Derived Categories Of Quasi-coherent Sheaves

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In this note we give a careful exposition of the basic properties of derived categories of quasi-coherent sheaves on a scheme. This includes Neeman's version of Grothendieck duality [Nee96] and the proof that every complex with quasi-coherent cohomology is isomorphic to a complex of quasi-coherent sheaves in the derived category.

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1 Introduction

All notation and conventions are from our notes on Derived Categories (DTC) or Derived Categories of Sheaves (DCOS). In particular we assume that every abelian category comes with canonical structures that allow us to define the cohomology of chain complexes in an unambiguous way. If we write *complex* we mean *cochain complex*. As usual we write $A = 0$ to indicate that A is a zero object (not necessarily the canonical one). We say that a complex is *exact* if all its cohomology objects are zero, and reserve the label *acyclic* for complexes described in (DTC2,Definition 4).

Given a scheme X we have the abelian category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X, whose derived category we denote by $\mathfrak{Dqcoh}(X)$. If X is concentrated (CON,Definition 3) then $\mathfrak{Qco}(X)$ is grothendieck abelian (MOS,Proposition 66) so this class of schemes is prevalent when dealing with derived functors defined on quasi-coherent sheaves. This is a very mild condition to put on a scheme: for example, any noetherian or affine scheme is concentrated.

In this note we develop the basic theory of the triangulated categories $\mathfrak{Dqcoh}(X)$. The major theorems are as follows:

- (Theorem 31) Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes. Then the following diagram commutes up to canonical trinatural equivalence

$$\begin{array}{ccc} \mathfrak{D}(X) & \xrightarrow{\mathbb{R}f_*} & \mathfrak{D}(Y) \\ \uparrow & & \uparrow \\ \mathfrak{Dqcoh}(X) & \xrightarrow[\mathbb{R}_q f_*]{} & \mathfrak{Dqcoh}(Y) \end{array}$$

  Outline of Proof: Show that both ways around the diagram are bounded (Corollary 22 and Proposition 30) thereby reducing to checking for quasi-coherent *sheaves*.

- (Theorem 39) Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes. The triangulated functor $\mathbb{R}_q f_* : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(Y)$ has a right adjoint. This is the Grothendieck duality theorem of Neeman [Nee96].

- (Theorem 42) For a quasi-compact semi-separated scheme X the canonical triangulated functor $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful, and induces an equivalence $\mathfrak{Dqcoh}(X) \cong \mathfrak{D}_{qc}(X)$. This is Bökstedt and Neeman's [BN93] Corollary 5.5.

- (Theorem 63) For a quasi-compact semi-separated scheme X with an ample family of invertible sheaves the triangulated category $\mathfrak{Dqcoh}(X)$ is compactly generated and the compact objects are precisely the perfect complexes. More generally [Nee96] shows how to remove the ampleness hypothesis.

- (Theorem 102) For an irreducible quasi-compact semi-separated scheme X with an ample family of invertible sheaves the units in $\mathfrak{Dqcoh}(X)$ under the derived tensor product are precisely the shifts of invertible sheaves. The ampleness hypothesis can again be removed with a little more background.

See (CON,Definition 4) for the definition of a *semi-separated* scheme, a *semi-separating cover* and a *semi-separating affine basis*. A quasi-compact semi-separated scheme is concentrated, and a separated scheme is semi-separated. For the definition of *hoinjective*, *hoprojective* and *hoflat* complexes see (DTC,Definition 24) and (DCOS,Definition 9). There are various other names for these complexes in current usage (DTC,Remark 20), the most common probably being *K-injective*, *K-projective* and *K-flat*.

1.1 Remarks on Noetherian Assumptions

The original work on Grothendieck duality [Har66] relied heavily on noetherian hypotheses, which were later removed after substantial effort by Lipman [Lip]. The approach we take in these notes is that of Neeman [Nee96], which goes via Brown representability and also manages to avoid any noetherian conditions. Since this is an important point in the literature, it is worth taking a moment here to describe what the noetherian condition buys you.

If X is a scheme then we have an exact functor $(-)|_U : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(U)$ and the induced coproduct preserving triangulated functor

$$(-)|_U : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(U)$$

If a complex $\mathscr{F}$ of quasi-coherent sheaves of modules on X is hoinjective as an object of $K(\mathfrak{Qco}(X))$, it is not necessarily hoinjective as an object of $K(X)$. Moreover, the restriction $\mathscr{F}|_U$ is not necessarily hoinjective in $K(\mathfrak{Qco}(U))$, because the left adjoint to restriction (extension by zero) does not necessarily preserve quasi-coherence. If we say that a quasi-coherent sheaf of modules is *injective*, then we mean it is injective in $\mathfrak{Mod}(X)$ unless there is some indication otherwise.

If X is a noetherian scheme then the grothendieck abelian category $\mathfrak{Qco}(X)$ is locally noetherian (MOS,Remark 10). Moreover a quasi-coherent sheaf $\mathscr{I}$ is injective in $\mathfrak{Qco}(X)$ if and only if it is injective in $\mathfrak{Mod}(X)$. If $\mathscr{I}$ is injective in $\mathfrak{Qco}(X)$ and $U \subseteq X$ is open, then $\mathscr{I}|_U$ is injective in $\mathfrak{Qco}(U)$ (MOS,Corollary 69).

It has been known for a long time, and particularly since [Har66] that quasi-coherent injectives on noetherian schemes are very well-behaved. They are so nice that we can use global resolutions and develop the properties of $\mathfrak{Dqcoh}(X)$ in exactly the same way as we studied $\mathfrak{D}(X)$ in our notes on Derived Categories of Sheaves (DCOS). See our notes on Derived Categories of Quasi-coherent Sheaves on a Noetherian Scheme (DCOQSN) for this development.

For non-noetherian schemes injectives and hoinjectives are less useful: they are not even stable under restriction. This explains why the general theory has a very different flavour. Whereas the proofs in the noetherian case rely on finding and using global resolutions, the proofs in the general case have the following recipe:

- Reduce to the affine case using induction on minimal affine covers, Čech resolutions (which to some extent replace hoinjective resolutions), and the Čech triangles of Section 4.2.
- In the affine case one relies on the homotopy theoretic techniques introduced by Bökstedt and Neeman [BN93], and the fact that $\mathfrak{Qco}(X) \cong R\mathbf{Mod}$ has exact products. This allows us to take homotopy inverse limits and reduce to sheaves instead of complexes.
- Once we have reduced to individual quasi-coherent sheaves the proof is typically finished by an application of some deep result of algebraic geometry. The reader can usually find these results in EGA or sometimes [Har77], and also in our notes on Cohomology of Sheaves (COS) and Higher Direct Images of Sheaves (HDIS). Since we are not working on noetherian schemes we usually need the "hard" version of these results.

1.2 Basic Properties

Lemma 1. *Let X be a scheme and $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism in $\mathfrak{Dqcoh}(X)$. If $\{V_i\}_{i \in I}$ is a nonempty open cover of X then ψ is an isomorphism if and only if $\psi|_{V_i}$ is an isomorphism in $\mathfrak{Dqcoh}(V_i)$ for every $i \in I$.*

Lemma 2. *Let X be a scheme, $\{\mathscr{F}_\lambda\}_{\lambda \in \Lambda}$ a nonempty family of complexes of quasi-coherent sheaves on X and $\{u_\lambda : \mathscr{F}_\lambda \longrightarrow \mathscr{F}\}_{\lambda \in \Lambda}$ a family of morphisms in $\mathfrak{Dqcoh}(X)$. If $\{V_i\}_{i \in I}$ is a nonempty open cover of X then the u_λ are a coproduct in $\mathfrak{Dqcoh}(X)$ if and only if the morphisms*

$$u_\lambda|_{V_i} : \mathscr{F}_\lambda|_{V_i} \longrightarrow \mathscr{F}|_{V_i}$$

are a coproduct in $\mathfrak{Dqcoh}(V_i)$ for each $i \in I$.

Remark 1. Let A be a commutative ring and X a scheme over $Spec(A)$. The abelian category $\mathfrak{Mod}(X)$ is A-linear and therefore for complexes $\mathscr{F}, \mathscr{G} \in \mathbf{C}(X)$ the complex $Hom^\bullet(\mathscr{F}, \mathscr{G})$ is canonically a complex of A-modules. That is, we have a functors additive in each variable

$$Hom(-,-) : \mathfrak{Mod}(X)^{\text{op}} \times \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$$
$$Hom^\bullet(-,-) : \mathbf{C}(X)^{\text{op}} \times \mathbf{C}(X) \longrightarrow \mathbf{C}(A)$$

The former functor is homlike, so define a functor triangulated in each variable (DTC2,Definition 16)

$$\mathbb{R}Hom^\bullet_A(-,-) : \mathfrak{D}(X)^{\text{op}} \times \mathfrak{D}(X) \longrightarrow \mathfrak{D}(A)$$

which we distinguish from the usual derived Hom with the subscript A. Given an assignment of hoinjectives $\mathcal{I}$ we have an equality $\mathbb{R}Hom^\bullet_{A,\mathcal{I}}(\mathscr{F}, \mathscr{G}) = \mathbb{R}Hom^\bullet_{\mathcal{I}}(\mathscr{F}, \mathscr{G})$ in $\mathfrak{D}(\mathbf{Ab})$ natural in both variables. With this introduction, observe that the second isomorphism of (DCOS,Proposition 69) is actually a natural isomorphism in $\mathfrak{D}(A)$.

2 Derived Structures

2.1 Derived Direct Image

Definition 1. Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes with X concentrated. Since $\mathfrak{Qco}(X)$ is grothendieck abelian the functor $f_* : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$ has a right derived functor

$$\mathbb{R}_q f_* : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(Y)$$

This is only determined up to canonical trinatural equivalence, but for an assignment $\mathcal{I}$ of hoinjective resolutions for $\mathfrak{Qco}(X)$ we have a canonical right derived functor $\mathbb{R}_{q,\mathcal{I}} f_*$. We introduce the subscript q to remind ourselves that in general, $\mathbb{R}_q f_*$ and $\mathbb{R} f_*$ are different functors.

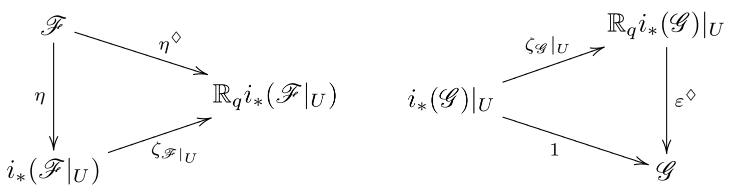
Lemma 3. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes with X concentrated and let $i : U \longrightarrow X$ be the inclusion of a quasi-compact open subset. There is a canonical trinatural equivalence*

$$\theta : \mathbb{R}_q(fi)_* \longrightarrow \mathbb{R}_q f_* \circ \mathbb{R}_q i_*$$

Proof. The functor $i_* : \mathfrak{Qco}(U) \longrightarrow \mathfrak{Qco}(X)$ has an exact left adjoint (restriction), so $K(i_*) : K(\mathfrak{Qco}(U)) \longrightarrow K(\mathfrak{Qco}(X))$ preserves hoinjectives (DTC,Lemma 62). From (DTC2,Theorem 6) we deduce the required trinatural equivalence. □

Lemma 4. *Let X be a concentrated scheme and $i : U \longrightarrow X$ the inclusion of a quasi-compact open subset. Then $\mathbb{R}_q i_* : \mathfrak{Dqcoh}(U) \longrightarrow \mathfrak{Dqcoh}(X)$ is fully faithful.*

Proof. Let $(\mathbb{R}_q i_*, \zeta)$ be any right derived functor. The functor $(-)|_U : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(U)$ is an exact left adjoint to i_* so it follows from (DTC2,Lemma 10) that $(-)|_U : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(U)$ is canonically left triadjoint to $\mathbb{R}_q i_*$. The unit $\eta^\diamond : 1 \longrightarrow \mathbb{R}_q(i_*) \circ (-)|_U$ and counit $\varepsilon^\diamond : (-)|_U \circ \mathbb{R}_q(i_*) \longrightarrow 1$ are the unique trinatural transformations making the following diagram commute for every complex $\mathscr{F}$ of quasi-coherent sheaves on X and complex $\mathscr{G}$ of quasi-coherent sheaves on U



One checks as in (DCOS,Lemma 17) that $\varepsilon^\diamond$ is a natural equivalence, and therefore $\mathbb{R}_q i_*$ is fully faithful. Further we deduce that $\zeta_{\mathscr{G}}|_U : \mathscr{G} \longrightarrow \mathbb{R}_q i_*(\mathscr{G})|_U$ is an isomorphism in $\mathfrak{Dqcoh}(U)$ for any complex $\mathscr{G}$ of quasi-coherent sheaves on U, and also that $\eta^\diamond|_U : \mathscr{F}|_U \longrightarrow \mathbb{R}_q i_*(\mathscr{F}|_U)|_U$ is an isomorphism in $\mathfrak{Dqcoh}(U)$ for any complex $\mathscr{F}$ of quasi-coherent sheaves on X. □

For the next result we introduce some notation. Let X be a concentrated scheme. For every affine open subset with inclusion $i : V \longrightarrow X$ we have a complex of quasi-coherent sheaves $\mathbb{R}_q i_*(\mathscr{F})$ on X for every complex of quasi-coherent sheaves $\mathscr{F}$ on V. Let $\Omega(X)$ denote the smallest triangulated subcategory of $\mathfrak{Dqcoh}(X)$ containing these $\mathbb{R}_q i_*(\mathscr{F})$ as V ranges over all affine open subsets of X. The proof of the following result is due to Neeman.

Proposition 5. *Let X be a quasi-compact semi-separated scheme. Then the triangulated category $\mathfrak{Dqcoh}(X)$ is generated by the objects of $\mathfrak{Dqcoh}(V)$ for affine open $V \subseteq X$. That is, we have $\mathfrak{Dqcoh}(X) = \Omega(X)$.*

Proof. Given a quasi-compact semi-separated scheme X let $n(X)$ denote the smallest number of affine open subsets that you can use to form a semi-separating cover of X. That is, the smallest integer $n \geq 1$ for which we can write $X = X_1 \cup \cdots \cup X_n$ for affine open subsets X_i such that the intersections $X_i \cap X_j$ are all affine. We proceed by induction on $n = n(X)$. The case $n = 1$ is trivial, so assume that $n(X) > 1$ with $X = X_1 \cup \cdots \cup X_n$ and set $U = X_1, V = X_2 \cup \cdots \cup X_n$. Then V is a quasi-compact semi-separated scheme with $n(V) < n$ so the result is true for V. Let $j : V \longrightarrow X$ and $i : U \longrightarrow X$ be the inclusions.

Let $\mathscr{G}$ be a complex of quasi-coherent sheaves on X. We have to show that $\mathscr{G} \in \Omega(X)$. It follows easily from Lemma 3 and (DTC,Remark 51) that $\mathbb{R}_q j_* \Omega(V) \subseteq \Omega(X)$ and in particular $\mathbb{R}_q j_*(\mathscr{G}|_V)$ belongs to $\Omega(X)$, since by the inductive hypothesis $\Omega(V) = \mathfrak{Dqcoh}(V)$. Extend the unit $\mathscr{G} \longrightarrow \mathbb{R}_q j_*(\mathscr{G}|_V)$ to a triangle in $\mathfrak{Dqcoh}(X)$

$$\mathscr{H} \longrightarrow \mathscr{G} \longrightarrow \mathbb{R}_q j_*(\mathscr{G}|_V) \longrightarrow \Sigma\mathscr{H} \tag{1}$$

Since $\Omega(X)$ is triangulated we need only show that $\mathscr{H}$ belongs to $\Omega(X)$, which we do by showing that the unit $\mathscr{H} \longrightarrow \mathbb{R}_q i_*(\mathscr{H}|_U)$ is an isomorphism. This restricts to an isomorphism on U, so by Lemma 2 it is enough to show that it restricts to an isomorphism in $\mathfrak{Dqcoh}(W)$ for every affine open $W \subseteq V$. From the triangle (1) we see that $\mathscr{H}|_V = 0$ in $\mathfrak{Dqcoh}(V)$, and in particular $\mathscr{H}|_W = 0$ in $\mathfrak{Dqcoh}(W)$, so what we have to show is that $\mathbb{R}_q i_*(\mathscr{H}|_U)|_W = 0$ in $\mathfrak{Dqcoh}(W)$.

Since the X_i form a semi-separating cover, the inclusion $i : U \longrightarrow X$ is affine and therefore $i_* : \mathfrak{Qco}(U) \longrightarrow \mathfrak{Qco}(X)$ is exact. We can therefore replace $\mathbb{R}_q i_*$ by the usual direct image i_*. In that case $i_*(\mathscr{H}|_U)|_W = k_*(\mathscr{H}|_{U\cap W}) = 0$ in $\mathfrak{Dqcoh}(W)$ where $k : U \cap W \longrightarrow W$ is the inclusion (which, as the pullback of i, is also affine). This is what we wanted to show, so the proof is complete. □

Remark 2. Let X be a quasi-compact *separated* scheme. Then for affine $V \subseteq X$ the inclusion $i : V \longrightarrow X$ is affine and therefore i_* is exact. It follows that in this case $\Omega(X)$ is the smallest triangulated subcategory of $\mathfrak{Dqcoh}(X)$ containing $i_*\mathscr{F}$ for every complex of quasi-coherent sheaves of modules $\mathscr{F}$ and inclusion $i : V \longrightarrow X$ of an affine open subset.

When we proved that the derived image functor was local in (DCOS,Lemma 19) and (DCOQSN,Lemma 3) the key point both times was that restricting some resolution on X was g_*-acyclic. Generally such a resolution is not available, but by being clever and reducing to the case where g is affine (so g_* is exact and *everything* is g_*-acyclic) we can avoid the need for one altogether.

