Derived Categories Of Sheaves

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We give a standard exposition of the elementary properties of derived categories of sheaves on a ringed space. This includes the derived direct and inverse image, the derived sheaf Hom, the derived tensor and the principal relations among these structures.

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1 Introduction

All notation and conventions are from our notes on Derived Functors and Derived Categories. In particular we assume that every abelian category comes with canonical structures that allow us to define the cohomology of cochain complexes in an unambiguous way. If we write *complex* we mean *cochain complex*, and we write $\mathbf{C}(\mathcal{A})$ for the abelian category of all complexes in $\mathcal{A}$. As usual we write $A = 0$ to indicate that A is a zero object (not necessarily the canonical one). We use the terms *preadditive category* and *additive category* as defined in (AC,Section 2). The reader should be familiar with our notes on Derived Categories (DTC) and also with the definition of derived functors given in Derived Categories Part II.

The standard references for derived categories of sheaves are probably [Har66] and Lipman's notes [Lip], with the latter being a more modern approach to the subject. These notes borrow heavily from [Lip]. To give just one example of the importance of [Lip], the reader should study [Spa88] which introduced K-flabby and K-limp complexes in order to study derived categories of sheaves. It is a clever observation of Lipman that allows us here to avoid such technical complications.

1.1 Basic Properties

Given a ringed space $(X, \mathcal{O}_X)$ we introduce the following notation

$$\mathbf{C}(X) = \mathbf{C}(\mathfrak{Mod}(X)), \quad K(X) = K(\mathfrak{Mod}(X)), \quad \mathfrak{D}(X) = \mathfrak{D}(\mathfrak{Mod}(X))$$

Let us review some important facts about these categories. The grothendieck abelian category $\mathfrak{Mod}(X)$ has enough injectives and hoinjectives (DTC,Remark 49). The triangulated categories $K(X)$ and $\mathfrak{D}(X)$ have coproducts and the triangulated functors

$$\mathbf{C}(X) \longrightarrow K(X), \quad K(X) \longrightarrow \mathfrak{D}(X)$$

preserve these coproducts (DTC,Proposition 44), (DTC,Lemma 42). The (portly) triangulated category $\mathfrak{D}(X)$ also has small morphism conglomerates (DTC,Corollary 114). If $U \subseteq X$ is an open subset then we have an exact functor $(-)|_U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)$ and induced coproduct preserving triangulated functors

$$(-)|_U : K(X) \longrightarrow K(U), \quad (-)|_U : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(U)$$

The restriction functor on sheaves has an exact left adjoint $i_!$ (MRS,Proposition 27), so the induced triangulated functor $i_! : K(U) \longrightarrow K(X)$ is left triadjoint to $(-)|_U : K(X) \longrightarrow K(U)$ (DTC,Lemma 25). In particular $(-)|_U$ preserves hoinjectives (DTC,Lemma 62).

Remark 1. Let $(X, \mathcal{O}_X)$ be a ringed space and $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of complexes of sheaves of modules on X. Then for any open subset $U \subseteq X$ there is a canonical isomorphism of complexes $C_\psi|_U \cong C_{\psi|_U}$ of the mapping cones.

Lemma 1. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism in $\mathfrak{D}(X)$. If $\{V_i\}_{i \in I}$ is a nonempty open cover of X then ψ is an isomorphism in $\mathfrak{D}(X)$ if and only if $\psi|_{V_i}$ is an isomorphism in $\mathfrak{D}(V_i)$ for every $i \in I$.*

Proof. First we show observe that if $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of complexes of sheaves of modules on X, then ψ is a quasi-isomorphism if and only if $\psi|_{V_i}$ is for every $i \in I$. Since a morphism is a quasi-isomorphism if and only if its mapping cone is exact this reduces to showing that the mapping cone commutes with restriction, which is clear from its construction.

Given this first step, let $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism in $\mathfrak{D}(X)$ represented by the following diagram in $K(X)$

$$\mathscr{F} \xleftarrow{\;b\;} \mathscr{H} \xrightarrow{\;a\;} \mathscr{G}$$

Then ψ is an isomorphism in $\mathfrak{D}(X)$ if and only if a is a quasi-isomorphism, and since the restriction of this diagram to $\mathfrak{D}(V_i)$ represents $\psi|_{V_i}$ the result follows from the previous paragraph. $\square$

Lemma 2. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\{\mathscr{F}_\lambda\}_{\lambda \in \Lambda}$ a nonempty family of complexes of sheaves of modules on X and $\{u_\lambda : \mathscr{F}_\lambda \longrightarrow \mathscr{F}\}_{\lambda \in \Lambda}$ a family of morphisms in $\mathfrak{D}(X)$. If $\{V_i\}_{i \in I}$ is a nonempty open cover of X then the u_λ are a coproduct in $\mathfrak{D}(X)$ if and only if the morphisms*

$$u_\lambda|_{V_i} : \mathscr{F}_\lambda|_{V_i} \longrightarrow \mathscr{F}|_{V_i}$$

are a coproduct in $\mathfrak{D}(V_i)$ for each $i \in I$.

Proof. One direction is easy, because the restriction functors $(-)|_{V_i} : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(V_i)$ all preserve coproducts. For the reverse inclusion we are given morphisms u_λ and we have to show that the canonical morphism $t : \bigoplus_\lambda \mathscr{F}_\lambda \longrightarrow \mathscr{F}$ is an isomorphism (we can assume the first coproduct is taken on the level of complexes). But by hypothesis $t|_{V_i}$ is an isomorphism for every $i \in I$, so the claim follows from Lemma 1. $\square$

Lemma 3. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a complex of sheaves of modules on X. Let $\mathscr{H}^n_{\mathscr{F}}$ be the sheaf associated to the presheaf of modules $U \mapsto H^n(\Gamma(U, \mathscr{F}))$. Then there is a canonical isomorphism of sheaves of modules*

$$\mathscr{H}^n_{\mathscr{F}} \longrightarrow H^n(\mathscr{F})$$

which is natural in $\mathscr{F}$.

Proof. Given an open set U we have the $\mathcal{O}_X(U)$-module $H^n(\Gamma(U, \mathscr{F}))$. An inclusion $V \subseteq U$ induces a morphism of complexes of abelian groups $\Gamma(U, \mathscr{F}) \longrightarrow \Gamma(V, \mathscr{F})$ and therefore a morphism $H^n(\Gamma(U, \mathscr{F})) \longrightarrow H^n(\Gamma(V, \mathscr{F}))$. This defines a presheaf of $\mathcal{O}_X$-modules that sheafifies to give $\mathscr{H}^n_{\mathscr{F}}$. By definition $H^n(\mathscr{F})$ is the cokernel of $Im\partial^{n-1} \longrightarrow Ker\partial^n$, and therefore also the cokernel of $\mathscr{F}^{n-1} \longrightarrow Ker\partial^n$. The canonical cokernel of this morphism is the sheafification of $U \mapsto Ker(\partial^n)_U / Im\partial^{n-1}_U = H^n(\Gamma(U, \mathscr{F}))$. Since $\mathscr{H}^n_{\mathscr{F}}$ and $H^n(\mathscr{F})$ are cokernels of the same morphism, we deduce the desired canonical isomorphism. Naturality is easily checked. $\square$

Remark 2. In particular this means that if a complex $\mathscr{F}$ of sheaves of modules is exact as a complex of *presheaves* then it is also exact as a complex of *sheaves*.

Lemma 4. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of complexes of sheaves of modules on X. If ψ is a quasi-isomorphism of complexes of presheaves, then it is also a quasi-isomorphism of complexes of sheaves.*

Proof. We can consider ψ as a morphism of complexes in the category $Mod(X)$ of presheaves of modules. We claim that if it is a quasi-isomorphism there, it is also a quasi-isomorphism as a morphism of complexes in $\mathfrak{Mod}(X)$. Checking for isomorphisms on cohomology amounts to checking exactness of the mapping cone, which is the same for presheaves and sheaves. Therefore the claim follows from Remark 2. $\square$

Lemma 5. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of complexes of sheaves of modules on X. If $\mathfrak{B}$ is a basis of X such that $\Gamma(V, \psi) : \Gamma(V, \mathscr{F}) \longrightarrow \Gamma(V, \mathscr{G})$ is a quasi-isomorphism of complexes of abelian groups for every $V \in \mathfrak{B}$, then ψ is a quasi-isomorphism.*

Proof. The morphism of complexes $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ is a quasi-isomorphism if and only if its mapping cone is exact. For any open set $V \subseteq X$ there is a canonical isomorphism of complexes of abelian groups $\Gamma(V, C_\psi) \cong C_{\Gamma(V,\psi)}$. Hence if all the $\Gamma(V, \psi)$ are quasi-isomorphisms, every complex $\Gamma(V, C_\psi)$ is exact and it then follows from Lemma 3 that C_ψ is exact. $\square$

Lemma 6. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism in $\mathfrak{D}(X)$. If $\mathfrak{B}$ is a basis of X such that $\mathbb{R}\Gamma(V, \psi)$ is an isomorphism in $\mathfrak{D}(\mathbf{Ab})$ for every $V \in \mathfrak{B}$, then ψ is an isomorphism in $\mathfrak{D}(X)$.*

Proof. Here $(\mathbb{R}\Gamma(V, -), \zeta)$ denotes an arbitrary right derived functor of $\Gamma(V, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$. We can reduce easily to the case where $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of complexes with both $\mathscr{F}, \mathscr{G}$ hoinjective. In this case the following diagram commutes in $\mathfrak{D}(\mathbf{Ab})$

$$\begin{array}{ccc} \Gamma(V, \mathscr{F}) & \xrightarrow{\zeta_{\mathscr{F}}} & \mathbb{R}\Gamma(V, \mathscr{F}) \\ {\scriptstyle \Gamma(V,\psi)} \downarrow & & \downarrow {\scriptstyle \mathbb{R}\Gamma(V,\psi)} \\ \Gamma(V, \mathscr{G}) & \xrightarrow[\zeta_{\mathscr{G}}]{} & \mathbb{R}\Gamma(V, \mathscr{G}) \end{array}$$

with the top and bottom row isomorphisms. This has the effect of reducing the claim to Lemma 5, which we already know. $\square$

The first thing one learns about the category $\mathfrak{Mod}(X)$ is that limits are computed pointwise, whereas colimits are computed by sheafifying the pointwise colimits. As one expects, the homotopy limit is also calculated pointwise.

Lemma 7. *Let $(X, \mathcal{O}_X)$ be a ringed space and suppose we have a sequence in $K(X)$*

$$\cdots \longrightarrow \mathscr{X}_3 \longrightarrow \mathscr{X}_2 \longrightarrow \mathscr{X}_1 \longrightarrow \mathscr{X}_0$$

Then for any open $U \subseteq X$ we have

$$\Gamma(U, \underleftarrow{holim}\,\mathscr{X}_i) = \underleftarrow{holim}\,\Gamma(U, \mathscr{X}_i)$$

Proof. A holimit in $K(X)$ is defined by a triangle in $K(X)$ of the form

$$\underleftarrow{holim}\,\mathscr{X}_i \longrightarrow \prod \mathscr{X}_i \xrightarrow{1-\nu} \prod \mathscr{X}_i \longrightarrow \Sigma \underleftarrow{holim}\,\mathscr{X}_i$$

The additive functor $\Gamma(U, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$ induces a triangulated functor $K(X) \longrightarrow K(\mathbf{Ab})$, which applied to this triangle yields a triangle in $K(\mathbf{Ab})$

$$\Gamma(U, \underleftarrow{holim}\,\mathscr{X}_i) \longrightarrow \prod \Gamma(U, \mathscr{X}_i) \longrightarrow \prod \Gamma(U, \mathscr{X}_i) \longrightarrow \Sigma\Gamma(U, \underleftarrow{holim}\,\mathscr{X}_i)$$

In other words, $\Gamma(U, \underleftarrow{holim}\,\mathscr{X}_i) = \underleftarrow{holim}\,\Gamma(U, \mathscr{X}_i)$, as claimed. □

Remark 3. Let $(X, \mathcal{O}_X)$ be a ringed space and set $A = \Gamma(X, \mathcal{O}_X)$. The abelian category $\mathfrak{Mod}(X)$ is A-linear in the sense of (AC,Definition 35), with action $(r \cdot \phi)_V(s) = r|_V \cdot \phi_V(s)$. As described in (DTC,Remark 11) the triangulated categories $K(X), \mathfrak{D}(X)$ are A-linear in the sense of (TRC,Definition 32) and the canonical quotient $K(X) \longrightarrow \mathfrak{D}(X)$ is an A-linear functor. Given an open set $U \subseteq X$ the canonical maps

$$Hom_{K(X)}(\mathscr{E}, \mathscr{F}) \longrightarrow Hom_{K(U)}(\mathscr{E}|_U, \mathscr{F}|_U)$$
$$Hom_{\mathfrak{D}(X)}(\mathscr{E}, \mathscr{F}) \longrightarrow Hom_{\mathfrak{D}(U)}(\mathscr{E}|_U, \mathscr{F}|_U)$$

send the action of $\Gamma(X, \mathcal{O}_X)$ to the action of $\Gamma(U, \mathcal{O}_X)$ in a way compatible with restriction.

1.2 Representing Cohomology

Let $(X, \mathcal{O}_X)$ be a ringed space and for open $U \subseteq X$ recall the definition of the sheaf of modules $\mathcal{O}_U$ (MRS,Section 1.5). These sheaves are flat, and taken together they generate the category $\mathfrak{Mod}(X)$. They also represent sections of sheaves, in the sense that there is a natural isomorphism

$$Hom_{\mathcal{O}_X}(\mathcal{O}_U, \mathscr{F}) \longrightarrow \Gamma(U, \mathscr{F})$$

from which we recover all the data present in a sheaf of modules $\mathscr{F}$ from the morphisms of the category $\mathfrak{Mod}(X)$. This idea of studying intrinsic structure using the morphisms of the category in which the object "lives" is a very powerful tool in modern algebra.

There are other invariants of a sheaf $\mathscr{F}$ that we would like to represent in this way: for example, its cohomology groups $H^i(X, \mathscr{F})$. This isn't possible in $\mathfrak{Mod}(X)$ (because in general cohomology doesn't commute with products), but by passing to the homotopy category $K(X)$ and derived category $\mathfrak{D}(X)$ we can achieve this goal. To begin with, we take the most obvious sheaves we can think of and study what happens when we put them in the homotopy and derived categories:

- The sheaf $\mathcal{O}_U$ in degree i for open $U \subseteq X$ represents the group $H^i(U, \mathscr{F})$.
- The sheaf $\mathcal{O}_Z$ in degree i for closed $Z \subseteq X$ represents the group $H^i_Z(X, \mathscr{F})$.
- There is a complex $C(\mathfrak{U})$ for an open cover $\mathfrak{U}$ of X with the property that $\Sigma^{-i}C(\mathfrak{U})$ represents the group $\check{H}^i(\mathfrak{U}, \mathscr{F})$.

Lemma 8. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a complex of sheaves of modules on X. Then for open $U \subseteq X$ there is a canonical isomorphism of complexes of abelian groups natural in $\mathscr{F}$*

$$Hom^\bullet(\mathcal{O}_U, \mathscr{F}) \longrightarrow \Gamma(U, \mathscr{F})$$

Proof. As we observed in (DTC2,Remark 7) the complex $Hom^\bullet(\mathcal{O}_U, \mathscr{F})$ is canonically isomorphic to $Hom(\mathcal{O}_U, \mathscr{F})$, and of course $Hom(\mathcal{O}_U, \mathscr{F})$ is canonically isomorphic to $\Gamma(U, \mathscr{F})$. □

Proposition 9. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a complex of sheaves of modules on X. Then for open $U \subseteq X$ there is a canonical isomorphism of abelian groups natural in U and $\mathscr{F}$*

$$\zeta : Hom_{K(X)}(\mathcal{O}_U, \Sigma^i \mathscr{F}) \longrightarrow H^i(\Gamma(U, \mathscr{F}))$$

Proof. We denote by $\mathcal{O}_U$ the sheaf of modules $i_!(\mathcal{O}_X|_U)$ on X, as described in (MRS,Section 1.5). Note that we can't apply (DTC,Proposition 57) directly in $K(X)$ because the $\mathcal{O}_U$ aren't projective (this is the whole point of sheaf cohomology). By (DTC,Lemma 31) there is an isomorphism

$$Hom_{\mathbf{C}(X)}(\Sigma^{-i}\mathcal{O}_U, \mathscr{F}) \longrightarrow Hom(\mathcal{O}_U, Ker\partial^i_{\mathscr{F}}) \cong \Gamma(U, Ker\partial^i_{\mathscr{F}})$$

identifying null-homotopic morphisms with those factoring through $\mathscr{F}^{i-1} \longrightarrow Ker\partial^i_X$, which corresponds to the subgroup $Im(\partial^{i-1}_{\mathscr{F}})_U$. Taking quotients by these subgroups we deduce a canonical isomorphism

$$Hom_{K(X)}(\mathcal{O}_U, \Sigma^i \mathscr{F}) \longrightarrow H^i(\Gamma(U, \mathscr{F}))$$

which is natural in U and $\mathscr{F}$, in the sense that for open sets $V \subseteq U$ the canonical monomorphism $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ makes the following diagram commute

$$\begin{array}{ccc} Hom_{K(X)}(\mathcal{O}_U, \Sigma^i \mathscr{F}) & \longrightarrow & H^i(\Gamma(U, \mathscr{F})) \\ \downarrow & & \downarrow \\ Hom_{K(X)}(\mathcal{O}_V, \Sigma^i \mathscr{F}) & \longrightarrow & H^i(\Gamma(V, \mathscr{F})) \end{array}$$

realising the complexes $\Sigma^i \mathcal{O}_U$ as something like "presheaf cohomology generators". Note that you can't hope to realise $\Gamma(U, H^i(\mathscr{F}))$ in this way, because cohomology in $\mathfrak{Mod}(X)$ doesn't commute with arbitrary products. □

Proposition 10. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a complex of sheaves of modules on X. For closed $Z \subseteq X$ there is a canonical isomorphism of abelian groups natural in Z and $\mathscr{F}$*

$$\tau : Hom_{K(X)}(\mathcal{O}_Z, \Sigma^i \mathscr{F}) \longrightarrow H^i(\Gamma_Z(X, \mathscr{F}))$$

Proof. We denote by $\mathcal{O}_Z$ the sheaf of modules $i_*(i^{-1}\mathcal{O}_X)$ described in (MRS,Section 1.5), where $i : Z \longrightarrow X$ is the inclusion. For a sheaf $\mathscr{F}$ the notation $\Gamma_Z(X, \mathscr{F})$ denotes all the global sections s of $\mathscr{F}$ with $s|_{X \setminus Z} = 0$. Applying this to a complex gives a complex of abelian groups, so it makes sense to take cohomology. By (DTC,Lemma 31) and (MRS,Proposition 34) there is an isomorphism

$$Hom_{\mathbf{C}(X)}(\Sigma^{-i}\mathcal{O}_Z, \mathscr{F}) \longrightarrow Hom(\mathcal{O}_Z, Ker\partial^i_{\mathscr{F}}) \cong \Gamma_Z(X, Ker\partial^i_{\mathscr{F}})$$

identifying null-homotopic morphisms with the elements in the image of $\Gamma_Z(X, \partial^{i-1}_{\mathscr{F}})$. Taking quotients by these subgroups we deduce the required natural isomorphism τ. Naturality in Z means that for closed sets $Z \subseteq Q$ the canonical morphism $\mathcal{O}_Q \longrightarrow \mathcal{O}_Z$ makes the following diagram commute

$$\begin{array}{ccc} Hom_{K(X)}(\mathcal{O}_Z, \Sigma^i \mathscr{F}) & \longrightarrow & H^i(\Gamma_Z(X, \mathscr{F})) \\ \downarrow & & \downarrow \\ Hom_{K(X)}(\mathcal{O}_Q, \Sigma^i \mathscr{F}) & \longrightarrow & H^i(\Gamma_Q(X, \mathscr{F})) \end{array}$$

which is easy to check. □

Corollary 11. *Let $(X, \mathcal{O}_X)$ be a quasi-noetherian ringed space and $U \subseteq X$ a quasi-compact open subset. The sheaves $\mathcal{O}_U$ and $\mathcal{O}_{X \setminus U}$ are compact as objects of $K(X)$.*

Proof. By (COS,Remark 8) the sheaves of modules $\mathcal{O}_U$ and $\mathcal{O}_{X\setminus U}$ are compact as objects of $\mathfrak{Mod}(X)$, and by (DTC,Lemma 58) this is enough to make them compact as objects of $K(X)$. $\square$

Theorem 12. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules on X. Then for open $U \subseteq X$ and $i \in \mathbb{Z}$ there is a canonical isomorphism of abelian groups natural in U and $\mathscr{F}$*

$$\alpha : Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i \mathscr{F}) \longrightarrow H^i(U, \mathscr{F})$$

For closed $Z \subseteq X$ there is a canonical isomorphism of abelian groups natural in Z and $\mathscr{F}$

$$\beta : Hom_{\mathfrak{D}(X)}(\mathcal{O}_Z, \Sigma^i \mathscr{F}) \longrightarrow H^i_Z(X, \mathscr{F})$$

Proof. If $i < 0$ then the right hand side is zero by convention and the left hand side is zero by (DTC,Lemma 32), so assume $i \geq 0$. Choose an injective resolution for $\mathscr{F}$ in $\mathfrak{Mod}(X)$: that is, a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{I}$ into a complex of injectives with $\mathscr{I}^j = 0$ for $j < 0$. Then we have a canonical isomorphism

$$\begin{aligned} Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i \mathscr{F}) &\cong Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i \mathscr{I}) \\ &\cong Hom_{K(X)}(\mathcal{O}_U, \Sigma^i \mathscr{I}) \\ &\cong H^i(\Gamma(U, \mathscr{I})) = H^i(U, \mathscr{F}) \end{aligned}$$

where we use (DTC,Corollary 50) and Proposition 9. Naturality in $\mathscr{F}$ is easily checked. By naturality in U we mean that for open sets $V \subseteq U$ the canonical monomorphism $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ makes the following diagram commute

$$\begin{array}{ccc} Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i \mathscr{F}) & \longrightarrow & H^i(U, \mathscr{F}) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(X)}(\mathcal{O}_V, \Sigma^i \mathscr{F}) & \longrightarrow & H^i(V, \mathscr{F}) \end{array}$$

where the morphism on the right hand side is the obvious one induced by restriction, as in (COS,Section 1.3). Commutativity of this diagram is also easily checked. The second claim is checked in the same way, where $H^i_Z(X, -)$ denotes the right derived functor of $\Gamma_Z(X, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$. $\square$

Remark 4. Here is an alternative proof of the first claim of Theorem 12. In the notation of (DTC2,Section 3.1) the group $Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i \mathscr{F})$ is $\underline{Ext}^i(\mathcal{O}_U, \mathscr{F})$, isomorphic to the usual Ext (DTC2,Lemma 28). By the argument given in (COS,Proposition 54) this is $H^i(U, \mathscr{F})$.

We have just shown how to represent the cohomology groups $H^i(U, \mathscr{F})$ by an object of the derived category. Since Čech cohomology is *defined* in terms of an explicit complex, it is not surprising that it is already represented by an object of the homotopy category. Let us now define this representing complex.

Definition 1. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a nonempty collection of open sets $U_i \subseteq X$ with totally ordered index set I. For any finite set of indices $i_0, \ldots, i_p \in I$ we denote the open intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U[i_0, \ldots, i_p]$. We define a complex as follows: for each $p \leq 0$, let

$$C^p(\mathfrak{U}) = \bigoplus_{i_0 < \cdots < i_p} \mathcal{O}_{U[i_0, \ldots, i_p]}$$

Given $p < 0$ and a sequence $i_0 < \cdots < i_p$ and $0 \leq k \leq p$ we write $i_0, \ldots, \widehat{i_k}, \ldots, i_p$ to denote the sequence with i_k omitted. We have an inclusion $U[i_0, \ldots, i_p] \subseteq U[i_0, \ldots, \widehat{i_k}, \ldots, i_p]$ and therefore a canonical morphism $\rho_{(i_0,\ldots,i_p),k} : \mathcal{O}_{U[i_0,\ldots,i_p]} \longrightarrow \mathcal{O}_{U[i_0,\ldots,\widehat{i_k},\ldots,i_p]}$. We define a morphism of sheaves of modules

$$d^p : C^p(\mathfrak{U}) \longrightarrow C^{p+1}(\mathfrak{U})$$
$$d^p u_{i_0,\ldots,i_p} = \sum_{k=0}^{p} (-1)^k u_{i_0,\ldots,\widehat{i_k},\ldots,i_p} \rho_{(i_0,\ldots,i_p),k}$$

It is straightforward to check that this makes $C(\mathfrak{U})$ into a complex of sheaves of modules

$$\cdots \longrightarrow \bigoplus_{i_0<i_{-1}<i_{-2}} \mathcal{O}_{U[i_0,i_{-1},i_{-2}]} \longrightarrow \bigoplus_{i_0<i_{-1}} \mathcal{O}_{U[i_0,i_{-1}]} \longrightarrow \bigoplus_{i_0} \mathcal{O}_{U[i_0]} \longrightarrow 0$$

Lemma 13. *Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a nonempty open cover of an open set $U \subseteq X$ with totally ordered index set I, and $\mathscr{F}$ a sheaf of modules on X. Then there is a canonical isomorphism of complexes of abelian groups natural in $\mathscr{F}$*

$$Hom^\bullet(C(\mathfrak{U}), \mathscr{F}) \longrightarrow C(\mathfrak{U}, \mathscr{F}|_U)$$

where $C(\mathfrak{U}, \mathscr{F}|_U)$ is the usual Čech complex.

Proof. See (COS,Section 4) for the definition of the Čech complex. For $p \geq 0$ we have a canonical isomorphism of abelian groups

$$\begin{aligned} Hom_{\mathcal{O}_X}\left(\oplus_{i_0<\cdots<i_{-p}} \mathcal{O}_{U[i_0,\ldots,i_{-p}]}, \mathscr{F}\right) &\cong \prod_{i_0<\cdots<i_{-p}} Hom_{\mathcal{O}_X}(\mathcal{O}_{U[i_0,\ldots,i_{-p}]}, \mathscr{F}) \\ &\cong \prod_{i_0<\cdots<i_{-p}} \Gamma(U[i_0,\ldots,i_{-p}], \mathscr{F}) \\ &= C^p(\mathfrak{U}, \mathscr{F}|_U) \end{aligned}$$

which one checks is an isomorphism of complexes. Naturality in $\mathscr{F}$ is also easily checked. □

Proposition 14. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathfrak{U} = \{U_i\}_{i\in I}$ a nonempty open cover of an open set $U \subseteq X$ with totally ordered index set I, and $\mathscr{F}$ a sheaf of modules on X. There is a canonical isomorphism of abelian groups natural in $\mathscr{F}$*

$$\mu : Hom_{K(X)}(C(\mathfrak{U}), \Sigma^i \mathscr{F}) \longrightarrow \check{H}^i(U, \mathscr{F}|_U)$$

Proof. This is immediate from Lemma 13 and (DTC2,Proposition 18). □

In the above we studied what happens when we translate the generators $\mathcal{O}_U$ of $\mathfrak{Mod}(X)$ into the homotopy category. Slightly less useful but still interesting is what happens to the *cogenerators*. If $(X, \mathcal{O}_X)$ is a ringed space and Λ_x an injective cogenerator of $\mathcal{O}_{X,x}$**Mod** for $x \in X$ then the sheaves $\lambda_x = Sky_x(\Lambda_x)$ form a family of injective cogenerators for $\mathfrak{Mod}(X)$ (MRS,Lemma 36). With this notation

Proposition 15. *Let $\mathscr{F}$ be a complex of sheaves of modules on X. For $x \in X$ there is a canonical isomorphism of abelian groups natural in $\mathscr{F}$*

$$\omega : Hom_{K(X)}(\Sigma^i \mathscr{F}, \lambda_x) \longrightarrow Hom_{\mathcal{O}_{X,x}}(H^i(\mathscr{F})_x, \Lambda_x)$$

Proof. Since λ_x is injective, we have by (DTC,Proposition 57) and adjointness a canonical isomorphism of abelian groups natural in $\mathscr{F}$

$$Hom_{K(X)}(\Sigma^i \mathscr{F}, \lambda_x) \cong Hom_{\mathcal{O}_X}(H^i(\mathscr{F}), Sky_x(\Lambda_x)) \cong Hom_{\mathcal{O}_{X,x}}(H^i(\mathscr{F})_x, \Lambda_x)$$

as claimed. □

Corollary 16. *Let $\mathcal{L}$ be the smallest colocalising subcategory of $K(X)$ containing the complexes $\{\lambda_x\}_{x\in X}$. Then ${}^\perp\mathcal{L} = \mathcal{Z}$.*

Proof. This follows from Proposition 15 or from the general statement of (DTC,Lemma 60). □

2 Derived Direct Image

Definition 2. Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. Since $\mathfrak{Mod}(X)$ is grothendieck abelian the functor $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ has a right derived functor (DTC2,Corollary 4)

$$\mathbb{R}f_* : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(Y)$$

which we call the *derived direct image functor*, or often just the *direct image functor*. This is only determined up to canonical trinatural equivalence, but if we fix an assignment $\mathcal{I}$ of hoinjective resolutions for $\mathfrak{Mod}(X)$ then we have a canonical right derived functor which we denote $\mathbb{R}_{\mathcal{I}} f_*$.

Lemma 17. *Let $(X, \mathcal{O}_X)$ be a ringed space and $i : U \longrightarrow X$ the inclusion of an open subset. Then $\mathbb{R}i_* : \mathfrak{D}(U) \longrightarrow \mathfrak{D}(X)$ is fully faithful.*

Proof. Let $(\mathbb{R}i_*, \zeta)$ be any right derived functor. The functor $(-)|_U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)$ is an exact left adjoint to i_* so it follows from (DTC2,Lemma 10) that $(-)|_U : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(U)$ is canonically left triadjoint to $\mathbb{R}i_*$. The unit $\eta^{\Diamond} : 1 \longrightarrow \mathbb{R}(i_*) \circ (-)|_U$ and counit $\varepsilon^{\Diamond} : (-)|_U \circ \mathbb{R}(i_*) \longrightarrow 1$ are the unique trinatural transformations making the following diagrams commute for every complex $\mathscr{F}$ of sheaves of modules on X and complex $\mathscr{G}$ of sheaves of modules on U

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\eta^{\Diamond}} & \mathbb{R}i_*(\mathscr{F}|_U) \\ {\scriptstyle \eta}\downarrow & \nearrow{\scriptstyle \zeta} & \\ i_*(\mathscr{F}|_U) & & \end{array} \qquad \begin{array}{ccc} & & \mathbb{R}i_*(\mathscr{G})|_U \\ & \nearrow{\scriptstyle \zeta_{\mathscr{G}}|_U} & \downarrow{\scriptstyle \varepsilon^{\Diamond}} \\ \mathscr{G} & \xrightarrow{1} & \mathscr{G} \end{array}$$

We claim that $\varepsilon^{\Diamond}$ is actually a natural equivalence. It suffices to check this on a hoinjective complex $\mathscr{I}$ in $K(U)$, in which case $\zeta_{\mathscr{I}}$ is an isomorphism so it is clear that $\varepsilon^{\Diamond}_{\mathscr{I}}$ is an isomorphism. It now follows by a standard argument (AC,Proposition 21) that $\mathbb{R}i_*$ must be fully faithful. We also observe that $\eta^{\Diamond}|_U : \mathscr{F}|_U \longrightarrow \mathbb{R}i_*(\mathscr{F}|_U)|_U$ is an isomorphism in $\mathfrak{D}(U)$ for any complex $\mathscr{F}$ of sheaves of modules on X. ☐

Lemma 18. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $i : U \longrightarrow X$ the inclusion of an open subset. There is a canonical trinatural equivalence*

$$\theta : \mathbb{R}(fi)_* \longrightarrow \mathbb{R}f_* \circ \mathbb{R}i_*$$

Proof. The functor $i_* : \mathfrak{Mod}(U) \longrightarrow \mathfrak{Mod}(X)$ has an exact left adjoint (restriction), so $K(i_*) : K(U) \longrightarrow K(X)$ preserves hoinjectives (DTC,Lemma 62). From (DTC2,Theorem 6) we deduce the required trinatural equivalence. We will return to the general case in Lemma 92. ☐

Lemma 19. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $V \subseteq Y$ an open subset. Then for any complex $\mathscr{F}$ of sheaves of modules on X there is a canonical isomorphism in $\mathfrak{D}(V)$ natural in $\mathscr{F}$*

$$\mu : (\mathbb{R}f_* \mathscr{F})|_V \longrightarrow \mathbb{R}g_*(\mathscr{F}|_U)$$

where $U = f^{-1}V$ and $g : U \longrightarrow V$ is the induced morphism of ringed spaces.

