcal{T}^c$, which is what we wanted to show. □

Remark 17. It might be worth reminding the reader that a projective R-module M is finitely generated if and only if it is finitely presented, so the characterisation in Proposition 51 of the compact objects could just as well have said that they are the bounded complexes of finitely presented projectives.

Theorem 52. *Let $\mathcal{C}$ be a grothendieck abelian category. Then $\mathfrak{D}(\mathcal{C})$ has products and an additive functor $\mathfrak{D}(\mathcal{C})^{op} \longrightarrow \mathbf{Ab}$ is representable if and only if it is homological and product preserving.*

Proof. Since $\mathfrak{D}(\mathcal{C})$ is mildly portly, we have already defined what we mean by a *representable* functor (TRC3,Definition 12). By the arguments of (DTC,Section 7), and in particular the proof of (DTC,Theorem 113), we can reduce by the Gabriel-Popescu theorem to the case where $\mathcal{C}$ is a localisation of a module category over a ringoid. Let $\mathcal{R}$ be a ringoid, J an additive topology on $\mathcal{R}$, $\mathcal{C} = \mathbf{Mod}(\mathcal{R}, J)$ the localisation with inclusion $i : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{R}$ and exact left adjoint a. Set $\mathcal{A} = \mathbf{Mod}\mathcal{R}$ and denote by $\mathbf{i} : \mathfrak{D}(\mathcal{C}) \longrightarrow \mathfrak{D}(\mathcal{A})$ the right triadjoint of $\mathfrak{D}(a)$ (DTC,Corollary 112). Let $F : \mathfrak{D}(\mathcal{C})^{\text{op}} \longrightarrow \mathbf{Ab}$ be a homological product preserving functor. The triangulated functor $\mathfrak{D}(a) : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{C})$ has a right adjoint and therefore preserves coproducts. From this we deduce that the composite

$$\mathfrak{D}(\mathcal{A})^{\text{op}} \xrightarrow{\mathfrak{D}(a)^{\text{op}}} \mathfrak{D}(\mathcal{C})^{\text{op}} \xrightarrow{F} \mathbf{Ab}$$

is homological and product preserving. Since the representability theorem holds for $\mathfrak{D}(\mathcal{A})$ by Corollary 48, the functor $F\mathfrak{D}(a)^{\text{op}}$ is representable, say by a complex $X \in \mathfrak{D}(\mathcal{A})$. That is, for $Y \in \mathfrak{D}(\mathcal{A})$ we have an isomorphism of (large) abelian groups natural in Y

$$Hom_{\mathfrak{D}(\mathcal{A})}(Y, X) \longrightarrow F\mathfrak{D}(a)Y$$

It is therefore clear that $X \in \mathcal{L}_J^{\perp}$, where $\mathcal{L}_J$ is the kernel of $\mathfrak{D}(a)$. We claim that the complex aX represents F. Given $Y \in \mathfrak{D}(\mathcal{C})$ we have an isomorphism of (large) abelian groups natural in Y

$$\begin{aligned} Hom_{\mathfrak{D}(\mathcal{C})}(Y, aX) &\cong Hom_{\mathfrak{D}(\mathcal{A})}(\mathbf{i}Y, \mathbf{i}\mathfrak{D}(a)X) \\ &\cong Hom_{\mathfrak{D}(\mathcal{A})}(\mathbf{i}Y, X) \\ &\cong F\mathfrak{D}(a)\mathbf{i}Y \cong F(Y) \end{aligned}$$

where we use the fact that $\mathbf{i}$ is fully faithful and $\mathbf{i}\mathfrak{D}(a)X \cong X$ since $X \in \mathcal{L}_J^{\perp}$ (DTC,Lemma 116). This proves that F is represented by aX and completes the proof that $\mathfrak{D}(\mathcal{C})$ satisfies the representability theorem. In particular it has all products. □

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