Lemma 6. *Let $f : X \longrightarrow Y$ be a morphism of separated schemes with X quasi-compact, and $V \subseteq Y$ a quasi-compact open subset. Then for any complex $\mathscr{F}$ of quasi-coherent sheaves of modules on X there is a canonical isomorphism in $\mathfrak{Dqcoh}(V)$ natural in $\mathscr{F}$*

$$\mu : (\mathbb{R}_q f_* \mathscr{F})|_V \longrightarrow \mathbb{R}_q g_*(\mathscr{F}|_U)$$

where $U = f^{-1}V$ and $g : U \longrightarrow V$ is the induced morphism of schemes.

Proof. To begin with all we assume is that we are given a concentrated morphism of schemes $f : X \longrightarrow Y$ and an open subset $V \subseteq Y$ such that X and $U = f^{-1}V$ are concentrated. Then we have a commutative diagram of additive functors

$$\begin{array}{ccc} \mathfrak{Qco}(X) & \xrightarrow{f_*} & \mathfrak{Qco}(Y) \\ {\scriptstyle (-)|_U}\downarrow & & \downarrow{\scriptstyle (-)|_V} \\ \mathfrak{Qco}(U) & \xrightarrow[g_*]{} & \mathfrak{Qco}(V) \end{array}$$

Choosing arbitrary right derived functors $(\mathbb{R}_q f_*, \zeta), (\mathbb{R}_q g_*, \omega)$ we have a diagram of triangulated functors

$$\begin{array}{ccc} \mathfrak{Dqcoh}(X) & \xrightarrow{\mathbb{R}_q f_*} & \mathfrak{Dqcoh}(Y) \\ {\scriptstyle (-)|_U}\downarrow & & \downarrow{\scriptstyle (-)|_V} \\ \mathfrak{Dqcoh}(U) & \xrightarrow[\mathbb{R}_q g_*]{} & \mathfrak{Dqcoh}(V) \end{array}$$

By (DTC2,Corollary 7) the pair $((-)|_V \mathbb{R}_q f_*, (-)_V \zeta)$ is a right derived functor of the functor $(-)|_V \circ f_* = g_* \circ (-)|_U$. On the other hand we have the trinatural transformation

$$\omega(-)|_U : Q_V K((-)|_V \circ f_*) = Q_V K(g_* \circ (-)|_U) \longrightarrow \mathbb{R}_q(g_*)(-)|_U Q_X$$

which induces a unique trinatural transformation $\mu : (-)|_V \mathbb{R}_q f_* \longrightarrow \mathbb{R}_q(g_*)(-)|_U$ such that $\mu Q_X \circ (-)|_V \zeta = \omega(-)|_U$. Observe that if g is affine then g_* is exact, so that every complex in $\mathfrak{Qco}(U)$ is g_*-acyclic and by the argument of (DCOS,Lemma 19) we deduce that μ is a natural equivalence. We want to show that under the hypotheses given in the statement of the lemma, μ is a natural equivalence.

The morphism f is quasi-compact and separated (in particular concentrated) by the standard results (CON,Proposition 4)(v) and (SPM,Proposition 3)(v) and the schemes X, U are concentrated. Let $\mathcal{S}$ be the full subcategory of $\mathfrak{Dqcoh}(X)$ consisting of complexes $\mathscr{F}$ such that $\mu_{\mathscr{F}}$ is an isomorphism in $\mathfrak{Dqcoh}(V)$. This is a triangulated subcategory of $\mathfrak{Dqcoh}(X)$ (TRC,Remark 30), so to show that μ is a natural equivalence it suffices by Proposition 5 to show that it is an isomorphism for any object of $\mathfrak{Dqcoh}(W)$ with $W \subseteq X$ affine.

So let $j : W \longrightarrow X$ be the inclusion of an affine open subset. Since X is separated this morphism is affine, and in particular if we form the following two pullback diagrams

$$\begin{array}{ccccc} W & \xrightarrow{j} & X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow & & \uparrow \\ U \cap W & \xrightarrow[k]{} & U & \xrightarrow[g]{} & V \end{array}$$

then k is affine. Moreover $f \circ j$ is a morphism from an affine scheme to a separated scheme, and is therefore affine (SPM,Proposition 21). We deduce from the outer pullback that $g \circ k$ is affine. Let $\mathscr{H}$ be a complex of quasi-coherent sheaves on W. Since $k, g \circ k$ are affine we have using (DCOQS,Lemma 3) an isomorphism in $\mathfrak{Dqcoh}(V)$

$$\begin{aligned} \mathbb{R}_q f_*(\mathbb{R}_q j_* \mathscr{H})|_V &\cong \mathbb{R}_q(fj)_*(\mathscr{H})|_V \\ &\cong \mathbb{R}_q(gk)_*(\mathscr{H}|_{U \cap W}) \\ &\cong \mathbb{R}_q g_*(\mathbb{R}_q k_*(\mathscr{H}|_{U \cap W})) \\ &\cong \mathbb{R}_q g_*(\mathbb{R}_q j_*(\mathscr{H})|_U) \end{aligned}$$

One checks that this is μ evaluated on $\mathbb{R}_q j_*(\mathscr{H})$, which shows that μ is a trinatural equivalence and completes the proof. $\square$

2.2 Derived Sheaf Hom

Throughout this section let X be a concentrated scheme. By (MOS,Definition 3) the inclusion $i : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ has a right adjoint Q called the *coherator*. Throughout this section we fix a particular right adjoint Q, and whenever we say that some construction on X is "canonical" we mean that it is canonical after this choice is fixed. In this sense we have a canonical triadjunction between the canonical functor $i : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ and $\mathbb{R}Q : \mathfrak{D}(X) \longrightarrow \mathfrak{Dqcoh}(X)$.

The problem with defining $\mathbb{R}\mathscr{H}om^\bullet(-,-)$ on $\mathfrak{Dqcoh}(X)$ is that in general $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is not quasi-coherent, even if both sheaves $\mathscr{F},\mathscr{G}$ are. Some further finiteness hypothesis on $\mathscr{F}$ is required. So to define the sheaf Hom on $\mathfrak{Dqcoh}(X)$ we take the sheaf Hom on $\mathfrak{D}(X)$ and coherate.

Definition 2. Fix an assignment of hoinjectives $\mathcal{I}$ for $\mathfrak{Mod}(X)$ and define $\mathbb{R}_{q,\mathcal{I}}\mathscr{H}om^\bullet(-,-)$ to be the following functor triangulated in each variable

$$\mathbb{R}Q \circ \mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(-,-) : \mathfrak{Dqcoh}(X)^{\mathrm{op}} \times \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(X)$$

As usual we will usually drop the subscript $\mathcal{I}$ and just write $\mathbb{R}_q\mathscr{H}om^\bullet(-,-)$ when there is no chance of confusion.

2.3 Derived Inverse Image

If $f : X \longrightarrow Y$ is a morphism of schemes then the additive functor $f^* : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ has a left derived functor $\mathbb{L}f^*$ (DCOS,Section 6). To show that $\mathbb{L}f^*$ existed we showed that every hoflat complex was acyclic for f^*, and also that every complex of sheaves of modules admitted a quasi-isomorphism from a hoflat complex. This latter fact follows from our ability to write every sheaf as the quotient of a flat sheaf (since the generators $\mathcal{O}_U$ of $\mathfrak{Mod}(X)$ are flat).

If $\mathscr{G}$ is a quasi-coherent sheaf on Y then $f^*\mathscr{G}$ is quasi-coherent, so we have an additive functor $f^* : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Qco}(X)$. In trying to define a left derived functor $\mathbb{L}_q f^*$ we have a problem: the sheaves $\mathcal{O}_U$ are not generally quasi-coherent, so it is not immediately clear how to write a quasi-coherent sheaf as a quotient of a flat quasi-coherent sheaf. At least on a quasi-compact semi-separated scheme this is always possible, and we direct the careful reader to the proof in Section 2.4.

Remark 3. Alternatively let X be a concentrated scheme that admits an ample family of invertible sheaves (AMF,Definition 2). Then every quasi-coherent sheaf can be written as a quotient of a locally free (hence flat and quasi-coherent) sheaf (AMF,Corollary 7). Note that any quasi-projective variety has an ample invertible sheaf (BU,Lemma 17), so the reader only interested in varieties can safely skip Section 2.4.

Definition 3. Let X be a scheme and $\mathscr{F}$ a complex of quasi-coherent sheaves. We say that $\mathscr{F}$ is *homotopy flat* (or *hoflat*) if it is hoflat as a complex of sheaves of modules in the sense of (DCOS,Definition 9). The hoflat complexes of quasi-coherent sheaves form a thick localising subcategory of $K(\mathfrak{Qco}(X))$. We say that X *has enough quasi-coherent hoflats* if every complex $\mathscr{F}$ in $\mathfrak{Qco}(X)$ admits a quasi-isomorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ a hoflat complex of quasi-coherent sheaves.

Lemma 7. *Let X be a concentrated scheme and $U \subseteq X$ a quasi-compact open subset. If X has enough quasi-coherent hoflats then so does U.*

Proof. The scheme U is concentrated and therefore so is the inclusion $i : U \longrightarrow X$. Let $\mathscr{F}$ be a complex of quasi-coherent sheaves on U. Then $i_*\mathscr{F}$ is a complex of quasi-coherent sheaves on X, and therefore admits a quasi-isomorphism $\mathscr{P} \longrightarrow i_*\mathscr{F}$ with $\mathscr{P}$ quasi-coherent and hoflat on X. Restricting to U we have a quasi-isomorphism $\mathscr{P}|_U \longrightarrow \mathscr{F}$ with $\mathscr{P}|_U$ quasi-coherent and hoflat, as required. ☐

Lemma 8. *Let X be either a quasi-compact semi-separated scheme or a concentrated scheme with an ample family of invertible sheaves. Then X has enough quasi-coherent hoflats.*

Proof. Either by Proposition 16 or (AMF,Corollary 7) every quasi-coherent sheaf $\mathscr{F}$ admits an epimorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ quasi-coherent and flat. Therefore the quasi-coherent flat sheaves are a smothering class for $\mathfrak{Qco}(X)$ in the sense of (DTC,Definition 30). Since the category of hoflat quasi-coherent complexes is localising in $K(\mathfrak{Qco}(X))$ it follows from (DTC,Proposition 71) that every complex in $\mathfrak{Qco}(X)$ admits a quasi-isomorphism from a hoflat quasi-coherent complex. In particular any affine scheme or quasi-projective variety over a field has enough quasi-coherent hoflats. □

Let $f : X \longrightarrow Y$ be a morphism of schemes where Y has enough quasi-coherent hoflats. Since $f^* : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Qco}(X)$ sends an exact hoflat complex to an exact complex (DCOS,Lemma 52) it follows as in the proof of (DCOS,Proposition 86) that every hoflat complex of quasi-coherent sheaves on Y is left f^*-acyclic. Therefore by (DTC2,Theorem 2) the functor f^* has a left derived functor.

Definition 4. Let $f : X \longrightarrow Y$ be a morphism of schemes where Y has enough quasi-coherent hoflats. The additive functor $f^* : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Qco}(X)$ has a left derived functor

$$\mathbb{L}_q f^* : \mathfrak{Dqcoh}(Y) \longrightarrow \mathfrak{Dqcoh}(X)$$

which we call the *derived inverse image functor*, or often just the *inverse image functor*. This is only determined up to canonical trinatural equivalence, but if we fix an assignment $\mathcal{F}$ of hoflat resolutions for $\mathfrak{Qco}(Y)$ then we have a canonical left derived functor which we denote $\mathbb{L}_{q,\mathcal{F}} f^*$.

Proposition 9. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes where X is concentrated and Y has enough quasi-coherent hoflats. There is a canonical triadjunction*

$$\mathfrak{Dqcoh}(X) \underset{\mathbb{L}_q f^*}{\overset{\mathbb{R}_q f_*}{\rightleftarrows}} \mathfrak{Dqcoh}(Y) \qquad \mathbb{L}_q f^* \longrightarrow\!\!\!\!\mid \mathbb{R}_q f_*$$

whose unit $\eta^\Diamond : 1 \longrightarrow \mathbb{R}_q f_* \circ \mathbb{L}_q f^*$ is the unique trinatural transformation making the following diagram commute for every complex $\mathscr{Y}$ of quasi-coherent sheaves on Y

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\eta^\Diamond} & \mathbb{R}_q f_*(\mathbb{L}_q f^* \mathscr{Y}) \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \mathbb{R}_q f_*(\omega)} \\ f_* f^*(\mathscr{Y}) & \xrightarrow[\zeta]{} & \mathbb{R}_q f_*(f^* \mathscr{Y}) \end{array} \tag{2}$$

Proof. This is a special case of (DTC2,Theorem 9). □

Remark 4. Let $f : X \longrightarrow Y$ be a morphism of schemes where Y has enough quasi-coherent hoflats and let $\mathbb{L}_q f^*$ be a left derived functor. The sheaf $\mathcal{O}_Y$ is quasi-coherent and flat, so it is hoflat as a complex and we have a canonical isomorphism $\mathbb{L}_q f^* \mathcal{O}_Y \cong f^* \mathcal{O}_Y \cong \mathcal{O}_X$ in $\mathfrak{Dqcoh}(X)$.

Lemma 10. *Let $f : X \longrightarrow Y$ be a morphism of schemes and $V \subseteq Y$ and $U \subseteq f^{-1}V$ open subsets where both Y, V have enough quasi-coherent hoflats. For any complex $\mathscr{G}$ of quasi-coherent sheaves on Y there is a canonical isomorphism in $\mathfrak{Dqcoh}(U)$ natural in $\mathscr{G}$*

$$\mu : (\mathbb{L}_q f^* \mathscr{G})|_U \longrightarrow \mathbb{L}_q g^*(\mathscr{G}|_V)$$

where $g : U \longrightarrow V$ is the induced morphism of schemes.

Proof. The proof is identical to (DCOS,Lemma 87). If $\theta : (-)|_U \circ f^* \longrightarrow g^* \circ (-)|_V$ is the canonical natural equivalence then we obtain a canonical trinatural equivalence $\mu = \mathbb{L}\theta : (-)|_U \circ \mathbb{L}_q f^* \longrightarrow \mathbb{L}_q g^* \circ (-)|_V$ making the following diagram commute

$$\begin{array}{ccc} (-)|_U \circ \mathbb{L}_q f^* \circ Q & \xrightarrow{\mu Q} & \mathbb{L}_q g^* \circ (-)|_V \circ Q \\ {\scriptstyle (-)|_U \zeta}\downarrow & & \downarrow{\scriptstyle \omega (-)|_V} \\ Q' \circ K((-)|_U \circ f^*) & \xrightarrow[Q' K(\theta)]{} & Q' \circ K(g^* \circ (-)|_V) \end{array}$$

as required. $\square$

It will take quite a bit of work in Theorem 31 to show that the quasi-coherent derived direct direct image agrees with the usual derived direct image. The analogous statement for the inverse image is very easy, because there is no question of flats being different in $\mathfrak{Qco}(X)$ and $\mathfrak{Mod}(X)$.

Proposition 11. *Let $f : X \longrightarrow Y$ be a morphism of schemes where Y has enough quasi-coherent hoflats. The diagram*

$$\begin{array}{ccc} \mathfrak{D}(Y) & \xrightarrow{\mathbb{L}f^*} & \mathfrak{D}(X) \\ {\scriptstyle U}\uparrow & & \uparrow{\scriptstyle u} \\ \mathfrak{Dqcoh}(Y) & \xrightarrow[\mathbb{L}_q f^*]{} & \mathfrak{Dqcoh}(X) \end{array}$$

commutes up to canonical trinatural equivalence.

Proof. Let u, U be the inclusions, so that we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{Mod}(Y) & \xrightarrow{f^*} & \mathfrak{Mod}(X) \\ {\scriptstyle U}\uparrow & & \uparrow{\scriptstyle u} \\ \mathfrak{Qco}(Y) & \xrightarrow[f^*]{} & \mathfrak{Qco}(X) \end{array}$$

and take arbitrary left derived functors $(\mathbb{L}f^*, \zeta), (\mathbb{L}_q f^*, \omega)$. The trinatural transformations are of the form

$$\zeta : \mathbb{L}f^* \circ Q \longrightarrow Q \circ K(f^*)$$
$$\omega : \mathbb{L}_q f^* \circ Q \longrightarrow Q \circ K(f^*)$$

and by (DTC2,Theorem 8) the pair $(u \circ \mathbb{L}_q f^*, u\omega)$ is a left derived functor of $u \circ f^* = f^* \circ U$. The trinatural transformation $\zeta K(U)$ induces a unique trinatural transformation $\mu : \mathbb{L}f^* \circ U \longrightarrow u \circ \mathbb{L}_q f^*$ making the following diagram commute

$$\begin{array}{ccc} \mathbb{L}f^* \circ U \circ Q & \xrightarrow{\mu Q} & u \circ \mathbb{L}_q f^* \circ Q \\ & {\scriptstyle \zeta K(U)} \searrow \quad \swarrow {\scriptstyle u\omega} & \\ & QK(u \circ f^*) & \end{array} \tag{3}$$

and we claim that μ is a trinatural equivalence. Because Y has enough quasi-coherent hoflats it suffices to check that $\mu_{\mathscr{Y}}$ is an isomorphism for $\mathscr{Y} \in \mathfrak{Dqcoh}(Y)$ hoflat. But this is clear from commutativity of (3), so the proof is complete. $\square$

2.4 Existence of Quasi-coherent Flats

To write a quasi-coherent sheaf as a quotient of a flat quasi-coherent sheaf, one essentially works locally over affines and then uses Čech complexes to patch together the resolutions on the affines. For this purpose the separatedness of the scheme is essential. Most of the proofs in this section are a special case of those given in [ATJLL97](1.2). Note that [ATJLL97](1.2) contains a serious gap, corrected in [ATJLL], but the gap is in a part of the proof that does not affect the discussion here.