Proof. We have a commutative diagram of additive functors

$$\begin{array}{ccc} \mathfrak{Mod}(X) & \xrightarrow{f_*} & \mathfrak{Mod}(Y) \\ {\scriptstyle (-)|_U}\downarrow & & \downarrow{\scriptstyle (-)|_V} \\ \mathfrak{Mod}(U) & \xrightarrow[g_*]{} & \mathfrak{Mod}(V) \end{array}$$

where the vertical functors are exact and preserve hoinjectives. Let $(\mathbb{R}f_*, \zeta), (\mathbb{R}g_*, \omega)$ be right derived functors. Since the lift of $(-)|_U$ to the derived category is its right derived functor, it follows from (DTC2, Theorem 6) that $\mathbb{R}(g_*(-)|_U) = \mathbb{R}g_* \circ (-)|_U$. More precisely, the pair $(\mathbb{R}g_* \circ (-)|_U, \omega(-)|_U)$ is a right derived functor of $g_*(-)|_U = (-)|_V f_*$. But by (DTC2, Corollary 7) the pair $((-)|_V \circ \mathbb{R}f_*, (-)|_V \zeta)$ is also a right derived functor of these equal composites. We deduce a unique trinatural equivalence

$$\mu : (-)|_V \circ \mathbb{R}f_* \longrightarrow \mathbb{R}g_* \circ (-)|_U$$

of triangulated functors $\mathfrak{D}(X) \longrightarrow \mathfrak{D}(V)$ making the following diagram of trinatural transformations commute

$$\begin{array}{ccc} Q_V g_*(-)|_U & \xrightarrow{(-)|_V \zeta} & (-)|_V \circ \mathbb{R}f_* \circ Q_X \\ & \searrow^{\omega(-)|_U} & \Downarrow \mu Q_X \\ & & \mathbb{R}g_* \circ (-)|_U \circ Q_X \end{array}$$

where $Q_X : K(X) \longrightarrow \mathfrak{D}(X)$ and $Q_V : K(V) \longrightarrow \mathfrak{D}(V)$ are the verdier quotients. Evaluating this on a complex $\mathscr{F}$ of sheaves of modules, we have the desired isomorphism. □

Lemma 20. *Let X be a ringed space and $U \subseteq V$ open subsets. There is a canonical trinatural transformation $\tau : \mathbb{R}i_{V*}((-)|_V) \longrightarrow \mathbb{R}i_{U*}((-)|_U)$ which is unique making the following diagram commute for every complex $\mathscr{F}$ of sheaves of modules on X*

$$\begin{array}{ccc} \mathbb{R}i_{V*}(\mathscr{F}|_V) & \xrightarrow{\tau} & \mathbb{R}i_{U*}(\mathscr{F}|_U) \\ \uparrow & & \uparrow \\ i_{V*}(\mathscr{F}|_V) & \longrightarrow & i_{U*}(\mathscr{F}|_U) \end{array}$$

the bottom morphism being defined by restriction.

Proof. Choose arbitrary right derived functors $(\mathbb{R}i_{V*}, \zeta)$ and $(\mathbb{R}i_{U*}, \omega)$. Let $k : U \longrightarrow V$ be the inclusion and choose a right derived functor $(\mathbb{R}k_*, \theta)$. As described in the proof of Lemma 17 there is a trinatural transformation $\eta^\diamondsuit : 1 \longrightarrow \mathbb{R}k_*((-)|_U)$ and by Lemma 18 a canonical trinatural equivalence

$$\mathbb{R}i_{U*} \longrightarrow \mathbb{R}i_{V*}\mathbb{R}k_*$$

from which we deduce a trinatural transformation

$$\mathbb{R}i_{V*}((-)|_V) \xrightarrow{\mathbb{R}i_{V*} \circ \eta^\diamondsuit \circ (-)|_V} \mathbb{R}i_{V*}\mathbb{R}k_*((-)|_U) \Longrightarrow \mathbb{R}i_{U*}((-)|_U)$$

one checks that this is independent of the chosen right derived functor of k, and that it makes the diagram commute. The uniqueness follows from the defining property of a right derived functor, since we observed in the proof of Lemma 19 that $\mathbb{R}i_{V*}((-)|_V) = \mathbb{R}(i_{V*} \circ (-)|_V)$ and $\mathbb{R}i_{U*}((-)|_U) = \mathbb{R}(i_{U*} \circ (-)|_U)$. □

Lemma 21 (Mayer-Vietoris triangle). *Let X be a ringed space with open cover $X = U \cup V$. For any complex $\mathscr{F}$ of sheaves of modules on X there is a canonical triangle in $\mathfrak{D}(X)$*

$$\mathscr{F} \longrightarrow \mathbb{R}i_{U*}(\mathscr{F}|_U) \oplus \mathbb{R}i_{V*}(\mathscr{F}|_V) \longrightarrow \mathbb{R}i_{U\cap V*}(\mathscr{F}|_{U\cap V}) \longrightarrow \Sigma\mathscr{F}$$

where $i_U : U \longrightarrow X, i_V : V \longrightarrow X$ and $i_{U\cap V} : U \cap V \longrightarrow X$ are the inclusions. This triangle is natural with respect to morphisms of complexes.

Proof. Let $\mathscr{F}$ be a sheaf of modules on X (not a complex). By definition of a sheaf there is a short exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\binom{\eta_U}{\eta_V}} i_{U*}(\mathscr{F}|_U) \oplus i_{V*}(\mathscr{F}|_V) \xrightarrow{(-h_U \ h_V)} i_{U\cap V*}(\mathscr{F}|_{U\cap V}) \longrightarrow 0$$

which is also known as the Čech resolution (COS,Lemma 32) associated to the cover $\{U, V\}$ and the sheaf $\mathscr{F}$. Here $\eta_U : \mathscr{F} \longrightarrow i_{U*}(\mathscr{F}|_U)$ and $h_U : i_{U*}(\mathscr{F}|_U) \longrightarrow i_{U\cap V*}(\mathscr{F}|_{U\cap V})$ are the canonical morphisms $(\eta_U)_W(a) = a|_{W\cap U}$ and $(h_U)_W(a) = a|_{W\cap U\cap V}$. This short exact sequence is natural in $\mathscr{F}$, so if we replace $\mathscr{F}$ by a complex of sheaves of modules we have a short exact sequence of complexes of the same form. From (DTC,Proposition 20) we deduce a canonical morphism $z : i_{U\cap V*}(\mathscr{F}|_{U\cap V}) \longrightarrow \Sigma\mathscr{F}$ in $\mathfrak{D}(X)$ fitting into a triangle

$$\mathscr{F} \longrightarrow i_{U*}(\mathscr{F}|_U) \oplus i_{V*}(\mathscr{F}|_V) \longrightarrow i_{U\cap V*}(\mathscr{F}|_{U\cap V}) \xrightarrow{-z} \Sigma\mathscr{F} \tag{1}$$

which is natural with respect to morphisms of complexes. Since the functor $(-)|_U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)$ is an exact left adjoint to i_{U*} it follows from (DTC2,Lemma 10) that $(-)|_U : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(U)$ is canonically left triadjoint to $\mathbb{R}i_{U*}$. In particular we have the unit morphisms

$$\eta_U^\diamondsuit : \mathscr{F} \longrightarrow \mathbb{R}i_{U*}(\mathscr{F}|_U)$$
$$\eta_V^\diamondsuit : \mathscr{F} \longrightarrow \mathbb{R}i_{V*}(\mathscr{F}|_V)$$

By Lemma 20 we have canonical trinatural transformations

$$\tau_U : (\mathbb{R}i_{U*})(-)|_U \longrightarrow (\mathbb{R}i_{U\cap V*})(-)|_{U\cap V}$$
$$\tau_V : (\mathbb{R}i_{V*})(-)|_V \longrightarrow (\mathbb{R}i_{U\cap V*})(-)|_{U\cap V}$$

We can now define two morphisms in $\mathfrak{D}(X)$ for any complex $\mathscr{F}$ of sheaves of modules on X

$$u = \begin{pmatrix} \eta_U^\diamondsuit \\ \eta_V^\diamondsuit \end{pmatrix} : \mathscr{F} \longrightarrow \mathbb{R}i_{U*}(\mathscr{F}|_U) \oplus \mathbb{R}i_{V*}(\mathscr{F}|_V)$$
$$v = \begin{pmatrix} -\tau_U & \tau_V \end{pmatrix} : \mathbb{R}i_{U*}(\mathscr{F}|_U) \oplus \mathbb{R}i_{V*}(\mathscr{F}|_V) \longrightarrow \mathbb{R}i_{U\cap V*}(\mathscr{F}|_{U\cap V})$$

One checks that these morphisms are both natural in $\mathscr{F}$. Given a complex $\mathscr{F}$, we can find a quasi-isomorphism of complexes $a : \mathscr{F} \longrightarrow \mathscr{I}$ with $\mathscr{I}$ hoinjective. There is a morphism in $\mathfrak{D}(X)$

$$w : \mathbb{R}i_{U\cap V*}(\mathscr{F}|_{U\cap V}) \Longrightarrow \mathbb{R}i_{U\cap V*}(\mathscr{I}|_{U\cap V}) \Longrightarrow i_{U\cap V*}(\mathscr{I}|_{U\cap V}) \xrightarrow{z_{\mathscr{I}}} \Sigma\mathscr{I} \Longrightarrow \Sigma\mathscr{F} \tag{2}$$

where $z_{\mathscr{I}}$ is the connecting morphism of (1) and we use the canonical isomorphism defined in (DTC2,Remark 2). We claim that w is canonical: it is independent of the choice of resolution a. Given another quasi-isomorphism $b : \mathscr{F} \longrightarrow \mathscr{I}'$ the composite $Q(b)Q(a)^{-1} : \mathscr{I} \longrightarrow \mathscr{I}'$ in $\mathfrak{D}(X)$ must by (DTC,Corollary 50) be of the form $Q(t)$ for some quasi-isomorphism of complexes $t : \mathscr{I} \longrightarrow \mathscr{I}'$. Using naturality of the constituents of (2) one observes that b yields the same morphism w, as claimed. Furthermore, w is natural with respect to morphisms of complexes.

We have now constructed a canonical sequence of morphisms in $\mathfrak{D}(X)$

$$\mathscr{F} \xrightarrow{u} \mathbb{R}i_{U*}(\mathscr{F}|_U) \oplus \mathbb{R}i_{V*}(\mathscr{F}|_V) \xrightarrow{v} \mathbb{R}i_{U\cap V*}(\mathscr{F}|_{U\cap V}) \xrightarrow{-w} \Sigma\mathscr{F}$$

which is natural with respect to morphisms of complexes. It remains to show that this sequence is a triangle. For this we can reduce immediately to the case where $\mathscr{F}$ is a hoinjective complex $\mathscr{I}$, in which case we have a diagram in $\mathfrak{D}(X)$ (DTC2,Remark 2)

$$\begin{array}{ccccccc}
\mathscr{I} & \xrightarrow{u} & \mathbb{R}i_{U*}(\mathscr{I}|_U) \oplus \mathbb{R}i_{V*}(\mathscr{I}|_V) & \xrightarrow{v} & \mathbb{R}i_{U\cap V*}(\mathscr{I}|_{U\cap V}) & \xrightarrow{-w} & \Sigma\mathscr{I} \\
{\scriptstyle 1}\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow{\scriptstyle 1} \\
\mathscr{I} & \longrightarrow & i_{U*}(\mathscr{I}|_U) \oplus i_{V*}(\mathscr{I}|_V) & \longrightarrow & i_{U\cap V*}(\mathscr{I}|_{U\cap V}) & \xrightarrow[-z]{} & \Sigma\mathscr{I}
\end{array}$$

Checking that this diagram commutes requires a little bit of work, but is straightforward. Since we know the bottom row is a triangle, so is the top row, which completes the proof. □

Lemma 22. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $X = U \cup V$ an open cover. For any complex $\mathscr{F}$ of sheaves of modules on X there is a canonical triangle in $\mathfrak{D}(Y)$*

$$\mathbb{R}f_*\mathscr{F} \longrightarrow \mathbb{R}f_{U*}(\mathscr{F}|_U) \oplus \mathbb{R}f_{V*}(\mathscr{F}|_V) \longrightarrow \mathbb{R}f_{U\cap V*}(\mathscr{F}|_{U\cap V}) \longrightarrow \Sigma\mathbb{R}f_*\mathscr{F}$$

where $f_U : U \longrightarrow Y, f_V : V \longrightarrow Y$ and $f_{U\cap V} : U \cap V \longrightarrow Y$ are the inclusions. This triangle is natural with respect to morphisms of complexes.

Proof. Let $\mathscr{F}$ be a complex of sheaves of modules on X and consider the triangle of Lemma 21

$$\mathscr{F} \longrightarrow \mathbb{R}i_{U*}(\mathscr{F}|_U) \oplus \mathbb{R}i_{V*}(\mathscr{F}|_V) \longrightarrow \mathbb{R}i_{U\cap V*}(\mathscr{F}|_{U\cap V}) \longrightarrow \Sigma\mathscr{F}$$

Applying $\mathbb{R}f_*$ and using Lemma 18 we obtain the desired triangle in $\mathfrak{D}(Y)$ natural with respect to morphisms of complexes. □

3 Derived Sheaf Hom

We already know from (DTC2,Section 3) that there exists a derived Hom functor

$$\mathbb{R}Hom^\bullet(-,-) : \mathfrak{D}(X)^{\mathrm{op}} \times \mathfrak{D}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

In this section we define the analogue of this functor for $\mathscr{H}om$. Throughout this section $(X, \mathcal{O}_X)$ is a ringed space and all sheaves of modules are over X unless otherwise specified.

Definition 3. We have functors additive in each variable

$$\begin{aligned}\mathscr{H}om(-,-) &: \mathfrak{Mod}(X)^{\mathrm{op}} \times \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)\\ \mathscr{H}om^\bullet(-,-) &: \mathbf{C}(X)^{\mathrm{op}} \times \mathbf{C}(X) \longrightarrow \mathbf{C}(X)\\ \mathscr{H}om^\bullet(-,-) &: K(X)^{\mathrm{op}} \times K(X) \longrightarrow K(X)\end{aligned}$$

For complexes $\mathscr{F}, \mathscr{G}$ the complex $\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})$ is defined as follows

$$\begin{aligned}\mathscr{H}om^n(\mathscr{F},\mathscr{G}) &= \prod_{i+j=n} \mathscr{H}om(\mathscr{F}^{-i}, \mathscr{G}^j) = \prod_{q\in\mathbb{Z}} \mathscr{H}om(\mathscr{F}^q, \mathscr{G}^{q+n})\\ \partial^n_U((f_p)_{p\in\mathbb{Z}})_q &= f_{q+1}\partial^q_{\mathscr{F}}|_U + (-1)^{n+1}\partial^{q+n}_{\mathscr{G}}|_U f_q\\ \mathscr{H}om^n(\varphi,\psi) &= \prod_{q\in\mathbb{Z}} \mathscr{H}om(\varphi^q, \psi^{q+n})\end{aligned}$$

Moreover it is clear that for any open $U \subseteq X$ we have an equality of complexes of $\Gamma(U, \mathcal{O}_X)$-modules natural in both variables $\Gamma(U, \mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})) = Hom^\bullet_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$, and an equality of complexes of sheaves of modules $\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})|_U = \mathscr{H}om^\bullet(\mathscr{F}|_U, \mathscr{G}|_U)$ natural in both variables.

Remark 5. Let $\mathscr{A}$ be a sheaf of modules on X and $\mathscr{Y}$ a complex of sheaves of modules on X. The complex $\mathscr{H}om^\bullet(\mathscr{A}, \mathscr{Y})$ is canonically isomorphic to the complex $\mathscr{H}om(\mathscr{A}, \mathscr{Y})$ with alternating signs on the differentials

$$\cdots \longrightarrow \mathscr{H}om(\mathscr{A}, \mathscr{Y}^0) \xrightarrow{-} \mathscr{H}om(\mathscr{A}, \mathscr{Y}^1) \longrightarrow \mathscr{H}om(\mathscr{A}, \mathscr{Y}^2) \xrightarrow{-} \cdots$$

which we denote by $(-1)^{\bullet+1}\mathscr{H}om(\mathscr{A}, \mathscr{Y})$. By (DTC2,Remark 7) this complex is canonically isomorphic in $\mathbf{C}(X)$ to $\mathscr{H}om(\mathscr{A}, \mathscr{Y})$ so finally we have a canonical natural isomorphism of complexes $\mathscr{H}om^\bullet(\mathscr{A}, \mathscr{Y}) \cong \mathscr{H}om(\mathscr{A}, \mathscr{Y})$.

Lemma 23. *The functor $\mathscr{H}om(-,-)$ is homlike.*

Proof. See (DTC2,Definition 13) for what we mean by *homlike*. Let $\mathscr{Z}, \mathscr{I}$ be complexes of sheaves of modules with $\mathscr{Z}$ exact and $\mathscr{I}$ hoinjective. We have to show that the complex $\mathscr{H}om^\bullet(\mathscr{Z}, \mathscr{I})$ is exact. For $n \in \mathbb{Z}$ the sheaf $H^n(\mathscr{H}om^\bullet(\mathscr{Z}, \mathscr{I}))$ is by Lemma 3 the sheafification of the presheaf

$$U \mapsto H^n(Hom^\bullet_{\mathcal{O}_X|_U}(\mathscr{Z}|_U, \mathscr{I}|_U))$$

But $\mathcal{Z}|_U$ is exact and $\mathscr{I}|_U$ hoinjective, so the complex $Hom^\bullet_{\mathcal{O}_X|_U}(\mathscr{Z}|_U, \mathscr{I}|_U)$ is exact, which completes the proof. $\square$

Definition 4. Setting $H = \mathscr{H}om(-,-)$ in (DTC2,Definition 16) we have functors additive in each variable

$$\mathbb{R}Hom^\bullet(-,-) : \mathfrak{D}(X)^{\mathrm{op}} \times \mathfrak{D}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$
$$\mathbb{R}\mathscr{H}om^\bullet(-,-) : \mathfrak{D}(X)^{\mathrm{op}} \times \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$

If we fix an assignment of hoinjectives $\mathcal{I}$ then these functors are canonically defined. If the chosen resolution of a complex $\mathscr{G}$ is $\mathscr{G} \longrightarrow I_{\mathscr{G}}$ then

$$\mathbb{R}Hom^\bullet(\mathscr{F}, \mathscr{G}) = Hom^\bullet(\mathscr{F}, I_{\mathscr{G}})$$
$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G}) = \mathscr{H}om^\bullet(\mathscr{F}, I_{\mathscr{G}})$$

and we have an equality of complexes of abelian groups

$$\Gamma(X, \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})) = \mathbb{R}_{\mathcal{I}}Hom^\bullet(\mathscr{F}, \mathscr{G})$$

As part of the data we have a morphism in $\mathfrak{D}(X)$ trinatural in both variables

$$\zeta : \mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})$$

which is an isomorphism if $\mathscr{G}$ is hoinjective.

Remark 6. With the notation of Definition 4 the partial functors

$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, -), \quad \mathbb{R}\mathscr{H}om^\bullet(-, \mathscr{G})$$

are canonically triangulated functors, and moreover these triangulated structures are compatible. That is, the isomorphisms in $\mathfrak{D}(X)$

$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \Sigma\mathscr{G}) \cong \Sigma\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})$$
$$\mathbb{R}\mathscr{H}om^\bullet(\Sigma^{-1}\mathscr{F}, \mathscr{G}) \cong \Sigma\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})$$

are natural in both variables.

Lemma 24. *Let $U \subseteq X$ be an open subset. For complexes of sheaves of modules $\mathscr{F}, \mathscr{G}$ we have a canonical isomorphism in $\mathfrak{D}(U)$ natural in both variables*

$$\mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{F}, \mathscr{G})|_U \longrightarrow \mathbb{R}\mathscr{H}om^\bullet_U(\mathscr{F}|_U, \mathscr{G}|_U)$$

Proof. Let $\mathscr{G} \longrightarrow I_{\mathscr{G}}$ be the chosen hoinjective resolution of $\mathscr{G}$. This restricts to a hoinjective resolution of $\mathscr{G}|_U$, so we have a canonical isomorphism in $\mathfrak{D}(U)$

$$\begin{aligned}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})|_U = \mathscr{H}om^\bullet(\mathscr{F}, I_{\mathscr{G}})|_U &= \mathscr{H}om^\bullet(\mathscr{F}|_U, I_{\mathscr{G}}|_U)\\ &\cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}|_U, I_{\mathscr{G}}|_U) \cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}|_U, \mathscr{G}|_U)\end{aligned}$$

which one checks is natural in both variables with respect to morphisms of $\mathfrak{D}(X)$. Observe that by construction the following diagram commutes

$$\begin{array}{ccc}\mathscr{H}om^\bullet_X(\mathscr{F}, \mathscr{G})|_U & \xrightarrow{\;1\;} & \mathscr{H}om^\bullet_U(\mathscr{F}|_U, \mathscr{G}|_U)\\ {\scriptstyle \zeta|_U}\downarrow & & \downarrow{\scriptstyle \zeta}\\ \mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{F}, \mathscr{G})|_U & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet_U(\mathscr{F}|_U, \mathscr{G}|_U)\end{array}$$

where the morphisms ζ come as part of the definition of the derived sheaf Hom. $\square$

Lemma 25. *Let $\mathscr{Y}$ be a complex of sheaves of modules on X. There is a canonical isomorphism in $\mathfrak{D}(X)$ natural in $\mathscr{Y}$*

$$\mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X, \mathscr{Y}) \longrightarrow \mathscr{Y}$$

Proof. If $\mathscr{Y} \longrightarrow I_{\mathscr{Y}}$ is the chosen hoinjective resolution of $\mathscr{Y}$ then we have by Remark 5 a canonical isomorphism $\mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X, \mathscr{Y}) = \mathscr{H}om^\bullet(\mathcal{O}_X, I_{\mathscr{Y}}) \cong I_{\mathscr{Y}} \cong \mathscr{Y}$ in $\mathfrak{D}(X)$ which is clearly natural in $\mathscr{Y}$. Observe that by definition the following diagram commutes

$$\begin{array}{ccc} \mathscr{H}om^\bullet(\mathcal{O}_X, \mathscr{Y}) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X, \mathscr{Y}) \\ & \searrow \quad \swarrow & \\ & \mathscr{Y} & \end{array}$$

and in particular the canonical morphism $\mathscr{H}om^\bullet(\mathcal{O}_X, \mathscr{Y}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X, \mathscr{Y})$ is an isomorphism. $\square$

Lemma 26. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules on X. There is a canonical morphism of complexes natural in both variables*

$$f_*\mathscr{H}om^\bullet_X(\mathscr{X}, \mathscr{Y}) \longrightarrow \mathscr{H}om^\bullet_Y(f_*\mathscr{X}, f_*\mathscr{Y})$$

Proof. For $q \in \mathbb{Z}$ we have a canonical morphism of sheaves of modules, using (MRS,Proposition 86)

$$\begin{aligned} f_*\mathscr{H}om^q_X(\mathscr{X}, \mathscr{Y}) = f_*\left(\prod_j \mathscr{H}om(\mathscr{X}^j, \mathscr{Y}^{j+q})\right) &= \prod_j f_*\mathscr{H}om(\mathscr{X}^j, \mathscr{Y}^{j+q}) \\ &\longrightarrow \prod_j \mathscr{H}om(f_*\mathscr{X}^j, f_*\mathscr{Y}^{j+q}) = \mathscr{H}om^q(f_*\mathscr{X}, f_*\mathscr{Y}) \end{aligned}$$

which is easily checked to define a morphism of complexes natural in both variables. $\square$

Lemma 27. *For sheaves of modules $\mathscr{F}, \mathscr{G}$ there are canonical isomorphisms natural in both variables*

$$H^i(\mathbb{R}Hom^\bullet(\mathscr{F}, \mathscr{G})) \longrightarrow Ext^i(\mathscr{F}, \mathscr{G})$$
$$H^i(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})) \longrightarrow \mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$$

Proof. To be precise, we mean that once you fix assignments of hoinjective and injective resolutions for $\mathfrak{Mod}(X)$ to calculate the various functors, there are canonical isomorphisms, where by convention the right hand sides are zero for $i < 0$. Let $\mathscr{G} \longrightarrow I_{\mathscr{G}}$ be the chosen injective resolution of $\mathscr{G}$ in $\mathfrak{Mod}(X)$. Then we have a canonical isomorphism

$$\begin{aligned} H^i(\mathbb{R}Hom^\bullet(\mathscr{F}, \mathscr{G})) &\cong H^i(\mathbb{R}Hom^\bullet(\mathscr{F}, I_{\mathscr{G}})) \\ &\cong H^i(Hom^\bullet(\mathscr{F}, I_{\mathscr{G}})) \\ &\cong Ext^i(\mathscr{F}, \mathscr{G}) \end{aligned}$$

since the complex $Hom^\bullet(\mathscr{F}, I_{\mathscr{G}})$ is canonically isomorphic to the complex $Hom(\mathscr{F}, I_{\mathscr{G}})$ with alternated signs on the differentials (DTC2,Example 1). Naturality in both variables is easily checked. One checks the second isomorphism in exactly the same way. $\square$

For arbitrary complexes of sheaves of modules $\mathscr{F}, \mathscr{G}$ we know that the cohomology of the complexes $Hom^\bullet(\mathscr{F}, \mathscr{G})$ and $\mathbb{R}Hom^\bullet(\mathscr{F}, \mathscr{G})$ calculate morphisms in $K(X)$ and $\mathfrak{D}(X)$ respectively (DTC2,Proposition 18) (DTC2,Lemma 26). It will turn out that the cohomology of the complexes $\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})$ and $\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{G})$ are sheaves which calculate morphisms in $K(U)$ and $\mathfrak{D}(U)$ for every open $U \subseteq X$. But first we need to make some definitions.

Definition 5. Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules and, using the module structures of Remark 3, define presheaves of $\mathcal{O}_X$-modules by

$$\Gamma(U, K\mathscr{H}om(\mathscr{X}, \mathscr{Y})) = Hom_{K(U)}(\mathscr{X}|_U, \mathscr{Y}|_U)$$
$$\Gamma(U, D\mathscr{H}om(\mathscr{X}, \mathscr{Y})) = Hom_{\mathfrak{D}(U)}(\mathscr{X}|_U, \mathscr{Y}|_U)$$

with the obvious restriction. Observe that, technically speaking, $\Gamma(U, D\mathscr{H}om(\mathscr{X}, \mathscr{Y}))$ is not even a set (it is a small conglomerate), but this is a pedantic distinction and we will safely ignore it. Both constructions are functorial in $\mathscr{X}, \mathscr{Y}$ in that morphisms of $K(X)$ or $\mathfrak{D}(X)$ in either variable induce morphisms of presheaves of $\mathcal{O}_X$-modules. Taking the sheafifications we have functorial sheaves of modules $\mathbb{K}\mathscr{H}om(\mathscr{X}, \mathscr{Y})$ and $\mathbb{D}\mathscr{H}om(\mathscr{X}, \mathscr{Y})$. For each open set $U \subseteq X$ we have a canonical morphism of $\Gamma(U, \mathcal{O}_X)$-modules

$$Hom_{K(U)}(\mathscr{X}|_U, \mathscr{Y}|_U) \longrightarrow Hom_{\mathfrak{D}(U)}(\mathscr{X}|_U, \mathscr{Y}|_U)$$

and this defines a canonical morphism of sheaves of modules natural in both variables

$$\mathbb{K}\mathscr{H}om(\mathscr{X}, \mathscr{Y}) \longrightarrow \mathbb{D}\mathscr{H}om(\mathscr{X}, \mathscr{Y})$$

which is an isomorphism if $\mathscr{Y}$ is hoinjective.

Proposition 28. *Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules and $n \in \mathbb{Z}$. There is a canonical isomorphism of sheaves of modules natural in both variables*

$$H^n \mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y}) \longrightarrow \mathbb{K}\mathscr{H}om(\mathscr{X}, \Sigma^n \mathscr{Y})$$

In other words, $H^n \mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y})$ is the sheafification of the presheaf

$$U \mapsto Hom_{K(U)}(\mathscr{X}|_U, \Sigma^n \mathscr{Y}|_U)$$

Proof. By Lemma 3 we know that $H^n \mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y})$ is canonically naturally isomorphic to the sheafification of the presheaf

$$\begin{aligned} U \mapsto H^n(\Gamma(U, \mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y}))) &= H^n(Hom^\bullet_{\mathcal{O}_X|_U}(\mathscr{X}|_U, \mathscr{Y}|_U)) \\ &\cong Hom_{K(U)}(\mathscr{X}|_U, \Sigma^n \mathscr{Y}|_U) \end{aligned}$$

using (DTC2,Proposition 18), and this sheaf is none other than $\mathbb{K}\mathscr{H}om(\mathscr{X}, \Sigma^n \mathscr{Y})$. Naturality in both variables is easily checked. □

Proposition 29. *Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules and $n \in \mathbb{Z}$. There is a canonical isomorphism of sheaves of modules natural in both variables*

$$H^n \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y}) \longrightarrow \mathbb{D}\mathscr{H}om(\mathscr{X}, \Sigma^n \mathscr{Y})$$

In other words, $H^n \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y})$ is the sheafification of the presheaf

$$U \mapsto Hom_{\mathfrak{D}(U)}(\mathscr{X}|_U, \Sigma^n \mathscr{Y}|_U)$$

Proof. If $\mathscr{Y} \longrightarrow I_{\mathscr{Y}}$ is the chosen hoinjective resolution of $\mathscr{Y}$, then we have using Proposition 28 a canonical isomorphism of sheaves of modules

$$\begin{aligned} H^n \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y}) &= H^n \mathscr{H}om^\bullet(\mathscr{X}, I_{\mathscr{Y}}) \\ &\cong \mathbb{K}\mathscr{H}om(\mathscr{X}, \Sigma^n I_{\mathscr{Y}}) \\ &\cong \mathbb{D}\mathscr{H}om(\mathscr{X}, \Sigma^n I_{\mathscr{Y}}) \\ &\cong \mathbb{D}\mathscr{H}om(\mathscr{X}, \Sigma^n \mathscr{Y}) \end{aligned}$$

since $\Sigma^n I_{\mathscr{Y}}$ is hoinjective. Naturality in both variables is easily checked. Observe that by construction the diagram

$$\begin{array}{ccc} H^n \mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y}) & \longrightarrow & \mathbb{K}\mathscr{H}om(\mathscr{X}, \Sigma^n \mathscr{Y}) \\ \downarrow & & \downarrow \\ H^n \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathscr{Y}) & \longrightarrow & \mathbb{D}\mathscr{H}om(\mathscr{X}, \Sigma^n \mathscr{Y}) \end{array}$$

commutes in $\mathfrak{Mod}(X)$. □

4 Derived Tensor

Throughout this section $(X, \mathcal{O}_X)$ is a ringed space, and all sheaves of modules are over X. We denote by $Q : K(X) \longrightarrow \mathfrak{D}(X)$ the verdier quotient. The tensor product defines a functor additive in each variable $- \otimes - : \mathfrak{Mod}(X) \times \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$. In this section we define the derived tensor product functor

$$- \underset{=}{\otimes} - : \mathfrak{D}(X) \times \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$

While we used hoinjective resolutions to define the derived sheaf Hom, we have to introduce the notion of a *hoflat* complex to define the derived tensor product.

4.1 Tensor Product of Complexes

As in Section (DTC2,Section 3) we develop the basic theory in the generality of a functor $T : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}$ additive in each variable.