Throughout this section X is a fixed quasi-compact semi-separated scheme and all sheaves of modules are over X, unless specified otherwise. Let $\mathfrak{U} = \{U_0, \ldots, U_n\}$ be a finite semi-separating open cover of X which is totally ordered, and let $\mathfrak{B}$ be the set of nonempty subsets of $\mathfrak{U}$. Given $\alpha = \{i_0, \ldots, i_n\}$ in $\mathfrak{B}$ we write U_α for the intersection $U_{i_0} \cap \cdots \cap U_{i_n}$.

Definition 5. An $\mathfrak{U}$-*module* is a collection $\mathscr{F} = \{\mathscr{F}_\alpha\}_{\alpha \in \mathfrak{B}}$ of sheaves of modules on U_α together with a morphism of sheaves of modules $\varphi_{\alpha,\beta} : \mathscr{F}_\beta|_{U_\alpha} \longrightarrow \mathscr{F}_\alpha$ whenever $\alpha \supseteq \beta$ in $\mathfrak{B}$, subject to the following conditions

(i) $\varphi_{\alpha,\alpha} = 1$ for any $\alpha \in \mathfrak{B}$.

(ii) Whenever $\alpha \supseteq \beta \supseteq \gamma$ in $\mathfrak{B}$ we have $\varphi_{\alpha,\gamma} = \varphi_{\alpha,\beta} \circ (\varphi_{\beta,\gamma}|_{U_\alpha})$.

A *morphism* of $\mathfrak{U}$-modules is a collection $\{\phi_\alpha\}_{\alpha \in \mathfrak{B}}$ of morphisms of sheaves of modules $\phi_\alpha : \mathscr{F}_\alpha \longrightarrow \mathscr{F}'_\alpha$ making the following diagram commute for every pair $\alpha \supseteq \beta$ in $\mathfrak{B}$

$$\begin{array}{ccc} \mathscr{F}_\beta|_{U_\alpha} & \xrightarrow{\phi_\beta|_{U_\alpha}} & \mathscr{F}'_\beta|_{U_\alpha} \\ {\scriptstyle \varphi_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \varphi'_{\alpha,\beta}} \\ \mathscr{F}_\alpha & \xrightarrow[\phi_\alpha]{} & \mathscr{F}'_\alpha \end{array}$$

This defines the preadditive category $\mathfrak{U}\mathbf{Mod}$ of $\mathfrak{U}$-modules. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of $\mathfrak{U}$-modules then we define modules $Ker(\phi)_\alpha = Ker(\phi_\alpha)$ and $Coker(\phi)_\alpha = Coker(\phi_\alpha)$ and these are the kernel and cokernel in $\mathfrak{U}\mathbf{Mod}$ respectively. The image is defined pointwise as well. Observe that ϕ is a monomorphism (resp. epimorphism) in $\mathfrak{U}\mathbf{Mod}$ if and only if ϕ_α has this property for every $\alpha \in \mathfrak{B}$. Given a nonempty family $\{\mathscr{F}^i\}_{i \in I}$ of $\mathfrak{U}$-modules we define a coproduct by $(\oplus_i \mathscr{F}^i)_\alpha = \oplus_i \mathscr{F}^i_\alpha$. It is now clear that $\mathfrak{U}\mathbf{Mod}$ is a cocomplete abelian category, in which a sequence is exact if and only if it is pointwise exact.

We say that a $\mathfrak{U}$-module $\mathscr{F}$ is *quasi-coherent* or *flat* if every $\mathscr{F}_\alpha$ has this property as a sheaf of modules on U_α. Suppose we are just given a family $\{\mathscr{F}_\alpha\}_{\alpha \in \mathfrak{B}}$ of sheaves of modules on U_α with no morphisms $\varphi_{\alpha,\beta}$. Define a new family of sheaves by

$$\mathscr{G}_\alpha = \bigoplus_{\alpha \supseteq \beta} \mathscr{F}_\beta|_{U_\alpha}$$

and let $\varphi_{\alpha,\beta}$ be the canonical morphism between the coproducts. This clearly defines an $\mathfrak{U}$-module, which we denote by $T(\mathscr{F})$. A family of morphisms $\phi_\alpha : \mathscr{F}_\alpha \longrightarrow \mathscr{F}'_\alpha$ determines a morphism of $\mathfrak{U}$-modules $T(\phi) : T(\mathscr{F}) \longrightarrow T(\mathscr{F}')$. Since coproducts in $\mathfrak{Mod}(X)$ are exact, if the ϕ_α are all epimorphisms then so is $T(\phi)$. If $\mathscr{F}$ is actually a $\mathfrak{U}$-module then there is a canonical epimorphism of $\mathfrak{U}$-modules $T(\mathscr{F}) \longrightarrow \mathscr{F}$ defined in the obvious way.

Lemma 12. *Every quasi-coherent $\mathfrak{U}$-module is a quotient of a flat quasi-coherent $\mathfrak{U}$-module.*

Proof. Let $\mathscr{F}$ be a quasi-coherent $\mathfrak{U}$-module. Since each U_α is affine we can find a quasi-coherent flat sheaf $\mathscr{Q}_\alpha$ and an epimorphism $\mathscr{Q}_\alpha \longrightarrow \mathscr{F}_\alpha$. Then $\mathscr{P} = T(\mathscr{Q})$ is a quasi-coherent flat $\mathfrak{U}$-module and there is an epimorphism $\mathscr{P} \longrightarrow \mathscr{F}$, as required. $\square$

Let $\mathscr{F}$ be a $\mathfrak{U}$-module and define for $p \geq 0$ a sheaf of modules on X by

$$\mathscr{C}^p(\mathscr{F}) = \prod_{i_0 < \cdots < i_p} f_* \mathscr{F}_{i_0,\ldots,i_p}$$

where $f : U_{i_0,\ldots,i_p} \longrightarrow X$ is always the inclusion. Observe that this is a finite product, hence also a coproduct. For $p \geq 0$ and a sequence $\alpha = \{i_0, \ldots, i_{p+1}\}$ we define a morphism of sheaves of modules

$$d^p : \mathscr{C}^p(\mathscr{F}) \longrightarrow \mathscr{C}^{p+1}(\mathscr{F})$$

$$(d^p)_V(x)_\alpha = \sum_{k=0}^{p+1} (-1)^k (\varphi_{\alpha,\beta_k})_{V \cap U_\alpha}(x_{\beta_k}|_{V \cap U_\alpha})$$

where $\beta_k = \{i_0, \dots, \widehat{i_k}, \dots, i_{p+1}\}$ is α with i_k deleted. This defines a complex $\mathscr{C}(\mathscr{F})$ of sheaves of modules on X. Given a morphism of $\mathfrak{U}$-modules $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ we have a morphism of complexes

$$\mathscr{C}(\phi) : \mathscr{C}(\mathscr{F}) \longrightarrow \mathscr{C}(\mathscr{G})$$
$$\mathscr{C}^p(\phi) = \prod_{i_0 < \cdots < i_p} f_* \phi_{i_0,\dots,i_p}$$

and this defines an additive functor $\mathscr{C}(-) : \mathfrak{U}\mathbf{Mod} \longrightarrow \mathbf{C}(X)$. Given a *complex* $\mathscr{X}$ of $\mathfrak{U}$-modules we have a bicomplex $M^{pq} = \mathscr{C}^q(\mathscr{X}^p)$ in $\mathfrak{Mod}(X)$ whose totalisation (DTC,Definition 33) we denote by $\mathscr{C}_{tot}(\mathscr{X})$. That is,

$$\mathscr{C}^n_{tot}(\mathscr{X}) = \bigoplus_{p+q=n} \mathscr{C}^q(\mathscr{X}^p)$$

The bicomplex is functorial in $\mathscr{X}$ and therefore so is its totalisation, so we have an additive functor $\mathscr{C}_{tot}(-) : \mathbf{C}(\mathfrak{U}) \longrightarrow \mathbf{C}(X)$. Given a complex $\mathscr{X}$ of $\mathfrak{U}$-modules there is a canonical isomorphism of complexes natural in $\mathscr{X}$

$$\rho : \mathscr{C}_{tot}(\Sigma\mathscr{X}) \longrightarrow \Sigma\mathscr{C}_{tot}(\mathscr{X})$$
$$\rho^n u_{pq} = u_{p+1,q}$$

Let $\varphi, \psi : \mathscr{X} \longrightarrow \mathscr{Y}$ be morphisms of complexes of $\mathfrak{U}$-modules and suppose that there exists a homotopy $\Sigma : \psi \longrightarrow \varphi$. Then $\Sigma^n : \mathscr{X}^n \longrightarrow \mathscr{Y}^{n-1}$ determines a morphism of complexes $\mathscr{C}(\mathscr{X}^n) \longrightarrow \mathscr{C}(\mathscr{Y}^{n-1})$ and this is a homotopy of the induced morphism of bicomplexes, in the sense of (SS,Definition 7). By (SS,Lemma 15) we deduce a homotopy between $\mathscr{C}_{tot}(\varphi)$ and $\mathscr{C}_{tot}(\psi)$, so there is an induced additive functor $\mathscr{C}_{tot}(-) : K(\mathfrak{U}) \longrightarrow K(X)$.

Lemma 13. *Let $u : \mathscr{X} \longrightarrow \mathscr{Y}$ be a morphism of complexes of $\mathfrak{U}$-modules. There is a canonical isomorphism of complexes $\mathscr{C}_{tot}(C_u) \cong C_{\mathscr{C}_{tot}(u)}$.*

Proof. For $n \in \mathbb{Z}$ we have a canonical isomorphism of sheaves of modules κ^n

$$\begin{aligned}
\mathscr{C}_{tot}(C_u)^n &= \oplus_{p+q=n} \mathscr{C}^q(C_u^p) = \oplus_{p+q=n} \mathscr{C}^q(\mathscr{X}^{p+1} \oplus \mathscr{Y}^p) \\
&= \oplus_{p+q=n} \left(\mathscr{C}^q(\mathscr{X}^{p+1}) \oplus \mathscr{C}^q(\mathscr{Y}^p) \right) \\
&= \left(\oplus_{p+q=n} \mathscr{C}^q(\mathscr{X}^{p+1}) \right) \oplus \left(\oplus_{p+q=n} \mathscr{C}^q(\mathscr{Y}^p) \right) \\
&= \mathscr{C}_{tot}(\Sigma\mathscr{X})^n \oplus \mathscr{C}_{tot}(\mathscr{Y})^n \\
&\cong \mathscr{C}_{tot}(\mathscr{X})^{n+1} \oplus \mathscr{C}_{tot}(\mathscr{Y})^n \\
&= C^n_{\mathscr{C}_{tot}(u)}
\end{aligned}$$

Here $C^n_{\mathscr{C}_{tot}(u)} = \mathscr{C}_{tot}(\mathscr{X})^{n+1} \oplus \mathscr{C}_{tot}(\mathscr{Y})^n$ and $\mathscr{C}^q(C_u^p) = \mathscr{C}^q(\mathscr{X}^{p+1}) \oplus \mathscr{C}^q(\mathscr{Y}^p)$, and with this biproduct structure understood we have

$$\kappa^n u_{pq} = u_{p+1,q} \oplus u_{pq}$$

One checks that this is an isomorphism of complexes $\mathscr{C}_{tot}(C_u) \longrightarrow C_{\mathscr{C}_{tot}(u)}$. ☐

Lemma 14. *The pair $(\mathscr{C}_{tot}(-), \rho)$ is a triangulated functor $K(\mathfrak{U}) \longrightarrow K(X)$ which sends exact complexes of quasi-coherent $\mathfrak{U}$-modules to exact complexes of quasi-coherent $\mathcal{O}_X$-modules. If $u : \mathscr{X} \longrightarrow \mathscr{Y}$ is a quasi-isomorphism of complexes of quasi-coherent $\mathfrak{U}$-modules then $\mathscr{C}_{tot}(u)$ is also a quasi-isomorphism.*

Proof. It is straightforward to check using Lemma 13 that the given pair is a triangulated functor. Suppose that $\mathscr{X}$ is an exact complex of $\mathfrak{U}$-modules. In the definition of $\mathscr{C}^p(\mathscr{F})$ the inclusions $f : U_{i_0,\dots,i_p} \longrightarrow X$ are all affine by definition of a semi-separating cover, so the induced functor f_* is exact on quasi-coherent sheaves. It follows that the rows of the bicomplex $M^{pq} = \mathscr{C}^q(\mathscr{X}^p)$ are exact, so by a spectral sequence argument (SS,Example 2) we deduce that $\mathscr{C}_{tot}(\mathscr{X})$ is exact, as required. ☐

Lemma 15. *Let $f : X \longrightarrow Y$ be a flat affine morphism of schemes and $\mathscr{F}$ a flat quasi-coherent sheaf of modules on X. Then $f_*\mathscr{F}$ is also flat and quasi-coherent.*

Proof. The question is local and f is affine, so we can assume $X = SpecA, Y = SpecB$ in which case f is induced by a flat morphism of rings $\varphi : B \longrightarrow A$. The result now follows from the fact that a module flat over A is also flat over B (MAT,Lemma 16). ☐

Proposition 16. *Let X be a quasi-compact semi-separated scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules. There is a flat quasi-coherent sheaf of modules $\mathscr{P}$ and an epimorphism $\mathscr{P} \longrightarrow \mathscr{F}$.*

Proof. We choose a finite semi-separating open cover of X and use the above notation. Let $\mathscr{F}'$ denote the $\mathfrak{U}$-module constructed from the restrictions of $\mathscr{F}$ in the obvious way. Then $\mathscr{C}_{tot}(\mathscr{F}')$ is canonically isomorphic (as a complex) to the usual Čech resolution of $\mathscr{F}$ with respect to $\mathfrak{U}$, so there is a canonical quasi-isomorphism of complexes $\mathscr{F} \longrightarrow \mathscr{C}_{tot}(\mathscr{F}')$.

Now let $\mathcal{A}$ be the abelian category of quasi-coherent $\mathfrak{U}$-modules. The class $\mathcal{P}$ of flat quasi-coherent $\mathfrak{U}$-modules forms by Lemma 12 a smothering class for $\mathcal{A}$ in the sense of (DTC,Definition 30). We can therefore find by (DTC,Proposition 69) a quasi-isomorphism of complexes of quasi-coherent $\mathfrak{U}$-modules $\mathscr{P} \longrightarrow \mathscr{F}'$, with $\mathscr{P}$ bounded above (in fact we can assume $\mathscr{P}^i = 0$ for $i > 0$) and each $\mathscr{P}^i$ flat. From Lemma 14 we conclude that there is a quasi-isomorphism $\mathscr{C}_{tot}(\mathscr{P}) \longrightarrow \mathscr{C}_{tot}(\mathscr{F}')$, with the terms of the first complex being both quasi-coherent and flat by Lemma 15. In other words, $\mathscr{F}$ is isomorphic in $\mathfrak{D}(X)$ to a bounded above complex $\mathscr{Q}$ of flat quasi-coherent sheaves.

If $\mathscr{F}$ is zero the result is trivial, so suppose otherwise and let $N \geq 0$ be such that $\mathscr{Q}^i = 0$ for $i > N$ and $\mathscr{Q}^N \neq 0$. The proof is by induction on N. If $N = 0$ then the existence of the isomorphism $\mathscr{F} \cong \mathscr{Q}$ in $\mathfrak{D}(X)$ exhibits $\mathscr{F}$ as a quotient of a flat quasi-coherent sheaf. If $N > 0$ then we have a short exact sequence

$$0 \longrightarrow Ker\partial_{\mathscr{Q}}^{N-1} \longrightarrow \mathscr{Q}^{N-1} \longrightarrow \mathscr{Q}^N \longrightarrow 0$$

from which it follows that $Ker\partial_{\mathscr{Q}}^{N-1}$ is flat and quasi-coherent. Hence the truncation $\mathscr{Q}_{\leq(N-1)}$ of (DTC,Definition 14) is a complex of flat quasi-coherent sheaves. Since $\mathscr{F} \cong \mathscr{Q}_{\leq(N-1)}$ in $\mathfrak{D}(X)$ we have reduced the problem by at least one degree, so by the inductive hypothesis we are done. ☐

2.5 Derived Tensor

Let X be a scheme. The tensor product of two quasi-coherent sheaves is quasi-coherent, so the tensor product on $\mathfrak{Mod}(X)$ restricts to a functor $- \otimes_{\mathcal{O}_X} - : \mathfrak{Qco}(X) \times \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(X)$ additive in each variable.

Definition 6. Let X be a scheme. Taking T to be the tensor product of quasi-coherent sheaves in (DCOS,Definition 6) we have a functor additive in each variable

$$- \otimes_{\mathcal{O}_X} - : \mathbf{C}(\mathfrak{Qco}(X)) \times \mathbf{C}(\mathfrak{Qco}(X)) \longrightarrow \mathbf{C}(\mathfrak{Qco}(X))$$

We drop the subscript on the tensor whenever it will not cause confusion. This is just the restriction of the tensor product on $\mathbf{C}(X)$ given in (DCOS,Definition 8), so in particular we have a canonical isomorphism of complexes $\tau : \mathscr{X} \otimes \mathscr{Y} \longrightarrow \mathscr{Y} \otimes \mathscr{X}$ natural in both variables. There is an induced functor additive in both variables

$$- \otimes_{\mathcal{O}_X} - : K(\mathfrak{Qco}(X)) \times K(\mathfrak{Qco}(X)) \longrightarrow K(\mathfrak{Qco}(X))$$

In particular for a complex $\mathscr{X}$ in $\mathfrak{Qco}(X)$ we have additive functors $-\otimes\mathscr{X}, \mathscr{X}\otimes- : K(\mathfrak{Qco}(X)) \longrightarrow K(\mathfrak{Qco}(X))$. For the reader's convenience we list the properties of this tensor product that are immediate from our earlier treatment of the tensor product on $\mathfrak{Mod}(X)$:

- Let $\mathscr{X}, \mathscr{Y}$ be complexes of quasi-coherent sheaves. There are canonical isomorphisms of complexes natural in both variables $\rho : \mathscr{X} \otimes (\Sigma\mathscr{Y}) \longrightarrow \Sigma(\mathscr{X} \otimes \mathscr{Y})$ and $\sigma : (\Sigma\mathscr{X}) \otimes \mathscr{Y} \longrightarrow \Sigma(\mathscr{X} \otimes \mathscr{Y})$.