Definition 6. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $T : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}$ a functor which is additive in each variable. For complexes X, Y in $\mathcal{A}$ we define a bicomplex $BT(X, Y)$ in $\mathcal{B}$ as follows. For $i, j \in \mathbb{Z}$ we have $BT(X, Y)^{ij} = T(X^i, Y^j)$ and we define the differentials by

$$\partial_1^{ij} = T(\partial_X^i, Y^j) : T(X^i, Y^j) \longrightarrow T(X^{i+1}, Y^j)$$
$$\partial_2^{ij} = T(X^i, \partial_Y^j) : T(X^i, Y^j) \longrightarrow T(X^i, Y^{j+1})$$

Given morphisms of complexes $\varphi : X \longrightarrow X'$ and $\psi : Y \longrightarrow Y'$ there are morphisms of bicomplexes

$$BT(\varphi, Y) : BT(X, Y) \longrightarrow BT(X', Y), \quad BT(\varphi, Y)^{ij} = T(\varphi^i, Y^j)$$
$$BT(X, \psi) : BT(X, Y) \longrightarrow BT(X, Y'), \quad BT(X, \psi)^{ij} = T(X^i, \psi^j)$$

It is clear that $BT(X', \psi)BT(\varphi, Y) = BT(\varphi, Y')BT(X, \psi)$ so we have defined a functor additive in each variable

$$BT : \mathbf{C}(\mathcal{A}) \times \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}^2(\mathcal{B})$$
$$BT(\varphi, \psi)^{ij} = T(\varphi^i, \psi^j)$$

Taking the totalisation (DTC,Definition 33) we have a functor $T^\bullet = Tot \circ BT$ additive in each variable

$$T^\bullet : \mathbf{C}(\mathcal{A}) \times \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$$
$$T^n(X, Y) = \bigoplus_{i+j=n} T(X^i, Y^j)$$

Lemma 30. *With the notation of Definition 6 suppose that we have homotopic morphisms of complexes $\varphi \simeq \varphi' : X \longrightarrow X'$ and $\psi \simeq \psi' : Y \longrightarrow Y'$ in $\mathcal{A}$. Then $T^\bullet(\varphi, \psi) \simeq T^\bullet(\varphi', \psi')$.*

Proof. It suffices to show that $T^\bullet(Z, \psi) \simeq T^\bullet(Z, \psi')$ and $T^\bullet(\varphi, Z) \simeq T^\bullet(\varphi', Z)$ for any complex Z and morphisms as in the statement of the Lemma. Let $\Sigma : \psi \longrightarrow \psi'$ be a homotopy and define a morphism $\Lambda^n : T^n(Z, Y) \longrightarrow T^n(Z, Y')$ by

$$\Lambda^n u_{ij} = (-1)^i u_{i(j-1)} T(Z^i, \Sigma^j)$$

One checks that Λ is a homotopy $T^\bullet(Z, \psi) \longrightarrow T^\bullet(Z, \psi')$. On the other hand if $\Sigma : \varphi \longrightarrow \varphi'$ is a homotopy then we define a morphism $\Lambda^n : T^n(X, Z) \longrightarrow T^{n-1}(X', Z)$ by

$$\Lambda^n u_{ij} = u_{(i-1)j} T(\Sigma^i, Z^j)$$

and one checks that Λ is a homotopy $T^\bullet(\varphi, Z) \longrightarrow T^\bullet(\varphi', Z)$. ∎

Definition 7. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $T : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}$ a functor which is additive in each variable. The functor $T^\bullet : \mathbf{C}(\mathcal{A}) \times \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$ extends to a functor additive in each variable $T^\bullet : K(\mathcal{A}) \times K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ which makes the following diagram commute

$$\begin{array}{ccc} \mathbf{C}(\mathcal{A}) \times \mathbf{C}(\mathcal{A}) & \longrightarrow & \mathbf{C}(\mathcal{B}) \\ \downarrow & & \downarrow \\ K(\mathcal{A}) \times K(\mathcal{A}) & \longrightarrow & K(\mathcal{B}) \end{array}$$

Proof. We define the new functor $T^\bullet$ on objects as before, and on morphisms by $T^\bullet([\varphi], [\psi]) = [T^\bullet(\varphi, \psi)]$ which is well-defined by Lemma 30. It is clear that this is a functor additive in each variable. □

Definition 8. Taking T to be the tensor product of sheaves of modules in Definition 6 we obtain a functor $- \otimes_{\mathcal{O}_X} - : \mathbf{C}(X) \times \mathbf{C}(X) \longrightarrow \mathbf{C}(X)$ additive in each variable. We drop the subscript on the tensor whenever it will not cause confusion. Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules. The complex of sheaves of modules $\mathscr{X} \otimes \mathscr{Y}$ is defined by

$$(\mathscr{X} \otimes \mathscr{Y})^n = \bigoplus_{i+j=n} \mathscr{X}^i \otimes \mathscr{Y}^j$$

$$\partial^n u_{ij} = u_{(i+1)j}(\partial^i_{\mathscr{X}} \otimes 1) + (-1)^i u_{i(j+1)}(1 \otimes \partial^j_{\mathscr{Y}})$$

This functor is defined on morphisms by $(\varphi \otimes \psi)^n = \bigoplus_{i+j=n} \varphi^i \otimes \psi^j$. There is a canonical isomorphism of complexes $\tau : \mathscr{X} \otimes \mathscr{Y} \longrightarrow \mathscr{Y} \otimes \mathscr{X}$ defined by

$$\tau^n u_{ij} = (-1)^{ij} u_{ji} \tau_{\mathscr{X}^i, \mathscr{Y}^j}$$

where $\tau_{\mathscr{F},\mathscr{G}} : \mathscr{F} \otimes \mathscr{G} \longrightarrow \mathscr{G} \otimes \mathscr{F}$ is the canonical twisting isomorphism for sheaves of modules. The isomorphism τ on complexes is natural in both variables. As in Definition 7 there is an induced functor additive in both variables $- \otimes - : K(X) \times K(X) \longrightarrow K(X)$. In particular for a complex $\mathscr{X}$ we have additive functors $- \otimes \mathscr{X}, \mathscr{X} \otimes - : K(X) \longrightarrow K(X)$ which are $\Gamma(X, \mathcal{O}_X)$-linear.

If R is a commutative ring and we take T to be the tensor product of modules in Definition 6 then we obtain a functor $- \otimes_R - : \mathbf{C}(R) \times \mathbf{C}(R) \longrightarrow \mathbf{C}(R)$ additive in each variable, where $\mathbf{C}(R) = \mathbf{C}(R\mathbf{Mod})$. In the same way we define a canonical twisting isomorphism $\tau : X \otimes Y \longrightarrow Y \otimes X$ natural in both variables, and there is an induced functor additive in both variables $- \otimes - : K(R) \times K(R) \longrightarrow K(R)$.

Lemma 31. *Let $\mathscr{X}$ be a complex of sheaves of modules and $\mathscr{F}$ a sheaf of modules, considered as a complex in degree zero. There is a canonical isomorphism of $\mathscr{X} \otimes \mathscr{F}$ with the complex*

$$\cdots \longrightarrow \mathscr{X}^{i-1} \otimes \mathscr{F} \longrightarrow \mathscr{X}^i \otimes \mathscr{F} \longrightarrow \mathscr{X}^{i+1} \otimes \mathscr{F} \longrightarrow \cdots$$

natural in both $\mathscr{X}$ and $\mathscr{F}$. Similarly $\mathscr{F} \otimes \mathscr{X}$ is canonically naturally isomorphic to the complex $\{\mathscr{F} \otimes \mathscr{X}^n\}_{n \in \mathbb{Z}}$. In particular there are isomorphisms $\mathscr{X} \otimes \mathcal{O}_X \longrightarrow \mathscr{X}$ and $\mathcal{O}_X \otimes \mathscr{X} \longrightarrow \mathscr{X}$ natural in $\mathscr{X}$.

Remark 7. Let $\mathscr{F}, \mathscr{G}$ be complexes of sheaves of modules. In (DTC2,Definition 12) we introduced a functor $\mathscr{Y} \mapsto (-1)^{\bullet+1}\mathscr{Y}$ on complexes which leaves the objects alone but alternates the signs on the differentials. We claim that there is a canonical isomorphism of complexes

$$\alpha : ((-1)^{\bullet+1}\mathscr{F}) \otimes \mathscr{G} \longrightarrow (-1)^{\bullet+1}(\mathscr{F} \otimes \mathscr{G})$$

natural in both variables. Given $q \in \mathbb{Z}$ we define $\alpha^q u_{ij} = (-1)^{ij + \frac{j(j+1)}{2}} u_{ij}$. This sign factor is necessary for α to be a morphism of complexes. One checks easily that this is an isomorphism natural in both variables.

Lemma 32. *Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules. There are canonical isomorphisms of complexes natural in both variables*

$$\begin{aligned} \rho : \mathscr{X} \otimes (\Sigma \mathscr{Y}) &\longrightarrow \Sigma(\mathscr{X} \otimes \mathscr{Y}), \quad \rho^n u_{ij} = (-1)^i u_{i(j+1)} \\ \sigma : (\Sigma \mathscr{X}) \otimes \mathscr{Y} &\longrightarrow \Sigma(\mathscr{X} \otimes \mathscr{Y}), \quad \sigma^n u_{ij} = u_{(i+1)j} \end{aligned}$$

which make the following diagram commute

$$\begin{array}{ccc} \mathscr{X} \otimes (\Sigma \mathscr{Y}) & \xrightarrow{\tau} & (\Sigma \mathscr{Y}) \otimes \mathscr{X} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \Sigma(\mathscr{X} \otimes \mathscr{Y}) & \xrightarrow[\Sigma\tau]{} & \Sigma(\mathscr{Y} \otimes \mathscr{X}) \end{array} \tag{3}$$

Remark 8. We say that an isomorphism is *trinatural* in a variable $\mathscr{X}$ if it is natural in $\mathscr{X}$ and whenever you substitute $\Sigma\mathscr{X}$ for $\mathscr{X}$ in both sides and apply ρ, σ (and other triangle isomorphisms) to commute the Σ out the front, you end up with a commutative diagram. For example, commutativity of (3) expresses trinaturality of the twisting isomorphism $\tau : \mathscr{X} \otimes \mathscr{Y} \longrightarrow \mathscr{Y} \otimes \mathscr{X}$.

Lemma 33. *Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules and $U \subseteq X$ open. There is a canonical isomorphism trinatural in both variables*

$$\delta : (\mathscr{X} \otimes \mathscr{Y})|_U \longrightarrow \mathscr{X}|_U \otimes \mathscr{Y}|_U$$

Proof. The existence of a natural isomorphism is easily checked. One also checks that the following diagrams commute

$$\begin{array}{ccc} ((\Sigma\mathscr{X}) \otimes \mathscr{Y})|_U & \xrightarrow{\delta} & (\Sigma\mathscr{X})|_U \otimes \mathscr{Y}|_U \\ {\scriptstyle\sigma|_U}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \Sigma(\mathscr{X} \otimes \mathscr{Y})|_U & \xrightarrow[\Sigma\delta]{} & \Sigma(\mathscr{X}|_U \otimes \mathscr{Y}|_U) \end{array} \qquad \begin{array}{ccc} (\mathscr{X} \otimes (\Sigma\mathscr{Y}))|_U & \xrightarrow{\delta} & \mathscr{X}|_U \otimes (\Sigma\mathscr{Y})|_U \\ {\scriptstyle\tau|_U}\downarrow & & \downarrow{\scriptstyle\tau} \\ \Sigma(\mathscr{X} \otimes \mathscr{Y})|_U & \xrightarrow[\Sigma\delta]{} & \Sigma(\mathscr{X}|_U \otimes \mathscr{Y}|_U) \end{array}$$

which is what we mean by *trinaturality* of δ. □

With x a point of our fixed ringed space $(X, \mathcal{O}_X)$ we have by (MRS,Section 1.1) a triple of adjoint functors between $\mathfrak{Mod}(X)$ and $\mathcal{O}_{X,x}\mathbf{Mod}$

$$(-)_x \longrightarrow Sky_x(-) \longrightarrow (-)^x$$

By (DTC,Lemma 24) there is a corresponding triple of adjoints between $\mathbf{C}(X)$ and $\mathbf{C}(\mathcal{O}_{X,x})$.

Lemma 34. *Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules. Then for $x \in X$ there is a canonical isomorphism of complexes of $\mathcal{O}_{X,x}$-modules natural in both variables*

$$\alpha : (\mathscr{X} \otimes \mathscr{Y})_x \longrightarrow \mathscr{X}_x \otimes \mathscr{Y}_x$$

Proof. For $n \in \mathbb{Z}$ we have a canonical isomorphism of $\mathcal{O}_{X,x}$-modules

$$(\mathscr{X} \otimes \mathscr{Y})^n_x = \left(\oplus_{i+j=n} \mathscr{X}^i \otimes \mathscr{Y}^j\right)_x \cong \bigoplus_{i+j=n} (\mathscr{X}^i \otimes \mathscr{Y}^j)_x \cong \bigoplus_{i+j=n} \mathscr{X}^i_x \otimes \mathscr{Y}^j_x = (\mathscr{X}_x \otimes \mathscr{Y}_x)^n$$

and one checks that these morphisms give the required isomorphism of complexes. □

Lemma 35. *Let $x \in X$ be a point, $\mathscr{F}$ a complex of sheaves of modules on X and M a complex of $\mathcal{O}_{X,x}$-modules. Then there is a canonical morphism of complexes of sheaves of modules natural in $\mathscr{F}$ and M*

$$\lambda : \mathscr{F} \otimes_{\mathcal{O}_X} Sky_x(M) \longrightarrow Sky_x(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} M)$$

with the property that λ_x is an isomorphism of complexes.

Proof. For $n \in \mathbb{Z}$ we have by (MRS,Lemma 14) a canonical morphism of sheaves of modules

$$\begin{aligned}\lambda^n : (\mathscr{F} \otimes Sky_x(M))^n &= \bigoplus_{i+j=n} (\mathscr{F}^i \otimes Sky_x(M^j)) \\ &\longrightarrow \bigoplus_{i+j=n} Sky_x(\mathscr{F}^i_x \otimes M^j) \\ &\cong Sky_x\left(\oplus_{i+j=n}\mathscr{F}^i_x \otimes M^j\right) \\ &= Sky_x(\mathscr{F}_x \otimes M)^n\end{aligned}$$

which gives a morphism of complexes with the desired properties. □

Lemma 36. *Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ be complexes of sheaves of modules. There is a canonical isomorphism of complexes of sheaves of modules trinatural in all three variables*

$$(\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z} \longrightarrow \mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z})$$

Proof. For $n \in \mathbb{Z}$ we have a canonical isomorphism of sheaves of modules

$$\begin{aligned}\varrho^n : \{(\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z}\}^n &= \bigoplus_{k+l+j=n} (\mathscr{X}^k \otimes \mathscr{Y}^l) \otimes \mathscr{Z}^j \\ &\cong \bigoplus_{k+l+j=n} \mathscr{X}^k \otimes (\mathscr{Y}^l \otimes \mathscr{Z}^j) \\ &= \{\mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z})\}^n\end{aligned}$$

defined by $\varrho^n u_{k,l,j} = u_{k,l,j} a_{\mathscr{X}^k,\mathscr{Y}^l,\mathscr{Z}^j}$ where a is the canonical associator for the tensor product of sheaves of modules. One checks that this is actually an isomorphism of complexes natural in all three variables. When we say that the isomorphism is *trinatural* in all three variables we mean that the three obvious diagrams commute. For example, trinaturality in $\mathscr{X}$ is commutativity of the following diagram

$$\begin{array}{ccc}(\Sigma\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z} & \longrightarrow & \Sigma\mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z}) \\ \downarrow & & \\ \Sigma(\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z} & & \downarrow \\ \downarrow & & \\ \Sigma((\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z}) & \longrightarrow & \Sigma(\mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z}))\end{array}$$

which is easily checked. Similarly one checks trinaturality in the other two variables. □

Lemma 37. *For any complex $\mathscr{Z}$ of sheaves of modules the additive functors $- \otimes \mathscr{Z}, \mathscr{Z} \otimes - :$ $\mathbf{C}(X) \longrightarrow \mathbf{C}(X)$ preserve colimits.*

Proof. Given the natural isomorphism τ it suffices to prove that $\mathscr{Z} \otimes -$ preserves colimits. This follows from the fact that tensor products and coproducts commute with colimits. □

Lemma 38. *Let $u : \mathscr{X} \longrightarrow \mathscr{Y}$ be a morphism of complexes of sheaves of modules. For any complex of sheaves of modules $\mathscr{Z}$ there is a canonical isomorphism of complexes $\mathscr{Z} \otimes C_u \cong C_{\mathscr{Z} \otimes u}$.*

Proof. For $n \in \mathbb{Z}$ we have a canonical isomorphism of sheaves of modules α^n

$$\begin{aligned}(\mathscr{Z} \otimes C_u)^n &= \bigoplus_{i+j=n} \mathscr{Z}^i \otimes C_u^j \\ &= \bigoplus_{i+j=n} \mathscr{Z}^i \otimes \{\mathscr{X}^{j+1} \oplus \mathscr{Y}^j\} \\ &\cong \bigoplus_{i+j=n} \left(\{\mathscr{Z}^i \otimes \mathscr{X}^{j+1}\} \oplus \{\mathscr{Z}^i \otimes \mathscr{Y}^j\}\right) \\ &\cong \left\{\oplus_{i+j=n}\mathscr{Z}^i \otimes \mathscr{X}^{j+1}\right\} \oplus \left\{\oplus_{i+j=n}\mathscr{Z}^i \otimes \mathscr{Y}^j\right\} \\ &\cong (\mathscr{Z} \otimes \mathscr{X})^{n+1} \oplus (\mathscr{Z} \otimes \mathscr{Y})^n = C^n_{\mathscr{Z} \otimes u}\end{aligned}$$

Writing α^n as a matrix $\left(\begin{smallmatrix} a^n \\ b^n \end{smallmatrix}\right)$ we have

$$\begin{aligned} a^n u_{ij} &= u_{i(j+1)}(\mathscr{Z}^i \otimes p_j^X) \\ b^n u_{ij} &= u_{ij}(\mathscr{Z}^i \otimes p_j^Y) \end{aligned}$$

where p_j^X, p_j^Y denote the projections from the mapping cone C_u^j. We need to modify the sign slightly on the first component. Set $A^n u_{ij} = (-1)^i a^n u_{ij}$ and define $\beta^n : (\mathscr{Z} \otimes C_u)^n \longrightarrow C^n_{\mathscr{Z}\otimes u}$ with components A^n, b^n. It is straightforward to check that β^n is an isomorphism, and that thus defined β is an isomorphism of complexes. □

Proposition 39. *For any complex $\mathscr{Z}$ of sheaves of modules the pairs $(\mathscr{Z} \otimes -, \rho), (- \otimes \mathscr{Z}, \sigma)$ are coproduct preserving triangulated functors $K(X) \longrightarrow K(X)$. There is a canonical trinatural equivalence $\tau : \mathscr{Z} \otimes - \longrightarrow - \otimes \mathscr{Z}$.*

Proof. The natural isomorphisms ρ, σ of Lemma 32 define natural equivalences $\rho : \mathscr{Z} \otimes \Sigma(-) \longrightarrow \Sigma(\mathscr{Z} \otimes -)$ and $\sigma : \Sigma(-) \otimes \mathscr{Z} \longrightarrow \Sigma(- \otimes \mathscr{Z})$ of additive endofunctors of $K(X)$. The twisting isomorphism τ gives a natural equivalence $\mathscr{Z} \otimes - \longrightarrow - \otimes \mathscr{Z}$ which commutes with these suspension functors, so it suffices to show that $(\mathscr{Z} \otimes -, \rho)$ is a coproduct preserving triangulated functor.

Given a morphism of complexes of sheaves of modules $u : \mathscr{X} \longrightarrow \mathscr{Y}$ and the induced triangle $\mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow C_u \longrightarrow \Sigma\mathscr{X}$ it is enough to show that the following candidate triangle in $K(X)$ is a triangle

$$\mathscr{Z} \otimes \mathscr{X} \longrightarrow \mathscr{Z} \otimes \mathscr{Y} \longrightarrow \mathscr{Z} \otimes C_u \longrightarrow \Sigma(\mathscr{Z} \otimes \mathscr{X})$$

Using the isomorphism of Lemma 38 we have a commutative diagram in $K(X)$

$$\begin{array}{ccccccc} \mathscr{Z} \otimes \mathscr{X} & \longrightarrow & \mathscr{Z} \otimes \mathscr{Y} & \longrightarrow & \mathscr{Z} \otimes C_u & \longrightarrow & \Sigma(\mathscr{Z} \otimes \mathscr{X}) \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle 1}\downarrow \\ \mathscr{Z} \otimes \mathscr{X} & \longrightarrow & \mathscr{Z} \otimes \mathscr{Y} & \longrightarrow & C_{\mathscr{Z}\otimes u} & \longrightarrow & \Sigma(\mathscr{Z} \otimes \mathscr{X}) \end{array}$$

which shows that $(\mathscr{Z} \otimes -, \rho)$ is a triangulated functor. Coproducts in $K(X)$ can be calculated in $\mathbf{C}(X)$, so Lemma 37 implies that the functor $\mathscr{Z} \otimes - : K(X) \longrightarrow K(X)$ preserves coproducts. □

Remark 9. Let R be a commutative ring. There is a parallel development of the properties of the tensor product of complexes of R-modules. To be precise:

(i) For any complex X of modules the additive functors $- \otimes X, X \otimes - : \mathbf{C}(R) \longrightarrow \mathbf{C}(R)$ preserve colimits.

(ii) Let X, Y be complexes of modules. There are canonical isomorphisms of complexes natural in both variables

$$\begin{aligned} \rho : X \otimes (\Sigma Y) \longrightarrow \Sigma(X \otimes Y), &\quad \rho^n u_{ij} = (-1)^i u_{i(j+1)} \\ \sigma : (\Sigma X) \otimes Y \longrightarrow \Sigma(X \otimes Y), &\quad \sigma^n u_{ij} = u_{(i+1)j} \end{aligned}$$

which make the following diagram commute

$$\begin{array}{ccc} X \otimes (\Sigma Y) & \xrightarrow{\ \tau\ } & (\Sigma Y) \otimes X \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \Sigma(X \otimes Y) & \xrightarrow[\Sigma\tau]{} & \Sigma(Y \otimes X) \end{array}$$

(iii) Let $u : X \longrightarrow Y$ be a morphism of complexes of modules. For any complex of modules Z there is a canonical isomorphism of complexes $Z \otimes C_u \cong C_{Z \otimes u}$.

(iv) For any complex Z of sheaves of modules the pairs $(Z \otimes -, \rho), (- \otimes Z, \sigma)$ are coproduct preserving triangulated functors $K(R) \longrightarrow K(R)$. There is a canonical trinatural equivalence $\tau : Z \otimes - \longrightarrow - \otimes Z$.

(v) For complexes of modules X, Y, Z there is a canonical isomorphism of complexes $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ natural in all three variables.

(vi) Let X be a complex of modules and F any R-module. There is a canonical isomorphism of complexes of $X \otimes c(F)$ with the complex with objects $X^i \otimes F$ and differentials $\partial_X^i \otimes F$. In particular there is a canonical isomorphism of complexes $X \otimes c(R) \cong X$.

(vii) Let $A \longrightarrow B$ be a ring morphism, X a complex of A-modules and Y a complex of B-modules. Then $X \otimes_A Y$ is canonically a complex of B-modules, and we have a canonical isomorphism of complexes of B-modules $(X \otimes_A Y) \otimes_B Z \cong X \otimes_A (Y \otimes_B Z)$ for any complex Z of B-modules. This is natural in all three variables.

(viii) Let $\mathfrak{p}$ be a prime ideal and X, Y complexes of R-modules. There is a canonical isomorphism of complexes of $A_{\mathfrak{p}}$-modules $(X \otimes_A Y)_{\mathfrak{p}} \cong X_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} Y_{\mathfrak{p}}$ natural in both variables.

4.2 Hoflat Complexes

Definition 9. We say that a complex $\mathscr{F}$ of sheaves of modules is *homotopy flat* (or *hoflat*) if for any exact complex $\mathscr{Z}$ of sheaves of modules, the complex $\mathscr{Z} \otimes \mathscr{F}$ is exact (equivalently, $\mathscr{F} \otimes \mathscr{Z}$ is exact). We denote by $K(F)$ the full subcategory of $K(X)$ consisting of all the hoflat complexes. It is clear that the tensor product of two hoflat complexes is hoflat.

Lemma 40. *A complex of sheaves of modules $\mathscr{F}$ is hoflat if and only if the complex of $\mathcal{O}_{X,x}$-modules $\mathscr{F}_x$ is hoflat for every $x \in X$.*

Proof. When we say that a complex M of modules over a commutative ring R is hoflat, we mean that the additive functor $- \otimes M : \mathbf{C}(R) \longrightarrow \mathbf{C}(R)$ sends exact complexes to exact complexes. Suppose that $\mathscr{F}_x$ is a hoflat complex of modules for every $x \in X$ and let an exact complex $\mathscr{Z}$ of sheaves of modules be given. From Lemma 34 we deduce that $(\mathscr{Z} \otimes \mathscr{F})_x$ is exact for every $x \in X$. Therefore $\mathscr{Z} \otimes \mathscr{F}$ is exact and $\mathscr{F}$ is hoflat.

Conversely, suppose that $\mathscr{F}$ is hoflat, fix $x \in X$ and let M be an exact complex of $\mathcal{O}_{X,x}$-modules. The functor $Sky_x(-) : \mathcal{O}_{X,x}\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ is exact, so by hoflatness the complex $\mathscr{F} \otimes Sky_x(M)$ is exact. From Lemma 35 we deduce that the complex of modules $Sky_x(\mathscr{F}_x \otimes M)_x$ is exact. This is isomorphic to $\mathscr{F}_x \otimes M$, which is therefore exact, showing that $\mathscr{F}_x$ is hoflat. □

Lemma 41. *If $\mathscr{F}$ is a hoflat complex of sheaves of modules on X, then for any open $U \subseteq X$ the complex $\mathscr{F}|_U$ is hoflat on U.*

Definition 10. Let $\mathscr{X}$ be a complex of sheaves of modules. A *homotopy flat* (or *hoflat*) *resolution* of $\mathscr{X}$ is a quasi-isomorphism of complexes $\vartheta : \mathscr{P} \longrightarrow \mathscr{X}$ with $\mathscr{P}$ a hoflat complex.

Lemma 42. *If $\mathscr{F}$ is a hoflat complex of sheaves of modules then the functors $\mathscr{F} \otimes -$ and $- \otimes \mathscr{F}$ preserve quasi-isomorphisms of complexes.*

Proof. Follows immediately from Proposition 39. □

Lemma 43. *The full subcategory $K(F)$ is a thick localising subcategory of $K(X)$.*

Proof. That is, the hoflat complexes form a triangulated subcategory of $K(X)$ which is closed under coproducts. Let $Q : K(X) \longrightarrow \mathfrak{D}(X)$ be the verdier quotient and $\mathscr{F}$ a complex of sheaves of modules. Then $\mathscr{F}$ is hoflat if and only if for every exact complex $\mathscr{Z}$ the coproduct preserving triangulated functor $Q \circ (\mathscr{Z} \otimes -)$ vanishes on $\mathscr{F}$. The kernel of a coproduct preserving triangulated functor is a thick, localising triangulated subcategory. All of these properties are preserved under arbitrary intersections, so it is clear that $K(F)$ is a localising subcategory of $K(X)$. □

In particular, arbitrary coproducts and direct summands of hoflats in $K(X)$ are hoflat, and hoflatness is stable under isomorphism. The next result is the analogue of (DTC,Lemma 49) for flatness.

Lemma 44. *A sheaf of modules $\mathscr{F}$ is flat if and only if it is hoflat considered as a complex concentrated in degree zero.*

Proof. We say that $\mathscr{F}$ is *flat* if the functor $\mathscr{F} \otimes -$ is exact on sheaves of modules (FMS,Definition 1). We denote by $c(\mathscr{F})$ the complex concentrated in degree zero constructed from $\mathscr{F}$. Given another complex $\mathscr{X}$ of sheaves of modules, the complex $\mathscr{X} \otimes c(\mathscr{F})$ is canonically isomorphic to the following complex

$$\cdots \longrightarrow \mathscr{X}^{i-1} \otimes \mathscr{F} \xrightarrow{\partial_{\mathscr{X}}^{i-1} \otimes 1} \mathscr{X}^{i} \otimes \mathscr{F} \xrightarrow{\partial_{\mathscr{X}}^{i} \otimes 1} \mathscr{X}^{i+1} \otimes \mathscr{F} \longrightarrow \cdots$$

So it is clear that $\mathscr{F}$ is flat if and only if $c(\mathscr{F})$ is hoflat. □

Remark 10. For sheaves of modules $\mathscr{F}, \mathscr{G}$ there is a canonical isomorphism of complexes natural in both variables $c(\mathscr{F} \otimes \mathscr{G}) \cong c(\mathscr{F}) \otimes c(\mathscr{G})$.

Example 1. If R is a ring then it is a projective generator for its category of modules. In particular it is flat. For a ringed space $(X, \mathcal{O}_X)$ the corresponding objects are the generators $\{\mathcal{O}_U \,|\, U \subseteq X\}$ of (MRS,Section 1.5). These sheaves are not necessarily projective, but they are certainly flat (FMS,Example 4) and therefore hoflat as complexes.

Lemma 45. *Let $\{\mathscr{F}_s, \mu_{st}\}_{s \in \Gamma}$ be a direct system in $\mathbf{C}(X)$ such that $\mathscr{F}_s$ is hoflat for every $s \in \Gamma$. Then the direct limit $\varinjlim_{s \in \Gamma} \mathscr{F}_s$ is also hoflat.*

Proof. Given an exact complex $\mathscr{Z}$ of sheaves of modules we have by Lemma 37

$$\mathscr{Z} \otimes \varinjlim_{s \in \Gamma} \mathscr{F}_s = \varinjlim_{s \in \Gamma} (\mathscr{Z} \otimes \mathscr{F}_s)$$

Since cohomology commutes with direct limits (DF,Lemma 68) we deduce that $\mathscr{Z} \otimes \varinjlim_{s \in \Gamma} \mathscr{F}_s$ is exact, as required. □

Proposition 46. *Any bounded above complex of flat sheaves of $\mathcal{O}_X$-modules is hoflat.*

Proof. The hoflat complexes form a triangulated subcategory $K(F)$ of $K(X)$, so it follows from (DTC,Lemma 79) and Lemma 44 that any *bounded* complex of flat sheaves is hoflat. Let $\mathscr{F}$ be a bounded above complex of flat sheaves. Then as in (DTC,Definition 17) the complex $\mathscr{F}$ is the direct limit in $\mathbf{C}(X)$ of bounded complexes of flat sheaves. From Lemma 45 we deduce that $\mathscr{F}$ is hoflat. □

Lemma 47. *Let $\mathscr{F}$ be a sheaf of modules. There is a canonical epimorphism $\psi : \ell(\mathscr{F}) \longrightarrow \mathscr{F}$ of sheaves of modules with $\ell(\mathscr{F})$ flat, which is functorial in $\mathscr{F}$.*

Proof. For any open set U there is a canonical isomorphism of abelian groups $Hom(\mathcal{O}_U, \mathscr{F}) \cong \mathscr{F}(U)$ (MRS,Proposition 30). Given a section $s \in \mathscr{F}(U)$ we also write s for the corresponding morphism of sheaves of modules $s : \mathcal{O}_U \longrightarrow \mathscr{F}$. We define the following coproduct

$$\ell(\mathscr{F}) = \bigoplus_{s \in \mathscr{F}(U)} \mathcal{O}_U$$

and let $\psi : \ell(\mathscr{F}) \longrightarrow \mathscr{F}$ be defined by $\psi u_s = s$. The sheaves of modules $\mathcal{O}_U$ generate $\mathfrak{Mod}(X)$ so it follows from a standard argument that ψ is an epimorphism. Each $\mathcal{O}_U$ is flat, and arbitrary coproducts of flat sheaves are flat, so the sheaf $\ell(\mathscr{F})$ is flat.

Given a morphism of sheaves of modules $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ we define a morphism $\ell(\alpha) : \ell(\mathscr{F}) \longrightarrow \ell(\mathscr{G})$ of sheaves of modules by $\ell(\alpha)u_s = u_{\alpha_U(s)}$. It is clear that the following diagram commutes

$$\begin{array}{ccc} \ell(\mathscr{F}) & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ \ell(\mathscr{G}) & \longrightarrow & \mathscr{G} \end{array}$$

One checks that $\ell(1) = 1$ and $\ell(\beta\alpha) = \ell(\beta)\ell(\alpha)$, so the proof is complete. □

Proposition 48. *Every complex $\mathscr{X}$ of sheaves of modules has a hoflat resolution.*

Proof. Let $\mathcal{P} \subseteq \mathcal{A}$ be the class of all flat sheaves of modules. This is a smothering class for $\mathfrak{Mod}(X)$ in the sense of (DTC,Definition 30). The existence of hoflat resolutions is now a consequence of (DTC,Proposition 71) and the following two facts: any bounded above complex of flat is sheaves is hoflat, and the hoflat complexes form a localising subcategory of $K(X)$. □

Lemma 49. *If $\mathscr{F}$ is a hoflat, exact complex of sheaves of modules then $\mathscr{F} \otimes \mathscr{X}$ is exact for any complex $\mathscr{X}$ of sheaves of modules.*

Proof. Since $\mathscr{F}$ is hoflat the functor $\mathscr{F} \otimes -$ preserves quasi-isomorphisms, so by Proposition 48 we may as well assume that $\mathscr{X}$ is also hoflat. But then $\mathscr{F}$ is exact, so by definition $\mathscr{F} \otimes \mathscr{X}$ must also be exact. □

Lemma 50. *Let $\mathscr{X}$ be a complex of sheaves of modules. Then any hoflat complex $\mathscr{F}$ is left acyclic for the triangulated functors $Q \circ (- \otimes \mathscr{X}), Q \circ (\mathscr{X} \otimes -) : K(X) \longrightarrow \mathfrak{D}(X)$.*

Proof. When we say that a complex is *left acyclic* we mean with respect to the category $\mathcal{Z}$ of exact complexes in $K(X)$ (TRC,Definition 52). The key observation is the following: if $\mathscr{P} \longrightarrow \mathscr{F}$ is a quasi-isomorphism of hoflat complexes then $\mathscr{P} \otimes \mathscr{X} \longrightarrow \mathscr{F} \otimes \mathscr{X}$ is a quasi-isomorphism for any complex $\mathscr{X}$. To prove this, observe that we have a triangle in $K(X)$ with $\mathscr{Z}$ exact

$$\mathscr{P} \longrightarrow \mathscr{F} \longrightarrow \mathscr{Z} \longrightarrow \Sigma\mathscr{P}$$

Since the hoflat complexes form a triangulated subcategory of $K(X)$, we deduce that $\mathscr{Z}$ is hoflat. Tensoring with $\mathscr{X}$ we obtain another triangle

$$\mathscr{P} \otimes \mathscr{X} \longrightarrow \mathscr{F} \otimes \mathscr{X} \longrightarrow \mathscr{Z} \otimes \mathscr{X} \longrightarrow \Sigma(\mathscr{P} \otimes \mathscr{X})$$

where $\mathscr{Z} \otimes \mathscr{X}$ is exact by Lemma 49. Therefore $\mathscr{P} \otimes \mathscr{X} \longrightarrow \mathscr{F} \otimes \mathscr{X}$ is a quasi-isomorphism, as claimed. To see that a hoflat complex $\mathscr{F}$ is left acyclic for $Q \circ (- \otimes \mathscr{X})$, suppose we are given a quasi-isomorphism $\mathscr{T} \longrightarrow \mathscr{F}$. By Proposition 48 there exists a quasi-isomorphism $\mathscr{P} \longrightarrow \mathscr{T}$ with $\mathscr{P}$ hoflat and by our observation the composite $\mathscr{P} \longrightarrow \mathscr{T} \longrightarrow \mathscr{F}$ is sent to an isomorphism by $Q \circ (- \otimes \mathscr{X})$. □

Remark 11. Let R be a commutative ring. Once again the above results translate easily to the tensor product of complexes of R-modules. To be precise:

(i) If F is a hoflat complex of modules then the functors $F \otimes -$ and $- \otimes F$ preserve quasi-isomorphisms of complexes.