- Let $u : \mathscr{X} \longrightarrow \mathscr{Y}$ be a morphism of complexes of quasi-coherent sheaves. For any complex $\mathscr{Z}$ of quasi-coherent sheaves there is a canonical isomorphism $\mathscr{Z} \otimes C_u \cong C_{\mathscr{Z} \otimes u}$.
- For any complex $\mathscr{Z}$ of quasi-coherent sheaves the pairs $(\mathscr{Z} \otimes -, \rho)$ and $(- \otimes \mathscr{Z}, \sigma)$ are coproduct preserving triangulated functors $K(\mathfrak{Qco}(X)) \longrightarrow K(\mathfrak{Qco}(X))$ and there is a canonical trinatural equivalence $\tau : \mathscr{Z} \otimes - \longrightarrow - \otimes \mathscr{Z}$.
- If $\mathscr{F}$ is a hoflat complex of quasi-coherent sheaves then the functors $\mathscr{F} \otimes -$ and $- \otimes \mathscr{F}$ preserve quasi-isomorphisms of complexes. If $\mathscr{F}$ is in addition exact then $\mathscr{F} \otimes \mathscr{X}$ is exact for any complex $\mathscr{X}$ of quasi-coherent sheaves.
- Suppose that X has enough quasi-coherent hoflats and let $\mathscr{X}$ be a complex of quasi-coherent sheaves. Then any hoflat complex $\mathscr{F}$ of quasi-coherent sheaves is left acyclic for the triangulated functors $Q \circ (- \otimes \mathscr{X}), Q \circ (\mathscr{X} \otimes -) : K(\mathfrak{Qco}(X)) \longrightarrow \mathfrak{Dqcoh}(X)$.

Definition 7. Let X be a scheme with enough quasi-coherent hoflats and $\mathscr{X}$ a complex of quasi-coherent sheaves. The triangulated functor $Q \circ (\mathscr{X} \otimes -) : K(\mathfrak{Qco}(X)) \longrightarrow \mathfrak{Dqcoh}(X)$ has a left derived functor

$$\mathscr{X} \underline{\underline{\otimes}} - : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(X)$$

To be precise, for each assignment $\mathcal{F}$ of quasi-coherent hoflat resolutions for X we have a canonical left derived functor $\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} -$ of $Q \circ (\mathscr{X} \otimes -)$. In particular $\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} = \mathscr{X} \otimes F_{\mathscr{Y}}$, where $F_{\mathscr{Y}} \longrightarrow \mathscr{Y}$ is the chosen hoflat resolution.

Let X be a scheme with enough quasi-coherent hoflats. We use the notation of Definition 7 and fix an assignment $\mathcal{F}$ of quasi-coherent hoflat resolutions. Given a morphism $\psi : \mathscr{X} \longrightarrow \mathscr{X}'$ in $K(\mathfrak{Qco}(X))$ we can define a trinatural transformation

$$\psi \otimes - : \mathscr{X} \otimes - \longrightarrow \mathscr{X}' \otimes -$$
$$(\psi \otimes -)_{\mathscr{Y}} = \psi \otimes \mathscr{Y}$$

This gives rise to a trinatural transformation $Q(\psi \otimes -) : Q(\mathscr{X} \otimes -) \longrightarrow Q(\mathscr{X}' \otimes -)$ which by (TRC,Definition 53) induces a canonical trinatural transformation

$$\psi \underline{\underline{\otimes}}^{\mathcal{F}} - : \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} - \longrightarrow \mathscr{X}' \underline{\underline{\otimes}}^{\mathcal{F}} -$$

which by (TRC,Lemma 127) must have the form $\psi \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} = Q(\psi \otimes F_{\mathscr{Y}})$ where $F_{\mathscr{Y}} \longrightarrow \mathscr{Y}$ is the chosen quasi-coherent hoflat resolution of $\mathscr{Y}$. For any complex $\mathscr{Y}$ of quasi-coherent sheaves we write $R_{\mathscr{Y}}$ for the additive functor $K(\mathfrak{Qco}(X)) \longrightarrow \mathfrak{Dqcoh}(X)$ defined on objects by $R_{\mathscr{Y}}(\mathscr{X}) = \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y}$ and on a morphism $\psi : \mathscr{X} \longrightarrow \mathscr{X}'$ by $R_{\mathscr{Y}}(\psi) = \psi \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y}$. In fact this is equal as an additive functor to the composite $Q(- \otimes F_{\mathscr{Y}}) : K(\mathfrak{Qco}(X)) \longrightarrow \mathfrak{Dqcoh}(X)$, so $R_{\mathscr{Y}}$ becomes a triangulated functor in a canonical way. Since $F_{\mathscr{Y}}$ is hoflat, the functor $R_{\mathscr{Y}}$ contains the exact complexes of $K(\mathfrak{Qco}(X))$ in its kernel, and therefore induces a triangulated functor

$$- \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(X)$$

One checks that for morphisms $\varphi : \mathscr{Y} \longrightarrow \mathscr{Y}'$ and $\psi : \mathscr{X} \longrightarrow \mathscr{X}'$ in $\mathfrak{Dqcoh}(X)$ we have

$$(\psi \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y}')(\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \varphi) = (\mathscr{X}' \underline{\underline{\otimes}}^{\mathcal{F}} \varphi)(\psi \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y}) \tag{4}$$

Definition 8. Let X be a scheme with enough quasi-coherent hoflats. Then for every assignment $\mathcal{F}$ of quasi-coherent hoflat resolutions for X there is a canonical functor additive in each variable

$$- \underline{\underline{\otimes}}^{\mathcal{F}} - : \mathfrak{Dqcoh}(X) \times \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(X)$$

with $\varphi \underline{\underline{\otimes}}^{\mathcal{F}} \psi$ defined to be the equal composites of (4). The partial functors in each variable are triangulated functors in a canonical way. To be explicit, for complexes $\mathscr{X}, \mathscr{Y}$ we have

$$\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} = \mathscr{X} \otimes F_{\mathscr{Y}}$$

As part of the data we have a morphism in $\mathfrak{Dqcoh}(X)$ trinatural in both variables

$$\zeta : \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} \longrightarrow \mathscr{X} \otimes \mathscr{Y}$$

which is an isomorphism if either of $\mathscr{X}, \mathscr{Y}$ is hoflat.

Remark 5. With the notation of Definition 8 the partial functors $\mathscr{X} \underline{\underline{\otimes}} -$ and $- \underline{\underline{\otimes}} \mathscr{Y}$ are canonically triangulated functors, and moreover these triangulated structures are compatible. That is, the isomorphisms in $\mathfrak{Dqcoh}(X)$

$$\mathscr{X} \underline{\underline{\otimes}} (\Sigma \mathscr{Y}) \cong \Sigma(\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y})$$
$$(\Sigma \mathscr{X}) \underline{\underline{\otimes}} \mathscr{Y} \cong \Sigma(\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y})$$

are natural in both variables. The structure sheaf $\mathcal{O}_X$ is also a unit for the tensor product, in the sense that the triangulated functors $\mathcal{O}_X \underline{\underline{\otimes}} -$ and $- \underline{\underline{\otimes}} \mathcal{O}_X$ are canonically trinaturally equivalent to the identity.

Remark 6. The analogues of (DCOS,Lemma 54), (DCOS,Lemma 55) and (DCOS,Lemma 56) hold for the quasi-coherent derived tensor product.

Lemma 17. *Let X be a scheme with enough quasi-coherent hoflats, and let $\mathcal{F}, \mathcal{F}'$ be assignments of quasi-coherent hoflat resolutions. For complexes $\mathscr{X}, \mathscr{Y}$ of quasi-coherent sheaves there is a canonical isomorphism in $\mathfrak{Dqcoh}(X)$ natural in both variables*

$$\mu : \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} \longrightarrow \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}'} \mathscr{Y}$$

which on the partial functors gives trinatural equivalences

$$\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} - \longrightarrow \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}'} - \tag{5}$$
$$- \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} \longrightarrow - \underline{\underline{\otimes}}^{\mathcal{F}'} \mathscr{Y} \tag{6}$$

Proof. By definition of a left derived functor we have trinatural transformations

$$\zeta : (\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} -)Q \longrightarrow Q(\mathscr{X} \otimes -)$$
$$\zeta' : (\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}'} -)Q \longrightarrow Q(\mathscr{X} \otimes -)$$

and therefore a trinatural equivalence $\mu : \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} - \longrightarrow \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}'} -$ which is the unique trinatural transformation making the following diagram commute

$$\begin{array}{ccc} (\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} -)Q & \xrightarrow{\ \mu Q\ } & (\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}'} -)Q \\ & {\scriptstyle \zeta} \searrow \quad \swarrow {\scriptstyle \zeta'} & \\ & Q(\mathscr{X} \otimes -) & \end{array} \tag{7}$$

This yields the desired isomorphism $\mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}} \mathscr{Y} \longrightarrow \mathscr{X} \underline{\underline{\otimes}}^{\mathcal{F}'} \mathscr{Y}$ which one checks is also natural in $\mathscr{X}$. It remains to check that the induced natural equivalence (6) is trinatural. For any quasi-coherent hoflat complex $\mathscr{Y}$ the morphisms ζ, ζ' of (7) are isomorphisms, and by hypothesis every object of $\mathfrak{Dqcoh}(X)$ is isomorphic to such a $\mathscr{Y}$, so it is easy to see that in fact μ is the unique *natural* transformation making (7) commute. Using this observation together with the fact that ζ, ζ' are actually trinatural in $\mathscr{X}$ as well as $\mathscr{Y}$, it is straightforward to check that (6) is trinatural. □

In the next result we justify our use of the same notation $\mathscr{X} \underline{\underline{\otimes}} \mathscr{Y}$ for the quasi-coherent derived tensor product and the derived tensor product of (DCOS,Definition 13).

Lemma 18. *Let X be a scheme with enough quasi-coherent hoflats. Then the following diagram of functors commutes up to canonical natural equivalence*

$$\begin{array}{ccc} \mathfrak{D}(X)\times\mathfrak{D}(X) & \xrightarrow{-\underline{\otimes}-} & \mathfrak{D}(X) \\ \uparrow & & \uparrow \\ \mathfrak{Dqcoh}(X)\times\mathfrak{Dqcoh}(X) & \xrightarrow[-\underline{\otimes}-]{} & \mathfrak{Dqcoh}(X) \end{array}$$

Proof. For the duration of the proof, write $\mathscr{X}\underline{\otimes}_{qc}\mathscr{Y}$ for the quasi-coherent derived tensor product. Let $\mathcal{F}$ be an assignment of hoflat resolutions for $\mathfrak{Mod}(X)$ and $\mathcal{F}'$ an assignment quasi-coherent hoflat resolutions for $\mathfrak{Qco}(X)$. Given complexes $\mathscr{X},\mathscr{Y}$ of quasi-coherent sheaves let $F_{\mathscr{Y}}\longrightarrow\mathscr{Y}$ and $F'_{\mathscr{Y}}\longrightarrow\mathscr{Y}$ be the hoflat resolutions of $\mathscr{Y}$ in $\mathfrak{Mod}(X),\mathfrak{Qco}(X)$ respectively. Then we have a canonical isomorphism in $\mathfrak{D}(X)$

$$\mathscr{X}\underline{\otimes}_{qc}^{\mathcal{F}'}\mathscr{Y}=\mathscr{X}\otimes F'_{\mathscr{Y}}\cong\mathscr{X}\underline{\otimes}^{\mathcal{F}}F'_{\mathscr{Y}}\cong\mathscr{X}\underline{\otimes}^{\mathcal{F}}\mathscr{Y}$$

where the first isomorphism comes from the fact that $F'_{\mathscr{Y}}$ is left acyclic for the tensor product of sheaves of modules. It is straightforward to check naturality in both variables. $\square$

3 Sheaves with Quasi-coherent Cohomology

Definition 9. If X is a scheme then the abelian subcategory $\mathfrak{Qco}(X)$ of $\mathfrak{Mod}(X)$ is plump (DTC,Definition 22) and closed under coproducts. Denote by $K_{qc}(X),\mathfrak{D}_{qc}(X)$ the corresponding localising subcategories of $K(X),\mathfrak{D}(X)$ respectively, consisting of complexes with quasi-coherent cohomology. If $\mathscr{F}$ is a complex of sheaves of modules on X with quasi-coherent cohomology then the same is true of the truncations $\mathscr{F}_{\le n},\mathscr{F}_{\ge n}$ for any $n\in\mathbb{Z}$.

We write $\mathfrak{D}_{qc}(X)^{\le n},\mathfrak{D}_{qc}(X)^{\ge n}$ for those complexes with quasi-coherent cohomology belonging to the subcategories $\mathfrak{D}(X)^{\le n},\mathfrak{D}(X)^{\ge n}$ respectively (DTC,Definition 20). It is clear that these are replete additive subcategories of $\mathfrak{D}_{qc}(X)$.

In (DTC,Proposition 75) we described a way to explicitly construct a hoinjective resolution of a complex. This construction is nice, because there is a strong connection between the resolution of the whole complex and the resolutions of the truncations, which allows us in many cases to reduce to complexes which are bounded below. Unfortunately (DTC,Proposition 75) was only proved under very restrictive hypotheses. In particular the construction does not apply to the category $\mathfrak{Mod}(X)$ of sheaves of modules on a scheme X. Nonetheless, we will show in the next result that this construction *can* be successfully carried out provided our complex of sheaves has quasi-coherent cohomology.

The next result is [BN93] Lemma 5.3. Although our proof is slightly different (in particular we can avoid spectral sequences) the underlying insight is the same: use the quasi-coherent cohomology to show that the presheaf cohomology of the $\mathscr{I}_n$ stabilises, then apply (DTC,Lemma 78).

Proposition 19. *Let X be a scheme and $\mathscr{F}$ a complex of sheaves of modules with quasi-coherent cohomology. Suppose we have a commutative diagram in $K(X)$*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathscr{F}_{\ge n} & \longrightarrow & \cdots & \longrightarrow & \mathscr{F}_{\ge -2} & \longrightarrow & \mathscr{F}_{\ge -1} & \longrightarrow & \mathscr{F}_{\ge 0} \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathscr{I}_n & \longrightarrow & \cdots & \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_0 \end{array} \qquad (8)$$

satisfying the following conditions

(i) Every vertical morphism is a quasi-isomorphism.

(ii) The $\mathscr{I}_n$ are hoinjective complexes.

Then the induced morphism $\vartheta : \mathscr{F} \longrightarrow \underleftarrow{holim}_{n \le 0} \mathscr{I}_n$ is a hoinjective resolution.

Proof. A holimit in $K(X)$ of the bottom row is defined by a triangle

$$\underleftarrow{holim} \mathscr{I}_n \longrightarrow \prod_{n \le 0} \mathscr{I}_n \xrightarrow{1-\nu} \prod_{n \le 0} \mathscr{I}_n \longrightarrow \Sigma \underleftarrow{holim} \mathscr{I}_n$$

The morphisms $\mathscr{F} \longrightarrow \mathscr{F}_{\ge n} \longrightarrow \mathscr{I}_n$ induce a morphism into the product $\prod \mathscr{I}_n$ which composes with $1-\nu$ to give zero, so we deduce a (non-canonical) factorisation $\vartheta : \mathscr{F} \longrightarrow \underleftarrow{holim} \mathscr{I}_n$ in $K(X)$. We claim that this is a quasi-isomorphism. For each $n \le -1$ we have a commutative square in $K(X)$

$$\begin{array}{ccc} \mathscr{F}_{\ge n} & \longrightarrow & \mathscr{F}_{\ge n+1} \\ \downarrow & & \downarrow \\ \mathscr{I}_n & \longrightarrow & \mathscr{I}_{n+1} \end{array}$$

Taking homotopy kernels of the horizontal morphisms, we have a morphism of triangles

$$\begin{array}{ccccccc} \mathscr{A}_n & \longrightarrow & \mathscr{F}_{\ge n} & \longrightarrow & \mathscr{F}_{\ge n+1} & \longrightarrow & \Sigma \mathscr{A}_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathscr{C}_n & \longrightarrow & \mathscr{I}_n & \longrightarrow & \mathscr{I}_{n+1} & \longrightarrow & \Sigma \mathscr{C}_n \end{array} \tag{9}$$

from which it is clear that $\mathscr{A}_n \longrightarrow \mathscr{C}_n$ is a hoinjective resolution. Let $U \subseteq X$ be affine. Since $\mathscr{C}_n$ is hoinjective it is acyclic for the additive functor $\Gamma(U, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$, so we have an isomorphism in $\mathfrak{D}(\mathbf{Ab})$

$$\Gamma(U, \mathscr{C}_n) \cong \mathbb{R}\Gamma(U, \mathscr{C}_n) \cong \mathbb{R}\Gamma(U, \mathscr{A}_n) \tag{10}$$