(ii) The hoflat complexes form a thick localising subcategory of $K(R)$.

(iii) A module F is flat if and only if it is hoflat as a complex concentrated in degree zero.

(iv) An arbitrary direct limit of hoflat complexes of modules is hoflat.

(v) Any bounded above complex of flat modules is hoflat.

(v) Every complex of modules has a hoflat resolution.

(vi) If F is an exact hoflat complex of modules then $F \otimes X$ is exact for *any* complex X of modules. It follows that any hoflat complex F is left acyclic for the triangulated functors $Q \circ (- \otimes X), Q \circ (X \otimes -) : K(R) \longrightarrow \mathfrak{D}(R)$.

Lemma 51. *Let R be a commutative ring and F a complex of R-modules. Then F is hoflat if and only if $F_\mathfrak{p}$ is hoflat as a complex of $R_\mathfrak{p}$-modules for every prime ideal $\mathfrak{p}$ of R.*

Proof. Given a prime ideal $\mathfrak{p}$ and a complex T of $R_\mathfrak{p}$-modules there is a canonical isomorphism of complexes of A-modules $N \otimes_A T \cong N_\mathfrak{p} \otimes_{A_\mathfrak{p}} T$ for any complex N of A-modules. Using this observation the claim is straightforward to check. □

Lemma 52. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{F}$ a hoflat complex of sheaves of modules on Y. Then*

(i) The complex $f^*\mathscr{F}$ is hoflat.

(ii) If further $\mathscr{F}$ is exact then $f^*\mathscr{F}$ is exact.

Proof. (i) By Lemma 40 it suffices to show that $(f^*\mathscr{F})_x$ is hoflat for every $x \in X$. In other words, we have to show that the complex $T = \mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ of $\mathcal{O}_{X,x}$-modules is hoflat. We have therefore reduced to proving the following statement of commutative algebra: if $A \longrightarrow B$ is a ring morphism and F a hoflat complex of A-modules then $F \otimes_A B$ is a hoflat complex of B-modules. This follows from the isomorphism of complexes $(F \otimes_A B) \otimes_B Z \cong F \otimes_A (B \otimes_B Z) \cong F \otimes_A Z$ for any exact complex Z of B-modules.

(ii) It suffices to show that T is exact for every $x \in X$. But $\mathscr{F}_{f(x)}$ is an exact hoflat complex of $\mathcal{O}_{Y,f(x)}$-modules by Lemma 40, so from Remark 11(vi) we deduce that the complex T is exact. □

4.3 Derived Tensor Product

Definition 11. An *assignment of hoflat resolutions for X* is an assignment to every complex of sheaves of modules $\mathscr{X}$ of a hoflat complex $F_\mathscr{X}$ and a quasi-isomorphism of complexes $F_\mathscr{X} \longrightarrow \mathscr{X}$.

Definition 12. Let $\mathscr{X}$ be a complex of sheaves of modules. From Lemma 50 and (TRC,Theorem 125) we infer that the triangulated functor $Q \circ (\mathscr{X} \otimes -) : K(X) \longrightarrow \mathfrak{D}(X)$ has a left derived functor

$$\mathscr{X} \underline{\underline{\otimes}} - : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$

To be precise, for each assignment $\mathcal{F}$ of hoflat resolutions for X we have a canonical left derived functor $\mathscr{X} \underline{\underline{\otimes}}^\mathcal{F} -$ of $Q \circ (\mathscr{X} \otimes -)$. In particular $\mathscr{X} \underline{\underline{\otimes}}^\mathcal{F} \mathscr{Y} = \mathscr{X} \otimes F_\mathscr{Y}$, where $F_\mathscr{Y} \longrightarrow \mathscr{Y}$ is the chosen hoflat resolution.

We use the notation of Definition 12 and fix an assignment $\mathcal{F}$ of hoflat resolutions. Given a morphism $\psi : \mathscr{X} \longrightarrow \mathscr{X}'$ in $K(X)$ we can define a trinatural transformation

$$\psi \otimes - : \mathscr{X} \otimes - \longrightarrow \mathscr{X}' \otimes -$$
$$(\psi \otimes -)_\mathscr{Y} = \psi \otimes \mathscr{Y}$$

This gives rise to a trinatural transformation $Q(\psi \otimes -) : Q(\mathscr{X} \otimes -) \longrightarrow Q(\mathscr{X}' \otimes -)$ which by (TRC,Definition 53) induces a canonical trinatural transformation

$$\psi \underline{\underline{\otimes}}^\mathcal{F} - : \mathscr{X} \underline{\underline{\otimes}}^\mathcal{F} - \longrightarrow \mathscr{X}' \underline{\underline{\otimes}}^\mathcal{F} -$$

which by (TRC,Lemma 127) must have the form $\psi \underline{\underline{\otimes}}^\mathcal{F} \mathscr{Y} = Q(\psi \otimes F_\mathscr{Y})$ where $F_\mathscr{Y} \longrightarrow \mathscr{Y}$ is the chosen hoflat resolution of $\mathscr{Y}$. Moreover we have

$$(\psi' \underline{\underline{\otimes}}^\mathcal{F} -)(\psi \underline{\underline{\otimes}}^\mathcal{F} -) = \psi'\psi \underline{\underline{\otimes}}^\mathcal{F} -$$
$$(\psi + \psi') \underline{\underline{\otimes}}^\mathcal{F} - = (\psi \underline{\underline{\otimes}}^\mathcal{F} -) + (\psi' \underline{\underline{\otimes}}^\mathcal{F} -)$$
$$1 \underline{\underline{\otimes}}^\mathcal{F} - = 1$$

For any complex $\mathscr{Y}$ of sheaves of modules we write $R_{\mathscr{Y}}$ for the additive functor $K(X) \longrightarrow \mathfrak{D}(X)$ defined on objects by $R_{\mathscr{Y}}(\mathscr{X}) = \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y}$ and on a morphism $\psi : \mathscr{X} \longrightarrow \mathscr{X}'$ by $R_{\mathscr{Y}}(\psi) = \psi \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y}$. In fact this is equal as an additive functor to the composite $Q(- \otimes F_{\mathscr{Y}}) : K(X) \longrightarrow \mathfrak{D}(X)$, so $R_{\mathscr{Y}}$ becomes by Lemma 39 a triangulated functor in a canonical way. Since $F_{\mathscr{Y}}$ is hoflat, the functor $R_{\mathscr{Y}}$ contains the exact complexes of $K(X)$ in its kernel, and therefore induces a triangulated functor

$$- \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$

Lemma 53. *For morphisms $\varphi : \mathscr{Y} \longrightarrow \mathscr{Y}'$ and $\psi : \mathscr{X} \longrightarrow \mathscr{X}'$ in $\mathfrak{D}(X)$ we have*

$$(\psi \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y}')(\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \varphi) = (\mathscr{X}' \underline{\underline{\otimes}}^{\mathcal{F}} \varphi)(\psi \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y})$$

Proof. This is straightforward but tedious. See the proof of (DTC2,Lemma 24) for the technique. □

Definition 13. Let $(X, \mathcal{O}_X)$ be a ringed space. Then for every assignment $\mathcal{F}$ of hoflat resolutions for X there is a canonical functor additive in each variable

$$- \underline{\underline{\otimes}}^{\mathcal{F}} - : \mathfrak{D}(X) \times \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$

with $\varphi \underline{\underline{\otimes}}^{\mathcal{F}} \psi$ defined to be the equal composites of Lemma 53. The partial functors in each variable are triangulated functors in a canonical way. To be explicit, for complexes $\mathscr{X}, \mathscr{Y}$ we have

$$\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} = \mathscr{X} \otimes F_{\mathscr{Y}}$$

As part of the data we have a morphism in $\mathfrak{D}(X)$ trinatural in both variables

$$\zeta : \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} \longrightarrow \mathscr{X} \otimes \mathscr{Y}$$

which is an isomorphism if either of $\mathscr{X}, \mathscr{Y}$ is hoflat.

Remark 12. With the notation of Definition 13 the partial functors $\mathscr{X} \underline{\underline{\otimes}} -$ and $- \underline{\underline{\otimes}} \mathscr{Y}$ are canonically $\Gamma(X, \mathcal{O}_X)$-linear triangulated functors, and moreover these triangulated structures are compatible. That is, the isomorphisms in $\mathfrak{D}(X)$

$$\mathscr{X} \underline{\underline{\otimes}} (\Sigma \mathscr{Y}) \cong \Sigma(\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y})$$
$$(\Sigma \mathscr{X}) \underline{\underline{\otimes}} \mathscr{Y} \cong \Sigma(\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y})$$

are natural in both variables. The structure sheaf $\mathcal{O}_X$ is also a unit for the tensor product, in the sense that the triangulated functors $\mathcal{O}_X \underline{\underline{\otimes}} -$ and $- \underline{\underline{\otimes}} \mathcal{O}_X$ are canonically trinaturally equivalent to the identity.

Lemma 54. *For complexes $\mathscr{X}, \mathscr{Y}$ of sheaves of modules there is a canonical isomorphism in $\mathfrak{D}(X)$ trinatural in both variables*

$$\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y} \longrightarrow \mathscr{Y} \underline{\underline{\otimes}} \mathscr{X}$$

In particular we have a canonical trinatural equivalence $\mathscr{X} \underline{\underline{\otimes}} - \longrightarrow - \underline{\underline{\otimes}} \mathscr{X}$.

Proof. Fix an assignment of hoflat resolutions $\mathcal{F}$ and let $F_{\mathscr{Y}} \longrightarrow \mathscr{Y}$ be the chosen hoflat resolution of $\mathscr{Y}$. Using ζ we have a canonical isomorphism in $\mathfrak{D}(X)$

$$\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} = \mathscr{X} \otimes F_{\mathscr{Y}} \cong F_{\mathscr{Y}} \otimes \mathscr{X} \cong F_{\mathscr{Y}} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{X} \cong \mathscr{Y} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{X}$$

which one checks is natural in both variables, with respect to morphisms of $\mathfrak{D}(X)$. It is worth checking that the composite $\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y} \longrightarrow \mathscr{Y} \underline{\underline{\otimes}} \mathscr{X} \longrightarrow \mathscr{X} \underline{\underline{\otimes}} \mathscr{Y}$ actually is the identity. Trinaturality in each variable is straightforward to verify. Observe that the diagram

$$\begin{array}{ccc} \mathscr{X} \underline{\underline{\otimes}} \mathscr{Y} & \longrightarrow & \mathscr{Y} \underline{\underline{\otimes}} \mathscr{X} \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \zeta} \\ \mathscr{X} \otimes \mathscr{Y} & \longrightarrow & \mathscr{Y} \otimes \mathscr{X} \end{array}$$

commutes in $\mathfrak{D}(X)$. □

Lemma 55. *For complexes $\mathscr{X}, \mathscr{Y}$ of sheaves of modules and $U \subseteq X$ open there is a canonical isomorphism in $\mathfrak{D}(U)$ natural in both variables*

$$\delta : (\mathscr{X} \underline{\otimes} \mathscr{Y})|_U \longrightarrow \mathscr{X}|_U \underline{\otimes} \mathscr{Y}|_U$$

Proof. If $\zeta : F_{\mathscr{Y}} \longrightarrow \mathscr{Y}$ is the chosen hoflat resolution of $\mathscr{Y}$ then $\zeta|_U$ is a hoflat resolution of $\mathscr{Y}|_U$ so using Lemma 33 we have a canonical isomorphism in $\mathfrak{D}(U)$

$$(\mathscr{X} \underline{\otimes} \mathscr{Y})|_U = (\mathscr{X} \otimes F_{\mathscr{Y}})|_U \cong \mathscr{X}|_U \otimes F_{\mathscr{Y}}|_U \cong \mathscr{X}|_U \underline{\otimes} F_{\mathscr{Y}}|_U \cong \mathscr{X}|_U \underline{\otimes} \mathscr{Y}|_U$$

which one checks is natural in both variables with respect to morphisms of $\mathfrak{D}(X)$. Observe that by construction the following diagram commutes

$$\begin{array}{ccc} (\mathscr{X} \underline{\otimes} \mathscr{Y})|_U & \xrightarrow{\delta} & \mathscr{X}|_U \underline{\otimes} \mathscr{Y}|_U \\ {\scriptstyle \zeta|_U}\downarrow & & \downarrow{\scriptstyle \zeta} \\ (\mathscr{X} \otimes \mathscr{Y})|_U & \longrightarrow & \mathscr{X}|_U \otimes \mathscr{Y}|_U \end{array}$$

where the morphisms ζ are the canonical morphisms forming part of the definition of the derived tensor products. □

Lemma 56. *For complexes $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ of sheaves of modules there is a canonical isomorphism in $\mathfrak{D}(X)$ trinatural in all three variables*

$$(\mathscr{X} \underline{\otimes} \mathscr{Y}) \underline{\otimes} \mathscr{Z} \longrightarrow \mathscr{X} \underline{\otimes} (\mathscr{Y} \underline{\otimes} \mathscr{Z})$$

Proof. We have using Lemma 36 a canonical isomorphism in $\mathfrak{D}(X)$

$$\begin{aligned} (\mathscr{X} \underline{\otimes} \mathscr{Y}) \underline{\otimes} \mathscr{Z} &= (\mathscr{X} \otimes F_{\mathscr{Y}}) \otimes F_{\mathscr{Z}} \\ &\cong \mathscr{X} \otimes (F_{\mathscr{Y}} \otimes F_{\mathscr{Z}}) \\ &\cong \mathscr{X} \underline{\otimes} (F_{\mathscr{Y}} \otimes F_{\mathscr{Z}}) \\ &\cong \mathscr{X} \underline{\otimes} (F_{\mathscr{Y}} \underline{\otimes} F_{\mathscr{Z}}) \\ &\cong \mathscr{X} \underline{\otimes} (\mathscr{Y} \underline{\otimes} \mathscr{Z}) \end{aligned}$$

since the tensor product $F_{\mathscr{Y}} \otimes F_{\mathscr{Z}}$ of two hoflat complexes is hoflat. It is easy to check that this isomorphism is natural in all three variables, with respect to morphisms of $\mathfrak{D}(X)$. Observe that the following diagram commutes

$$\begin{array}{ccc} (\mathscr{X} \underline{\otimes} \mathscr{Y}) \underline{\otimes} \mathscr{Z} & \longrightarrow & \mathscr{X} \underline{\otimes} (\mathscr{Y} \underline{\otimes} \mathscr{Z}) \\ \downarrow & & \downarrow \\ (\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z} & \longrightarrow & \mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z}) \end{array} \tag{4}$$

To check trinaturality in all three variables, first reduce to $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ hoflat. Then use the compatibility diagram (4) to reduce to trinaturality of the isomorphism $(\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z} \cong \mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z})$ which we have already checked. □

There are many compatibility diagrams between the associator, twist and restriction isomorphisms for the derived tensor product that one could conceivably want to check. Rather than write them all down here, we leave most up to the reader to check when they become necessary. The technique is always the same: use the compatibility diagrams between $- \underline{\otimes} -$ and $- \otimes -$ to reduce to the ordinary tensor product of complexes, in which case these compatibilities are easily checked.

The experienced reader can avoid some of these verifications by appealing to a suitable coherence theorem for monoidal categories, but when we begin to mix $- \underline{\otimes} -$ and derived functors such as $\mathbb{R}f_*, \mathbb{L}f^*$ such coherence theorems will be unavailable.

Remark 13. Let R be a commutative ring. In exactly the same way as we defined the derived tensor product on $\mathfrak{Mod}(X)$ above, we define the derived tensor product on $R\mathbf{Mod}$. That is, for every assignment of hoflat resolutions for $R\mathbf{Mod}$ we have a canonical functor additive in each variable

$$- \underline{\underline{\otimes}}^{\mathcal{F}} - : \mathfrak{D}(R) \times \mathfrak{D}(R) \longrightarrow \mathfrak{D}(R)$$

triangulated in each variable, defined by $X \underline{\underline{\otimes}}^{\mathcal{F}} Y = X \otimes F_Y$. We have a morphism in $\mathfrak{D}(R)$ trinatural in both variables

$$\zeta : X \underline{\underline{\otimes}}^{\mathcal{F}} Y \longrightarrow X \otimes Y$$

which is an isomorphism if either of X, Y is hoflat.

Lemma 57. *For complexes $\mathscr{X}, \mathscr{Y}$ of sheaves of modules and a point $x \in X$ there is a canonical isomorphism in $\mathfrak{D}(\mathcal{O}_{X,x})$ natural in both variables*

$$(\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y})_x \longrightarrow \mathscr{X}_x \underline{\underline{\otimes}} \mathscr{Y}_x$$

Proof. We have using Lemma 34 a canonical isomorphism in $\mathfrak{D}(\mathcal{O}_{X,x})$

$$\begin{aligned}(\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y})_x = (\mathscr{X} \otimes F_{\mathscr{Y}})_x &\cong \mathscr{X}_x \otimes (F_{\mathscr{Y}})_x \\ &\cong \mathscr{X}_x \underline{\underline{\otimes}} (F_{\mathscr{Y}})_x \cong \mathscr{X}_x \underline{\underline{\otimes}} \mathscr{Y}_x\end{aligned}$$

since $(F_{\mathscr{Y}})_x$ is hoflat. Naturality in both variables with respect to morphisms of $\mathfrak{D}(X)$ is easily checked. Observe that by construction the following diagram commutes

$$\begin{array}{ccc} (\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y})_x & \longrightarrow & \mathscr{X}_x \underline{\underline{\otimes}} \mathscr{Y}_x \\ {\scriptstyle \zeta_x}\downarrow & & \downarrow{\scriptstyle \zeta} \\ (\mathscr{X} \otimes \mathscr{Y})_x & \longrightarrow & \mathscr{X}_x \otimes \mathscr{Y}_x \end{array}$$

where the vertical morphisms are the canonical ones. □

Lemma 58. *Let $x \in X$ be a point, $\mathscr{F}$ a complex of sheaves of modules and M a complex of $\mathcal{O}_{X,x}$-modules. There is a canonical morphism in $\mathfrak{D}(X)$ natural in both variables*

$$\lambda : \mathscr{F} \underline{\underline{\otimes}}_{\mathcal{O}_X} Sky_x(M) \longrightarrow Sky_x(\mathscr{F}_x \underline{\underline{\otimes}}_{\mathcal{O}_{X,x}} M)$$

with λ_x an isomorphism in $\mathfrak{D}(\mathcal{O}_{X,x})$. If x is a closed point then λ is an isomorphism.

Proof. Choose an isomorphism $\mathscr{H} \cong \mathscr{F}$ in $\mathfrak{D}(X)$ with $\mathscr{H}$ hoflat. Using Lemma 35 we have a morphism in $\mathfrak{D}(X)$

$$\begin{aligned}\mathscr{F} \underline{\underline{\otimes}}_{\mathcal{O}_X} Sky_x(M) &\cong \mathscr{H} \underline{\underline{\otimes}}_{\mathcal{O}_X} Sky_x(M) \cong \mathscr{H} \otimes_{\mathcal{O}_X} Sky_x(M) \\ &\longrightarrow Sky_x(\mathscr{H}_x \otimes_{\mathcal{O}_{X,x}} M) \cong Sky_x(\mathscr{H}_x \underline{\underline{\otimes}}_{\mathcal{O}_{X,x}} M) \cong Sky_x(\mathscr{F}_x \underline{\underline{\otimes}}_{\mathcal{O}_{X,x}} M)\end{aligned}$$

which one checks is independent of the chosen isomorphism, and is therefore canonical. Naturality in both variables is easily checked, and λ_x is an isomorphism by virtue of the analogous statement in Lemma 35. The last claim is immediate. □

4.4 Change of Rings

Throughout this section let R, S be commutative rings and B an R-S-bimodule. Then for any R-module M the module $M \otimes_R B$ is an R-S-bimodule in a canonical way, and this defines an additive functor $- \otimes_R B : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$.

Now suppose that B is a *complex* of R-S-bimodules, by which we mean that each B^i is an R-S-bimodule and the differentials $B^i \longrightarrow B^{i+1}$ are morphisms of both R-modules and S-modules.

Given a complex M of R-modules, the complex $M \otimes_R B$ of R-modules is a complex of R-S-bimodules in a canonical way, and this defines an additive functor

$$- \otimes_R B : \mathbf{C}(R) \longrightarrow \mathbf{C}(S)$$

called the *change of rings* functor. This clearly extends to a triangulated functor $K(R) \longrightarrow K(S)$, and any hoflat complex of R-modules is acyclic for $Q \circ (- \otimes_R B)$ so we can define as above a derived change of rings functor

$$- \underline{\underline{\otimes}}_R B : \mathfrak{D}(R) \longrightarrow \mathfrak{D}(S)$$

For any complex M of R-modules we have a canonical morphism in $\mathfrak{D}(S)$ trinatural in M

$$M \underline{\underline{\otimes}}_R B \longrightarrow M \otimes_R B$$

which is an isomorphism if M is hoflat.

4.5 Tor for Sheaves

Definition 14. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ sheaves of modules. For $n \in \mathbb{Z}$ and an assignment of hoflat resolutions $\mathcal{F}$ we have a canonically defined sheaf of modules

$$\mathscr{T}\!or_n^{\mathcal{F}}(\mathscr{F}, \mathscr{G}) = H^{-n}(\mathscr{F} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{G})$$

which is covariant and additive in both variables. Usually we will drop the hoflat assignment from the notation, and call $\mathscr{T}\!or_n(\mathscr{F}, \mathscr{G})$ the *nth Tor sheaf* of the pair $\mathscr{F}, \mathscr{G}$. Replacing $\mathscr{F}$ with a resolution $\mathscr{X}$ by flat sheaves, we find that $\mathscr{T}\!or_n(\mathscr{F}, \mathscr{G})$ is canonically isomorphic to the nth homology of the chain complex

$$\cdots \longrightarrow \mathscr{X}^2 \otimes \mathscr{G} \longrightarrow \mathscr{X}^1 \otimes \mathscr{G} \longrightarrow \mathscr{X}^0 \otimes \mathscr{G} \longrightarrow 0 \longrightarrow \cdots$$

Similarly replacing $\mathscr{G}$ with a flat resolution $\mathscr{Y}$ we find that $\mathscr{T}\!or_n(\mathscr{F}, \mathscr{G})$ is canonically isomorphic to the nth homology of the chain complex

$$\cdots \longrightarrow \mathscr{F} \otimes \mathscr{Y}^2 \longrightarrow \mathscr{F} \otimes \mathscr{Y}^1 \longrightarrow \mathscr{F} \otimes \mathscr{Y}^0 \longrightarrow 0 \longrightarrow \cdots$$

In particular it is clear that $\mathscr{T}\!or_n(\mathscr{F}, \mathscr{G}) = 0$ for $n < 0$. As part of the definition of the derived tensor we have a morphism in $\mathfrak{D}(X)$ natural in both variables

$$\mathscr{F} \underline{\underline{\otimes}} \mathscr{G} \longrightarrow \mathscr{F} \otimes \mathscr{G}$$

Taking cohomology we have a canonical morphism of sheaves of modules $\mathscr{T}\!or_0(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{F} \otimes \mathscr{G}$ natural in both variables, which by replacing either sheaf by a flat resolution one can check is an isomorphism. Given a short exact sequence $0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$ there is by (DTC,Proposition 20) a canonical morphism $z : \mathscr{G}'' \longrightarrow \Sigma\mathscr{G}'$ in $\mathfrak{D}(X)$ fitting into a triangle

$$\mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \xrightarrow{-z} \Sigma\mathscr{G}'$$

For $n \geq 0$ we have a canonical morphism $\omega^{n+1} : \mathscr{T}\!or_{n+1}(\mathscr{F}, \mathscr{G}'') \longrightarrow \mathscr{T}\!or_n(\mathscr{F}, \mathscr{G}')$ defined to be $H^{-n-1}(\phi \circ (\mathscr{F} \underline{\underline{\otimes}} -z))$ where $\phi : \mathscr{F} \underline{\underline{\otimes}} \Sigma\mathscr{G}' \longrightarrow \Sigma(\mathscr{F} \underline{\underline{\otimes}} \mathscr{G}')$ is canonical. These connecting morphisms fit into a long exact sequence

$$\begin{array}{l}\cdots \longrightarrow \mathscr{T}\!or_2(\mathscr{F}, \mathscr{G}') \longrightarrow \mathscr{T}\!or_2(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{T}\!or_2(\mathscr{F}, \mathscr{G}'') \longrightarrow \\ \mathscr{T}\!or_1(\mathscr{F}, \mathscr{G}') \longrightarrow \mathscr{T}\!or_1(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{T}\!or_1(\mathscr{F}, \mathscr{G}'') \longrightarrow \\ \mathscr{F} \otimes \mathscr{G}' \longrightarrow \mathscr{F} \otimes \mathscr{G} \longrightarrow \mathscr{F} \otimes \mathscr{G}'' \longrightarrow 0\end{array}$$

Similarly a short exact sequence in the first variable induces a long exact sequence of $\mathscr{T}\!or(-,-)$ sheaves. Both long exact sequences are natural in the short exact sequence and also the sheaf in the other variable.

Remark 14. Let R be a commutative ring. Using the derived tensor structure on $\mathfrak{D}(R)$ we can introduce the R-modules $\underline{T}or_n^{\mathcal{F}}(M,N) = H^{-n}(M \underline{\otimes}^{\mathcal{F}} N)$ for $n \in \mathbb{Z}$ as above. All the same statements go through. In fact this definition is technically more convenient than the classical one given in terms of projective resolutions.

Lemma 59. *For a commutative ring R there is a canonical natural equivalence of bifunctors $\underline{T}or_n(-,-) \cong Tor_n(-,-)$ where the second bifunctor is defined classically (TOR,Definition 1).*

Proof. Fix an assignment of projective resolutions for R**Mod** and define the bifunctor $Tor_n(-,-)$: R**Mod**$\times R$**Mod** $\longrightarrow R$**Mod** as in (TOR,Section 5). An assignment of hoflat resolutions for R**Mod** also fixes a bifunctor $\underline{T}or_n(-,-)$. Given a pair of R-modules M,N let P be the chosen projective resolution of M. We have a canonical isomorphism

$$\begin{aligned}\underline{T}or_n(M,N) &= H^{-n}(M \underline{\otimes} N) \cong H^{-n}(P \underline{\otimes} N)\\ &\cong H^{-n}(P \otimes N) \cong H_n(\cdots \longrightarrow P_1 \otimes N \longrightarrow P_0 \otimes N \longrightarrow 0 \longrightarrow \cdots)\\ &= Tor_n(M,N)\end{aligned}$$

This is clearly natural in both variables. □

Lemma 60. *Let $(X,\mathcal{O}_X)$ be a ringed space and $\mathscr{F},\mathscr{G}$ sheaves of modules. For $n \geq 0$ and $x \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules natural in both variables*

$$\mathscr{T}or_n(\mathscr{F},\mathscr{G})_x \longrightarrow \underline{T}or_n(\mathscr{F}_x,\mathscr{G}_x)$$

Proof. Using Lemma 57 we have a canonical isomorphism

$$\begin{aligned}\mathscr{T}or_n(\mathscr{F},\mathscr{G})_x &= H^{-n}(\mathscr{F} \underline{\otimes} \mathscr{G})_x \cong H^{-n}((\mathscr{F} \underline{\otimes} \mathscr{G})_x)\\ &\cong H^{-n}(\mathscr{F}_x \underline{\otimes} \mathscr{G}_x) = \underline{T}or_n(\mathscr{F}_x,\mathscr{G}_x)\end{aligned}$$

natural in both variables, as required. □

Lemma 61. *Let $(X,\mathcal{O}_X)$ be a ringed space, $U \subseteq X$ an open subset and $\mathscr{F},\mathscr{G}$ sheaves of modules on X. For $n \geq 0$ there is a canonical isomorphism of sheaves of modules natural in both variables*

$$\mathscr{T}or_n^X(\mathscr{F},\mathscr{G})|_U \longrightarrow \mathscr{T}or_n^U(\mathscr{F}|_U,\mathscr{G}|_U)$$

Proof. Using Lemma 55 we have a canonical isomorphism

$$\begin{aligned}\mathscr{T}or_n^X(\mathscr{F},\mathscr{G})|_U &= H^{-n}(\mathscr{F} \underline{\otimes} \mathscr{G})|_U \cong H^{-n}((\mathscr{F} \underline{\otimes} \mathscr{G})|_U)\\ &\cong H^{-n}(\mathscr{F}|_U \underline{\otimes} \mathscr{G}|_U) = \mathscr{T}or_n^U(\mathscr{F}|_U,\mathscr{G}|_U)\end{aligned}$$

natural in both variables, as required. □

Definition 15. Let $(X,\mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules. Given an integer $n \geq -1$ we say that $flat.dim.\mathscr{F} \leq n$ if for every $\mathscr{G} \in \mathfrak{Mod}(X)$ we have

$$\mathscr{T}or_i(\mathscr{F},\mathscr{G}) = 0, \qquad \forall i > n$$

Clearly $\mathscr{F} = 0$ if and only if $flat.dim.\mathscr{F} \leq -1$. If no integer $n \geq -1$ exists with $flat.dim.\mathscr{F} \leq n$ then we set $flat.dim.\mathscr{F} = \infty$. Otherwise there is a least such n and we define $flat.dim.\mathscr{F} = n$. Then for $n \geq 0$ we have $flat.dim.\mathscr{F} = n$ if and only if $\mathscr{T}or_i(\mathscr{F},-)$ is zero for $i > n$ but some $\mathscr{T}or_n(\mathscr{F},\mathscr{G})$ is nonzero. So $flat.dim.\mathscr{F}$ is an element of the set $\{-1,0,1,\ldots,\infty\}$ and is equal to -1 if and only if $\mathscr{F}$ is zero.

We say that a sheaf of modules $\mathscr{F}$ has *finite flat dimension* if $flat.dim.\mathscr{F} < \infty$. Using the long exact $\mathscr{T}or(-,-)$ sequences it is clear that $\mathscr{F}$ is flat if and only if $flat.dim.\mathscr{F} \leq 0$.

Throughout this section let $(X,\mathcal{O}_X)$ be a fixed ringed space. Unless specified otherwise all sheaves of modules are over X. Our first observation is that the usual dimension shifting arguments go through.

Lemma 62. *Suppose we have an exact sequence of sheaves of modules*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{P}_m \longrightarrow \cdots \longrightarrow \mathscr{P}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

with each $\mathscr{P}_i$ flat. For $n > 0$ and any sheaf of modules $\mathscr{G}$ there is a canonical isomorphism $\mathscr{T}or_{n+m+1}(\mathscr{F},\mathscr{G}) \cong \mathscr{T}or_n(\mathscr{F}',\mathscr{G})$ natural in $\mathscr{G}$ and also in the exact sequence. There is a canonical exact sequence

$$0 \longrightarrow \mathscr{T}or_{m+1}(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{F}' \otimes \mathscr{G} \longrightarrow \mathscr{P}_m \otimes \mathscr{G}$$

natural in both $\mathscr{G}$ and the exact sequence.

Proof. Dividing the long exact sequence up into a series of short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{P}_m \longrightarrow \mathscr{K}_m \longrightarrow 0 \\ 0 \longrightarrow \mathscr{K}_m \longrightarrow \mathscr{P}_{m-1} \longrightarrow \mathscr{K}_{m-1} \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow \mathscr{K}_1 \longrightarrow \mathscr{P}_0 \longrightarrow \mathscr{F} \longrightarrow 0 \end{gathered}$$

we can reduce to the case where $m = 0$. The exact sequence therefore induces a long exact sequence on $\mathscr{T}or(-,-)$ involving pieces of the form

$$\cdots \longrightarrow \mathscr{T}or_{n+1}(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{T}or_n(\mathscr{F}',\mathscr{G}) \longrightarrow \mathscr{T}or_n(\mathscr{P}_0,\mathscr{G}) \longrightarrow \mathscr{T}or_n(\mathscr{F},\mathscr{G}) \longrightarrow \cdots$$

By hypothesis the sheaves $\mathscr{T}or_n(\mathscr{P}_0,\mathscr{G})$ are zero for $n > 0$ and we have therefore a canonical isomorphism $\mathscr{T}or_{n+1}(\mathscr{F},\mathscr{G}) \cong \mathscr{T}or_n(\mathscr{F}',\mathscr{G})$, as claimed. This is clearly natural in both the exact sequence and $\mathscr{G}$. The second claim also follows from this long exact sequence, so the proof is complete. □

Proposition 63. *Given a sheaf of modules $\mathscr{F}$ and $n \geq 0$ the following conditions are equivalent:*

(a) $flat.dim.\mathscr{F} \leq n$.