Now in $\mathfrak{D}(X)$ there is a canonical triangle (DTC,Lemma 27)

$$c_n H^n(\mathscr{F}) \longrightarrow \mathscr{F}_{\ge n} \longrightarrow \mathscr{F}_{\ge n+1} \longrightarrow \Sigma c_n H^n(\mathscr{F})$$

and therefore an isomorphism $\mathscr{A}_n \cong c_n H^n(\mathscr{F})$ in $\mathfrak{D}(X)$. Combining this with (10) we have an isomorphism $\Gamma(U, \mathscr{C}_n) \cong \Sigma^{-n} \mathbb{R}\Gamma(U, H^n(\mathscr{F}))$ in $\mathfrak{D}(\mathbf{Ab})$, and therefore by (DTC2,Corollary 12) an isomorphism of abelian groups for $m \ge n$

$$H^m \Gamma(U, \mathscr{C}_n) \cong H^{m-n} \mathbb{R}\Gamma(U, H^n(\mathscr{F})) \cong H^{m-n}(U, H^n(\mathscr{F}))$$

which is of course zero for $m > n$ by Serre's theorem (COS,Theorem 14), because $\mathscr{F}$ has quasi-coherent cohomology. Now applying $\Gamma(U, -)$ to the triangle in the bottom row of (9) we have a triangle in $K(\mathbf{Ab})$, whose long exact cohomology sequence includes the following for $m > n$

$$H^m \Gamma(U, \mathscr{C}_n) \longrightarrow H^m \Gamma(U, \mathscr{I}_n) \longrightarrow H^m \Gamma(U, \mathscr{I}_{n+1}) \longrightarrow H^{m+1} \Gamma(U, \mathscr{C}_n)$$

Therefore $H^m \Gamma(U, \mathscr{I}_n) \longrightarrow H^m \Gamma(U, \mathscr{I}_{n+1})$ is an isomorphism for $m > n$. In other words, for any $m \in \mathbb{Z}$ the sequence

$$\cdots \longrightarrow H^m \Gamma(U, \mathscr{I}_{-2}) \longrightarrow H^m \Gamma(U, \mathscr{I}_{-1}) \longrightarrow H^m \Gamma(U, \mathscr{I}_0)$$

eventually consists entirely of isomorphisms: the presheaf cohomology of the $\mathscr{I}_n$ stabilises over open affines. Applying (DTC,Lemma 78) with $\mathcal{A} = \mathbf{Ab}$ and using (DCOS,Lemma 7) we find that for $n \le m$ we have an isomorphism

$$H^m \Gamma(U, \underleftarrow{holim} \mathscr{I}_n) = H^m(\underleftarrow{holim} \Gamma(U, \mathscr{I}_n)) \longrightarrow H^m \Gamma(U, \mathscr{I}_n)$$

Sheafifying and using (DCOS,Lemma 3) we infer that

$$H^m(\underleftarrow{holim} \mathscr{I}_n) \longrightarrow H^m(\mathscr{I}_n)$$

is an isomorphism for $n \leq m$. For any $m \in \mathbb{Z}$ and $n \leq 0$ we have commutative diagrams in $K(X)$ and $\mathfrak{Mod}(X)$ respectively

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\vartheta} & \underleftarrow{holim}\,\mathscr{I}_n \\
\downarrow & & \downarrow \\
\mathscr{F}_{\geq n} & \longrightarrow & \mathscr{I}_n
\end{array}
\qquad\qquad
\begin{array}{ccc}
H^m(\mathscr{F}) & \xrightarrow{H^m(\vartheta)} & H^m(\underleftarrow{holim}\,\mathscr{I}_n) \\
\downarrow & & \downarrow \\
H^m(\mathscr{F}_{\geq n}) & \longrightarrow & H^m(\mathscr{I}_n)
\end{array}
$$

Taking $n = m$ in the right hand diagram, it is clear that $H^m(\vartheta) : H^m(\mathscr{F}) \longrightarrow H^m(\underleftarrow{holim}\,\mathscr{I}_n)$ is an isomorphism. Since $m \in \mathbb{Z}$ was arbitrary, this shows that $\vartheta : \mathscr{F} \longrightarrow \underleftarrow{holim}\,\mathscr{I}_n$ is a quasi-isomorphism. Since $\underleftarrow{holim}\,\mathscr{I}_n$ is the homotopy limit of hoinjective complexes it is clearly hoinjective, so the proof is complete. □

Remark 7. The theory of resolutions of unbounded complexes has been independently discovered several times from different directions. The analogue of Proposition 19 was first stated and proved for the inverse limit, not the homotopy limit, in Spaltenstein [Spa88] Proposition 3.13. We give this proof in the Appendix.

For the homotopy limit the result is [BN93] Lemma 5.3, which introduced homotopy limits and colimits to the algebraists. In fact comparing the proofs of Proposition 19 and Proposition 104 one begins to see why holimits and hocolimits are the "right" tool. This is the important insight of Bökstedt and Neeman in [BN93].

The two approaches, using the inverse limit and the homotopy limit, have almost the same content because in this case the two constructions agree (up to quasi-isomorphism). This is the observation of Corollary 105. However, Proposition 19 is more flexible because it works in the homotopy category, not on the level of complexes. This becomes essential, for example, in the proof of Proposition 41.

Remark 8. Given a complex $\mathscr{F}$ with quasi-coherent cohomology, a commutative diagram (8) with the properties $(i), (ii)$ exists by the inductive construction given in (DTC,Proposition 75). In fact this construction yields a commutative diagram of *complexes* having much more specific properties (see the conditions of Proposition 104).

Corollary 20. *Let X be a scheme and $\mathscr{F}$ a complex of sheaves of modules with quasi-coherent cohomology. There exists a hoinjective resolution $\mathscr{F} \longrightarrow \mathscr{I}$ with $\mathscr{I}$ a complex of injectives.*

Proof. Using the inductive construction of (DTC,Proposition 75) we can find a commutative diagram of complexes whose image in $K(X)$ is of the type studied in Proposition 19, with each $\mathscr{I}_n$ a bounded below complex of injectives. If we take $\mathscr{I} = \underleftarrow{holim}\,\mathscr{I}_n$ to be the canonical holimit on the level of complexes (DTC,Definition 29) then it is clear from the definition of the homotopy kernel that $\mathscr{I}$ is a complex of injectives. Since it is also a hoinjective resolution of $\mathscr{F}$ by Proposition 19, the proof is complete. □

3.1 Generalising Serre's Theorem

Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes with Y quasi-compact, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then the quasi-coherent sheaves $R^i f_*(\mathscr{F})$ are the relative version of the cohomology groups $H^i(X, \mathscr{F})$. So Serre's theorem (COS,Theorem 14) reaches its final form in the statement that there exists $d \geq 0$ such that $R^i f_*(\mathscr{F}) = 0$ for $i > d$ and any quasi-coherent $\mathscr{F}$ (HDIS,Proposition 33) (taking $Y = Spec\mathbb{Z}$ and $d = 0$ recovers Serre's theorem).

Thinking in terms of complexes, this says that if you start with a complex concentrated in degree zero and apply $\mathbb{R}f_*$, the cohomology of the resulting complex is bounded above. Moreover this "growth" is bounded by a single integer for every quasi-coherent sheaf. The next result extends this to arbitrary complexes, and can therefore be considered as the ultimate generalisation of Serre's theorem.

Theorem 21. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes. Then*

$$\mathbb{R}f_*(\mathfrak{D}_{qc}(X)) \subseteq \mathfrak{D}_{qc}(Y) \tag{11}$$

$$\mathbb{R}f_*(\mathfrak{D}_{qc}(X)^{\geq n}) \subseteq \mathfrak{D}_{qc}(Y)^{\geq n} \quad n \in \mathbb{Z} \tag{12}$$

If Y is quasi-compact there exists $d \geq 0$ such that for every $n \in \mathbb{Z}$

$$\mathbb{R}f_*(\mathfrak{D}_{qc}(X)^{\leq n}) \subseteq \mathfrak{D}_{qc}(Y)^{\leq (n+d)} \tag{13}$$

Proof. In words, if $\mathscr{F}$ is a complex of sheaves of modules on X with quasi-coherent cohomology, then $\mathbb{R}f_*\mathscr{F}$ also has quasi-coherent cohomology. Moreover there exists $d \geq 0$ such that if whenever the cohomology of $\mathscr{F}$ vanishes above n the cohomology of $\mathbb{R}f_*\mathscr{F}$ vanishes above $n+d$. Our proof follows [Lip] (3.9.2) which appears to be based on the proof of [Spa88] (3.13), but we use holimits instead of inverse limits.

As a right derived functor it is trivial that $\mathbb{R}f_*$ has lower dimension zero, and in particular sends objects of $\mathfrak{D}(X)^{\geq n}$ to $\mathfrak{D}(Y)^{\geq n}$ (DTC2,Lemma 33). We claim that if $\mathscr{F}$ is a complex of sheaves of modules with quasi-coherent cohomology and $\mathscr{F} \in \mathfrak{D}(X)^{\geq n}$ for some $n \in \mathbb{Z}$, then $\mathbb{R}f_*\mathscr{F}$ has quasi-coherent cohomology. By (DTC2,Proposition 40) it would be enough to show that $\mathbb{R}f_*\mathscr{F}$ has quasi-coherent cohomology for any quasi-coherent *sheaf* $\mathscr{F}$ on X. But from (DTC2,Corollary 12) we know that $H^i(\mathbb{R}f_*\mathscr{F}) = 0$ for $i < 0$ and for $i \geq 0$ there is an isomorphism of sheaves of modules

$$H^i(\mathbb{R}f_*\mathscr{F}) \longrightarrow R^if_*(\mathscr{F})$$

Since f is concentrated the sheaf $R^if_*\mathscr{F}$ is quasi-coherent (HDIS,Corollary 31), and therefore $\mathbb{R}f_*\mathscr{F} \in \mathfrak{D}_{qc}(Y)$ as claimed. This establishes the formula (12).

By (DCOS,Lemma 19) the functor $\mathbb{R}f_*$ is local, so to prove (11) we can reduce to the case where Y is quasi-compact. Assume that Y is quasi-compact, so that by (HDIS,Proposition 33) there exists an integer $d \geq 0$ with $R^if_*(\mathscr{G}) = 0$ for every $i > d$ and quasi-coherent sheaf $\mathscr{G}$ on X. Let $\mathscr{F}$ be a complex of sheaves of modules on X with quasi-coherent cohomology. The proof is modelled on that of Proposition 19. We can construct a commutative diagram of complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathscr{F}_{\geq n} & \longrightarrow & \cdots & \longrightarrow & \mathscr{F}_{\geq -2} & \longrightarrow & \mathscr{F}_{\geq -1} & \longrightarrow & \mathscr{F}_{\geq 0} \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathscr{I}_n & \longrightarrow & \cdots & \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_0
\end{array}$$

with the properties $(i), (ii)$ described in Proposition 19. As in the proof of Proposition 19 we deduce a morphism of triangles in $K(X)$ for $n < 0$

$$\begin{array}{ccccccc}
\mathscr{A}_n & \longrightarrow & \mathscr{F}_{\geq n} & \longrightarrow & \mathscr{F}_{\geq n+1} & \longrightarrow & \Sigma\mathscr{A}_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathscr{C}_n & \longrightarrow & \mathscr{I}_n & \longrightarrow & \mathscr{I}_{n+1} & \longrightarrow & \Sigma\mathscr{C}_n
\end{array}$$

Given an open affine subset $V \subseteq Y$ we set $U = f^{-1}V$ and from the isomorphism $\Gamma(U, \mathscr{C}_n) \cong \Sigma^{-n}\mathbb{R}\Gamma(U, H^n(\mathscr{F}))$ in $\mathfrak{D}(\mathbf{Ab})$ we deduce an isomorphism of abelian groups for $m > n + d$

$$H^m\Gamma(U, \mathscr{C}_n) \cong H^{m-n}(U, H^n(\mathscr{F})) \cong \Gamma(V, R^{m-n}f_*(H^n(\mathscr{F}))) = 0$$

using (HDIS,Corollary 31). The argument given in the proof of Proposition 19 applies here to show that for $m \geq n + d$ we have an isomorphism

$$H^m\Gamma(U, \underleftarrow{holim}\mathscr{I}_n) \longrightarrow H^m\Gamma(U, \mathscr{I}_n)$$

Sheafifying and using (DCOS,Lemma 3) together with the fact that $\underleftarrow{holim}\mathscr{I}_n$ and $\mathscr{I}_n$ are hoinjective we have isomorphisms for $m \geq n + d$

$$H^m(f_*\underleftarrow{holim}\mathscr{I}_n) \longrightarrow H^m(f_*\mathscr{I}_n)$$

$$H^m(\mathbb{R}f_*\underleftarrow{holim}\mathscr{I}_n) \longrightarrow H^m(\mathbb{R}f_*\mathscr{I}_n)$$

For any $m \in \mathbb{Z}$ and $n \leq 0$ we have commutative diagrams in $K(X)$ and $\mathfrak{Mod}(Y)$ respectively

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \underleftarrow{holim}\mathscr{I}_n \\ \downarrow & & \downarrow \\ \mathscr{F}_{\geq n} & \longrightarrow & \mathscr{I}_n \end{array} \qquad \begin{array}{ccc} H^m(\mathbb{R}f_*\mathscr{F}) & \longrightarrow & H^m(\mathbb{R}f_*\underleftarrow{holim}\mathscr{I}_n) \\ \downarrow & & \downarrow \\ H^m(\mathbb{R}f_*\mathscr{F}_{\geq n}) & \longrightarrow & H^m(\mathbb{R}f_*\mathscr{I}_n) \end{array}$$

Taking $m \geq n + d$ in the right hand diagram, and using the fact that by Proposition 19 the morphism $\mathscr{F} \longrightarrow \underleftarrow{holim}\mathscr{I}_n$ is a quasi-isomorphism, we have an isomorphism

$$H^m(\mathbb{R}f_*\mathscr{F}) \longrightarrow H^m(\mathbb{R}f_*(\mathscr{F}_{\geq n}))$$

We already know from the first part of the proof that $\mathbb{R}f_*(\mathscr{F}_{\geq n})$ has quasi-coherent cohomology, so $H^m(\mathbb{R}f_*\mathscr{F})$ is quasi-coherent. Since n was arbitrary, this shows that $\mathbb{R}f_*\mathscr{F}$ has quasi-coherent cohomology, thus proving (11).

If $\mathscr{F}$ belongs to $\mathfrak{D}_{qc}(X)^{\leq -1}$ then for $m \geq d$ we have an isomorphism

$$H^m(\mathbb{R}f_*\mathscr{F}) \cong H^m(\mathbb{R}f_*(\mathscr{F}_{\geq 0})) = 0$$

which shows that $\mathbb{R}f_*\mathscr{F} \in \mathfrak{D}_{qc}(Y)^{\leq (d-1)}$, establishing (13) and completing the proof. □

Corollary 22. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes with Y quasi-compact. Then the restricted functor $\mathbb{R}f_* : \mathfrak{D}_{qc}(X) \longrightarrow \mathfrak{D}(Y)$ is bounded.*

Proof. See (DTC2,Definition 19) for what we mean by a *bounded* triangulated functor. Let $d \geq 0$ be such that $R^i f_*(\mathscr{F}) = 0$ for any $i > d$ and quasi-coherent sheaf $\mathscr{F}$ on X. Then the proof of Theorem 21 shows that $dim^- T = 0$ and $dim^+ T \leq d$, and in particular T is bounded. □

3.2 Local Cohomology

Definition 10. Let $(X, \mathcal{O}_X)$ be a ringed space, $Z \subseteq X$ a closed subset with open complement U and $\mathscr{F}$ a sheaf of modules on X. We define a submodule $\Gamma_Z(\mathscr{F})$ of $\mathscr{F}$ by

$$\Gamma(V, \Gamma_Z(\mathscr{F})) = \{s \in \Gamma(V, \mathscr{F}) \,|\, s|_{U \cap V} = 0\}$$

for open $V \subseteq X$, called the *submodule of sections with support in Z*. This construction defines an additive functor

$$\Gamma_Z(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$$

together with a natural transformation $\Gamma_Z \longrightarrow 1$ given by the inclusions. In fact $\Gamma_Z(\mathscr{F})$ is the kernel of the canonical morphism $\mathscr{F} \longrightarrow j_*(\mathscr{F}|_U)$ where $j : U \longrightarrow X$ is the inclusion. That is, we have an exact sequence

$$0 \longrightarrow \Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*(\mathscr{F}|_U)$$

Remark 9. In (DCOS,Section 1.2) we encountered the functor $\Gamma_Z(X, -)$ and its derived functors $H^i_Z(X, -)$, which define the *cohomology with support in Z*. We saw that these cohomology groups are represented in the derived category by the suspensions of the sheaf $\mathcal{O}_Z$.

Remark 10. With the notation of Definition 10 suppose that $\mathscr{F}$ is flasque. Then it is easy to check that $\mathscr{F} \longrightarrow j_*(\mathscr{F}|_U)$ is an epimorphism, so we have a short exact sequence

$$0 \longrightarrow \Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*(\mathscr{F}|_U) \longrightarrow 0$$

In the next result we construct the local cohomology triangle of Grothendieck. The key technical point is the existence of a hoinjective resolution consisting of injective sheaves. Such a resolution exists for arbitrary sheaves by a result of Spaltenstein [Spa88] Theorem 4.5, but the proof of this result would take us on a lengthy detour. Instead we restrict ourselves to complexes with quasi-coherent cohomology, where the existence of such a resolution is just Proposition 19.

Lemma 23 (Local cohomology triangle). *Let X be a scheme and $j : U \longrightarrow X$ the inclusion of an open subset with complement Z. For any complex $\mathscr{F}$ of sheaves of modules on X with quasi-coherent cohomology, there is a canonical triangle in $\mathfrak{D}(X)$*

$$\mathbb{R}\Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \mathbb{R}j_*(\mathscr{F}|_U) \longrightarrow \Sigma\mathbb{R}\Gamma_Z(\mathscr{F})$$

This triangle is natural with respect to morphisms of complexes.