(b) *$\mathscr{T}or_{n+1}(\mathscr{F},-)$ is the zero functor.*

(c) *$\mathscr{T}or_n(\mathscr{F},-)$ is left exact.*

(d) *For any exact sequence of sheaves of modules*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{P}_{n-1} \longrightarrow \cdots \longrightarrow \mathscr{P}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

if each $\mathscr{P}_i$ is flat then so is $\mathscr{F}'$.

(e) *There is an exact sequence of sheaves of modules with each $\mathscr{P}_i$ flat*

$$0 \longrightarrow \mathscr{P}_n \longrightarrow \cdots \longrightarrow \mathscr{P}_0 \longrightarrow \mathscr{F} \longrightarrow 0 \tag{5}$$

Proof. $(a) \Rightarrow (b)$ is trivial and $(b) \Rightarrow (c)$ follows from the long exact sequence. For $(c) \Rightarrow (d)$ suppose we are given an exact sequence of the stated form (we may assume $n > 0$ since otherwise (d) is trivial). Let $\mathscr{G} \longrightarrow \mathscr{G}'$ be a monomorphism of sheaves of modules. From the dimension shifting lemma (Lemma 62) we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathscr{T}or_n(\mathscr{F},\mathscr{G}) & \longrightarrow & \mathscr{F}' \otimes \mathscr{G} & \longrightarrow & \mathscr{P}_{n-1} \otimes \mathscr{G} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{T}or_n(\mathscr{F},\mathscr{G}') & \longrightarrow & \mathscr{F}' \otimes \mathscr{G}' & \longrightarrow & \mathscr{P}_{n-1} \otimes \mathscr{G}' \end{array}$$

Applying the Five Lemma we deduce that $\mathscr{F}' \otimes \mathscr{G} \longrightarrow \mathscr{F}' \otimes \mathscr{G}'$ is a monomorphism so $\mathscr{F}'$ is flat. $(d) \Rightarrow (e)$ Take any flat resolution $\mathscr{P}$ of $\mathscr{F}$ and apply (d) to the image of $\mathscr{P}_n \longrightarrow \mathscr{P}_{n-1}$. $(e) \Rightarrow (a)$ follows by dimension shifting. □

Remark 15. In particular, the flat dimension of any nonzero sheaf of modules $\mathscr{F}$ is the least integer $n \geq 0$ for which there exists a flat resolution (5) of length n and is infinite if and only if no such finite resolution exists.

Remark 16. Let $\mathscr{F}$ be a nonzero sheaf of modules and take an arbitrary, infinite flat resolution of $\mathscr{F}$. With the kernels included, the relevant diagram is of the form

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & \mathscr{P}_2 & & \longrightarrow & & \mathscr{P}_1 & & \longrightarrow & & \mathscr{P}_0 & \longrightarrow & \mathscr{F} \longrightarrow 0 \\ & & & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\ & & & & \mathscr{K}_2 & & & & \mathscr{K}_1 & & & & \end{array}$$

From Proposition 63(d) we learn the following: consider the sequence of sheaves of modules

$$\mathscr{F} = \mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2, \ldots$$

Either none of these are flat, in which case we set $n = \infty$, or there is a least integer $n \geq 0$ such that $\mathscr{K}_n$ is flat, and moreover every $\mathscr{K}_i$ with $i \geq n$ is then by necessity also flat. Then $flat.dim.\mathscr{F} = n$ and if this is finite we have a flat resolution of $\mathscr{F}$ of minimal length

$$0 \longrightarrow \mathscr{K}_n \longrightarrow \mathscr{P}_{n-1} \longrightarrow \cdots \longrightarrow \mathscr{P}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

In other words, *any* flat resolution contains within it a minimal flat resolution, and you can read the flat dimension of $\mathscr{F}$ of an arbitrary flat resolution by looking at the sequence of kernels.

Lemma 64. *Suppose we have an exact sequence of sheaves of modules*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{P}_m \longrightarrow \cdots \longrightarrow \mathscr{P}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

with each $\mathscr{P}_i$ flat. Then $flat.dim.\mathscr{F} \leq flat.dim.\mathscr{F}' + m + 1$ with equality provided $\mathscr{F}'$ is not flat.

Proof. The inequality is trivial unless $\mathscr{F}$ is nonzero and $flat.dim.\mathscr{F}'$ is finite. Given $i > flat.dim.\mathscr{F}' + m + 1$ and a sheaf of modules $\mathscr{G}$ we have $\mathscr{T}or_i(\mathscr{F}, \mathscr{G}) \cong \mathscr{T}or_{i-m-1}(\mathscr{F}', \mathscr{G}) = 0$ by Lemma 62 as required. Now suppose that $\mathscr{F}'$ is not flat, and take an arbitrary flat resolution

$$\cdots \longrightarrow \mathscr{P}_{m+2} \longrightarrow \mathscr{P}_{m+1} \longrightarrow \mathscr{F}' \longrightarrow 0 \tag{6}$$

Attaching this to the original exact sequence we have an infinite flat resolution of $\mathscr{F}$.If $flat.dim.\mathscr{F}'$ is infinite then none of the kernels in this infinite resolution can be flat, so by Remark 16 we have $proj.dim.\mathscr{F} = \infty$ also.

If $flat.dim.\mathscr{F}'$ is finite then we have identified the position at which all the kernels of (6) become flat. Passing to the longer resolution of $\mathscr{F}$ we conclude that $flat.dim.\mathscr{F} = flat.dim.\mathscr{F}' + m + 1$ as required. Alternatively one can prove this directly using Lemma 62. $\square$

Lemma 65. *Given a sheaf of modules $\mathscr{F}$ we have*

$$flat.dim.\mathscr{F} = \sup\{flat.dim_{\mathcal{O}_{X,x}}\mathscr{F}_x \,|\, x \in X\}$$

Proof. To prove the inequality $\leq$ we may assume $e = \sup\{flat.dim_{\mathcal{O}_{X,x}}\mathscr{F}_x \,|\, x \in X\}$ is finite and we have to show that $\mathscr{T}or_n(\mathscr{F}, \mathscr{G}) = 0$ for all sheaves $\mathscr{G}$ and $n > e$. But in such a situation we have

$$\mathscr{T}or_n(\mathscr{F}, \mathscr{G})_x \cong \underline{T}or_n(\mathscr{F}_x, \mathscr{G}_x) = 0$$

because $flat.dim_{\mathcal{O}_{X,x}}\mathscr{F}_x \leq e$. Hence $\mathscr{T}or_n(\mathscr{F}, \mathscr{G}) = 0$ as claimed. The reverse inequality is trivial if $flat.dim.\mathscr{F} = d$ is infinite, so assume it is finite. We have by Proposition 63 a flat resolution

$$0 \longrightarrow \mathscr{P}_d \longrightarrow \cdots \longrightarrow \mathscr{P}_1 \longrightarrow \mathscr{P}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

Passing to stalks gives a flat resolution of $\mathscr{F}_x$ for each $x \in X$, whence $flat.dim_{\mathcal{O}_{X,x}}\mathscr{F}_x \leq d$ for each $x \in X$, as required. $\square$

Lemma 66. *Given a sheaf of modules $\mathscr{F}$ and an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X we have*

$$flat.dim.\mathscr{F} = \sup\{flat.dim.\mathscr{F}|_{U_\alpha} \,|\, \alpha \in \Lambda\}$$

Proof. The proof is identical to the proof of Lemma 65, with stalks replaced by restriction and using Lemma 61. □

5 Adjunctions

In this section we seek to upgrade the usual Hom-tensor adjunction in $\mathfrak{Mod}(X)$ to an adjunction in the derived category. Since there are various meanings one can ascribe to "adjunction", "tensor" and "Hom" in our current context (for example instead of an adjunction with $Hom(-,-)$ on the outside one can have $\mathbb{R}Hom^\bullet(-,-)$ or even $\mathbb{R}\mathscr{H}om^\bullet(-,-)$) the reader will be faced with several adjunction results and compatibility statements relating the different adjunctions. Such compatibilities become essential when we come to actually calculate using the adjunction in $\mathfrak{D}(X)$. Throughout this section $(X, \mathcal{O}_X)$ is a fixed ringed space and all sheaves of modules are over X, unless specified otherwise.

Proposition 67. *For complexes of sheaves of modules $\mathscr{F}, \mathscr{G}, \mathscr{H}$ there is a canonical isomorphism of complexes of sheaves of modules natural in all three variables*

$$\kappa : \mathscr{H}om^\bullet(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \longrightarrow \mathscr{H}om^\bullet(\mathscr{F}, \mathscr{H}om^\bullet(\mathscr{G}, \mathscr{H}))$$

Taking global sections gives a canonical isomorphism natural in all three variables

$$Hom^\bullet(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \longrightarrow Hom^\bullet(\mathscr{F}, \mathscr{H}om^\bullet(\mathscr{G}, \mathscr{H}))$$

Proof. For $n \in \mathbb{Z}$ we have using (MRS,Proposition 77) a canonical isomorphism of sheaves of modules

$$\begin{aligned}
\mathscr{H}om^n(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) &= \prod_q \mathscr{H}om((\mathscr{F} \otimes \mathscr{G})^q, \mathscr{H}^{q+n}) \\
&= \prod_q \mathscr{H}om(\bigoplus_{i+j=q} \mathscr{F}^i \otimes \mathscr{G}^j, \mathscr{H}^{q+n}) \\
&\cong \prod_q \prod_{i+j=q} \mathscr{H}om(\mathscr{F}^i \otimes \mathscr{G}^j, \mathscr{H}^{q+n}) \\
&\cong \prod_{i,j} \mathscr{H}om(\mathscr{F}^i, \mathscr{H}om(\mathscr{G}^j, \mathscr{H}^{i+j+n})) \\
&\cong \prod_i \mathscr{H}om(\mathscr{F}^i, \prod_j \mathscr{H}om(\mathscr{G}^j, \mathscr{H}^{i+j+n})) \\
&= \prod_i \mathscr{H}om(\mathscr{F}^i, \mathscr{H}om^{i+n}(\mathscr{G}, \mathscr{H})) \\
&= \mathscr{H}om^n(\mathscr{F}, \mathscr{H}om^\bullet(\mathscr{G}, \mathscr{H}))
\end{aligned}$$

Unfortunately this does not define a morphism of complexes, so we have to manually introduce some signs. Define for open $U \subseteq X$ and morphisms $f_q : (\mathscr{F} \otimes \mathscr{G})^q|_U \longrightarrow \mathscr{H}^{q+n}|_U$

$$\kappa^n_U((f_q)_{q\in\mathbb{Z}}) = ((-1)^{\lambda(i,n)} G_i)_{i\in\mathbb{Z}}, \qquad \lambda(i,n) = in + \frac{i(i+1)}{2} + \frac{n(n+1)}{2}$$

where G_i is defined as follows: since $(\mathscr{F} \otimes \mathscr{G})^q = \bigoplus_{i+j=q} \mathscr{F}^i \otimes \mathscr{G}^j$ the sequence $(f_q)_{q\in\mathbb{Z}}$ determines morphisms $f_{i,j} : (\mathscr{F}^i \otimes \mathscr{G}^j)|_U \longrightarrow \mathscr{H}^{i+j+n}|_U$ for $i, j \in \mathbb{Z}$. The adjunction formula of (MRS,Proposition 77) maps this to a morphism $g_{i,j} : \mathscr{F}^i|_U \longrightarrow \mathscr{H}om(\mathscr{G}^j, \mathscr{H}^{i+j+n})|_U$ and as j varies this induces a morphism $G_i : \mathscr{F}^i|_U \longrightarrow \mathscr{H}om^{i+n}(\mathscr{G}, \mathscr{H})|_U$. One now checks that κ gives an isomorphism of complexes (this is where the unusual factor of $\lambda(i,n)$ becomes necessary) natural in all three variables. □

Remark 17. The morphism κ of Proposition 67 is local, in the sense that for an open set $U \subseteq X$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{H}om^\bullet(\mathscr{F}\otimes\mathscr{G},\mathscr{H})|_U & \xrightarrow{\kappa|_U} & \mathscr{H}om^\bullet(\mathscr{F},\mathscr{H}om^\bullet(\mathscr{G},\mathscr{H}))|_U \\ \Downarrow & & \Downarrow 1 \\ \mathscr{H}om^\bullet(\mathscr{F}|_U\otimes\mathscr{G}|_U,\mathscr{H}|_U) & \xrightarrow[\kappa]{} & \mathscr{H}om^\bullet(\mathscr{F}|_U,\mathscr{H}om^\bullet(\mathscr{G}|_U,\mathscr{H}|_U)) \end{array}$$

Corollary 68. *Let $\mathscr{G},\mathscr{H}$ be complexes of sheaves of modules with $\mathscr{G}$ hoflat and $\mathscr{H}$ hoinjective. Then $\mathscr{H}om^\bullet(\mathscr{G},\mathscr{H})$ is hoinjective.*

Proof. Taking $\mathscr{F}$ exact in Proposition 67 this follows from (DTC2,Corollary 19). □

Proposition 69. *For complexes of sheaves of modules $\mathscr{X},\mathscr{Y},\mathscr{Z}$ there is a canonical isomorphism in $\mathfrak{D}(X)$ natural in all three variables*

$$\mu : \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}\underline{\underline{\otimes}}\mathscr{Y},\mathscr{Z}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y},\mathscr{Z}))$$

Similarly we have a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$ natural in all three variables

$$\mathbb{R}Hom^\bullet(\mathscr{X}\underline{\underline{\otimes}}\mathscr{Y},\mathscr{Z}) \longrightarrow \mathbb{R}Hom^\bullet(\mathscr{X},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y},\mathscr{Z}))$$

Proof. When we say that the isomorphism is canonical, we mean that once you fix an assignment $\mathcal{F}$ of hoflat resolutions to calculate $-\underline{\underline{\otimes}}^{\mathcal{F}}-$ and an assignment of hoinjective resolutions to calculate $\mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(-,-)$ there are no further choices required. By definition $\mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{X},-)$ is the right derived functor of the composite $Q'\circ\mathscr{H}om^\bullet(\mathscr{X},-) : K(X) \longrightarrow \mathfrak{D}(X)$, so as part of the data we have a trinatural transformation

$$\zeta : Q'\circ\mathscr{H}om^\bullet(-,-) \longrightarrow \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(-,-)\circ Q$$

with $\zeta_{\mathscr{S},\mathscr{T}}$ an isomorphism for any hoinjective $\mathscr{T}$. In particular we have isomorphisms in $\mathfrak{D}(X)$

$$\mathscr{H}om^\bullet(\mathscr{X},\mathscr{H}om^\bullet(F_{\mathscr{Y}},I_{\mathscr{Z}})) \longrightarrow \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{X},\mathscr{H}om^\bullet(F_{\mathscr{Y}},I_{\mathscr{Z}}))$$
$$\mathscr{H}om^\bullet(F_{\mathscr{Y}},I_{\mathscr{Z}}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(F_{\mathscr{Y}},I_{\mathscr{Z}})$$

where $F_{\mathscr{Y}},I_{\mathscr{Z}}$ denote the chosen resolutions. Finally using Proposition 67 we have a canonical isomorphism in $\mathfrak{D}(X)$

$$\begin{aligned} \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{X}\underline{\underline{\otimes}}^{\mathcal{F}}\mathscr{Y},\mathscr{Z}) &= \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}\otimes F_{\mathscr{Y}},\mathscr{Z}) \\ &= \mathscr{H}om^\bullet(\mathscr{X}\otimes F_{\mathscr{Y}},I_{\mathscr{Z}}) \\ &\cong \mathscr{H}om^\bullet(\mathscr{X},\mathscr{H}om^\bullet(F_{\mathscr{Y}},I_{\mathscr{Z}})) \\ &\cong \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{X},\mathscr{H}om^\bullet(F_{\mathscr{Y}},I_{\mathscr{Z}})) \\ &= \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{X},\mathbb{R}\mathscr{H}om^\bullet(F_{\mathscr{Y}},\mathscr{Z})) \\ &\cong \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{X},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y},\mathscr{Z})) \end{aligned}$$

Naturality in $\mathscr{X},\mathscr{Y},\mathscr{Z}$ with respect to morphisms of complexes is now straightforward to check, using the explicit construction of the derived functors given in (TRC,Remark 75) and (TRC,Remark 80). One then upgrades to naturality with respect to morphisms in $\mathfrak{D}(X)$. The canonical isomorphism for $\mathbb{R}Hom^\bullet(-,-)$ is defined in the same way. □

Corollary 70. *For complexes of sheaves of modules $\mathscr{X},\mathscr{Y},\mathscr{Z}$ there are canonical isomorphisms of abelian groups natural in all three variables*

$$Hom_{\mathbf{C}(X)}(\mathscr{X}\otimes\mathscr{Y},\mathscr{Z}) \longrightarrow Hom_{\mathbf{C}(X)}(\mathscr{X},\mathscr{H}om^\bullet(\mathscr{Y},\mathscr{Z})) \tag{7}$$
$$Hom_{K(X)}(\mathscr{X}\otimes\mathscr{Y},\mathscr{Z}) \longrightarrow Hom_{K(X)}(\mathscr{X},\mathscr{H}om^\bullet(\mathscr{Y},\mathscr{Z})) \tag{8}$$
$$Hom_{\mathfrak{D}(X)}(\mathscr{X}\underline{\underline{\otimes}}\mathscr{Y},\mathscr{Z}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{X},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y},\mathscr{Z})) \tag{9}$$

That is, we have adjoint pairs

$$\mathbf{C}(X) \underset{-\otimes \mathscr{Y}}{\overset{\mathscr{H}om^\bullet(\mathscr{Y},-)}{\rightleftarrows}} \mathbf{C}(X) \qquad -\otimes \mathscr{Y} \longrightarrow\!\!\!\!| \ \mathscr{H}om^\bullet(\mathscr{Y},-)$$

$$K(X) \underset{-\otimes \mathscr{Y}}{\overset{\mathscr{H}om^\bullet(\mathscr{Y},-)}{\rightleftarrows}} K(X) \qquad -\otimes \mathscr{Y} \longrightarrow\!\!\!\!| \ \mathscr{H}om^\bullet(\mathscr{Y},-)$$

$$\mathfrak{D}(X) \underset{-\underline{\otimes} \mathscr{Y}}{\overset{\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y},-)}{\rightleftarrows}} \mathfrak{D}(X) \qquad -\underline{\otimes} \mathscr{Y} \longrightarrow\!\!\!\!| \ \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y},-)$$

Proof. For the second isomorphism and corresponding adjunction we apply the cohomology functor $H^0(-)$ to the second natural isomorphism of Proposition 67 and then use (DTC2,Proposition 18). For the first isomorphism and adjunction, take kernels instead of cohomology. For the third, we apply the cohomology functor $H^0(-)$ to the second natural isomorphism of Proposition 69 and then use (DTC2,Lemma 26). To be precise, we mean naturality with respect to morphisms of $\mathfrak{D}(X)$ in all three variables. Observe that the maps (7), (8) and (7) are actually isomorphisms of $\Gamma(X,\mathcal{O}_X)$-modules. □

Remark 18. In the context of Corollary 70 it is worth writing down exactly what the adjunction map does to a morphism of complexes $f : \mathscr{X} \otimes \mathscr{Y} \longrightarrow \mathscr{Z}$. Given $q \in \mathbb{Z}$ this is a morphism

$$f^q : (\mathscr{X} \otimes \mathscr{Y})^q = \bigoplus_{i+j=q} \mathscr{X}^i \otimes \mathscr{Y}^j \longrightarrow \mathscr{Z}^q$$

which has components $f_{ij} : \mathscr{X}^i \otimes \mathscr{Y}^j \longrightarrow \mathscr{Z}^q$. Each of these components determines by the adjunction of (MRS,Proposition 76) a morphism $\mathscr{X}^i \longrightarrow \mathscr{H}om(\mathscr{Y}^j, \mathscr{Z}^q)$, which determines a morphism $G_i : \mathscr{X}^i \longrightarrow \mathscr{H}om^i(\mathscr{Y}, \mathscr{Z})$ into the product over all $j \in \mathbb{Z}$. Then the adjoint partner of f is the morphism of complexes $g : \mathscr{X} \longrightarrow \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})$ defined by $g^i = (-1)^{\frac{i(i+1)}{2}} G_i$. Given this explicit description it is easy to check that the adjunction is *local*. That is, given an open set $U \subseteq X$ the following diagram commutes

$$\begin{array}{ccc} Hom_{\mathbf{C}(X)}(\mathscr{X} \otimes \mathscr{Y}, \mathscr{Z}) & \longrightarrow & Hom_{\mathbf{C}(X)}(\mathscr{X}, \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \\ \downarrow & & \downarrow \\ Hom_{\mathbf{C}(U)}(\mathscr{X}|_U \otimes \mathscr{Y}|_U, \mathscr{Z}|_U) & \longrightarrow & Hom_{\mathbf{C}(U)}(\mathscr{X}|_U, \mathscr{H}om^\bullet(\mathscr{Y}|_U, \mathscr{Z}|_U)) \end{array}$$

and similarly with $\mathbf{C}(-)$ replaced by $K(-)$.

In the next result we check that the adjunctions of Proposition 67 and Proposition 69 are compatible, which implies compatibility of the adjunctions in $K(X)$ and $\mathfrak{D}(X)$.

Lemma 71. *For complexes of sheaves of modules $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ the following diagrams commute in $\mathfrak{D}(X)$ and $\mathfrak{D}(\mathbf{Ab})$ respectively*

$$\begin{array}{ccc} \mathscr{H}om^\bullet(\mathscr{X} \otimes \mathscr{Y}, \mathscr{Z}) & \longrightarrow & \mathscr{H}om^\bullet(\mathscr{X}, \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \\ \downarrow & & \downarrow \\ \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X} \underline{\otimes} \mathscr{Y}, \mathscr{Z}) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \end{array} \tag{10}$$

$$\begin{array}{ccc} Hom^\bullet(\mathscr{X} \otimes \mathscr{Y}, \mathscr{Z}) & \longrightarrow & Hom^\bullet(\mathscr{X}, \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \\ \downarrow & & \downarrow \\ \mathbb{R}Hom^\bullet(\mathscr{X} \underline{\otimes} \mathscr{Y}, \mathscr{Z}) & \longrightarrow & \mathbb{R}Hom^\bullet(\mathscr{X}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \end{array} \tag{11}$$

Taking cohomology we have a commutative diagram of abelian groups

$$\begin{array}{ccc} Hom_{K(X)}(\mathscr{X} \otimes \mathscr{Y}, \mathscr{Z}) & \longrightarrow & Hom_{K(X)}(\mathscr{X}, \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(X)}(\mathscr{X} \underset{=}{\otimes} \mathscr{Y}, \mathscr{Z}) & \longrightarrow & Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \end{array} \tag{12}$$

Proof. In defining these diagrams, we make use of the canonical morphisms

$$Hom^\bullet(-,-) \longrightarrow \mathbb{R}Hom^\bullet(-,-)$$
$$\mathscr{H}om^\bullet(-,-) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(-,-)$$
$$(-) \underset{=}{\otimes} (-) \longrightarrow (-) \otimes (-)$$

that form part of the definition of these derived functors. The third diagram is (up to isomorphism) obtained by applying $H^0(-)$ to the second, so it suffices to check commutativity of (10), (11). If one looks at the explicit definition of the adjunction isomorphisms in Proposition 69 this is straightforward, if a little tedious. □

This result makes dealing with the adjunction of the derived functors much easier, because we can reduce to the adjunction on the level of complexes. For example, the following lemma has a direct proof but it is much more cumbersome.

Lemma 72. *The adjunction of Proposition 69 is local. That is, for complexes of sheaves of modules $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ and open $U \subseteq X$ the following diagrams commute in $\mathfrak{D}(U)$ and* **Ab** *respectively*

$$\begin{array}{ccc} \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X} \underset{=}{\otimes} \mathscr{Y}, \mathscr{Z})|_U & \xrightarrow{\mu|_U} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}))|_U \\ \Downarrow & & \Downarrow \\ \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}|_U \underset{=}{\otimes} \mathscr{Y}|_U, \mathscr{Z}|_U) & \xrightarrow[\mu]{} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}|_U, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}|_U, \mathscr{Z}|_U)) \end{array} \tag{13}$$

$$\begin{array}{ccc} Hom_{\mathfrak{D}(X)}(\mathscr{X} \underset{=}{\otimes} \mathscr{Y}, \mathscr{Z}) & \longrightarrow & Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(U)}(\mathscr{X}|_U \underset{=}{\otimes} \mathscr{Y}|_U, \mathscr{Z}|_U) & \longrightarrow & Hom_{\mathfrak{D}(U)}(\mathscr{X}|_U, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}|_U, \mathscr{Z}|_U)) \end{array} \tag{14}$$

Proof. Both diagrams are natural in $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$, so to check that they commute we can assume $\mathscr{Y}$ hoflat and $\mathscr{Z}$ hoinjective. In that case the vertical morphisms in Lemma 71 are isomorphisms. Commutativity of (13) then reduces to commutativity of the diagram in Remark 17, and commutativity of (14) reduces to commutativity of the diagram in Remark 18. □

Lemma 73. *The adjunction isomorphisms of Corollary 70 are trinatural.*

Proof. We already know that the isomorphisms (7), (8) and (9) are natural in all three variables.

For the isomorphism (7) trinaturality means that the following diagrams commute

$$\begin{array}{ccc}
Hom_{\mathbf{C}(X)}((\Sigma\mathscr{X})\otimes\mathscr{Y},\mathscr{Z}) & \longrightarrow & Hom_{\mathbf{C}(X)}(\Sigma\mathscr{X},\mathscr{H}om^{\bullet}(\mathscr{Y},\mathscr{Z})) \\
\downarrow & & \downarrow \\
Hom_{\mathbf{C}(X)}(\Sigma(\mathscr{X}\otimes\mathscr{Y}),\mathscr{Z}) & & Hom_{\mathbf{C}(X)}(\mathscr{X},\Sigma^{-1}\mathscr{H}om^{\bullet}(\mathscr{Y},\mathscr{Z})) \\
\downarrow & & \downarrow \\
Hom_{\mathbf{C}(X)}(\mathscr{X}\otimes\mathscr{Y},\Sigma^{-1}\mathscr{Z}) & \longrightarrow & Hom_{\mathbf{C}(X)}(\mathscr{X},\mathscr{H}om^{\bullet}(\mathscr{Y},\Sigma^{-1}\mathscr{Z}))
\end{array}$$

$$\begin{array}{ccc}
Hom_{\mathbf{C}(X)}(\mathscr{X}\otimes(\Sigma\mathscr{Y}),\mathscr{Z}) & \longrightarrow & Hom_{\mathbf{C}(X)}(\mathscr{X},\mathscr{H}om^{\bullet}(\Sigma\mathscr{Y},\mathscr{Z})) \\
\downarrow & & \downarrow \\
Hom_{\mathbf{C}(X)}(\Sigma(\mathscr{X}\otimes\mathscr{Y}),\mathscr{Z}) & & Hom_{\mathbf{C}(X)}(\mathscr{X},\Sigma^{-1}\mathscr{H}om^{\bullet}(\mathscr{Y},\mathscr{Z})) \\
\downarrow & & \downarrow \\
Hom_{\mathbf{C}(X)}(\mathscr{X}\otimes\mathscr{Y},\Sigma^{-1}\mathscr{Z}) & \longrightarrow & Hom_{\mathbf{C}(X)}(\mathscr{X},\mathscr{H}om^{\bullet}(\mathscr{Y},\Sigma^{-1}\mathscr{Z}))
\end{array}$$

One checks commutativity of these diagrams using the explicit description of Remark 18. It is then clear that the analogous diagrams for $K(X)$ commute. To check that the third isomorphism (9) is trinatural reduce to $\mathscr{Y}$ hoflat and $\mathscr{Z}$ hoinjective and use Lemma 71. The trinaturality in $\mathscr{X}$ means that in particular these adjunctions are triadjunctions in the sense of (TRC,Theorem 42). *Warning:* Not all conceivable diagrams of the above type commute. For example if you take $\Sigma^{-1}\mathscr{Y}$ in the second variable, pull the Σ^{-1} out of the tensor and push it back into the first variable, you end up with a diagram which *anticommutes*. □

Corollary 74. *For a complex $\mathscr{X}$ of sheaves of modules on X the triangulated functors $\mathscr{X}\underline{\underline{\otimes}}-$ and $-\underline{\underline{\otimes}}\mathscr{X}$ preserve coproducts.*

Proof. By Corollary 70 the triangulated functor $-\underline{\underline{\otimes}}\mathscr{X}$ has a right adjoint, and therefore must preserve coproducts. It then follows from Lemma 54 that the functor $\mathscr{X}\underline{\underline{\otimes}}-$ preserves coproducts. □

Fix a complex $\mathscr{X}$ of sheaves of modules on X. For an open set $U\subseteq X$ we have a diagram of triangulated functors

$$\begin{array}{ccccc}
K(X) & \xrightarrow{\mathscr{H}om^{\bullet}(\mathscr{X},-)} & K(X) & \xrightarrow{\Gamma(U,-)} & K(\mathbf{Ab}) \\
{\scriptstyle Q}\downarrow & & {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q'} \\
\mathfrak{D}(X) & \xrightarrow[\mathbb{R}\mathscr{H}om^{\bullet}(\mathscr{X},-)]{} & \mathfrak{D}(X) & \xrightarrow[\mathbb{R}\Gamma(U,-)]{} & \mathfrak{D}(\mathbf{Ab})
\end{array}$$

where the composite in the top row is $Hom^{\bullet}_U(\mathscr{X}|_U,-)$. For convenience set $F=\mathscr{H}om^{\bullet}(\mathscr{X},-)$ so that $Hom^{\bullet}_U(\mathscr{X}|_U,-)=\Gamma(U,-)\circ F$. If the right derived functors are the pairs $(\mathbb{R}F,\zeta)$, $(\mathbb{R}\Gamma(U,-),\omega)$ and $(\mathbb{R}(\Gamma(U,-)\circ F),\xi)$ then we have a trinatural transformation

$$Q'\Gamma(U,-)F\xrightarrow{\omega F}\mathbb{R}\Gamma(U,-)QF\xrightarrow{\mathbb{R}\Gamma(U,-)\zeta}\mathbb{R}\Gamma(U,-)\mathbb{R}(F)Q$$

which we denote by μ. By definition of a right derived functor there is an induced trinatural transformation $\theta:\mathbb{R}(\Gamma(U,-)F)\longrightarrow\mathbb{R}\Gamma(U,-)\mathbb{R}(F)$. If we have a morphism of complexes $\mathscr{X}'\longrightarrow\mathscr{X}$ this gives rise to a trinatural transformation $\alpha:F\longrightarrow F'$ and one checks that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{R}(\Gamma(U,-)F) & \xrightarrow{\theta} & \mathbb{R}\Gamma(U,-)\mathbb{R}(F) \\
{\scriptstyle \mathbb{R}(\Gamma(U,-)\alpha)}\downarrow & & \downarrow{\scriptstyle \mathbb{R}\Gamma(U,-)\mathbb{R}(\alpha)} \\
\mathbb{R}(\Gamma(U,-)F') & \xrightarrow[\theta']{} & \mathbb{R}\Gamma(U,-)\mathbb{R}(F')
\end{array} \tag{15}$$

With this notation, we have the following.

Proposition 75. *For complexes of sheaves of modules $\mathscr{X}, \mathscr{Y}$ and an open set $U \subseteq X$ there is a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$ natural in both variables*

$$\mathbb{R}Hom^{\bullet}_U(\mathscr{X}|_U, \mathscr{Y}|_U) \longrightarrow \mathbb{R}\Gamma\left(U, \mathbb{R}\mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})\right)$$

which makes the following diagram commute in $\mathfrak{D}(\mathbf{Ab})$

$$\begin{array}{ccc}
Hom^{\bullet}_U(\mathscr{X}|_U, \mathscr{Y}|_U) & \xrightarrow{\ 1\ } & \Gamma(U, \mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})) \\
\Big\downarrow & & \Big\downarrow \\
 & & \mathbb{R}\Gamma(U, \mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})) \\
 & & \Big\downarrow \\
\mathbb{R}Hom^{\bullet}_U(\mathscr{X}|_U, \mathscr{Y}|_U) & \longrightarrow & \mathbb{R}\Gamma(U, \mathbb{R}\mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y}))
\end{array}$$

Proof. Fix assignments of hoinjective resolutions for $\mathfrak{Mod}(X)$ and $\mathfrak{Mod}(U)$ so that we have canonical right derived functors

$$\mathbb{R}F = \mathbb{R}\mathscr{H}om^{\bullet}(\mathscr{X}, -) : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X) \tag{16}$$

$$\mathbb{R}Hom^{\bullet}_U(\mathscr{X}|_U, -) : \mathfrak{D}(U) \longrightarrow \mathfrak{D}(\mathbf{Ab}) \tag{17}$$

and let $(\mathbb{R}\Gamma(U,-), \omega)$ be an arbitrary right derived functor. Observe that composing (17) with the restriction $(-)|_U : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(U)$ we obtain a right derived functor $(\mathbb{R}(\Gamma(U,-)F), \xi)$. We claim that the induced trinatural transformation $\theta : \mathbb{R}(\Gamma(U,-)F) \longrightarrow \mathbb{R}\Gamma(U,-)\mathbb{R}(F)$ described above is a natural equivalence.