Proof. Let $\mathscr{F}$ be a complex of flasque sheaves of modules on X. We have a canonical short exact sequence of complexes in $\mathfrak{Mod}(X)$

$$0 \longrightarrow \Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*(\mathscr{F}|_U) \longrightarrow 0$$

and therefore a canonical triangle in $\mathfrak{D}(X)$ by (DTC,Proposition 20) which is natural with respect to morphisms of complexes

$$\Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*(\mathscr{F}|_U) \xrightarrow{-z} \Sigma\Gamma_Z(\mathscr{F}) \tag{14}$$

As described in the proof of (DCOS,Lemma 21) there is a canonical trinatural transformation $\eta^\diamond : \mathscr{F} \longrightarrow \mathbb{R}j_*(\mathscr{F}|_U)$ defined for any complex $\mathscr{F}$ in $\mathfrak{Mod}(X)$. The natural transformation $\Gamma_Z(-) \longrightarrow 1$ induces a trinatural transformation

$$\kappa : \mathbb{R}\Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F}$$

Given a complex $\mathscr{F}$ in $\mathfrak{Mod}(X)$ with quasi-coherent cohomology, we can by Corollary 20 find a hoinjective resolution $\mathscr{I}$ of $\mathscr{F}$ with each $\mathscr{I}^i$ injective, therefore flasque. There is a morphism in $\mathfrak{D}(X)$

$$w : \mathbb{R}j_*(\mathscr{F}|_U) \cong \mathbb{R}j_*(\mathscr{I}|_U) \cong j_*(\mathscr{I}|_U) \longrightarrow \Sigma\Gamma_Z(\mathscr{I}) \cong \Sigma\mathbb{R}\Gamma_Z(\mathscr{I}) \cong \Sigma\mathbb{R}\Gamma_Z(\mathscr{F})$$

using the connecting morphism $z_{\mathscr{I}} : j_*(\mathscr{I}|_U) \longrightarrow \Sigma\Gamma_Z(\mathscr{I})$ of (14). One checks as in the proof of (DCOS,Lemma 21) that w does not actually depend on the choice of resolution $\mathscr{I}$, and that it is natural in $\mathscr{F}$ with respect to morphisms of complexes.

We have now constructed a canonical sequence of morphisms in $\mathfrak{D}(X)$

$$\mathbb{R}\Gamma_Z(\mathscr{F}) \xrightarrow{\kappa} \mathscr{F} \xrightarrow{\eta^\diamond} \mathbb{R}j_*(\mathscr{F}|_U) \xrightarrow{-w} \Sigma\mathbb{R}\Gamma_Z(\mathscr{F})$$

which is natural with respect to morphisms of complexes. It remains to show that this sequence is a triangle. For this we can reduce immediately to the case where $\mathscr{F}$ is a hoinjective complex $\mathscr{I}$ with each $\mathscr{I}^i$ injective, in which case we have a commutative diagram in $\mathfrak{D}(X)$

$$\begin{array}{ccccccc}
\mathbb{R}\Gamma_Z(\mathscr{I}) & \longrightarrow & \mathscr{I} & \longrightarrow & \mathbb{R}j_*(\mathscr{I}|_U) & \longrightarrow & \Sigma\mathbb{R}\Gamma_Z(\mathscr{I}) \\
\Downarrow & & \Downarrow 1 & & \Downarrow & & \Downarrow \\
\Gamma_Z(\mathscr{I}) & \longrightarrow & \mathscr{I} & \longrightarrow & j_*(\mathscr{I}|_U) & \longrightarrow & \Sigma\Gamma_Z(\mathscr{I})
\end{array}$$

Since we know the bottom row is a triangle, so is the top row, which completes the proof. □

4 Grothendieck Duality

Classical Serre duality states that for a nonsingular projective scheme X of finite dimension n over a field k, there exists a coherent sheaf ω_X° together with canonical isomorphisms of k-modules for any coherent sheaf $\mathscr{F}$ of the form

$$\theta^i : H^{n-i}(X, \mathscr{F})^\vee \longrightarrow Ext^i(\mathscr{F}, \omega_X^\circ)$$

Actually something slightly more general is true, but we direct the reader to [Har77] or our notes on the subject (SDT,Theorem 7) for the precise statement. If we define a complex $\varpi_X = \Sigma^n \omega_X^\circ$ and use the derived category we can rewrite this as an isomorphism

$$H^k(X, \mathscr{F})^\vee \longrightarrow Hom_{\mathfrak{D}(X)}(\Sigma^k \mathscr{F}, \varpi_{X/k}) \tag{15}$$

A far reaching generalisation of Serre duality is Grothendieck duality, first described in Hartshorne's book [Har77] and later clarified by many authors, including Neeman [Nee96] who gave the first purely formal proof using Brown representability. It turns out that we can recover classical Serre duality from a powerful abstract statement about the existence of adjoints (Theorem 39).

In this section we first introduce a class of quasi-coherent sheaves called the *dilute* sheaves. Intuitively these play the role of injective or flasque quasi-coherent sheaves. They have enough good properties that we can use dilute resolutions in Corollary 29 to calculate explicitly the derived direct image functor. This is one of the major inputs into the proof of Theorem 31. Next we study the *Čech triangles* which are a useful tool in the theory of derived categories of quasi-coherent sheaves, since they allow us to efficiently reduce global problems to local problems. Using these results we conclude the section with the proof of Neeman's version of Grothendieck duality.

4.1 Dilute Resolutions

Throughout this section let $\mathcal{A}$ be an abelian category. Let X be a complex in $\mathcal{A}$, $M \geq 0$ an integer and suppose that for every $n \in \mathbb{Z}$ we have an exact sequence

$$0 \longrightarrow X^n \longrightarrow A^{n,0} \longrightarrow A^{n,1} \longrightarrow \cdots \longrightarrow A^{n,M} \longrightarrow 0$$

which fit into a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{n-1,M} & \longrightarrow & A^{n,M} & \longrightarrow & A^{n+1,M} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{n-1,1} & \longrightarrow & A^{n,1} & \longrightarrow & A^{n+1,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{n-1,0} & \longrightarrow & A^{n,0} & \longrightarrow & A^{n+1,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^{n} & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Write A for the bicomplex formed by the objects $A^{n,q}$, so that we have a canonical morphism of complexes $X \longrightarrow A^{\bullet,0}$. Taking the totalisation $Tot(A)$ of this bicomplex (DTC,Definition 33) we have a canonical morphism of complexes $\psi : X \longrightarrow Tot(A)$.

Lemma 24. *The morphism $\psi : X \longrightarrow Tot(A)$ is a quasi-isomorphism.*

Proof. Here is the quick proof using spectral sequences. There is a canonical spectral sequence (SS,Proposition 8)

$$E_2^{pq} = H^p(H_I^{\bullet,q}(A)) \Longrightarrow H^{p+q}(Tot(A))$$

and of course $H_I^{n,q}(A) = 0$ for $q > 0$ because the columns are exact, so this spectral sequence degenerates and we deduce an isomorphism $H^p(X) \cong H^p(Tot(A))$. Of course one has to actually check that this is equal to $H^p(\psi)$.

There is another proof by diagram chasing, which has the advantage of making the statement intuitively "obvious" after working through the proof. Firstly, all involved coproducts are finite so we can apply an embedding theorem to reduce to the case $\mathcal{A} = \mathbf{Ab}$ (DCAC,Theorem 1). Suppose we are given a sequence $(a_{ij})_{i+j=n}$ in the kernel of $\partial^n : Tot(A)^n \longrightarrow Tot(A)^{n+1}$. Since the bicomplex vanishes above M we must have $a_{n-M,M} \in Ker\partial_2^{n-M,M}$, and since the columns are exact we can choose $b_{n-M,M-1} \in A^{n-M,M-1}$ with

$$(-1)^{n-M}\partial_2^{n-M,M-1}(b_{n-M,M-1}) = a_{n-M,M}$$

But then $\partial_1^{n-M,M-1}(b_{n-M,M-1})$ and $a_{n-M+1,M-1}$ map to the same element under $\partial_2^{n-M+1,M-1}$. We deduce an element $b_{n-M+1,M-2} \in A^{n-M+1,M-2}$ with

$$(-1)^{n-M+1}\partial_2^{n-M+1,M-2}(b_{n-M+1,M-2}) + \partial_1^{n-M,M-1}(b_{n-M,M-1}) = a_{n-M+1,M-1}$$

Proceeding in this way, we construct a sequence $(b_{ij})_{i+j=n-1} \in Tot(A)^{n-1}$ together with an element $x \in Ker(\partial_X^n)$ such that

$$(a_{ij})_{i+j=n} - \partial^{n-1}(b_{ij})_{i+j=n-1} = \psi^n(x)$$

In other words, the morphism $H^n(\psi) : H^n(X) \longrightarrow H^n(Tot(A))$ is surjective. An easy diagram chase shows that it is also injective, and the proof is complete. ☐

Remark 11. Let X be a quasi-compact semi-separated scheme (CON,Definition 4). Equivalently, X is a scheme which admits a finite open cover $\mathfrak{U}$ by affine open sets with affine pairwise intersections. Suppose that $\mathfrak{U}$ contains $d \geq 1$ open affines. If $\mathscr{F}$ is a quasi-coherent sheaf on X, then we have the canonical Čech resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots \longrightarrow \mathscr{C}^{d-1}(\mathfrak{U}, \mathscr{F}) \longrightarrow 0 \tag{16}$$

which is an exact sequence of quasi-coherent sheaves, such that the sheaf $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is $\Gamma(X, -)$-acyclic for $p \geq 0$ (COS,Theorem 35). More generally, if $f : X \longrightarrow Y$ is a morphism of schemes with Y semi-separated then $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is f_*-acyclic (HDIS,Proposition 9).

Let $V \subseteq X$ be an open subset whose inclusion $i : V \longrightarrow X$ is affine and let $\mathfrak{U}|_V = \{U \cap V\}_{U \in \mathfrak{U}}$ denote the restricted affine open cover of V, which still has affine pairwise intersections. It is clear that $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})|_V = \mathscr{C}^p(\mathfrak{U}|_V, \mathscr{F}|_V)$, so the restriction to V of the Čech resolution (16) is the Čech resolution for $V, \mathfrak{U}|_V$ and $\mathscr{F}|_V$.

Definition 11. Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. We say that $\mathscr{F}$ is *predilute* if it has the following properties:

(a) $H^i(X, \mathscr{F}) = 0$ for any $i > 0$.

(b) $R^i f_*(\mathscr{F}) = 0$ for any $i > 0$ and morphism of schemes $f : X \longrightarrow Y$ with Y quasi-compact and semi-separated.

A quasi-coherent sheaf of modules $\mathscr{F}$ on X is *dilute* if it is predilute and the sheaf $\mathscr{F}|_V$ is predilute for every open subset $V \subseteq X$ with affine inclusion $V \longrightarrow X$. This property is stable under isomorphism. If $\mathscr{F}$ is dilute then so is $\mathscr{F}|_V$ for any open set $V \subseteq X$ with affine inclusion.

Remark 12. Any flasque quasi-coherent sheaf is dilute, and this is the motivation for the definition of a dilute sheaf. In the category of quasi-coherent sheaves injective objects are not well behaved (in general they are not even stable under restriction). A dilute sheaf has many of the good properties of an injective sheaf, but it is much more robust.

Lemma 25. *Let X be a quasi-compact semi-separated scheme, $\mathscr{F}$ a quasi-coherent sheaf of modules on X and $\mathfrak{U}$ a finite semi-separating cover of X. For $p \geq 0$ the sheaves $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ are dilute.*

Proof. It is clear from Remark 11 that $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is predilute. For the same reason, if $V \subseteq X$ is an open subset with affine inclusion $V \longrightarrow X$ then $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})|_V = \mathscr{C}^p(\mathfrak{U}|_V, \mathscr{F}|_V)$ is predilute. Therefore $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is dilute. ☐

Remark 13. It follows from Lemma 25 that every quasi-coherent sheaf on a quasi-compact semi-separated scheme admits a finite resolution (16) by dilute sheaves.

Lemma 26. *Let X be a quasi-compact semi-separated scheme. Then*

(i) If $\{\mathscr{D}_i\}_{i\in I}$ is a family of dilute sheaves on X then $\bigoplus_{i\in I}\mathscr{D}_i$ is dilute.

(ii) If $\{\mathscr{D}_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in\Lambda}$ is a direct system of dilute sheaves on X then $\varinjlim \mathscr{D}_\alpha$ is dilute.

Proof. The diluteness property is stable under restriction along affine inclusions, so it suffices in both cases to show that the given sheaves are predilute. The scheme X is concentrated, and therefore its underlying topological space is quasi-noetherian, so on X both the cohomology $H^i(X,-)$ and derived direct image $R^if_*(-)$ commute with coproducts and direct limits (COS,Theorem 26) (HDIS,Corollary 38). $\square$

Proposition 27. *Let X be a quasi-compact semi-separated scheme and $\mathscr{F}$ a complex of quasi-coherent sheaves on X. There is quasi-isomorphism $\mathscr{F}\longrightarrow\mathscr{D}$ with $\mathscr{D}$ a complex of dilute sheaves.*

Proof. Let $\mathfrak{U}$ be a finite semi-separating cover of X, with $d\geq 1$ elements. Then every quasi-coherent sheaf on X has a Čech resolution of the same length (16) and moreover these resolutions are natural in the sheaf, so we have a commutative diagram with exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathscr{C}^{d-1}(\mathfrak{U},\mathscr{F}^{n-1}) & \longrightarrow & \mathscr{C}^{d-1}(\mathfrak{U},\mathscr{F}^{n}) & \longrightarrow & \mathscr{C}^{d-1}(\mathfrak{U},\mathscr{F}^{n+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathscr{C}^{1}(\mathfrak{U},\mathscr{F}^{n-1}) & \longrightarrow & \mathscr{C}^{1}(\mathfrak{U},\mathscr{F}^{n}) & \longrightarrow & \mathscr{C}^{1}(\mathfrak{U},\mathscr{F}^{n+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathscr{C}^{0}(\mathfrak{U},\mathscr{F}^{n-1}) & \longrightarrow & \mathscr{C}^{0}(\mathfrak{U},\mathscr{F}^{n}) & \longrightarrow & \mathscr{C}^{0}(\mathfrak{U},\mathscr{F}^{n+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathscr{F}^{n-1} & \longrightarrow & \mathscr{F}^{n} & \longrightarrow & \mathscr{F}^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with every $\mathscr{C}^i(\mathfrak{U},\mathscr{F}^n)$ dilute by Lemma 25. Dropping the complex $\mathscr{F}$ from this diagram and totalising the remaining bicomplex, we have a complex $\mathscr{D}$ of dilute sheaves and a canonical quasi-isomorphism $\mathscr{F}\longrightarrow\mathscr{D}$ as defined in Lemma 24. $\square$

Definition 12. Let X be a quasi-compact semi-separated scheme, $\mathfrak{U}$ a finite semi-separating cover and $\mathscr{F}$ a complex of quasi-coherent sheaves on X. We write $\mathscr{C}_{tot}(\mathfrak{U},\mathscr{F})$ for the canonical complex $\mathscr{D}$ of dilute sheaves given in Proposition 27. There is a canonical quasi-isomorphism

$$\mathscr{F}\longrightarrow\mathscr{C}_{tot}(\mathfrak{U},\mathscr{F}) \tag{17}$$

called the *Čech resolution* of $\mathscr{F}$ with respect to $\mathfrak{U}$. For $n\in\mathbb{Z}$ we have

$$\mathscr{C}^n_{tot}(\mathfrak{U},\mathscr{F}) = \bigoplus_{i+j=n}\mathscr{C}^j(\mathfrak{U},\mathscr{F}^i)$$

The resolution (17) is natural in $\mathscr{F}$. Given a morphism of complexes $\mathscr{F}\longrightarrow\mathscr{G}$ in $\mathfrak{Qco}(X)$ the morphisms $\mathscr{C}^i(\mathfrak{U},\mathscr{F}^n)\longrightarrow\mathscr{C}^i(\mathfrak{U},\mathscr{G}^n)$ induce a canonical morphism of complexes $\mathscr{C}_{tot}(\mathfrak{U},\mathscr{F})\longrightarrow\mathscr{C}_{tot}(\mathfrak{U},\mathscr{G})$ making the following diagram commute

$$\begin{array}{ccc}
\mathscr{F} & \longrightarrow & \mathscr{C}_{tot}(\mathfrak{U},\mathscr{F}) \\
\downarrow & & \downarrow \\
\mathscr{G} & \longrightarrow & \mathscr{C}_{tot}(\mathfrak{U},\mathscr{G})
\end{array}$$

Proposition 28. *Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes. Then any complex of dilute sheaves on X is acyclic for the additive functor $f_*^{qc} : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$.*

Proof. First, observe that f is concentrated because it is a morphism of concentrated schemes (CON,Lemma 16). Let $\mathscr{X}$ be a complex of dilute sheaves on X. We claim that $\mathscr{X}$ is right acyclic for f_*^{qc} in the sense of (DTC2,Definition 4). Given Proposition 27 it suffices to show that if $\mathscr{X}$ is an *exact* complex of dilute sheaves, then $f_*^{qc}(\mathscr{X})$ is exact. Let $n \in \mathbb{Z}$ be arbitrary and set $\mathscr{K} = Ker\partial^n_{\mathscr{X}}$ so that we have a resolution of $\mathscr{K}$ by dilute sheaves

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{X}^n \longrightarrow \mathscr{X}^{n+1} \longrightarrow \mathscr{X}^{n+2} \longrightarrow \cdots$$

By definition every dilute sheaf is acyclic for $f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$, so by (DTC2,Remark 14) or (DF,Proposition 54) we can use this resolution to calculate $R^i f_*$. That is, for $k > 0$ we have

$$H^{n+k} f_*(\mathscr{X}) \cong R^k f_*(\mathscr{K})$$

Since f is concentrated and Y quasi-compact, we can find by (HDIS,Proposition 33) an integer $d \geq 0$ such that $R^i f_*(\mathscr{F}) = 0$ for $i > d$ and $\mathscr{F}$ quasi-coherent. Therefore $H^i f_*(\mathscr{K}) = 0$ for $i > n + d$. But n was arbitrary, so we conclude that $f_*(\mathscr{X})$ must be exact, as required. □

In our study of the higher direct image functors (HDIS,Proposition 9) played a central role. It tells us how to calculate the higher direct image of a quasi-coherent sheaf using the Čech complex. We are now in a position to prove the analogue for the derived direct image.

Corollary 29. *Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes and $\mathscr{F}$ a complex of quasi-coherent sheaves on X. For a finite semi-separating cover $\mathfrak{U}$ of X there is a canonical isomorphism in $\mathfrak{Dqcoh}(Y)$ natural in $\mathscr{F}$*

$$\mathbb{R}_q f_*(\mathscr{F}) \longrightarrow f_*(\mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F}))$$

Proof. By definition we have a canonical quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F})$, and the complex $\mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F})$ is acyclic for $f_* : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$ by Proposition 28. We have therefore a canonical isomorphism in $\mathfrak{Dqcoh}(Y)$ natural in $\mathscr{F}$

$$\mathbb{R}_q f_*(\mathscr{F}) \cong \mathbb{R}_q f_*(\mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F})) \cong f_*(\mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F}))$$

using the isomorphism of (DTC2,Remark 2). □

Proposition 30. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes with X concentrated. Then for $n \in \mathbb{Z}$*

$$\mathbb{R}_q f_*(\mathfrak{Dqcoh}(X)^{\geq n}) \subseteq \mathfrak{Dqcoh}(Y)^{\geq n}$$

If X, Y are quasi-compact and semi-separated there exists $d \geq 0$ such that for every $n \in \mathbb{Z}$

$$\mathbb{R}_q f_*(\mathfrak{Dqcoh}(X)^{\leq n}) \subseteq \mathfrak{Dqcoh}(Y)^{\leq (n+d)}$$

and in particular the triangulated functor $\mathbb{R}_q f_* : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(Y)$ is bounded.

Proof. As a right derived functor it is trivial that $\mathbb{R}_q f_*$ has lower dimension zero, and in particular sends objects of $\mathfrak{Dqcoh}(X)^{\geq n}$ to $\mathfrak{Dqcoh}(Y)^{\geq n}$ (DTC2,Lemma 33). For the rest of the proof we assume that X, Y are quasi-compact and semi-separated.

Let $\mathfrak{U}$ be a finite semi-separating cover of X with $d \geq 1$ elements and let a complex $\mathscr{F}$ in $\mathfrak{Dqcoh}(X)^{\leq n}$ be given. We may as well assume that $\mathscr{F}^i = 0$ for $i > n$. Consider the bicomplex of Čech sheaves given in Proposition 27. It is zero above row $d - 1$ and beyond column n. Therefore when we totalise to produce the Čech resolution $\mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F})$ the higher diagonals must all vanish. To be precise, we have $\mathscr{C}^i(\mathfrak{U}, \mathscr{F}) = 0$ for $i > n + d - 1$, and therefore

$$f_*(\mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F})) \in \mathfrak{Dqcoh}(Y)^{\leq (n+d-1)}$$

By Corollary 29 we have an isomorphism $\mathbb{R}_q f_*(\mathscr{F}) \cong f_*(\mathscr{C}_{tot}(\mathfrak{U}, \mathscr{F}))$ from which it follows that $\mathbb{R}_q f_*(\mathscr{F}) \in \mathfrak{Dqcoh}(Y)^{\leq (n+d-1)}$ as required. This shows in particular that $\mathbb{R}_q f_*$ is bounded in the sense of (DTC2,Definition 19). □

Theorem 31. *Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes. The diagram*

$$\begin{array}{ccc} \mathfrak{D}(X) & \xrightarrow{\mathbb{R}f_*} & \mathfrak{D}(Y) \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle U} \\ \mathfrak{Dqcoh}(X) & \xrightarrow[\mathbb{R}_q f_*]{} & \mathfrak{Dqcoh}(Y) \end{array}$$

commutes up to canonical trinatural equivalence.

Proof. Let u, U be the inclusions, so that we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{Mod}(X) & \xrightarrow{f_*} & \mathfrak{Mod}(Y) \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle U} \\ \mathfrak{Qco}(X) & \xrightarrow[f_*]{} & \mathfrak{Qco}(Y) \end{array}$$

and take arbitrary right derived functors $(\mathbb{R}f_*, \zeta), (\mathbb{R}_q f_*, \omega)$. The trinatural transformations are of the form

$$\zeta : Q \circ K(f_*) \longrightarrow \mathbb{R}f_* \circ Q$$
$$\omega : Q \circ K(f_*) \longrightarrow \mathbb{R}_q f_* \circ Q$$

and by (DTC2,Corollary 7) the pair $(U \circ \mathbb{R}_q f_*, U\omega)$ is a right derived functor of $U \circ f_* = f_* \circ u$. The trinatural transformation $\zeta K(u)$ induces a unique trinatural transformation $\mu : U \circ \mathbb{R}_q f_* \longrightarrow \mathbb{R}f_* \circ u$ making the following diagram commute

$$\begin{array}{ccccc} & & QK(f_* \circ u) & & \\ & {\scriptstyle U\omega}\swarrow & & \searrow{\scriptstyle \zeta K(u)} & \\ U \circ \mathbb{R}_q f_* \circ Q & & \xrightarrow[\mu Q]{} & & \mathbb{R}f_* \circ u \circ Q \end{array}$$

and we claim that μ is a trinatural equivalence. The triangulated functor $\mathbb{R}f_* \circ u$ is bounded by (DCOQS,Corollary 22), and $U \circ \mathbb{R}_q f_*$ is bounded by Proposition 30. Therefore by (DTC2,Proposition 38) it is enough to show that

$$\mu : \mathbb{R}_q f_*(\mathscr{F}) \longrightarrow \mathbb{R}f_*(\mathscr{F})$$

is an isomorphism in $\mathfrak{D}(Y)$ for any quasi-coherent *sheaf* $\mathscr{F}$ on X. Let $\mathfrak{U}$ be a finite semi-separating cover of X and consider the Čech complex

$$\mathscr{C}(\mathfrak{U}, \mathscr{F}) : 0 \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots$$

This is a complex of quasi-coherent sheaves, and we have a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{C}(\mathfrak{U}, \mathscr{F})$. We already know that $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ is acyclic for $f_*^{qc} : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$ from Proposition 28. Each sheaf $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is acyclic for $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ by definition of a dilute sheaf, so the complex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ is acyclic for $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ by (DTC2,Corollary 43). We deduce a commutative diagram in $\mathfrak{D}(Y)$

$$\begin{array}{ccccc} \mathbb{R}_q f_*(\mathscr{F}) & \Longrightarrow & \mathbb{R}_q f_*(\mathscr{C}(\mathfrak{U}, \mathscr{F})) & \Longrightarrow & f_*(\mathscr{C}(\mathfrak{U}, \mathscr{F})) \\ {\scriptstyle \mu}\downarrow & & {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathbb{R}f_*(\mathscr{F}) & \Longrightarrow & \mathbb{R}f_*(\mathscr{C}(\mathfrak{U}, \mathscr{F})) & \Longrightarrow & f_*(\mathscr{C}(\mathfrak{U}, \mathscr{F})) \end{array}$$

from which it is clear that $\mu : \mathbb{R}_q f_*(\mathscr{F}) \longrightarrow \mathbb{R}f_*(\mathscr{F})$ is an isomorphism, as required. □

Remark 14. Let $f : X \longrightarrow Y$ be an affine morphism of schemes with X concentrated and Y quasi-compact. Then the conclusion of Theorem 31 holds.

4.2 The Čech Triangles

Throughout this section X is a fixed quasi-compact semi-separated scheme and all sheaves of modules are over X, unless specified otherwise. Let $\mathscr{F}$ be a quasi-coherent sheaf of modules and $\mathfrak{U}$ a finite semi-separating open cover of X which is totally ordered. Then we have an exact sequence in $\mathfrak{Qco}(X)$ (COS,Theorem 35)

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \longrightarrow \cdots$$

called the *Čech resolution* of $\mathscr{F}$. We can split this into a series of short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{D}^1(\mathfrak{U},\mathscr{F}) \longrightarrow 0 \\ 0 \longrightarrow \mathscr{D}^1(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{D}^2(\mathfrak{U},\mathscr{F}) \longrightarrow 0 \\ \vdots \end{gathered}$$

That is, we define for $i \geq 1$ a canonical quasi-coherent sheaf of modules

$$\begin{aligned} \mathscr{D}^i(\mathfrak{U},\mathscr{F}) &= Im\left(\mathscr{C}^{i-1}(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^i(\mathfrak{U},\mathscr{F})\right) \\ &= Ker\left(\mathscr{C}^i(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^{i+1}(\mathfrak{U},\mathscr{F})\right) \end{aligned}$$

In order to simplify some statements, we set $\mathscr{D}^0(\mathfrak{U},\mathscr{F}) = \mathscr{F}$. The Čech resolution is functorial in $\mathscr{F}$ and therefore so are the sheaves $\mathscr{D}^i$. That is, for $i \geq 0$ we have additive functors

$$\begin{aligned} \mathscr{C}^i(\mathfrak{U},-) &: \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(X) \\ \mathscr{D}^i(\mathfrak{U},-) &: \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(X) \end{aligned}$$

together with a natural transformation $\mathscr{D}^i(\mathfrak{U},-) \longrightarrow \mathscr{C}^i(\mathfrak{U},-)$.

Proposition 32. *For $i \geq 0$ the additive functors $\mathscr{C}^i(\mathfrak{U},-)$ and $\mathscr{D}^i(\mathfrak{U},-)$ are exact and coproduct preserving.*

Proof. Suppose we have an exact sequence $\mathscr{X}$ of quasi-coherent sheaves

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

To show that $\mathscr{C}^i(\mathfrak{U},\mathscr{X})$ or $\mathscr{D}^i(\mathfrak{U},\mathscr{X})$ are exact, it suffices to show that these complexes are exact after we apply $\Gamma(V,-)$ for every V belonging to a semi-separating affine *basis* of X (DCOS,Lemma 3). Note that for any such V, the inclusion $V \longrightarrow X$ is affine, and in particular $\Gamma(V,\mathscr{C}^i(\mathfrak{U},\mathscr{X}))$ is the product of a finite number of complexes of the form

$$0 \longrightarrow \Gamma(V \cap U_{i_0,\ldots,i_p},\mathscr{F}') \longrightarrow \Gamma(V \cap U_{i_0,\ldots,i_p},\mathscr{F}) \longrightarrow \Gamma(V \cap U_{i_0,\ldots,i_p},\mathscr{F}'') \longrightarrow 0$$

which is exact because $V \cap U_{i_0,\ldots,i_p}$ is affine. Therefore $\mathscr{C}^i(\mathfrak{U},-)$ is an exact functor. It remains to check that the sequence

$$0 \longrightarrow \Gamma(V,\mathscr{D}^i(\mathfrak{U},\mathscr{F}')) \longrightarrow \Gamma(V,\mathscr{D}^i(\mathfrak{U},\mathscr{F})) \longrightarrow \Gamma(V,\mathscr{D}^i(\mathfrak{U},\mathscr{F}'')) \longrightarrow 0 \qquad (18)$$

is exact. Since V itself is affine the functor $\Gamma(V,-)$ is exact on $\mathfrak{Qco}(X)$, and we have a commutative diagram in which the top and bottom rows are exact, the top triple of vertical morphisms are epimorphisms and the bottom triple are monomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(V,\mathscr{C}^{i-1}(\mathfrak{U},\mathscr{F}')) & \longrightarrow & \Gamma(V,\mathscr{C}^{i-1}(\mathfrak{U},\mathscr{F})) & \longrightarrow & \Gamma(V,\mathscr{C}^{i-1}(\mathfrak{U},\mathscr{F}'')) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(V,\mathscr{D}^i(\mathfrak{U},\mathscr{F}')) & \longrightarrow & \Gamma(V,\mathscr{D}^i(\mathfrak{U},\mathscr{F})) & \longrightarrow & \Gamma(V,\mathscr{D}^i(\mathfrak{U},\mathscr{F}'')) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(V,\mathscr{C}^i(\mathfrak{U},\mathscr{F}')) & \longrightarrow & \Gamma(V,\mathscr{C}^i(\mathfrak{U},\mathscr{F})) & \longrightarrow & \Gamma(V,\mathscr{C}^i(\mathfrak{U},\mathscr{F}'')) & \longrightarrow & 0 \end{array}$$

Since kernels are left exact it is easy to see that the middle row must also be exact, as required. It is straightforward to check from the definition that $\mathscr{C}^i(\mathfrak{U}, -)$ is coproduct preserving for $i \geq 0$, and then since coproducts are exact we deduce that the $\mathscr{D}^i(\mathfrak{U}, -)$ are coproduct preserving as well. ☐

Definition 13 (Derived Čech complexes). For $p \geq 0$ the exact functors of Proposition 32 extend to coproduct preserving triangulated functors

$$\mathscr{C}^p(\mathfrak{U}, -), \mathscr{D}^p(\mathfrak{U}, -) : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(X)$$

Given a complex $\mathscr{F}$ of quasi-coherent sheaves we call $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ the pth *derived Čech complex*. Given indices $i_0, \ldots, i_p$ of $\mathfrak{U}$ we write $U_{i_0,\ldots,i_p}$ for $U_{i_0} \cap \cdots \cap U_{i_p}$, and by abuse of notation write $f : U_{i_0,\ldots,i_p} \longrightarrow X$ for any inclusion. These are all affine morphisms, so f_* is exact and therefore

$$\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \bigoplus_{i_0 < \cdots < i_p} f_*(\mathscr{F}|_{U_{i_0,\ldots,i_p}}) = \bigoplus_{i_0 < \cdots < i_p} \mathbb{R}_q f_*(\mathscr{F}|_{U_{i_0,\ldots,i_p}})$$

Proposition 33 (Čech Triangles). *For any complex $\mathscr{F}$ of quasi-coherent sheaves on X there is a canonical triangle in $\mathfrak{Dqcoh}(X)$ natural in $\mathscr{F}$*

$$\mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{F}$$

and for $p \geq 1$ another canonical triangle natural in $\mathscr{F}$

$$\mathscr{D}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^{p+1}(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{D}^p(\mathfrak{U}, \mathscr{F})$$

Proof. From the short exact sequences of complexes

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow 0$$
$$0 \longrightarrow \mathscr{D}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow 0$$

we deduce canonical triangles in $\mathfrak{Dqcoh}(X)$

$$\mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{F}$$
$$\mathscr{D}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{D}^p(\mathfrak{U}, \mathscr{F})$$

which are clearly natural in $\mathscr{F}$. ☐

Remark 15. Suppose our cover $\mathfrak{U}$ contains $d > 1$ open affines. Then $\mathscr{C}^p(\mathfrak{U}, -) = 0$ for $p \geq d$ and therefore the functor $\mathscr{D}^p(\mathfrak{U}, -)$ is zero for $p \geq d$. By definition we have $\mathscr{D}^{d-1}(\mathfrak{U}, \mathscr{F}) = \mathscr{C}^{d-1}(\mathfrak{U}, \mathscr{F})$ so our sequence of triangles is of the form

$$\mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{F}$$
$$\mathscr{D}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{D}^2(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{D}^1(\mathfrak{U}, \mathscr{F})$$
$$\vdots$$
$$\mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^{d-2}(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^{d-1}(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{F})$$

The derived Čech complexes are very simple objects. Once we establish some property for them, we can climb our way back up the sequence of triangles and show that $\mathscr{F}$ possesses this property. See for example the proof of Theorem 42. If our cover $\mathfrak{U}$ consists of two open affines U, V then the sequence consists of one triangle

$$\mathscr{F} \longrightarrow \mathbb{R}_q i_{U*}(\mathscr{F}|_U) \oplus \mathbb{R}_q i_{V*}(\mathscr{F}|_V) \longrightarrow \mathbb{R}_q i_{U \cap V*}(\mathscr{F}|_{U \cap V}) \longrightarrow \Sigma\mathscr{F}$$

4.3 Applications of the Čech Triangles

The derived Čech triangles are a very simple and useful technical tool. They will be the key to our proof of unbounded Grothendieck duality and also to the proof of the equivalence $\mathfrak{Dqcoh}(X) \cong \mathfrak{D}_{qc}(X)$. In this section we use the derived Čech triangles to strengthen and simplify the proof of Proposition 5.

Definition 14. Let X be a quasi-compact semi-separated scheme with finite semi-separating cover $\mathfrak{U}$. Let $\Omega(\mathfrak{U}, X)$ denote the smallest triangulated subcategory of $\mathfrak{Dqcoh}(X)$ containing the complexes $\mathbb{R}_q f_*(\mathscr{F})$, as $f : W \longrightarrow X$ ranges over the inclusions of finite intersections $W = U_{i_0} \cap \cdots \cap U_{i_p}$ of elements of $\mathfrak{U}$, and $\mathscr{F}$ ranges over all complexes of quasi-coherent sheaves on W. Clearly $\Omega(\mathfrak{U}, X) \subseteq \Omega(X)$ in the notation of Section 1.2.

Remark 16. With the notation of Definition 14 any of the inclusions $f : W \longrightarrow X$ is an affine morphism, so $\mathbb{R}_q f_*(\mathscr{F}) \cong f_*(\mathscr{F})$.

Proposition 34. *Let X be a quasi-compact semi-separated scheme with finite semi-separating cover $\mathfrak{U}$. Then the triangulated category $\mathfrak{Dqcoh}(X)$ is generated by the objects of $\mathfrak{Dqcoh}(U_{i_0,\ldots,i_p})$ for sequences $i_0 < \cdots < i_p$ of indices of $\mathfrak{U}$. That is, we have $\mathfrak{Dqcoh}(X) = \Omega(\mathfrak{U}, X)$.*

Proof. Given $p \geq 0$ and a complex $\mathscr{F}$ of quasi-coherent sheaves on X, it is clear that the derived Čech complex $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ belongs to $\Omega(\mathfrak{U}, X)$, because by definition it is a finite coproduct of complexes of the form $\mathbb{R}_q f_*(\mathscr{F}|_W)$. Climbing up the sequence of derived Čech triangles of Remark 15 we conclude that $\mathscr{F} \in \Omega(\mathfrak{U}, \mathscr{F})$, as required. □

We can now give a very slight improvement of Lemma 6 which we will make use of in our proof of unbounded Grothendieck duality.