If $F_{\mathscr{X}} \longrightarrow \mathscr{X}$ is a hoflat resolution then the induced trinatural transformations

$$\mathscr{H}om^{\bullet}(\mathscr{X}, -) \longrightarrow \mathscr{H}om^{\bullet}(F_{\mathscr{X}}, -), \quad Hom^{\bullet}_U(\mathscr{X}|_U, -) \longrightarrow Hom^{\bullet}_U(F_{\mathscr{X}}|_U, -)$$

are quasi-isomorphisms when you evaluate on them on hoinjectives (since both $Hom(-,-)$ and $\mathscr{H}om(-,-)$ are homlike), so it follows from (TRC,Lemma 118) that in (15) the vertical morphisms are natural equivalences. We can therefore reduce to the case where $\mathscr{X}$ is hoflat, in which case the triangulated functor $\mathscr{H}om^{\bullet}(\mathscr{X}, -)$ preserves hoinjectives by Corollary 68, so the fact that θ is a natural equivalence is a consequence of (TRC,Theorem 113).

This completes the proof of our claim, so that for fixed $\mathscr{X}$ we have a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$ natural in $\mathscr{Y}$ (with respect to morphisms of $\mathfrak{D}(X)$)

$$\mathbb{R}Hom^{\bullet}_U(\mathscr{X}|_U, \mathscr{Y}|_U) \longrightarrow \mathbb{R}\Gamma\left(U, \mathbb{R}\mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})\right)$$

We deduce from commutativity of (15) that this isomorphism is also natural in $\mathscr{X}$, at least with respect to morphisms of complexes. As usual, it is straightforward to upgrade this to naturality in $\mathfrak{D}(X)$. □

Given a hoinjective complex $\mathscr{Y}$ it is not generally the case that $\mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})$ is hoinjective (unless $\mathscr{X}$ is hoflat). Nonetheless it is always the case that $\mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})$ is acyclic for $\Gamma(U,-)$.

Corollary 76. *Let $\mathscr{X}, \mathscr{Y}$ be complexes of sheaves of modules with $\mathscr{Y}$ hoinjective, and $U \subseteq X$ an open set. Then $\mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})$ is right acyclic for $\Gamma(U,-)$. That is, the canonical morphism*

$$\Gamma(U, \mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})) \longrightarrow \mathbb{R}\Gamma(U, \mathscr{H}om^{\bullet}(\mathscr{X}, \mathscr{Y})) \tag{18}$$

is an isomorphism in $\mathfrak{D}(\mathbf{Ab})$.

Proof. The complex $\mathscr{Y}|_U$ is hoinjective, so in the compatibility diagram of Proposition 75 every morphism except possibly (18) is an isomorphism. Hence (18) is an isomorphism as well. This result should be compared with (SS,Lemma 13). □

Remark 19. In Definition 4 we have written the expression

$$\Gamma(X, \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(\mathscr{F},\mathscr{G})) = \mathbb{R}_{\mathcal{I}}Hom^\bullet(\mathscr{F},\mathscr{G}) \tag{19}$$

which seems to suggest that in Proposition 75 the $\mathbb{R}\Gamma(U,-)$ on the right hand side could be replaced by $\Gamma(U,-)$. The content of this observation is just Corollary 76, which in particular shows that we have a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$

$$\begin{aligned}\mathbb{R}\Gamma(U, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X},\mathscr{Y})) &= \mathbb{R}\Gamma(U, \mathscr{H}om^\bullet(\mathscr{X}, I_{\mathscr{Y}}))\\ &\cong \Gamma(U, \mathscr{H}om^\bullet(\mathscr{X}, I_{\mathscr{Y}}))\\ &= \Gamma(U, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X},\mathscr{Y}))\end{aligned}$$

While the right hand side *can* be made natural in $\mathscr{X},\mathscr{Y}$ with respect to morphisms of $\mathfrak{D}(X)$ the whole situation seems a little distasteful: derived functors do not want to live on the inside of underived functors.

Remark 20. Let $\mathscr{F}$ be a sheaf of modules on X and $U \subseteq X$ an open subset. We know from Theorem 12 that there is a canonical isomorphism $Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i\mathscr{F}) \cong H^i(U,\mathscr{F})$. We can observe this from another direction as follows. By (DTC2,Lemma 26) we have a canonical isomorphism

$$Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i\mathscr{F}) \longrightarrow H^i(\mathbb{R}Hom^\bullet(\mathcal{O}_U,\mathscr{F}))$$

since $Hom^\bullet(\mathcal{O}_U,\mathscr{F})$ is canonically isomorphic to $\Gamma(U,\mathscr{I})$ for an injective resolution $\mathscr{I}$ of $\mathscr{F}$ (modulo some sign changes that don't affect cohomology) we find another proof of Theorem 12.

Definition 16. Let $\mathscr{X}$ be a complex of sheaves of modules, and define a complex of sheaves of modules

$$\mathscr{X}^{\mathbb{R}\vee} = \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathcal{O}_X)$$

which we call the *derived dual complex*, or often just the *dual complex*. Taking duals defines a contravariant triangulated functor

$$(-)^{\mathbb{R}\vee} : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$

This is only determined up to canonical isomorphism, but if we fix an assignment $\mathcal{I}$ of hoinjective resolutions for $\mathfrak{Mod}(X)$ then we have a canonical dual complex which we denote $\mathscr{X}^{\mathbb{R}_{\mathcal{I}}\vee}$. Often we simply write $\mathscr{X}^\vee$ for the derived dual. This introduces some possible ambiguity when $\mathscr{X}$ is a sheaf in degree zero, but as we will see in a moment the danger is slight.

Remark 21. Let $\mathscr{F}$ be a sheaf of modules. Then by Lemma 27 we have a canonical isomorphism

$$H^i(\mathscr{F}^{\mathbb{R}\vee}) \cong \begin{cases} 0 & i < 0 \\ \mathscr{F}^\vee & i = 0 \\ \mathscr{E}xt^i(\mathscr{F},\mathcal{O}_X) & i > 0 \end{cases}$$

If $\mathscr{F}$ is locally finitely free then the functor $\mathscr{H}om(\mathscr{F},-)$ is exact (MOS,Lemma 37) and therefore $\mathscr{E}xt^i(\mathscr{F},\mathcal{O}_X) = 0$ for $i > 0$. Hence the dual sheaf $\mathscr{F}^\vee$ and the derived dual complex $\mathscr{F}^{\mathbb{R}\vee}$ are isomorphic in the derived category. Since we are usually only interested in the dual sheaf of locally finitely free sheaves, this means that there is no real danger in writing $\mathscr{X}^\vee$ for the derived dual throughout.

5.1 Units and Counits

The adjunctions of Corollary 70 determine unit and counit morphisms. To be precise, the adjunction (7) determines canonical morphisms of complexes

$$\begin{aligned}\eta &: \mathscr{X} \longrightarrow \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X}\otimes\mathscr{Y})\\ \varepsilon &: \mathscr{H}om^\bullet(\mathscr{Y},\mathscr{Z})\otimes\mathscr{Y} \longrightarrow \mathscr{Z}\end{aligned}$$

which are trinatural in $\mathscr{X}$ and $\mathscr{Z}$ respectively. Similarly the adjunction (8) determines canonical morphisms in $\mathfrak{D}(X)$

$$\eta : \mathscr{X} \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \underline{\otimes} \mathscr{Y})$$
$$\varepsilon : \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \mathscr{Y} \longrightarrow \mathscr{Z}$$

which are trinatural in $\mathscr{X}$ and $\mathscr{Z}$ respectively. One checks that these two sets of units and counits are compatible, in the sense that the following diagrams commute in $\mathfrak{D}(X)$

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\eta} & \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \mathscr{Y}) \\ {\scriptstyle\eta}\downarrow & & \downarrow \\ \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \underline{\otimes} \mathscr{Y}) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \mathscr{Y}) \end{array} \tag{20}$$

$$\begin{array}{ccc} \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \mathscr{Y} & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \mathscr{Y} \\ \downarrow & & \downarrow{\scriptstyle\varepsilon} \\ \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \otimes \mathscr{Y} & \xrightarrow[\varepsilon]{} & \mathscr{Z} \end{array} \tag{21}$$

and moreover both sets of units and counits are *local*, in the sense that if you restrict them to an open set they are (after composing with the necessary canonical isomorphisms) the units and counits there. This is a consequence of Remark 18 and Lemma 72.

5.1.1 Properties of the Unit

Using Remark 18 one checks that the unit $\eta : \mathscr{X} \longrightarrow \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \mathscr{Y})$ is for $q \in \mathbb{Z}$ the morphism

$$\eta^q : \mathscr{X}^q \longrightarrow \prod_j \mathscr{H}om(\mathscr{Y}^j, (\mathscr{X} \otimes \mathscr{Y})^{j+q})$$

with components $p_j \eta^q = (-1)^{\frac{q(q+1)}{2}} \overline{u}_j$ where $\overline{u}_j$ corresponds under (MRS,Proposition 76) to the injection $\mathscr{X}^q \otimes \mathscr{Y}^j \longrightarrow (\mathscr{X} \otimes \mathscr{Y})^{j+q}$.

The two units are natural in $\mathscr{X}$, but in order to reduce from the derived unit to the ordinary unit it is necessary to have naturality in both variables (at least for isomorphisms). Naturality in $\mathscr{Y}$ is expressed by the commutativity of the following diagrams for a morphism $\alpha : \mathscr{Y} \longrightarrow \mathscr{Y}'$ in $\mathbf{C}(X)$ and $\mathfrak{D}(X)$ respectively

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\eta} & \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \mathscr{Y}) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle \mathscr{H}om^\bullet(\mathscr{Y}, 1 \otimes \alpha)} \\ \mathscr{H}om^\bullet(\mathscr{Y}', \mathscr{X} \otimes \mathscr{Y}') & \xrightarrow[\mathscr{H}om^\bullet(\alpha, \mathscr{X} \otimes \mathscr{Y}')]{} & \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \mathscr{Y}') \end{array}$$

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\eta} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \underline{\otimes} \mathscr{Y}) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, 1 \underline{\otimes} \alpha)} \\ \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}', \mathscr{X} \underline{\otimes} \mathscr{Y}') & \xrightarrow[\mathbb{R}\mathscr{H}om^\bullet(\alpha, \mathscr{X} \underline{\otimes} \mathscr{Y}')]{} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \underline{\otimes} \mathscr{Y}') \end{array}$$

In particular if α is an isomorphism in $\mathfrak{D}(X)$ then in the second diagram the bottom and right morphisms are isomorphisms, and we have a commutative diagram expressing the two units as related by an isomorphism.

The units are also *trinatural* in $\mathscr{Y}$, in the sense that the following diagrams commute

$$\begin{array}{ccc} & & \mathscr{H}om^\bullet(\Sigma\mathscr{Y}, \mathscr{X} \otimes \Sigma\mathscr{Y}) \\ & \overset{\eta}{\nearrow} & \Downarrow \\ & & \Sigma^{-1}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \Sigma\mathscr{Y}) \\ & & \Downarrow \\ \mathscr{X} & & \Sigma^{-1}\mathscr{H}om^\bullet(\mathscr{Y}, \Sigma(\mathscr{X} \otimes \mathscr{Y})) \\ & & \Downarrow \\ & & \Sigma^{-1}\Sigma(\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \mathscr{Y})) \\ & \underset{\eta}{\searrow} & \Downarrow \\ & & \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \otimes \mathscr{Y}) \end{array} \qquad \begin{array}{ccc} & & \mathbb{R}\mathscr{H}om^\bullet(\Sigma\mathscr{Y}, \mathscr{X} \underline{\otimes} \Sigma\mathscr{Y}) \\ & \overset{\eta}{\nearrow} & \Downarrow \\ & & \Sigma^{-1}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \underline{\otimes} \Sigma\mathscr{Y}) \\ & & \Downarrow \\ \mathscr{X} & & \Sigma^{-1}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \Sigma(\mathscr{X} \underline{\otimes} \mathscr{Y})) \\ & & \Downarrow \\ & & \Sigma^{-1}\Sigma(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \underline{\otimes} \mathscr{Y})) \\ & \underset{\eta}{\searrow} & \Downarrow \\ & & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{X} \underline{\otimes} \mathscr{Y}) \end{array}$$

Commutativity of the first is straightforward using our explicit description of the unit, or alternatively the diagrams of Lemma 73. Commutativity of the second diagram also follows from Lemma 73. If instead of $\Sigma\mathscr{Y}$ we have $\Sigma^{-1}\mathscr{Y}$ then there are two similar diagrams, which also commute.

5.1.2 Properties of the Counit

Using Remark 18 one checks that the counit $\varepsilon : \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \otimes \mathscr{Y} \longrightarrow \mathscr{Z}$ is for $q \in \mathbb{Z}$ the morphism

$$\varepsilon^q : \bigoplus_{i+j=q} \mathscr{H}om^i(\mathscr{Y}, \mathscr{Z}) \otimes \mathscr{Y}^j \longrightarrow \mathscr{Z}^q$$

with components $\varepsilon_{ij} = (-1)^{\frac{i(i+1)}{2}}\overline{p}_j$ where $\overline{p}_j$ corresponds under the bijection of (MRS,Proposition 76) to the canonical projection $p_j : \mathscr{H}om^i(\mathscr{Y}, \mathscr{Z}) \longrightarrow \mathscr{H}om(\mathscr{Y}^j, \mathscr{Z}^{i+j})$.

The two counits are natural in $\mathscr{Z}$, but again we need naturality in both variables. Naturality in $\mathscr{Y}$ is expressed by the commutativity of the following diagrams for a morphism $\alpha : \mathscr{Y} \longrightarrow \mathscr{Y}'$ in $\mathbf{C}(X)$ and $\mathfrak{D}(X)$ respectively

$$\begin{array}{ccc} \mathscr{H}om^\bullet(\mathscr{Y}', \mathscr{Z}) \otimes \mathscr{Y} & \xrightarrow{\mathscr{H}om^\bullet(\alpha, \mathscr{Z}) \otimes 1} & \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \otimes \mathscr{Y} \\ {\scriptstyle 1\otimes\alpha}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \mathscr{H}om^\bullet(\mathscr{Y}', \mathscr{Z}) \otimes \mathscr{Y}' & \xrightarrow[\varepsilon]{} & \mathscr{Z} \end{array}$$

$$\begin{array}{ccc} \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}', \mathscr{Z}) \underline{\otimes} \mathscr{Y} & \xrightarrow{\mathbb{R}\mathscr{H}om^\bullet(\alpha, \mathscr{Z}) \underline{\otimes} 1} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \mathscr{Y} \\ {\scriptstyle 1\underline{\otimes}\alpha}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}', \mathscr{Z}) \underline{\otimes} \mathscr{Y}' & \xrightarrow[\varepsilon]{} & \mathscr{Z} \end{array}$$

The counits are trinatural $\mathscr{Y}$, in the sense that the following diagrams commute

$$\begin{array}{ccc} \mathscr{H}om^\bullet(\Sigma\mathscr{Y}, \mathscr{Z}) \otimes \Sigma\mathscr{Y} & & \\ \Downarrow & \overset{\varepsilon}{\searrow} & \\ \Sigma^{-1}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \otimes \Sigma\mathscr{Y} & & \\ \Downarrow & & \\ \Sigma\left(\Sigma^{-1}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \otimes \mathscr{Y}\right) & & \mathscr{Z} \\ \Downarrow & & \\ \Sigma\Sigma^{-1}\left(\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \otimes \mathscr{Y}\right) & & \\ \Downarrow & \underset{\varepsilon}{\nearrow} & \\ \mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \otimes \mathscr{Y} & & \end{array} \qquad \begin{array}{ccc} \mathbb{R}\mathscr{H}om^\bullet(\Sigma\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \Sigma\mathscr{Y} & & \\ \Downarrow & \overset{\varepsilon}{\searrow} & \\ \Sigma^{-1}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \Sigma\mathscr{Y} & & \\ \Downarrow & & \\ \Sigma\left(\Sigma^{-1}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \mathscr{Y}\right) & & \mathscr{Z} \\ \Downarrow & & \\ \Sigma\Sigma^{-1}\left(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \mathscr{Y}\right) & & \\ \Downarrow & \underset{\varepsilon}{\nearrow} & \\ \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}) \underline{\otimes} \mathscr{Y} & & \end{array}$$

Commutativity of the first is straightforward to check using our explicit description of the counit. For the second we use naturality of the counits in $\mathscr{Y}, \mathscr{Z}$ to reduce to $\mathscr{Y}$ hoflat and $\mathscr{Z}$ hoinjective. The claim then follows from commutativity of the first diagram and the compatibility diagram (21). *Warning:* The order in which you extract the Σ's in the vertical isomorphisms *does* matter, due to sign issues. If instead of $\Sigma\mathscr{Y}$ we have $\Sigma^{-1}\mathscr{Y}$ then there are two similar diagrams, which actually *anticommute*.

Remark 22. By Lemma 25 there is a canonical isomorphism $\mathcal{O}_X \longrightarrow \mathcal{O}_X^\vee = \mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X, \mathcal{O}_X)$ in $\mathfrak{D}(X)$. One checks that under the canonical adjunction this corresponds to the product morphism $\mathcal{O}_X \underset{=}{\otimes} \mathcal{O}_X \longrightarrow \mathcal{O}_X$.

5.2 Adjoint Constructions

The main thing to take away from the next result is the existence of a canonical morphism from a complex $\mathscr{F}$ to its double derived dual $(\mathscr{F}^\vee)^\vee$ in the derived category. In the study of the properties of this morphism, however, it becomes convenient to have worked out some details in greater generality.

Lemma 77. *Given complexes $\mathscr{F}, \mathscr{E}$ of sheaves of modules there are canonical morphisms in $\mathbf{C}(X)$ and $\mathfrak{D}(X)$ respectively, trinatural in $\mathscr{F}$*

$$\tau : \mathscr{F} \longrightarrow \mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E})$$
$$\tau' : \mathscr{F} \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E})$$

and also natural in $\mathscr{E}$, in the sense that the following diagrams commute for a morphism $\alpha : \mathscr{E} \longrightarrow \mathscr{E}'$ in $\mathbf{C}(X)$ or $\mathfrak{D}(X)$ respectively

$$\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\tau} & \mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) \\
{\scriptstyle\tau}\downarrow & & \downarrow \\
\mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}'), \mathscr{E}') & \longrightarrow & \mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}')
\end{array}$$

$$\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\tau'} & \mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) \\
{\scriptstyle\tau'}\downarrow & & \downarrow \\
\mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}'), \mathscr{E}') & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}')
\end{array}$$

The morphisms τ, τ' are compatible in the sense that the following diagram commutes in $\mathfrak{D}(X)$

$$\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\tau} & \mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) \\
{\scriptstyle\tau'}\downarrow & & \downarrow \\
\mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E})
\end{array} \tag{22}$$

Taking $\mathscr{E} = \mathcal{O}_X$ we have a canonical trinatural morphism $\mathscr{F} \longrightarrow (\mathscr{F}^\vee)^\vee$ in $\mathfrak{D}(X)$.

Proof. Composing the counit $\varepsilon : \mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}) \otimes \mathscr{F} \longrightarrow \mathscr{E}$ with the twisting isomorphism, we have a canonical morphism $\mathscr{F} \otimes \mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}) \longrightarrow \mathscr{E}$ which by the adjunction corresponds to a morphism τ of the required form. Similarly one defines τ' using the derived counit and twisting isomorphism for the derived tensor. The naturality of the adjunction isomorphisms in all three variables means that both of these morphisms are natural in $\mathscr{F}$ and $\mathscr{E}$.

To see that (22) commutes one uses commutativity of (21), the naturality of the adjunction of Corollary 70 in the middle variable, and the commutativity of the diagram (12) of Lemma 71. Commutativity of (22) is important, because it allows us to reduce from the derived double dual to the double dual on complexes, and thereby to the double dual of sheaves.

Using the remarks of Section 5.1.2 we can describe the morphism τ explicitly. Given integers $n, q \in \mathbb{Z}$ we have from (MRS,Proposition 74) a canonical morphism of sheaves of modules

$$\tau_{n,q} : \mathscr{F}^n \longrightarrow \mathscr{H}om(\mathscr{H}om(\mathscr{F}^n, \mathscr{E}^{n+q}), \mathscr{E}^{n+q})$$

For an open set $U \subseteq X$ and $t \in \Gamma(U, \mathscr{F}^n)$ we have $\tau^n_U(t) = ((-1)^{\delta(q,n)} f_q)_{q \in \mathbb{Z}}$ where the sign factor is $\delta(q,n) = qn + \frac{q(q+1)}{2} + \frac{n(n+1)}{2}$ and $f_q : \prod_p \mathscr{H}om(\mathscr{F}^p|_U, \mathscr{E}^{p+q}|_U) \longrightarrow \mathscr{E}^{n+q}|_U$ is the composite

$$\textstyle\prod_p \mathscr{H}om(\mathscr{F}^p|_U, \mathscr{E}^{p+q}|_U) \longrightarrow \mathscr{H}om(\mathscr{F}^n|_U, \mathscr{E}^{n+q}|_U) \xrightarrow{\tau_{n,q,U}(t)} \mathscr{E}^{n+q}|_U$$

It only remains to check that the morphisms τ, τ' are *trinatural* in $\mathscr{F}$, by which we mean that the following diagrams commute

$$\begin{array}{ccc} \Sigma\mathscr{F} & \xrightarrow{\tau} & \mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\Sigma\mathscr{F}, \mathscr{E}), \mathscr{E}) \\ & & \Downarrow \\ & & \mathscr{H}om^\bullet(\Sigma^{-1}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) \\ & \searrow^{\Sigma\tau} & \Downarrow \\ & & \Sigma\mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) \end{array} \qquad \begin{array}{ccc} \Sigma\mathscr{F} & \xrightarrow{\tau'} & \mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\Sigma\mathscr{F}, \mathscr{E}), \mathscr{E}) \\ & & \Downarrow \\ & & \mathbb{R}\mathscr{H}om^\bullet(\Sigma^{-1}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) \\ & \searrow^{\Sigma\tau'} & \Downarrow \\ & & \Sigma\mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E}) \end{array}$$

To check commutativity of the first diagram use the explicit description of τ. Since τ' is natural in $\mathscr{E}$ in checking the second diagram we can assume $\mathscr{E}$ hoinjective. Commutativity then follows from (22) and commutativity of the first diagram. ☐

Remark 23. The morphisms of Lemma 77 are local, in the sense that $\tau_{\mathscr{F},\mathscr{E}}|_U = \tau_{\mathscr{F}|_U, \mathscr{E}|_U}$ for any open $U \subseteq X$ and the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}|_U & \xrightarrow{\tau'_{\mathscr{F}}|_U} & \mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}), \mathscr{E})|_U \\ & \searrow^{\tau'_{\mathscr{F}|_U}} & \Downarrow \\ & & \mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}|_U, \mathscr{E}|_U), \mathscr{E}|_U) \end{array}$$

which is a consequence of Lemma 72.

Lemma 78. *Given complexes $\mathscr{E}, \mathscr{F}, \mathscr{G}$ of sheaves of modules there are canonical morphisms in* $\mathbf{C}(X)$ *and* $\mathfrak{D}(X)$ *respectively, natural in all three variables*

$$\xi : \mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \otimes \mathscr{G} \longrightarrow \mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F} \otimes \mathscr{G})$$
$$\xi' : \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \underline{\otimes} \mathscr{G} \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F} \underline{\otimes} \mathscr{G})$$

The morphisms ξ, ξ' are compatible in the sense that the following diagram commutes in $\mathfrak{D}(X)$

$$\begin{array}{ccc} \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \underline{\otimes} \mathscr{G} & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F} \underline{\otimes} \mathscr{G}) \\ \uparrow & & \downarrow \\ \mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \underline{\otimes} \mathscr{G} & & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F} \otimes \mathscr{G}) \\ \downarrow & & \uparrow \\ \mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \otimes \mathscr{G} & \longrightarrow & \mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F} \otimes \mathscr{G}) \end{array} \tag{23}$$

Proof. We have a canonical morphism $\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \otimes \mathscr{E} \longrightarrow \mathscr{F}$, and tensoring with $\mathscr{G}$ yields a canonical morphism of complexes of sheaves of modules with codomain $\mathscr{F} \otimes \mathscr{G}$ and domain

$$\begin{aligned} (\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \otimes \mathscr{E}) \otimes \mathscr{G} &\cong \mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \otimes (\mathscr{E} \otimes \mathscr{G}) \\ &\cong \mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \otimes (\mathscr{G} \otimes \mathscr{E}) \\ &\cong (\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \otimes \mathscr{G}) \otimes \mathscr{E} \end{aligned}$$

By adjointness this morphism must correspond to a canonical morphism ξ of complexes of sheaves of modules of the desired form. Similarly one defines ξ' using the derived twist and derived associator. It is straightforward to check that ξ is natural in all three variables with respect to morphisms of complexes, and ξ' is natural in all three variables with respect to morphisms of $\mathfrak{D}(X)$. To check commutativity of (23) one uses adjointness and the fact that by Lemma 71 the two types of adjunction are compatible.

One checks that ξ, ξ' are actually *trinatural* in $\mathscr{E}$ and $\mathscr{G}$. For example, trinaturality in $\mathscr{E}$ means that the following diagrams commute in $\mathbf{C}(X)$ and $\mathfrak{D}(X)$ respectively

$$\begin{array}{ccc}
\mathscr{H}om^\bullet(\Sigma^{-1}\mathscr{E},\mathscr{F})\otimes\mathscr{G} & \xrightarrow{\ \xi\ } & \mathscr{H}om^\bullet(\Sigma^{-1}\mathscr{E},\mathscr{F}\otimes\mathscr{G}) \\
\Downarrow & & \Downarrow \\
\Sigma\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F})\otimes\mathscr{G} & & \\
\Downarrow & & \\
\Sigma(\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F})\otimes\mathscr{G}) & \xrightarrow[\Sigma\xi]{} & \Sigma\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F}\otimes\mathscr{G})
\end{array}$$

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om^\bullet(\Sigma^{-1}\mathscr{E},\mathscr{F})\underline{\underline{\otimes}}\mathscr{G} & \xrightarrow{\ \xi'\ } & \mathbb{R}\mathscr{H}om^\bullet(\Sigma^{-1}\mathscr{E},\mathscr{F}\underline{\underline{\otimes}}\mathscr{G}) \\
\Downarrow & & \Downarrow \\
\Sigma\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F})\underline{\underline{\otimes}}\mathscr{G} & & \\
\Downarrow & & \\
\Sigma(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F})\underline{\underline{\otimes}}\mathscr{G}) & \xrightarrow[\Sigma\xi']{} & \Sigma\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F}\underline{\underline{\otimes}}\mathscr{G})
\end{array}$$

This verification is straightforward, but it involves checking several compatibility diagrams that we have not yet encountered. One has to be careful to note that some of these compatibility diagrams *anticommute* individually, but together they lead to the required commutative diagrams. Trinaturality in $\mathscr{G}$ is easier. $\square$

Remark 24. The morphisms of Lemma 78 are local, in the sense that if you restrict ξ or ξ' to an open set $U \subseteq X$ and compose with the obvious canonical isomorphisms, you end up with ξ and ξ' respectively for the complexes $\mathscr{E}|_U, \mathscr{F}|_U$ and $\mathscr{G}|_U$.

Remark 25. Using Section 5.1.2 we can describe explicitly the morphism ξ. For $q \in \mathbb{Z}$ it is a morphism

$$\xi^q : \bigoplus_{i+j=q} \mathscr{H}om^i(\mathscr{E},\mathscr{F})\otimes\mathscr{G}^j \longrightarrow \prod_t \mathscr{H}om(\mathscr{E}^t,(\mathscr{F}\otimes\mathscr{G})^{q+t})$$

whose component $p_t\xi^q u_{ij}$ is $(-1)^{\frac{q(q+1)}{2}+jt}\overline{\lambda}$ where $\overline{\lambda}$ is the morphism corresponding under the bijection of (MRS,Proposition 76) to the following composite λ

$$\begin{array}{ccccc}
(\mathscr{H}om^i(\mathscr{E},\mathscr{F})\otimes\mathscr{G}^j)\otimes\mathscr{E}^t & \Longrightarrow & \mathscr{H}om^i(\mathscr{E},\mathscr{F})\otimes(\mathscr{G}^j\otimes\mathscr{E}^t) & \Longrightarrow & \mathscr{H}om^i(\mathscr{E},\mathscr{F})\otimes(\mathscr{E}^t\otimes\mathscr{G}^j) \\
 & & & & \Downarrow \\
(\mathscr{F}\otimes\mathscr{G})^{q+t} & \longleftarrow & \mathscr{F}^{i+t}\otimes\mathscr{G}^j & \xleftarrow[\varepsilon_{it}\otimes 1]{} & (\mathscr{H}om^i(\mathscr{E},\mathscr{F})\otimes\mathscr{E}^t)\otimes\mathscr{G}^j
\end{array}$$

5.3 Commutative Diagrams

There are various diagrams one can construct out of the morphisms we have defined in previous sections. In applications one uses commutativity of many such diagrams, so it is worth writing down the main ones here. The reader is advised to skip this section and return to the results as needed. In verifying these and other diagrams the trick is to use the compatibility diagrams relating the derived and underived morphisms to reduce to checking commutativity of a diagram of *complexes* and then of *sheaves*. Essentially the only difficulty is in checking that the various complicated sign factors combine in the correct way.

Throughout this section $(X, \mathcal{O}_X)$ is a ringed space and all sheaves of modules are over X. Unlabelled morphisms are the canonical ones defined in Section 5 or earlier. Once again we encourage the reader who wants to skip these verifications to learn about coherence in closed monoidal categories.

Lemma 79. *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}$ be complexes of sheaves of modules. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F}) \underline{\underline{\otimes}} (\mathscr{G} \underline{\underline{\otimes}} \mathscr{H}) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F} \underline{\underline{\otimes}} (\mathscr{G} \underline{\underline{\otimes}} \mathscr{H})) \\
\Downarrow & & \\
(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F}) \underline{\underline{\otimes}} \mathscr{G}) \underline{\underline{\otimes}} \mathscr{H} & & \Downarrow \\
\downarrow{\scriptstyle \xi' \otimes 1} & & \\
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F} \underline{\underline{\otimes}} \mathscr{G}) \underline{\underline{\otimes}} \mathscr{H} & \xrightarrow[\zeta']{} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},(\mathscr{F} \underline{\underline{\otimes}} \mathscr{G}) \underline{\underline{\otimes}} \mathscr{H})
\end{array}$$

Proof. Using the adjunction of Corollary 70 we change to question to commutativity of a diagram beginning with $(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F}) \underline{\underline{\otimes}} (\mathscr{G} \underline{\underline{\otimes}} \mathscr{H})) \underline{\underline{\otimes}} \mathscr{E}$. This reduces to commutativity of a diagram of complexes beginning with $(\mathscr{H}om(\mathscr{E},\mathscr{F}) \otimes (\mathscr{G} \otimes \mathscr{H})) \otimes \mathscr{E}$ that one checks explicitly. $\square$

Lemma 80. *Let $\mathscr{F}, \mathscr{G}$ be complexes of sheaves of modules. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathscr{G}) \underline{\underline{\otimes}} \mathcal{O}_X & \xrightarrow{\zeta'} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathscr{G} \underline{\underline{\otimes}} \mathcal{O}_X) \\
& \searrow\!\!\!\searrow \quad \swarrow\!\!\!\swarrow & \\
& \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathscr{G}) &
\end{array}$$

Lemma 81. *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}$ be complexes of sheaves of modules. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G})) \underline{\underline{\otimes}} \mathscr{H} & \xrightarrow{\zeta'} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G}) \underline{\underline{\otimes}} \mathscr{H}) \\
& & \downarrow{\scriptstyle \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\zeta')} \\
\Downarrow & & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G} \underline{\underline{\otimes}} \mathscr{H})) \\
& & \Downarrow \\
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G}) \underline{\underline{\otimes}} \mathscr{H} & \xrightarrow[\zeta']{} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G} \underline{\underline{\otimes}} \mathscr{H})
\end{array}$$

Proof. One first reduces to the corresponding diagram of complexes, by observing that we can assume $\mathscr{G}$ hoinjective and $\mathscr{H}, \mathscr{E}$ hoflat. Commutativity then follows from a calculation involving the explicit definition of ζ' and the counit ε. $\square$

Lemma 82. *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}$ be complexes of sheaves of modules. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G})) \underline{\underline{\otimes}} (\mathscr{F} \underline{\underline{\otimes}} \mathscr{E}) & \Longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G}) \underline{\underline{\otimes}} (\mathscr{F} \underline{\underline{\otimes}} \mathscr{E}) \\
\Downarrow & & \\
\left(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G})) \underline{\underline{\otimes}} \mathscr{F}\right) \underline{\underline{\otimes}} \mathscr{E} & & \downarrow{\scriptstyle \varepsilon} \\
\downarrow{\scriptstyle \varepsilon \underline{\underline{\otimes}} \mathscr{E}} & & \\
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G}) \underline{\underline{\otimes}} \mathscr{E} & \xrightarrow[\varepsilon]{} & \mathscr{G}
\end{array}$$

Remark 26. Let $\mathscr{F}, \mathscr{E}, \mathscr{G}$ be complexes of sheaves of modules. The canonical isomorphism

$$\alpha : \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{G})) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\otimes} \mathscr{E}, \mathscr{G})$$

corresponds itself under the adjunction isomorphism of (DCOS,Corollary 70) to a morphism

$$\overline{\alpha} : \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{G})) \underline{\otimes} (\mathscr{F} \underline{\otimes} \mathscr{E}) \longrightarrow \mathscr{G}$$

One can check that $\overline{\alpha}$ is actually the following composite, built out of the associativity isomorphism and two applications of the counit

$$\begin{aligned}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{G})) \underline{\otimes} (\mathscr{F} \underline{\otimes} \mathscr{E}) &\cong \big(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{G})) \underline{\otimes} \mathscr{F}\big) \underline{\otimes} \mathscr{E} \\ &\longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{G}) \underline{\otimes} \mathscr{E} \longrightarrow \mathscr{G}\end{aligned}$$

Lemma 83. *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}$ be complexes of sheaves of modules. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\otimes} \mathscr{G}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}, \mathscr{E})) & \Longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{G}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}, \mathscr{E}))) \\
\Downarrow & & \\
\mathbb{R}\mathscr{H}om^\bullet((\mathscr{F} \underline{\otimes} \mathscr{G}) \underline{\otimes} \mathscr{H}, \mathscr{E}) & & \Downarrow \\
\Downarrow & & \\
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\otimes} (\mathscr{G} \underline{\otimes} \mathscr{H}), \mathscr{E}) & \Longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{G} \underline{\otimes} \mathscr{H}, \mathscr{E}))
\end{array}$$

Lemma 84. *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}$ be complexes of sheaves of modules. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \underline{\otimes} \mathscr{G}) \underline{\otimes} \mathscr{E} & \xrightarrow{\zeta' \underline{\otimes} 1} & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F} \underline{\otimes} \mathscr{G}) \underline{\otimes} \mathscr{E} \\
\Downarrow & & \\
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \underline{\otimes} (\mathscr{G} \underline{\otimes} \mathscr{E}) & & \\
\Downarrow & & \downarrow \varepsilon \\
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \underline{\otimes} (\mathscr{E} \underline{\otimes} \mathscr{G}) & & \\
\Downarrow & & \\
(\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}) \underline{\otimes} \mathscr{E}) \underline{\otimes} \mathscr{G} & \xrightarrow[\varepsilon \underline{\otimes} 1]{} & \mathscr{F} \underline{\otimes} \mathscr{G}
\end{array}$$

Lemma 85. *Let $\mathscr{E}$ be a complex of sheaves of modules. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathscr{E} \underline{\otimes} \mathscr{E}^\vee & \xrightarrow{\tau' \underline{\otimes} 1} & (\mathscr{E}^\vee)^\vee \underline{\otimes} \mathscr{E}^\vee \\
\Downarrow & & \downarrow \varepsilon_{\mathscr{E}^\vee} \\
\mathscr{E}^\vee \underline{\otimes} \mathscr{E} & \xrightarrow[\varepsilon_{\mathscr{E}}]{} & \mathcal{O}_X
\end{array}$$

Proof. Here the ε denote the counit of Section 5.1 with $\mathscr{Z} = \mathcal{O}_X$ and $\mathscr{Y} = \mathscr{E}, \mathscr{Y} = \mathscr{E}^\vee$ respectively. Using the adjunction we have to check that the morphisms $\mathscr{E} \longrightarrow (\mathscr{E}^\vee)^\vee$ induced by each direction around the square are equal. But this is true by definition of τ'. $\square$

6 Derived Inverse Image

Proposition 86. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. The additive functor $f^* : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ has a left derived functor $\mathbb{L}f^*$ and there is a canonical triadjunction*

$$\mathfrak{D}(X) \underset{\mathbb{L}f^*}{\overset{\mathbb{R}f_*}{\rightleftarrows}} \mathfrak{D}(Y) \qquad \mathbb{L}f^* \longrightarrow\!\!| \; \mathbb{R}f_*$$

whose unit $\eta^\diamondsuit : 1 \longrightarrow \mathbb{R}f_* \circ \mathbb{L}f^*$ is the unique trinatural transformation making the following diagram commute for every complex $\mathscr{Y}$ of sheaves of modules on Y

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\eta^\diamondsuit} & \mathbb{R}f_*(\mathbb{L}f^*\mathscr{Y}) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\mathbb{R}f_*(\omega)} \\ f_*f^*(\mathscr{Y}) & \xrightarrow[\zeta]{} & \mathbb{R}f_*(f^*\mathscr{Y}) \end{array} \tag{24}$$

Proof. Let $Q : K(X) \longrightarrow \mathfrak{D}(X)$ and $Q' : K(Y) \longrightarrow \mathfrak{D}(Y)$ be the verdier quotients. First we show that any hoflat complex of sheaves of modules $\mathscr{F}$ on Y is left f^*-acyclic in the sense of (DTC2,Definition 4). As in the proof of Lemma 50 the key point is that f^* sends an exact hoflat complex to an exact complex. This was the content of Lemma 52, so any hoflat complex is left f^*-acyclic. Therefore Proposition 48 implies that every complex $\mathscr{X}$ of sheaves of modules on Y admits a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{X}$ with $\mathscr{F}$ left f^*-acyclic, and so f^* has a left derived functor $(\mathbb{L}f^*, \omega)$ by (DTC2,Theorem 2).

The additive functor $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ always has a right derived functor $(\mathbb{R}f_*, \zeta)$ since $\mathfrak{Mod}(X)$ has enough hoinjectives. The existence of a canonical triadjunction $\mathbb{L}f^* \longrightarrow\!\!| \, \mathbb{R}f_*$ now follows from (DTC2,Theorem 9). □

Definition 17. Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. The additive functor $f^* : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ has a left derived functor $\mathbb{L}f^*$

$$\mathbb{L}f^* : \mathfrak{D}(Y) \longrightarrow \mathfrak{D}(X)$$

which we call the *derived inverse image functor*, or often just the *inverse image functor*. This is only determined up to canonical trinatural equivalence, but if we fix an assignment $\mathcal{F}$ of hoflat resolutions for $\mathfrak{Mod}(Y)$ then we have a canonical left derived functor which we denote $\mathbb{L}_{\mathcal{F}}f^*$.

Lemma 87. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $V \subseteq Y$ and $U \subseteq f^{-1}V$ open sets. Then for any complex $\mathscr{G}$ of sheaves of modules on Y there is a canonical isomorphism in $\mathfrak{D}(U)$ natural in $\mathscr{G}$*

$$\mu : (\mathbb{L}f^*\mathscr{G})|_U \longrightarrow \mathbb{L}g^*(\mathscr{G}|_V)$$

where $g : U \longrightarrow V$ is the induced morphism of ringed spaces.

Proof. Let $(\mathbb{L}f^*, \zeta), (\mathbb{L}g^*, \omega)$ be arbitrary left derived functors. By Lemma 41 restriction preserves hoflat complexes, so we deduce from (DTC2,Theorem 8) that the pairs

$$((-)|_U \circ \mathbb{L}f^*, (-)|_U\zeta), \quad (\mathbb{L}g^* \circ (-)|_V, \omega(-)|_V)$$

are left derived functors of $(-)|_U \circ f^*$ and $g^* \circ (-)|_V$ respectively. Let $\theta : (-)|_U \circ f^* \longrightarrow g^* \circ (-)|_V$ be the canonical natural equivalence. By (DTC2,Definition 3) this induces a canonical trinatural equivalence $\mu = \mathbb{L}\theta : (-)|_U \circ \mathbb{L}f^* \longrightarrow \mathbb{L}g^* \circ (-)|_V$ making the following diagram commute

$$\begin{array}{ccc} (-)|_U \circ \mathbb{L}f^* \circ Q & \xrightarrow{\mu Q} & \mathbb{L}g^* \circ (-)|_V \circ Q \\ {\scriptstyle(-)|_U\zeta}\downarrow & & \downarrow{\scriptstyle\omega(-)|_V} \\ Q' \circ K((-)|_U \circ f^*) & \xrightarrow[Q'K(\theta)]{} & Q' \circ K(g^* \circ (-)|_V) \end{array}$$

Evaluating μ on a complex $\mathscr{G}$ gives the desired isomorphism. □

Remark 27. Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules on X, Y respectively. Then the two adjunction isomorphisms fit into a commutative diagram

$$\begin{array}{ccc} Hom_{K(X)}(f^*\mathscr{Y}, \mathscr{X}) & \longrightarrow & Hom_{K(Y)}(\mathscr{Y}, f_*\mathscr{X}) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(X)}(f^*\mathscr{Y}, \mathscr{X}) & & Hom_{\mathfrak{D}(Y)}(\mathscr{Y}, f_*\mathscr{X}) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(X)}(\mathbb{L}f^*\mathscr{Y}, \mathscr{X}) & \longrightarrow & Hom_{\mathfrak{D}(Y)}(\mathscr{Y}, \mathbb{R}f_*\mathscr{X}) \end{array} \tag{25}$$

where the vertical maps are induced by composition with the canonical morphisms $\mathbb{L}f^*\mathscr{Y} \longrightarrow f^*\mathscr{Y}$ and $f_*\mathscr{X} \longrightarrow \mathbb{R}f_*\mathscr{X}$. We claim moreover that the adjunction of the derived functors is *local*, by which we mean that given an open set $V \subseteq U$ if we set $U = f^{-1}V$ and let $g : U \longrightarrow V$ be the induced morphism then the following diagram commutes

$$\begin{array}{ccc} Hom_{\mathfrak{D}(X)}(\mathbb{L}f^*\mathscr{Y}, \mathscr{X}) & \longrightarrow & Hom_{\mathfrak{D}(Y)}(\mathscr{Y}, \mathbb{R}f_*\mathscr{X}) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(U)}(\mathbb{L}f^*(\mathscr{Y})|_U, \mathscr{X}|_U) & & Hom_{\mathfrak{D}(V)}(\mathscr{Y}|_V, \mathbb{R}f_*(\mathscr{X})|_V) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(U)}(\mathbb{L}g^*(\mathscr{Y}|_V), \mathscr{X}|_U) & \longrightarrow & Hom_{\mathfrak{D}(V)}(\mathscr{Y}|_V, \mathbb{R}g_*(\mathscr{X}|_U)) \end{array}$$

To check this reduce to $\mathscr{Y}$ hoflat and $\mathscr{X}$ hoinjective and use (25). In particular the unit morphism $\eta^{\diamond}|_V : \mathscr{Y}|_V \longrightarrow \mathbb{R}f_*(\mathbb{L}f^*\mathscr{Y})|_V$ composed with $\mathbb{R}f_*(\mathbb{L}f^*\mathscr{Y})|_V \cong \mathbb{R}g_*(\mathbb{L}g^*(\mathscr{Y}|_V))$ is the unit morphism for $\mathscr{Y}|_V$.

Here is something clever from Lipman's notes [Lip] (3.2.2). Recall that if a complex $\mathscr{X}$ is hoinjective then the map $Hom_{K(X)}(\mathscr{Y}, \mathscr{X}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{Y}, \mathscr{X})$ is an isomorphism. If we replace $\mathscr{X}$ by $f_*\mathscr{X}$ then this is still true, provided we assume something about the complex $\mathscr{Y}$.

Lemma 88. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules on X, Y respectively with $\mathscr{X}$ hoinjective and $\mathscr{Y}$ hoflat. Then the canonical map*

$$\nu : Hom_{K(Y)}(\mathscr{Y}, f_*\mathscr{X}) \longrightarrow Hom_{\mathfrak{D}(Y)}(\mathscr{Y}, f_*\mathscr{X})$$

is an isomorphism. As a consequence, the canonical morphism in $\mathfrak{D}(\mathbf{Ab})$

$$Hom^\bullet(\mathscr{Y}, f_*(\mathscr{X})) \longrightarrow \mathbb{R}Hom^\bullet(\mathscr{Y}, f_*(\mathscr{X})) \tag{26}$$

is an isomorphism.

Proof. Fix derived functors $(\mathbb{R}f_*, \zeta)$ and $(\mathbb{L}f^*, \omega)$. Commutativity of the following diagram (the two nonobvious maps being composition with ζ and ω) is an immediate consequence of commutativity of (24)

$$\begin{array}{ccc} Hom_{K(X)}(f^*\mathscr{Y}, \mathscr{X}) & \longrightarrow & Hom_{K(Y)}(\mathscr{Y}, f_*\mathscr{X}) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(X)}(f^*\mathscr{Y}, \mathscr{X}) & & Hom_{\mathfrak{D}(Y)}(\mathscr{Y}, f_*\mathscr{X}) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(X)}(\mathbb{L}f^*\mathscr{Y}, \mathscr{X}) & \longrightarrow & Hom_{\mathfrak{D}(Y)}(\mathscr{Y}, \mathbb{R}f_*\mathscr{X}) \end{array}$$

Since every map other than ν is an isomorphism, we have the desired result. This together with (DTC2,Lemma 27) implies that (26) is an isomorphism. ☐

Our next task is to upgrade the adjunction isomorphism of Proposition 86 to an isomorphism on the level of Hom complexes, and then to derived Hom complexes. The first step is trivial.

Lemma 89. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. For complexes of sheaves of modules $\mathscr{X}, \mathscr{Y}$ on X, Y respectively we have a canonical isomorphism of complexes of abelian groups natural in both variables*

$$Hom^\bullet(f^*(\mathscr{Y}), \mathscr{X}) \longrightarrow Hom^\bullet(\mathscr{Y}, f_*(\mathscr{X}))$$

Proposition 90. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. For complexes of sheaves of modules $\mathscr{X}, \mathscr{Y}$ on X, Y respectively we have a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$ natural in both variables*

$$\mathbb{R}Hom^\bullet_{\mathcal{O}_X}(\mathbb{L}f^*(\mathscr{Y}), \mathscr{X}) \longrightarrow \mathbb{R}Hom^\bullet_{\mathcal{O}_Y}(\mathscr{Y}, \mathbb{R}f_*(\mathscr{X}))$$

Proof. Fix assignments of hoinjective resolutions to $\mathfrak{Mod}(X)$ and $\mathfrak{Mod}(Y)$, which we use to calculate the derived Hom functors $\mathbb{R}Hom^\bullet(-,-)$ and also $\mathbb{R}f_*$. Fix also an assignment of hoflat resolutions $\mathcal{F}$ to $\mathfrak{Mod}(Y)$ which we use to calculate $\mathbb{L}f^*$. Let $F_{\mathscr{Y}} \longrightarrow \mathscr{Y}$ be the chosen hoflat resolution of $\mathscr{Y}$ and $I \longrightarrow I_{\mathscr{X}}$ the chosen hoinjective resolution of $\mathscr{X}$. By Lemma 88 the canonical morphism in $\mathfrak{D}(\mathbf{Ab})$

$$Hom^\bullet(F_{\mathscr{Y}}, f_*(I_{\mathscr{X}})) \longrightarrow \mathbb{R}Hom^\bullet(F_{\mathscr{Y}}, f_*(I_{\mathscr{X}}))$$

is an isomorphism. Combining this isomorphism with the one in Lemma 89 we have a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$

$$\begin{aligned}
\mathbb{R}Hom^\bullet(\mathbb{L}f^*(\mathscr{Y}), \mathscr{X}) &= Hom^\bullet(f^*(F_{\mathscr{Y}}), I_{\mathscr{X}})\\
&\cong Hom^\bullet(F_{\mathscr{Y}}, f_*(I_{\mathscr{X}}))\\
&\cong \mathbb{R}Hom^\bullet(F_{\mathscr{Y}}, f_*(I_{\mathscr{X}}))\\
&\cong \mathbb{R}Hom^\bullet(\mathscr{Y}, \mathbb{R}f_*(\mathscr{X}))
\end{aligned}$$

One checks that this isomorphism is natural with respect to morphisms of complexes in $\mathscr{X}, \mathscr{Y}$, and then as usual one upgrades to naturality with respect to morphisms in $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$ respectively. ∎

Lemma 91. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $V \subseteq Y$ an open subset. For a complex $\mathscr{F}$ of sheaves of modules on X there is a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$ natural in $\mathscr{F}$*

$$\mathbb{R}\Gamma(f^{-1}V, \mathscr{F}) \longrightarrow \mathbb{R}\Gamma(V, \mathbb{R}f_*(\mathscr{F}))$$

Proof. To be clear, we have additive functors $\Gamma(V, -) : \mathfrak{Mod}(Y) \longrightarrow \mathbf{Ab}$ and $\Gamma(f^{-1}V, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$ and therefore right derived functors $\mathbb{R}\Gamma(V, -)$ and $\mathbb{R}\Gamma(f^{-1}V, -)$ fitting into a diagram of triangulated functors

$$\begin{array}{ccc}
\mathfrak{D}(X) & \xrightarrow{\mathbb{R}f_*} & \mathfrak{D}(Y)\\
& {\scriptstyle\mathbb{R}\Gamma(f^{-1}V,-)}\searrow \quad \swarrow{\scriptstyle\mathbb{R}\Gamma(V,-)} & \\
& \mathfrak{D}(\mathbf{Ab}) &
\end{array} \tag{27}$$

which we claim commutes up to canonical trinatural equivalence. We prove the claim by showing that if $\mathscr{I}$ is a hoinjective complex on X then $f_*(\mathscr{I})$ is right $\Gamma(V, -)$-acyclic.

By Lemma 8 we have a canonical natural equivalence $Hom^\bullet(\mathcal{O}_V, -) \cong \Lambda \circ \Gamma(V, -)$ of triangulated functors $K(Y) \longrightarrow K(\mathbf{Ab})$ (using the notation of (DTC2,Definition 12)). We infer that there is a canonical trinatural equivalence $\mathbb{R}Hom^\bullet(\mathcal{O}_V, -) \cong \Lambda \circ \mathbb{R}\Gamma(V, -)$ (TRC,Lemma 117). In particular for our hoinjective complex $\mathscr{I}$ we have a commutative diagram in $\mathfrak{D}(\mathbf{Ab})$

$$\begin{array}{ccc}
Hom^\bullet(\mathcal{O}_V, f_*(\mathscr{I})) & \longrightarrow & \mathbb{R}Hom^\bullet(\mathcal{O}_V, f_*(\mathscr{I}))\\
\Downarrow & & \Downarrow\\
\Lambda\Gamma(V, f_*(\mathscr{I})) & \longrightarrow & \Lambda\mathbb{R}\Gamma(V, f_*(\mathscr{I}))
\end{array}$$

From Lemma 88 and the fact that $\mathscr{O}_V$ is hoflat and $\mathscr{I}$ hoinjective we deduce that the top row is an isomorphism. Therefore so is the bottom row, which means that $\Gamma(V, f_*(\mathscr{I})) \longrightarrow \mathbb{R}\Gamma(V, f_*(\mathscr{I}))$ is an isomorphism in $\mathfrak{D}(\mathbf{Ab})$. This shows that $f_*(\mathscr{I})$ is right $\Gamma(V,-)$-acyclic, as claimed.

Since $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ sends hoinjective complexes to $\Gamma(V,-)$-acyclic ones, it is now a formal consequence of (DTC2,Theorem 6) that (27) commutes up to canonical trinatural equivalence. □

Lemma 92. *Let $f : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ and $g : (Y, \mathscr{O}_Y) \longrightarrow (Z, \mathscr{O}_Z)$ be morphisms of ringed spaces. For a complex $\mathscr{F}$ of sheaves of modules on X there is a canonical isomorphism in $\mathfrak{D}(Z)$ natural in $\mathscr{F}$*

$$\mathbb{R}(gf)_*(\mathscr{F}) \longrightarrow \mathbb{R}g_*(\mathbb{R}f_*(\mathscr{F}))$$

Proof. The key point is to show that if $\mathscr{I}$ is a hoinjective complex on X then $f_*(\mathscr{I})$ is right g_*-acyclic. Let a quasi-isomorphism $s : f_*(\mathscr{I}) \longrightarrow \mathscr{G}$ be given, and find a quasi-isomorphism $t : \mathscr{G} \longrightarrow \mathscr{J}$ with $\mathscr{J}$ hoinjective on Y. We have to show that $g_*(ts)$ is a quasi-isomorphism. By Lemma 5 it suffices to show that

$$\Gamma(W, g_*(ts)) = \Gamma(g^{-1}W, ts) \tag{28}$$

is a quasi-isomorphism of complexes of abelian groups for every open $W \subseteq Z$. But we know that $f_*(\mathscr{I})$ is $\Gamma(g^{-1}W, -)$-acyclic, from which it follows easily that $\Gamma(g^{-1}W, ts)$ must be a quasi-isomorphism (see the proof of (TRC,Theorem 116)(ii)). Therefore $g_*(ts)$ is a quasi-isomorphism, and $f_*(\mathscr{I})$ is right g_*-acyclic. It now follows formally from (DTC2,Theorem 6) that there is a canonical trinatural equivalence $\mathbb{R}(gf)_* \longrightarrow \mathbb{R}g_* \circ \mathbb{R}f_*$ as required. □

Lemma 93. *Let $f : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ and $g : (Y, \mathscr{O}_Y) \longrightarrow (Z, \mathscr{O}_Z)$ be morphisms of ringed spaces. For a complex $\mathscr{G}$ of sheaves of modules on Z there is a canonical isomorphism in $\mathfrak{D}(X)$ natural in $\mathscr{G}$*

$$\mathbb{L}f^*(\mathbb{L}g^*(\mathscr{G})) \longrightarrow \mathbb{L}(gf)^*(\mathscr{G})$$

Proof. Combining Lemma 52 and (DTC2,Theorem 8) we deduce that given arbitrary left derived functors $(\mathbb{L}(gf)^*, \xi), (\mathbb{L}f^*, \omega)$ and $(\mathbb{L}g^*, \zeta)$ there is a canonical trinatural equivalence $\theta : \mathbb{L}(f^*)\mathbb{L}(g^*) \longrightarrow \mathbb{L}(f^*g^*) = \mathbb{L}(gf)^*$. In fact θ is the unique trinatural transformation making the following diagram commute

$$\begin{array}{ccc}
\mathbb{L}(f^*)Q_Y K(g^*) & \xrightarrow{\ \omega K(g^*)\ } & Q_X K(f^*g^*) \\
\uparrow{\scriptstyle \mathbb{L}(f^*)\zeta} & & \uparrow{\scriptstyle \xi'} \\
\mathbb{L}(f^*)\mathbb{L}(g^*)Q_Z & \xrightarrow[\ \theta Q_Z\ ]{} & \mathbb{L}(gf)^*Q_Z
\end{array}$$

where ξ' is the composite of $\xi : \mathbb{L}(gf)^*Q_Z \longrightarrow Q_X K(gf)^*$ with the trinatural equivalence $Q_X K(gf)^* \cong Q_X K(f^*g^*)$ induced by the canonical natural equivalence $(gf)^* \cong f^*g^*$. □

Lemma 94. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. For complexes of sheaves of modules $\mathscr{E}, \mathscr{F}$ on X we have a canonical morphism of complexes natural in both variables*

$$f_*\mathscr{H}om^\bullet_X(\mathscr{E}, \mathscr{F}) \longrightarrow \mathscr{H}om^\bullet_Y(f_*\mathscr{E}, f_*\mathscr{F})$$

Proof. For $q \in \mathbb{Z}$ we have a morphism of sheaves of modules, using (MRS,Proposition 86)

$$\begin{aligned}
f_*\mathscr{H}om^q(\mathscr{E}, \mathscr{F}) = f_* \prod_j \mathscr{H}om(\mathscr{E}^j, \mathscr{F}^{j+q}) &= \prod_j f_*\mathscr{H}om(\mathscr{E}^j, \mathscr{F}^{j+q}) \\
&\longrightarrow \prod_j \mathscr{H}om(f_*\mathscr{E}^j, f_*\mathscr{F}^{j+q}) = \mathscr{H}om^q(f_*\mathscr{E}, f_*\mathscr{F})
\end{aligned}$$

which is clearly a morphism of complexes natural in both variables. One also checks this morphism is trinatural in $\mathscr{E}$. □

Lemma 95. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. For complexes of sheaves of modules $\mathscr{X}, \mathscr{Y}$ on X, Y respectively we have a canonical isomorphism of complexes natural in both variables*

$$f_* \mathscr{H}om^\bullet_X(f^*\mathscr{Y}, \mathscr{X}) \longrightarrow \mathscr{H}om^\bullet_Y(\mathscr{Y}, f_*\mathscr{X})$$

Proof. The isomorphism is defined as in Lemma 94, but using (MRS,Corollary 87). Again one checks that this isomorphism is natural in both variables and trinatural in $\mathscr{Y}$. □

Lemma 96. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. For complexes of sheaves of modules $\mathscr{E}, \mathscr{F}$ on Y there is a canonical morphism of complexes natural in both variables*

$$f^* \mathscr{H}om^\bullet_Y(\mathscr{E}, \mathscr{F}) \longrightarrow \mathscr{H}om^\bullet_X(f^*\mathscr{E}, f^*\mathscr{F})$$

Proof. Using the isomorphism of Lemma 95 we have a canonical morphism of complexes of sheaves of modules on Y

$$\mathscr{H}om^\bullet_Y(\mathscr{E}, \mathscr{F}) \xrightarrow{\mathscr{H}om^\bullet_Y(\mathscr{E},\eta)} \mathscr{H}om^\bullet_Y(\mathscr{E}, f_*f^*\mathscr{F}) \Longrightarrow f_*\mathscr{H}om^\bullet_X(f^*\mathscr{E}, f^*\mathscr{F})$$

which corresponds under the adjunction to a morphism of the desired form. Naturality in both variables is easily checked. □

Remark 28. Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $V \subseteq Y$ an open subset and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules on X, Y respectively with $\mathscr{X}$ hoinjective and $\mathscr{Y}$ hoflat. Then $\mathscr{H}om^\bullet_X(f^*\mathscr{Y}, \mathscr{X})$ is hoinjective by Lemma 52 and Corollary 68. It therefore follows from the proof of Lemma 91 that $f_*\mathscr{H}om^\bullet_X(f^*\mathscr{Y}, \mathscr{X})$ is right acyclic for the additive functor $\Gamma(V, -) : \mathfrak{Mod}(Y) \longrightarrow \mathbf{Ab}$. Finally from Lemma 95 we deduce that the same must be true of the complex $\mathscr{H}om^\bullet_Y(\mathscr{Y}, f_*(\mathscr{X}))$. In particular the canonical morphism in $\mathfrak{D}(Y)$

$$\Gamma(V, \mathscr{H}om^\bullet_Y(\mathscr{Y}, f_*(\mathscr{X}))) \longrightarrow \mathbb{R}\Gamma(V, \mathscr{H}om^\bullet_Y(\mathscr{Y}, f_*(\mathscr{X})))$$

is an isomorphism.

Using this observation we can upgrade Lemma 88 to its final form.

Lemma 97. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules on X, Y respectively with $\mathscr{X}$ hoinjective and $\mathscr{Y}$ hoflat. The canonical morphism in $\mathfrak{D}(Y)$*

$$\mathscr{H}om^\bullet(\mathscr{Y}, f_*(\mathscr{X})) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, f_*(\mathscr{X}))$$

is an isomorphism.

Proof. It suffices by Lemma 6 to show that for every open $V \subseteq Y$ this is an isomorphism in $\mathfrak{D}(\mathbf{Ab})$ after applying $\mathbb{R}\Gamma(V, -)$. But for each such open set we have by Proposition 75 a commutative diagram in $\mathfrak{D}(\mathbf{Ab})$

$$\begin{array}{ccc}
Hom^\bullet_V(\mathscr{Y}|_V, g_*(\mathscr{X}|_U)) & \xrightarrow{1} & \Gamma(V, \mathscr{H}om^\bullet(\mathscr{Y}, f_*(\mathscr{X}))) \\
\downarrow & & \downarrow \\
 & & \mathbb{R}\Gamma(V, \mathscr{H}om^\bullet(\mathscr{Y}, f_*(\mathscr{X}))) \\
 & & \downarrow \\
\mathbb{R}Hom^\bullet_V(\mathscr{Y}|_V, g_*(\mathscr{X}|_U)) & \longrightarrow & \mathbb{R}\Gamma(V, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, f_*(\mathscr{X})))
\end{array}$$

where $U = f^{-1}V$ and $g : U \longrightarrow V$ is the induced morphism. Using Remark 28 we have reduced to showing that the left hand vertical morphism is an isomorphism, which is Lemma 88. □

Proposition 98. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. For complexes of sheaves of modules $\mathscr{X}, \mathscr{Y}$ on X, Y respectively we have a canonical isomorphism in $\mathfrak{D}(Y)$ natural in both variables*

$$\aleph : \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{Y}, \mathscr{X}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y}, \mathbb{R}f_*\mathscr{X})$$

which makes the following diagram commute in $\mathfrak{D}(Y)$

$$\begin{array}{ccc} f_*\mathscr{H}om^\bullet_X(f^*\mathscr{Y},\mathscr{X}) & \longrightarrow & \mathscr{H}om^\bullet_Y(\mathscr{Y},f_*\mathscr{X}) \\ \downarrow & & \downarrow \\ \mathbb{R}f_*\mathscr{H}om^\bullet_X(f^*\mathscr{Y},\mathscr{X}) & & \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y},f_*\mathscr{X}) \\ \downarrow & & \\ \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(f^*\mathscr{Y},\mathscr{X}) & & \downarrow \\ \downarrow & & \\ \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{Y},\mathscr{X}) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y},\mathbb{R}f_*\mathscr{X}) \end{array} \tag{29}$$

Proof. First of all assume that $\mathscr{Y}$ is hoflat and $\mathscr{X}$ hoinjective. Then $\mathscr{H}om^\bullet_X(f^*\mathscr{Y},\mathscr{X})$ is hoinjective and using Lemma 95 and Lemma 97 we have a canonical isomorphism in $\mathfrak{D}(Y)$

$$\begin{aligned} \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{Y},\mathscr{X}) &\cong \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(f^*\mathscr{Y},\mathscr{X}) \\ &\cong \mathbb{R}f_*\mathscr{H}om^\bullet_X(f^*\mathscr{Y},\mathscr{X}) \\ &\cong f_*\mathscr{H}om^\bullet_X(f^*\mathscr{Y},\mathscr{X}) \\ &\cong \mathscr{H}om^\bullet_Y(\mathscr{Y},f_*\mathscr{X}) \\ &\cong \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y},f_*\mathscr{X}) \\ &\cong \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y},\mathbb{R}f_*\mathscr{X}) \end{aligned}$$

Given arbitrary complexes $\mathscr{X},\mathscr{Y}$ let $\mathscr{X}'\cong\mathscr{X}$ and $\mathscr{Y}'\cong\mathscr{Y}$ be arbitrary isomorphisms in $\mathfrak{D}(X),\mathfrak{D}(Y)$ respectively, with $\mathscr{X}'$ hoinjective and $\mathscr{Y}'$ hoflat, and define $\aleph_{\mathscr{X},\mathscr{Y}}$ to be the composite

$$\begin{aligned} \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{Y},\mathscr{X}) &\cong \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{Y}',\mathscr{X}') \\ &\cong \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y}',\mathbb{R}f_*\mathscr{X}') \cong \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y},\mathbb{R}f_*\mathscr{X}) \end{aligned}$$

this does not depend on the choice of isomorphisms, and is therefore canonical. This isomorphism is easily checked to be natural with respect to morphisms of $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$. The diagram (29) is natural in both variables, so in checking commutativity we may assume $\mathscr{Y}$ hoflat and $\mathscr{X}$ hoinjective, and in this case the claim is a tautology.

One also checks that $\aleph$ is local, in the following sense: given an open set $V\subseteq Y$ set $U=f^{-1}V$ and let $g:U\longrightarrow V$ be the induced morphism of ringed spaces. We claim that the following diagram commutes in $\mathfrak{D}(V)$

$$\begin{array}{ccc} \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{Y},\mathscr{X})|_V & \xrightarrow{\aleph|_V} & \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{Y},\mathbb{R}f_*\mathscr{X})|_V \\ \downarrow & & \downarrow \\ \mathbb{R}g_*\mathbb{R}\mathscr{H}om^\bullet_U(\mathbb{L}g^*(\mathscr{Y}|_V),\mathscr{X}|_U) & \xrightarrow[\aleph]{} & \mathbb{R}\mathscr{H}om^\bullet_V(\mathscr{Y}|_V,\mathbb{R}g_*(\mathscr{X}|_U)) \end{array}$$

which is straightforward to check. With a little work one can verify that $\aleph$ is trinatural in $\mathscr{Y}$, which we will make use of below. □

Lemma 99. *Let $f:X\longrightarrow Y$ be a morphism of ringed spaces. For complexes of sheaves of modules $\mathscr{E},\mathscr{F}$ on Y there is a canonical morphism in $\mathfrak{D}(X)$ natural in both variables*

$$\mathbb{L}f^*\mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{E},\mathscr{F})\longrightarrow\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{E},\mathbb{L}f^*\mathscr{F})$$

Proof. There is a canonical triadjunction between $\mathbb{L}f^*$ and $\mathbb{R}f_*$ which is determined by its unit, a trinatural transformation $\eta^\diamondsuit:1\longrightarrow\mathbb{R}f_*\circ\mathbb{L}f^*$. Using Proposition 98 we have a canonical morphism in $\mathfrak{D}(Y)$

$$\begin{array}{ccc} \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{E},\mathscr{F}) & \xrightarrow{\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\eta^\diamondsuit)} & \mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{E},\mathbb{R}f_*\mathbb{L}f^*\mathscr{F}) \\ & & \Downarrow \\ & & \mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{E},\mathbb{L}f^*\mathscr{F}) \end{array}$$

which corresponds under the adjunction to a morphism of the desired form. Naturality in both variables is easily checked. One also checks that this morphism is local, in the following sense: given an open set $V \subseteq Y$ set $U = f^{-1}V$ and let $g : U \longrightarrow V$ be the induced morphism of ringed spaces. Then one checks that

$$\begin{array}{ccc} \mathbb{L}f^*\mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{E},\mathscr{F})|_U & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{E},\mathbb{L}f^*\mathscr{F})|_U \\ \downarrow & & \downarrow \\ \mathbb{L}g^*\mathbb{R}\mathscr{H}om^\bullet_V(\mathscr{E}|_V,\mathscr{F}|_V) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet_U(\mathbb{L}g^*(\mathscr{E}|_V),\mathbb{L}g^*(\mathscr{F}|_V)) \end{array}$$

commutes in $\mathfrak{D}(U)$. Further one can verify that the morphism is trinatural in $\mathscr{E}$. □

7 Stalks and Skyscrapers

Throughout this section let $(X, \mathcal{O}_X)$ be a fixed ringed space and $x \in X$ a point, and we assume that all sheaves of modules are over X unless specified otherwise. We have by (MRS,Section 1.1) a pair of adjoint functors between $\mathfrak{Mod}(X)$ and $\mathcal{O}_{X,x}\mathbf{Mod}$

$$\mathcal{O}_{X,x}\mathbf{Mod} \underset{(-)_x}{\overset{Sky_x(-)}{\rightleftarrows}} \mathfrak{Mod}(X) \qquad (-)_x \dashv Sky_x(-) \tag{30}$$

Both of these functors are exact, so they extend to the derived categories and we have a canonical triadjunction (DTC2,Theorem 9)

$$\mathfrak{D}(\mathcal{O}_{X,x}) \underset{(-)_x}{\overset{Sky_x(-)}{\rightleftarrows}} \mathfrak{D}(X) \qquad (-)_x \dashv Sky_x(-) \tag{31}$$

One can interpret this as the adjunction between $\mathbb{L}f^*$ and $\mathbb{R}f_*$ where $f : \{x\} \longrightarrow X$ is the inclusion of a point. In this sense, the results of this section consist of translations of earlier results into a slightly different notation. But first we need to make some preliminary remarks about derived Hom for commutative rings.

7.1 Remarks on Rings

Let A be a commutative ring and set $\mathcal{A} = A\mathbf{Mod}$. This is an abelian category, so as always we have functors

$$Hom^\bullet(-,-) : \mathbf{C}(A)^{\text{op}} \times \mathbf{C}(A) \longrightarrow \mathbf{C}(\mathbf{Ab})$$
$$\mathbb{R}Hom^\bullet(-,-) : \mathfrak{D}(A)^{\text{op}} \times \mathfrak{D}(A) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

Of course given A-modules M, N the abelian group $Hom_A(M,N)$ is an A-module, and this defines a functor additive in each variable

$$Hom_A(-,-) : \mathcal{A}^{\text{op}} \times \mathcal{A} \longrightarrow \mathcal{A}$$

which is clearly homlike, in the sense of (DTC2,Definition 13). We can therefore define functors additive in each variable (DTC2,Definition 16)

$$Hom^\bullet_A(-,-) : \mathbf{C}(A)^{\text{op}} \times \mathbf{C}(A) \longrightarrow \mathbf{C}(A)$$
$$\mathbb{R}Hom^\bullet_A(-,-) : \mathfrak{D}(A)^{\text{op}} \times \mathfrak{D}(A) \longrightarrow \mathfrak{D}(A)$$

which we distinguish from the usual derived Hom with the subscript A. However this is hardly necessary, because given an assignment of hoinjectives $\mathcal{I}$ we have an equality $\mathbb{R}Hom^\bullet_{A,\mathcal{I}}(M,N) = \mathbb{R}Hom^\bullet_{\mathcal{I}}(M,N)$ in $\mathfrak{D}(\mathbf{Ab})$ natural in both variables. We have already introduced the tensor product, hoflatness and derived tensor product for a commutative ring: see Remark 9, Remark 11 and Remark 13.

Remark 29. Given a complex M of A-modules the complex $Hom^\bullet_A(A,M)$ is canonically naturally isomorphic to M. As in Lemma 25 we deduce a canonical isomorphism $\mathbb{R}Hom^\bullet_A(A,M) \cong M$ in $\mathfrak{D}(A)$ natural in M which fits into a commutative diagram

$$\begin{array}{ccc} Hom^\bullet_A(A,M) & \longrightarrow & \mathbb{R}Hom^\bullet_A(A,M) \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

and in particular the canonical morphism $Hom^\bullet_A(A,M) \longrightarrow \mathbb{R}Hom^\bullet_A(A,M)$ is an isomorphism.

Remark 30. Let $\varphi : A \longrightarrow B$ be a morphism of commutative rings and $\varphi_* : B\mathbf{Mod} \longrightarrow A\mathbf{Mod}$ the restrictions of scalars functor. Given complexes M, N of B-modules there is a canonical morphism of complexes of A-modules natural in both variables

$$\varphi_* Hom^\bullet_B(M,N) \longrightarrow Hom^\bullet_A(\varphi_* M, \varphi_* N) \tag{32}$$

Now assume that φ is an isomorphism, so that φ_* is an isomorphism of categories and (32) is an isomorphism of complexes. Observe that φ_* lifts to an isomorphism of triangulated categories $\mathfrak{D}(B) \longrightarrow \mathfrak{D}(A)$ and we have a canonical isomorphism in $\mathfrak{D}(A)$ natural in both variables

$$\varphi_* \mathbb{R}Hom^\bullet_B(M,N) \longrightarrow \mathbb{R}Hom^\bullet_A(\varphi_* M, \varphi_* N)$$

Suppose we are given a ringed space $(X, \mathcal{O}_X)$, a point $x \in X$ and an open set $x \in U \subseteq X$. Given complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules on X the canonical isomorphism $\varphi : (\mathcal{O}_X|_U)_x \longrightarrow \mathcal{O}_{X,x}$ induces a canonical isomorphism in $\mathfrak{D}(\mathcal{O}_{X,x})$

$$\varphi_* \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \longrightarrow \mathbb{R}Hom^\bullet_{(\mathcal{O}_X|_U)_x}(\varphi_* \mathscr{E}_x, \varphi_* \mathscr{F}_x) \cong \mathbb{R}Hom^\bullet_{(\mathcal{O}_X|_U)_x}((\mathscr{E}|_U)_x, (\mathscr{F}|_U)_x)$$

natural in both variables which fits into a commutative diagram

$$\begin{array}{ccc} \varphi_* Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) & \longrightarrow & Hom^\bullet_{(\mathcal{O}_X|_U)_x}((\mathscr{E}|_U)_x, (\mathscr{F}|_U)_x) \\ \downarrow & & \downarrow \\ \varphi_* \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) & \longrightarrow & \mathbb{R}Hom^\bullet_{(\mathcal{O}_X|_U)_x}((\mathscr{E}|_U)_x, (\mathscr{F}|_U)_x) \end{array}$$

Copying the proofs of Proposition 67, Proposition 69, Corollary 70 and Lemma 71 we have the following results.

Proposition 100. *For complexes of A-modules F, G, H there is a canonical isomorphism of complexes of A-modules natural in all three variables*

$$Hom^\bullet_A(F \otimes G, H) \longrightarrow Hom^\bullet_A(F, Hom^\bullet_A(G,H))$$

Proposition 101. *For complexes of A-modules X, Y, Z there is a canonical isomorphism in $\mathfrak{D}(A)$ natural in all three variables*

$$\mathbb{R}Hom^\bullet_A(X \overset{\mathbb{L}}{\otimes} Y, Z) \longrightarrow \mathbb{R}Hom^\bullet_A(X, \mathbb{R}Hom^\bullet_A(Y,Z))$$

Corollary 102. *For complexes of A-modules X, Y, Z there are canonical isomorphisms of abelian groups natural in all three variables*

$$Hom_{\mathbf{C}(A)}(X \otimes Y, Z) \longrightarrow Hom_{\mathbf{C}(A)}(X, Hom^\bullet_A(Y,Z))$$
$$Hom_{K(A)}(X \otimes Y, Z) \longrightarrow Hom_{K(A)}(X, Hom^\bullet_A(Y,Z))$$
$$Hom_{\mathfrak{D}(A)}(X \overset{\mathbb{L}}{\otimes} Y, Z) \longrightarrow Hom_{\mathfrak{D}(A)}(X, \mathbb{R}Hom^\bullet_A(Y,Z))$$

Lemma 103. *For complexes of A-modules X, Y, Z the following diagram commutes in $\mathfrak{D}(A)$*

$$\begin{array}{ccc} Hom^\bullet_A(X \otimes Y, Z) & \longrightarrow & Hom^\bullet_A(X, Hom^\bullet_A(Y,Z)) \\ \downarrow & & \downarrow \\ \mathbb{R}Hom^\bullet_A(X \underline{\otimes} Y, Z) & \longrightarrow & \mathbb{R}Hom^\bullet_A(X, \mathbb{R}Hom^\bullet_A(Y,Z)) \end{array}$$

Taking cohomology we have a commutative diagram of abelian groups

$$\begin{array}{ccc} Hom_{K(A)}(X \otimes Y, Z) & \longrightarrow & Hom_{K(A)}(X, Hom^\bullet_A(Y,Z)) \\ \downarrow & & \downarrow \\ Hom_{\mathfrak{D}(A)}(X \underline{\otimes} Y, Z) & \longrightarrow & Hom_{\mathfrak{D}(A)}(X, \mathbb{R}Hom^\bullet_A(Y,Z)) \end{array}$$

Remark 31. Let $(X, \mathcal{O}_X)$ be a ringed space, $x \in X$ a point and $\mathscr{E}, \mathscr{F}$ complexes of sheaves of modules. There is a canonical morphism of complexes of abelian groups

$$Hom^\bullet_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \longrightarrow Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \tag{33}$$

which for $q \in \mathbb{Z}$ is the product of the canonical maps $Hom_{\mathcal{O}_X}(\mathscr{E}^j, \mathscr{F}^{j+q}) \longrightarrow Hom_{\mathcal{O}_{X,x}}(\mathscr{E}^j_x, \mathscr{F}^{j+q}_x)$. In fact in (33) the left hand side is canonically a complex of $\Gamma(X, \mathcal{O}_X)$-modules and the right hand side is canonically a complex of $\mathcal{O}_{X,x}$-modules, and our morphism sends the one action to the other. Given an open set $U \subseteq X$ the diagram of complexes of abelian groups

$$\begin{array}{ccc} Hom^\bullet_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) & \longrightarrow & Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \\ \downarrow & & \Downarrow \\ Hom^\bullet_{\mathcal{O}_X|_U}(\mathscr{E}|_U, \mathscr{F}|_U) & \longrightarrow & Hom^\bullet_{(\mathcal{O}_X|_U)_x}((\mathscr{E}|_U)_x, (\mathscr{F}|_U)_x) \end{array}$$

is commutative. The morphism (33) is also compatible with (DTC2,Proposition 18) in the sense that for $n \in \mathbb{Z}$ the diagram

$$\begin{array}{ccc} H^n Hom^\bullet_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) & \longrightarrow & H^n Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \\ \downarrow & & \downarrow \\ Hom_{K(X)}(\mathscr{E}, \Sigma^n \mathscr{F}) & \longrightarrow & Hom_{K(\mathcal{O}_{X,x})}(\mathscr{E}_x, \Sigma^n \mathscr{F}_x) \end{array}$$

is commutative.

7.2 Adjunctions

Lemma 104. *Given a complex M of $\mathcal{O}_{X,x}$-modules and a complex $\mathscr{F}$ of sheaves of modules there is a canonical isomorphism of complexes natural in both variables*

$$Sky_x Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) \longrightarrow \mathscr{H}om^\bullet_X(\mathscr{F}, Sky_x(M))$$

Proof. The proof is identical to Lemma 94, *mutatis mutandis* and using (MRS,Lemma 16). One also checks that the isomorphism is trinatural in $\mathscr{F}$. □

Lemma 105. *Given complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules there is a canonical morphism of complexes of $\mathcal{O}_{X,x}$-modules natural in both variables*

$$\mathscr{H}om^\bullet_X(\mathscr{E}, \mathscr{F})_x \longrightarrow Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x)$$

Proof. The proof is identical to Lemma 96, *mutatis mutandis* and using Lemma 104. Observe that for an open neighborhood $x \in U$ we have $\Gamma(U, \mathscr{H}om^\bullet_X(\mathscr{E},\mathscr{F})) = Hom^\bullet_{\mathcal{O}_X|_U}(\mathscr{E}|_U, \mathscr{F}|_U)$ and the following diagram commutes

$$\begin{array}{ccc} Hom^\bullet_{\mathcal{O}_X|_U}(\mathscr{E}|_U, \mathscr{F}|_U) & \longrightarrow & Hom^\bullet_{(\mathcal{O}_X|_U)_x}((\mathscr{E}|_U)_x, (\mathscr{F}|_U)_x) \\ \downarrow & & \Vert \\ \mathscr{H}om^\bullet_X(\mathscr{E},\mathscr{F})_x & \longrightarrow & Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \end{array}$$

where the top morphism is the one defined in Remark 31 □

Remark 32. The adjunction (31) is particularly simple, because both triangulated functors lift exact additive functors. If we write $\eta : \mathscr{X} \longrightarrow Sky_x(\mathscr{X}_x)$ and $\varepsilon : Sky_x(M)_x \longrightarrow M$ for the unit and counit of the adjunction (30) then the unit and counit of (31) are defined by $\eta^\Diamond_{\mathscr{X}} = QK(\eta_{\mathscr{X}})$ and $\varepsilon^\Diamond_M = QK(\varepsilon_M)$.

Remark 33. The skyscraper functor $Sky_x(-) : \mathcal{O}_{X,x}\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ has an exact left adjoint, and therefore the induced functor on complexes sends hoinjective complexes of $\mathcal{O}_{X,x}$-modules to hoinjective complexes of sheaves of modules (DTC,Lemma 62). In particular given a hoinjective complex M of $\mathcal{O}_{X,x}$-modules and a complex $\mathscr{F}$ of sheaves of modules, the canonical morphisms in $\mathfrak{D}(X), \mathfrak{D}(\mathbf{Ab})$ and $\mathbf{Ab}$ respectively

$$\mathscr{H}om^\bullet_X(\mathscr{F}, Sky_x(M)) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{F}, Sky_x(M)) \tag{34}$$
$$Hom^\bullet_X(\mathscr{F}, Sky_x(M)) \longrightarrow \mathbb{R}Hom^\bullet_X(\mathscr{F}, Sky_x(M)) \tag{35}$$
$$Hom_{K(X)}(\mathscr{F}, Sky_x(M)) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{F}, Sky_x(M)) \tag{36}$$

are all isomorphisms.

Proposition 106. *Given a complex M of $\mathcal{O}_{X,x}$-modules and a complex $\mathscr{F}$ of sheaves of modules there is a canonical isomorphism in $\mathfrak{D}(X)$ natural in both variables*

$$Sky_x \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{F}, Sky_x(M))$$

which makes the following diagram commute in $\mathfrak{D}(X)$

$$\begin{array}{ccc} Sky_x Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) & \longrightarrow & \mathscr{H}om^\bullet_X(\mathscr{F}, Sky_x(M)) \\ \downarrow & & \downarrow \\ Sky_x \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, Sky_x(M)) \end{array} \tag{37}$$

Proof. First of all assume that M is hoinjective. Then using Lemma 104 we have a canonical isomorphism in $\mathfrak{D}(X)$

$$\begin{aligned} Sky_x \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) &\cong Sky_x Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{F}_x, M) \\ &\cong \mathscr{H}om^\bullet_X(\mathscr{F}, Sky_x(M)) \cong \mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{F}, Sky_x(M)) \end{aligned}$$

Given an arbitrary complex M let $M \cong M'$ be an arbitrary isomorphism in $\mathfrak{D}(\mathcal{O}_{X,x})$ with M' hoinjective, and define the morphism for M to be the composite of the obvious isomorphisms with the morphism defined for M' above. Clearly this does not depend on the choice of isomorphism, so we have defined our canonical isomorphism. Naturality in both variables is easily checked. One also checks that the isomorphism is *trinatural* in $\mathscr{F}$, by first reducing to M hoinjective and then using (37) to reduce to the trinaturality of Lemma 104. □

Lemma 107. *Given complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules there is a canonical morphism in $\mathfrak{D}(\mathcal{O}_{X,x})$ natural in both variables*

$$\mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E},\mathscr{F})_x \longrightarrow \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x)$$

which makes the following diagram commute in $\mathfrak{D}(\mathcal{O}_{X,x})$

$$\begin{array}{ccc} \mathscr{H}om^\bullet_X(\mathscr{E},\mathscr{F})_x & \longrightarrow & Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x) \\ \downarrow & & \downarrow \\ \mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E},\mathscr{F})_x & \longrightarrow & \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x) \end{array}$$

Proof. Using Proposition 106 and the unit we have a canonical morphism in $\mathfrak{D}(X)$

$$\begin{array}{ccc} \mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E},\mathscr{F}) & \xrightarrow{\mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E},\eta^\diamond)} & \mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E},Sky_x(\mathscr{F}_x)) \\ & & \Downarrow \\ & & Sky_x\mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x) \end{array}$$

which corresponds under the adjunction to a morphism of the desired form. Naturality in both variables is easily checked. Trinaturality of this isomorphism in $\mathscr{E}$ follows from trinaturality of Proposition 106 in $\mathscr{F}$. ☐

Remark 34. The morphism of Lemma 107 is local, in the following sense: let an open neighborhood $x \in U$ be given, and let $\varphi : (\mathcal{O}_X|_U)_x \longrightarrow \mathcal{O}_{X,x}$ be the canonical isomorphism of rings. The following diagrams then commute up to canonical (tri)natural equivalence

$$\begin{array}{ccc} \mathfrak{Mod}(X) & \longrightarrow & \mathcal{O}_{X,x}\mathbf{Mod} \\ {\scriptstyle (-)|_U}\downarrow & & \Downarrow{\scriptstyle \varphi_*} \\ \mathfrak{Mod}(U) & \longrightarrow & (\mathcal{O}_X|_U)_x\mathbf{Mod} \end{array} \qquad \begin{array}{ccc} \mathfrak{D}(X) & \longrightarrow & \mathfrak{D}(\mathcal{O}_{X,x}) \\ {\scriptstyle (-)|_U}\downarrow & & \Downarrow{\scriptstyle \varphi_*} \\ \mathfrak{D}(U) & \longrightarrow & \mathfrak{D}((\mathcal{O}_X|_U)_x) \end{array}$$

We claim that the following diagram commutes in $\mathfrak{D}((\mathcal{O}_X|_U)_x)$

$$\begin{array}{ccc} \varphi_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E},\mathscr{F})_x & \longrightarrow & \varphi_*\mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x) \\ \Downarrow & & \Downarrow \\ \mathbb{R}\mathscr{H}om^\bullet_U(\mathscr{E}|_U,\mathscr{F}|_U)_x & \longrightarrow & \mathbb{R}Hom^\bullet_{(\mathcal{O}_X|_U)_x}((\mathscr{E}|_U)_x,(\mathscr{F}|_U)_x) \end{array}$$

where the right hand side is the isomorphism of Remark 30. This whole diagram is natural in $\mathscr{F}$, so in checking commutativity we may assume $\mathscr{F}$ hoinjective. In that case we can use the compatibility diagrams of Lemma 107 and Remark 30 to reduce to checking commutativity of a diagram in $\mathbf{C}(U)$, which involves some calculation but is straightforward.

8 Hypercohomology

Hypercohomology extends the definition of sheaf cohomology to a *complex* of sheaves. Historically hypercohomology was first defined using a Cartan-Eilenberg resolution of the complex. The more modern definition is to take the derived functor of the global sections functor. The main technical advantage of the Cartan-Eilenberg hypercohomology is the existence of the hypercohomology spectral sequences.

Throughout this section $(X, \mathcal{O}_X)$ is a fixed ringed space and all sheaves of modules are over X, unless specified otherwise.

Definition 18. Let $U \subseteq X$ be an open subset, $\mathscr{X}$ a complex of sheaves of modules and $\Gamma(U,-) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$ the sections functor. There exists by (SS,Lemma 9) a Cartan-Eilenberg resolution $\mathscr{I}$ of $\mathscr{X}$, and we define the *Cartan-Eilenberg hypercohomology complex* of $\mathscr{X}$ over U to be

$$^h\mathbb{H}(U,\mathscr{X}) = Tot\Gamma(U,\mathscr{I})$$

where we use the coproduct totalisation of a bicomplex as defined in (DTC,Definition 33). For each $n \in \mathbb{Z}$ we define an additive functor

$$
\begin{aligned}
{}^h\mathbb{H}^n(U,-) &: \mathbf{C}(X) \longrightarrow \mathbf{Ab} \\
{}^h\mathbb{H}^n(U,\mathscr{X}) &= H^n(Tot\Gamma(U,\mathscr{I}))
\end{aligned}
$$

which in the notation of (SS,Definition 8) is the n-th hyperderived functor of $\Gamma(U,-)$. This definition depends on a choice of assignment of Cartan-Eilenberg resolutions, but is independent of this choice up to canonical natural equivalence.

Proposition 108 (Hypercohomology spectral sequences). *Given a complex of sheaves of modules $\mathscr{X}$ and an open set $U \subseteq X$ there are canonical spectral sequences ${}'E, {}''E$ starting on page zero, with*

$$
\begin{aligned}
{}'E_2^{pq} &= H^p(H^q(U,\mathscr{X})) \Longrightarrow {}^h\mathbb{H}^{p+q}(U,\mathscr{X}) \\
{}''E_2^{pq} &= H^p(U,H^q(\mathscr{X})) \Longrightarrow {}^h\mathbb{H}^{p+q}(U,\mathscr{X})
\end{aligned}
$$

Proof. This is a special case of (SS,Proposition 18). □

Now we come to the derived functor hypercohomology.

Definition 19. Let $U \subseteq X$ be an open subset, $\Gamma(U,-) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$ the sections functor and $\mathbb{R}\Gamma(U,-) : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$ a right derived functor. Given a complex $\mathscr{X}$ of sheaves of modules we define the *derived cohomology complex* of $\mathscr{X}$ over U to be

$$\mathbb{H}(U,\mathscr{X}) = \mathbb{R}\Gamma(U,\mathscr{X})$$

That is, $\mathbb{H}(U,-) : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$ is a synonym for $\mathbb{R}\Gamma(U,-)$. This triangulated functor is only determined up to canonical trinatural equivalence, but if we fix an assignment $\mathcal{I}$ of hoinjective resolutions for $\mathfrak{Mod}(X)$ then we have a canonical functor $\mathbb{H}_{\mathcal{I}}(U,-)$. As usual given $n \in \mathbb{Z}$ we write $\mathbb{H}^n(U,\mathscr{X})$ for $H^n(\mathbb{H}(U,\mathscr{X}))$. Of course if $\mathscr{I}$ is the chosen hoinjective resolution of $\mathscr{X}$ then we have $\mathbb{H}(U,\mathscr{X}) = \Gamma(U,\mathscr{I})$. By definition of a derived functor we have a natural morphism in $\mathfrak{D}(\mathbf{Ab})$

$$\zeta : \Gamma(U,\mathscr{X}) \longrightarrow \mathbb{H}(U,\mathscr{X})$$

We observed in (DCOS,Theorem 12) that in the derived category $\mathfrak{D}(X)$ the single object complex $\mathcal{O}_X$ in degree i represents the cohomology functor $H^i(X,-)$ on individual sheaves. It is therefore not surprising that it represents hypercohomology on complexes of sheaves.

Proposition 109. *Let $\mathscr{X}$ be a complex of sheaves of modules on X. For open $U \subseteq X$ and $i \in \mathbb{Z}$ there is a canonical isomorphism of abelian groups natural in $\mathscr{X}$*

$$\alpha : Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i\mathscr{X}) \longrightarrow \mathbb{H}^i(U,\mathscr{X})$$

Proof. Choose an isomorphism $\mathscr{I} \cong \mathscr{X}$ in $\mathfrak{D}(X)$ with $\mathscr{I}$ hoinjective. Then by (DTC,Corollary 50) and Proposition 9 we have a canonical isomorphism

$$
\begin{aligned}
Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i\mathscr{X}) &\cong Hom_{\mathfrak{D}(X)}(\mathcal{O}_U, \Sigma^i\mathscr{I}) \\
&\cong Hom_{K(X)}(\mathcal{O}_U, \Sigma^i\mathscr{I}) \\
&\cong H^i\Gamma(U,\mathscr{I}) \cong \mathbb{H}^i(U,\mathscr{I}) \\
&\cong \mathbb{H}^i(U,\mathscr{X})
\end{aligned}
$$

One checks that this is independent of the chosen isomorphism, and therefore canonical. Naturality is also easily checked. □

Lemma 110. *Let $U \subseteq X$ be an open subset and $\mathscr{F}$ a sheaf of modules on X. There is a canonical isomorphism of abelian groups $H^i(U,\mathscr{F}) \longrightarrow \mathbb{H}^i(U,\mathscr{F})$ natural in $\mathscr{F}$.*

There are several occasions earlier in these notes where we have already used hypercohomology. Take for example Proposition 75 and Lemma 91. We now give a different proof of the latter result as an example of how Proposition 109 is useful.

Lemma 111. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{X}$ a complex of sheaves of modules on X. Then for open $V \subseteq Y$ there is a canonical isomorphism of abelian groups natural in $\mathscr{X}$*

$$\mathbb{H}^i(V, \mathbb{R}f_*\mathscr{X}) \longrightarrow \mathbb{H}^i(f^{-1}V, \mathscr{X})$$

In particular if $\mathscr{F}$ is a sheaf of modules we have a canonical isomorphism

$$\mathbb{H}^i(V, \mathbb{R}f_*\mathscr{F}) \longrightarrow H^i(f^{-1}V, \mathscr{F})$$

Proof. Since $\mathcal{O}_V$ is flat, we have

$$\begin{aligned}\mathbb{H}^i(V, \mathbb{R}f_*\mathscr{X}) &\cong Hom_{\mathfrak{D}(Y)}(\Sigma^{-i}\mathcal{O}_V, \mathbb{R}f_*\mathscr{X})\\ &\cong Hom_{\mathfrak{D}(X)}(\mathbb{L}f^*(\Sigma^{-i}\mathcal{O}_V), \mathscr{X})\\ &\cong Hom_{\mathfrak{D}(X)}(\Sigma^{-i}f^*\mathcal{O}_V, \mathscr{X})\\ &\cong Hom_{\mathfrak{D}(X)}(\mathcal{O}_{f^{-1}V}, \Sigma^i\mathscr{X})\\ &\cong \mathbb{H}^i(f^{-1}V, \mathscr{X})\end{aligned}$$

as claimed. □

Proposition 112 (Mayer-Vietoris sequence). *Let $X = U \cup V$ be an open cover and $\mathscr{X}$ a complex of sheaves of modules. There is a canonical triangle in $\mathfrak{D}(\mathbf{Ab})$ natural in $\mathscr{X}$*

$$\mathbb{H}(X, \mathscr{X}) \longrightarrow \mathbb{H}(U, \mathscr{X}|_U) \oplus \mathbb{H}(V, \mathscr{X}|_V) \longrightarrow \mathbb{H}(U \cap V, \mathscr{X}|_{U\cap V}) \longrightarrow \Sigma\mathbb{H}(X, \mathscr{X}) \qquad (38)$$

and therefore a natural long exact sequence of abelian groups

$$\begin{aligned}\cdots \longrightarrow \mathbb{H}^n(X, \mathscr{X}) \longrightarrow \mathbb{H}^n(U, \mathscr{X}|_U) \oplus \mathbb{H}^n(V, \mathscr{X}|_V)&\\ \longrightarrow \mathbb{H}^n(U \cap V, \mathscr{X}|_{U\cap V}) \longrightarrow \mathbb{H}^{n+1}(X, \mathscr{X}) \longrightarrow \cdots&\end{aligned}$$

Proof. Let $W \subseteq X$ be an open subset with inclusion $i : W \longrightarrow X$. We checked in the proof of Lemma 91 that i_* sends hoinjective complexes to $\Gamma(X, -)$-acyclic ones, so it follows by the standard argument that $\mathbb{H}(X, -) \circ \mathbb{R}i_* = \mathbb{H}(W, -)$. If we now apply $\mathbb{H}(X, -)$ to the triangle of Lemma 21 we obtain the desired triangle (38). Taking cohomology we deduce the long exact sequence. □

We have now defined the Cartan-Eilenberg and derived functor hypercohomology and given some properties of each. It is useful to know when these two types of hypercohomology agree. It is known that they do not agree in general (see the appendix to [Wei94]), but for bounded below complexes they agree. Actually over a quasi-compact separated scheme the two types of hypercohomology agree for all complexes with quasi-coherent cohomology [Kel98] but we do not include the proof here.

Proposition 113. *Let $\mathcal{A}$ be an abelian category with enough injectives and Y a bounded below complex in $\mathcal{A}$. Given a Cartan-Eilenberg resolution I of Y the canonical morphism $Y \longrightarrow Tot(I)$ is a hoinjective resolution.*

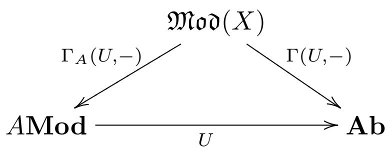
Proof. Any two Cartan-Eilenberg resolutions yield totalisations isomorphic in $K(\mathcal{A})$, so we may as well assume that the columns $I^{p,\bullet}$ are zero for all sufficiently large negative p. Then $Tot(I)^n$ is a finite coproduct of injectives, and is therefore itself injective, so $Tot(I)$ is a bounded below complex of injectives and hence hoinjective. For the proof that the canonical morphism $Y \longrightarrow Tot(I)$ is a quasi-isomorphism, we refer to [Ver96] Proposition III 4.6.8. □

Proposition 114. *Let $U \subseteq X$ be an open subset and $\mathscr{X}$ a bounded below complex of sheaves of modules. There is an isomorphism ${}^h\mathbb{H}^n(U, \mathscr{X}) \longrightarrow \mathbb{H}^n(U, \mathscr{X})$ natural in $\mathscr{X}$.*

Proof. Fix an assignment of Cartan-Eilenberg resolutions for $\mathfrak{Mod}(X)$ and also an assignment of hoinjective resolutions, to calculate the two additive functors ${}^h\mathbb{H}^n(U,-), \mathbb{H}^n(U,-)$ between $\mathbf{C}(X)$ and $\mathbf{Ab}$. Then we claim these functors are canonically naturally equivalent. We may as well assume that the Cartan-Eilenberg resolution assigned to any bounded below complex is itself bounded below in the same way.

Let $\mathscr{X}$ be a complex in $\mathcal{A}$ with chosen hoinjective resolution $\mathscr{X} \longrightarrow J$ and chosen Cartan-Eilenberg resolution I. By Proposition 113 we have a hoinjective resolution $\mathscr{X} \longrightarrow Tot(I)$, and therefore by (DTC2,Remark 3) a canonical isomorphism $\mathbb{H}(U,\mathscr{X}) \cong \Gamma(U, Tot(I))$ in $\mathfrak{D}(\mathbf{Ab})$. Since all involved coproducts are finite we have $\Gamma(U, Tot(I)) \cong Tot\Gamma(U, I)$ (as complexes) and therefore an isomorphism $\mathbb{H}(U,\mathscr{X}) \cong Tot\Gamma(U,I) = {}^h\mathbb{H}(U,\mathscr{X})$ in $\mathfrak{D}(\mathbf{Ab})$. Taking cohomology yields the desired natural isomorphisms. □

Remark 35. In our notes on Cohomology of Sheaves (COS) there is a detailed discussion of the induced module structure on sheaf cohomology groups (COS,Section 1.2). For hypercohomology the issue is completely trivial. Given open $U \subseteq X$ we set $A = \Gamma(U, \mathcal{O}_X)$ so that we have a commutative diagram



Let $\mathbb{H}_A(U,-)$ and $\mathbb{H}(U,-)$ denote right derived functors of $\Gamma_A(U,-)$ and $\Gamma(U,-)$ respectively. Since the forgetful functor $U : A\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is exact it lifts to the derived category, and we deduce a canonical trinatural equivalence $\mathbb{H}(U,-) \cong U \circ \mathbb{H}_A(U,-)$ (DTC2,Corollary 7). Thus in calculating hypercohomology it is irrelevant whether we use A-modules or abelian groups (observe that this deduction is much more cumbersome for Cartan-Eilenberg hypercohomology). We also remark that Proposition 109 can be upgraded to an isomorphism of A-modules in the case where $U = X$.

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