Lemma 35. *Let $f : X \longrightarrow Y$ be a morphism of semi-separated schemes with X quasi-compact, and $V \subseteq Y$ a quasi-compact open subset. Then for any complex $\mathscr{F}$ of quasi-coherent sheaves of modules on X there is a canonical isomorphism in $\mathfrak{Dqcoh}(V)$ natural in $\mathscr{F}$*

$$\mu : (\mathbb{R}_q f_* \mathscr{F})|_V \longrightarrow \mathbb{R}_q g_*(\mathscr{F}|_U)$$

where $U = f^{-1}V$ and $g : U \longrightarrow V$ is the induced morphism of schemes.

Proof. It follows from (CON,Proposition 15) that f is concentrated, so as described at the beginning of the proof of Lemma 6 we have a canonical trinatural transformation $\mu : (-)|_V \mathbb{R}_q f_* \longrightarrow \mathbb{R}_q(g_*)(-)|_U$. Let $\mathcal{S}$ be the full subcategory of $\mathfrak{Dqcoh}(X)$ consisting of complexes $\mathscr{F}$ such that $\mu_{\mathscr{F}}$ is an isomorphism in $\mathfrak{Dqcoh}(V)$. This is a triangulated subcategory of $\mathfrak{Dqcoh}(X)$, and so by Proposition 34 it suffices to show that it is an isomorphism for any object of $\mathfrak{Dqcoh}(W)$ with $W = U_{i_0,\ldots,i_p}$ for a sequence of indices $i_0 < \cdots < i_p$ of a finite semi-separating open cover $\mathfrak{U}$ of X.

In this case the inclusion $j : W \longrightarrow X$ is affine, and one checks that any morphism from an affine scheme to a semi-separated scheme is affine, so the rest of the proof is the same as for Lemma 6. □

Proposition 36. *Let X be a quasi-compact semi-separated scheme with finite semi-separating cover $\mathfrak{U}$. A complex $\mathscr{X} \in \mathfrak{Dqcoh}(X)$ is compact in $\mathfrak{Dqcoh}(X)$ if and only if $\mathscr{X}|_V$ is compact in $\mathfrak{Dqcoh}(V)$ for every open set $V \in \mathfrak{U}$.*

Proof. Let $V = U_{i_0} \cap \cdots \cap U_{i_p}$ be a finite intersection of open sets in the cover $\mathfrak{U}$. Then V is affine and the inclusion $f : V \longrightarrow X$ is an affine morphism, so $f_* : \mathfrak{Qco}(V) \longrightarrow \mathfrak{Qco}(X)$ is exact and $\mathbb{R}_q f_* = \mathfrak{D}(f_*)$. In particular $\mathbb{R}_q f_*$ preserves coproducts (HDIS,Proposition 37), from which we deduce that $(-)|_V : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(V)$ preserves compactness (TRC3,Lemma 22).

Now suppose that $\mathscr{X}|_V$ is compact in $\mathfrak{Dqcoh}(V)$ for every open set $V \in \mathfrak{U}$. It is clear that the same is true whenever V is a finite intersection $V = U_{i_0} \cap \cdots \cap U_{i_p}$ of elements in the cover. Let

$d > 1$ be the number of open affines in $\mathfrak{U}$. Suppose we are given a family of complexes $\{\mathscr{F}_i\}_{i \in I}$ in $\mathfrak{Dqcoh}(X)$. By Proposition 33 we have canonical triangles in $\mathfrak{Dqcoh}(X)$ for each $i \in I$

$$\mathscr{F}_i \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \mathscr{D}^1(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \Sigma \mathscr{F}_i$$
$$\mathscr{D}^1(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \mathscr{D}^2(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \Sigma \mathscr{D}^1(\mathfrak{U}, \mathscr{F}_i)$$
$$\vdots$$
$$\mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \mathscr{C}^{d-2}(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \mathscr{C}^{d-1}(\mathfrak{U}, \mathscr{F}_i) \longrightarrow \Sigma \mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{F}_i)$$

We prove that $\mathscr{X}$ is compact by climbing the sequence and using the local compactness. Observe that for $p \geq 0$ we have (all Homs taken in $\mathfrak{Dqcoh}(X)$ or the local $\mathfrak{Dqcoh}(V)$)

$$\begin{aligned}
Hom(\mathscr{X}, \oplus_i \mathscr{C}^p(\mathfrak{U}, \mathscr{F}_i)) &\cong Hom(\mathscr{X}, \mathscr{C}^p(\mathfrak{U}, \oplus_i \mathscr{F}_i)) \\
&\cong Hom(\mathscr{X}, \oplus_{i_0 < \cdots < i_p} \mathbb{R}_q f_*(\oplus_i \mathscr{F}_i|_{U_{i_0,\ldots,i_p}})) \\
&\cong \oplus_{i_0 < \cdots < i_p} Hom(\mathscr{X}, \mathbb{R}_q f_*(\oplus_i \mathscr{F}_i|_{U_{i_0,\ldots,i_p}})) \\
&\cong \oplus_{i_0 < \cdots < i_p} Hom(\mathscr{X}|_{U_{i_0,\ldots,i_p}}, \oplus_i \mathscr{F}_i|_{U_{i_0,\ldots,i_p}}) \\
&\cong \oplus_{i_0 < \cdots < i_p} \oplus_i Hom(\mathscr{X}|_{U_{i_0,\ldots,i_p}}, \mathscr{F}_i|_{U_{i_0,\ldots,i_p}}) \\
&\cong \oplus_{i_0 < \cdots < i_p} \oplus_i Hom(\mathscr{X}, \mathbb{R}_q f_*(\mathscr{F}_i|_{U_{i_0,\ldots,i_p}})) \\
&\cong \oplus_i Hom(\mathscr{X}, \mathscr{C}^p(\mathfrak{U}, \mathscr{F}_i))
\end{aligned}$$

That is, the composite $Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathscr{C}^p(\mathfrak{U}, -))$ preserves coproducts. Using the final Čech triangle and the Five Lemma we deduce that the canonical map

$$\oplus_i Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{F}_i)) \longrightarrow Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \oplus_i \mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{F}_I))$$

is an isomorphism. Then from the second last Čech triangle we deduce that the analogous map for $\mathscr{D}^{d-1}(\mathfrak{U}, -)$ is an isomorphism. Climbing up the sequence we end up showing that the map

$$\oplus_i Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathscr{F}_i) \longrightarrow Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \oplus_i \mathscr{F}_i)$$

is an isomorphism. That is, $\mathscr{X}$ is compact in $\mathfrak{Dqcoh}(X)$, as required. □

4.4 Proof of Grothendieck Duality

The result of this section asserts the existence of a certain right adjoint, which we deduce from Brown representability following [Nee96].

Proposition 37. *If $h : X \longrightarrow Y$ is a morphism of semi-separated schemes with X quasi-compact then the triangulated functor $\mathbb{R}_q h_* : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(Y)$ preserves coproducts.*

Proof. Let $\mathfrak{V}$ be a semi-separating cover of Y. For each $V \in \mathfrak{V}$ the inclusion $V \longrightarrow Y$ is affine, and therefore by pullback so is $h^{-1}V \longrightarrow X$, so $h^{-1}V$ is quasi-compact semi-separated. Coproducts in the derived category are local by Lemma 2, and $\mathbb{R}_q h_*$ is local by Lemma 35, so we can reduce to the case where Y is affine.

If X is also affine then $h : SpecB \longrightarrow SpecA$ is induced by a morphism of rings $A \longrightarrow B$. In this case the additive functor $h_* : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$ is exact: it is just restriction of scalars. Therefore $\mathbb{R}_q h_* = \mathfrak{D}(h_*)$ acts by the usual direct image on complexes, and trivially preserves coproducts.

For the general case let $\mathfrak{U}$ be a finite semi-separating cover of X with $d > 1$ elements. The idea is to apply $\mathbb{R}_q h_*$ to the derived Čech triangles of Proposition 33 and use the affine case of the previous paragraph to climb the sequence of triangles until we reach $\mathbb{R}_q h_*(-)$. More precisely we have for $i \geq 0$ a pair of triangulated functors

$$\mathbb{R}_q h_* \circ \mathscr{D}^i(\mathfrak{U}, -), \mathbb{R}_q h_* \circ \mathscr{C}^i(\mathfrak{U}, -) : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(Y)$$

Using Lemma 3 we have for each $\mathscr{F} \in \mathfrak{Dqcoh}(X)$ a canonical isomorphism natural in $\mathscr{F}$

$$\begin{aligned} \mathbb{R}_q h_* \left(\mathscr{C}^i(\mathfrak{U}, \mathscr{F}) \right) &= \mathbb{R}_q h_* \left(\oplus_{i_0 < \cdots < i_p} \mathbb{R}_q f_* (\mathscr{F}|_{U_{i_0,\ldots,i_p}}) \right) \\ &\cong \oplus_{i_0 < \cdots < i_p} \mathbb{R}_q (hf)_* (\mathscr{F}|_{U_{i_0,\ldots,i_p}}) \end{aligned}$$

Since each $U_{i_0,\ldots,i_p}$ is affine the functors $\mathbb{R}_q(hf)$ preserve coproducts, and it follows that the composite $\mathbb{R}_q h_* \circ \mathscr{C}^i(\mathfrak{U}, -)$ preserves coproducts. Given a nonempty family $\{\mathscr{F}_\lambda\}_{\lambda \in \Lambda}$ in $\mathfrak{Dqcoh}(X)$ if we set $\mathscr{F} = \bigoplus_\lambda \mathscr{F}_\lambda$ then the last Čech triangle of Remark 15 yields a morphism of triangles

$$\begin{array}{ccccccc} \bigoplus_\lambda \mathbb{R}_q h_* \mathscr{D}^{d-2}(\mathscr{F}_\lambda) & \to & \bigoplus_\lambda \mathbb{R}_q h_* \mathscr{C}^{d-2}(\mathscr{F}_\lambda) & \to & \bigoplus_\lambda \mathbb{R}_q h_* \mathscr{C}^{d-1}(\mathscr{F}_\lambda) & \to & \Sigma \bigoplus_\lambda \mathbb{R}_q h_* \mathscr{D}^{d-2}(\mathscr{F}_\lambda) \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \downarrow \\ \mathbb{R}_q h_* \mathscr{D}^{d-2}(\mathscr{F}) & \longrightarrow & \mathbb{R}_q h_* \mathscr{C}^{d-2}(\mathscr{F}) & \longrightarrow & \mathbb{R}_q h_* \mathscr{C}^{d-1}(\mathscr{F}) & \longrightarrow & \Sigma \mathbb{R}_q h_* \mathscr{D}^{d-2}(\mathscr{F}) \end{array}$$

where we have dropped $\mathfrak{U}$ from the notation to fit the triangles on the page. Our earlier discussion shows that β, γ are isomorphisms, and therefore α is an isomorphism. In other words, the triangulated functor $\mathbb{R}_q h_* \circ \mathscr{D}^{d-2}(\mathfrak{U}, -)$ preserves coproducts.

Now from the second last Čech triangle we deduce using the same argument that $\mathbb{R}_q h_* \circ \mathscr{D}^{d-3}(\mathfrak{U}, -)$ preserves coproducts, and so on until we reach the first Čech triangle. The argument applied to this triangle shows that $\mathbb{R}_q h_*$ preserves coproducts, and completes the proof. □

Corollary 38. *Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes. The composite of $\mathbb{R}f_* : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(Y)$ with the inclusion $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is a coproduct preserving triangulated functor.*

Proof. By Theorem 31 we have a diagram of triangulated functors which commutes up to canonical trinatural equivalence

$$\begin{array}{ccc} \mathfrak{D}(X) & \xrightarrow{\mathbb{R}f_*} & \mathfrak{D}(Y) \\ u \uparrow & & \uparrow U \\ \mathfrak{Dqcoh}(X) & \xrightarrow[\mathbb{R}_q f_*]{} & \mathfrak{Dqcoh}(Y) \end{array}$$

The triangulated functor $\mathbb{R}_q f_*$ preserves coproducts by Proposition 37, and the functor U preserves coproducts because the inclusion $\mathfrak{Qco}(Y) \longrightarrow \mathfrak{Mod}(X)$ does. Hence the composite $\mathbb{R}f_* \circ u$ must preserve coproducts, as required. □

Theorem 39. *Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes. The triangulated functor $\mathbb{R}_q f_* : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(Y)$ has a right adjoint.*

Proof. Both schemes are concentrated, so $\mathfrak{Qco}(X), \mathfrak{Qco}(Y)$ are grothendieck abelian and therefore their derived categories are mildly portly (DTC,Corollary 114). By (DTC2,Theorem 52) the triangulated category $\mathfrak{Dqcoh}(X)$ satisfies the representability theorem, and by Proposition 37 the functor $\mathbb{R}_q f_*$ preserves coproducts, so it follows from (TRC3,Corollary 27) that $\mathbb{R}_q f_*$ has a right adjoint. □

5 A Comparison of Derived Categories

One of the main results of [BN93] is that for a quasi-compact and separated scheme X the canonical functor $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful, with essential image consisting of those complexes with quasi-coherent cohomology. In this section we prove this result, following [BN93].

Let X be an arbitrary scheme and $\mathfrak{Qco}(X)$ the abelian category of quasi-coherent sheaves on X. This is an abelian subcategory of $\mathfrak{Mod}(X)$, so there is a canonical triangulated functor $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$. First we study the case where X is affine.

Proposition 40. *Let $X = Spec(R)$ be an affine scheme. Then the canonical triangulated functor $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful.*

Proof. The abelian category $\mathfrak{Qco}(X)$ is a full replete subcategory of $\mathfrak{Mod}(X)$, so we already know that $K(\mathfrak{Qco}(X)) \longrightarrow K(X)$ is fully faithful (DTC,Lemma 38). Given complexes $\mathscr{X}, \mathscr{Y}$ of quasi-coherent sheaves, we have to show that the map

$$Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathscr{Y}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathscr{Y}) \tag{19}$$

is a bijection. The first step is to reduce to the case where $\mathscr{Y}$ is bounded below. We do this by writing $\mathscr{Y}$ as a holimit of bounded below complexes in two ways (one for $\mathfrak{Qco}(X)$ and one for $\mathfrak{Mod}(X)$) and then comparing these limits. Some care is necessary because products in $\mathfrak{Qco}(X)$ and $\mathfrak{Mod}(X)$ do not agree. Throughout the proof we use a superscript $\underleftarrow{holim}^{qc}$ to denote a limit or holimit taken in $\mathfrak{Qco}(X)$.

We can apply the technique of (DTC,Proposition 75) to construct a commutative diagram of complexes in $\mathfrak{Qco}(X)$

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathscr{Y}_{\geq -2} & \longrightarrow & \mathscr{Y}_{\geq -1} & \longrightarrow & \mathscr{Y}_{\geq 0} \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_{0} \end{array} \tag{20}$$

with each vertical morphism a quasi-isomorphism and the $\mathscr{I}_n$ bounded below complexes of injectives in $\mathfrak{Qco}(X)$. Now consider the bottom row of this diagram as a sequence of morphisms of complexes in $\mathfrak{Mod}(X)$, which we can inductively resolve in $\mathfrak{Mod}(X)$ to produce a commutative diagram of complexes

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_{0} \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & \mathscr{J}_{-2} & \longrightarrow & \mathscr{J}_{-1} & \longrightarrow & \mathscr{J}_{0} \end{array} \tag{21}$$

which vertical quasi-isomorphisms and the $\mathscr{J}_n$ bounded below complexes of injectives in $\mathfrak{Mod}(X)$. That is, so that the composite of the two diagrams (20) and (21) is a diagram with all the good properties listed in Proposition 19.

Take the triangle defining $\underleftarrow{holim}^{qc}\mathscr{I}_n$ and map it into $K(X)$. It is still a triangle, but the involved products are not necessarily products in $\mathfrak{Mod}(X)$. Nonetheless, we deduce a morphism of triangles from this to the triangle defining $\underleftarrow{holim}\,\mathscr{J}_n$

$$\begin{array}{ccccccc} \underleftarrow{holim}^{qc}\mathscr{I}_n & \longrightarrow & \prod^{qc}\mathscr{I}_n & \longrightarrow & \prod^{qc}\mathscr{I}_n & \longrightarrow & \Sigma\underleftarrow{holim}^{qc}\mathscr{I}_n \\ {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underleftarrow{holim}\,\mathscr{J}_n & \longrightarrow & \prod\mathscr{J}_n & \longrightarrow & \prod\mathscr{J}_n & \longrightarrow & \Sigma\underleftarrow{holim}\,\mathscr{J}_n \end{array} \tag{22}$$

We claim that α is a quasi-isomorphism. Since $\mathfrak{Qco}(X) \cong R\mathbf{Mod}$ has exact products and a projective generator, we know from (DTC,Proposition 75) that there is a canonical quasi-isomorphism $\mathscr{Y} \longrightarrow \underleftarrow{holim}^{qc}\mathscr{I}_n$. The composite

$$\mathscr{Y} \longrightarrow \underleftarrow{holim}^{qc}\mathscr{I}_n \longrightarrow \underleftarrow{holim}\,\mathscr{J}_n$$

in $K(X)$ is a factorisation of the induced morphism $\mathscr{Y} \longrightarrow \prod\mathscr{J}_n$ through the holimit, and therefore by the proof of Proposition 19 must be a quasi-isomorphism. It is now immediate that α is a quasi-isomorphism as well.

We know that $\mathscr{Y}$ is quasi-isomorphic to both of the holimits in (22), so we can use this morphism of triangles to compare the Hom sets in $\mathfrak{Dqcoh}(X)$ and $\mathfrak{D}(X)$. We do this by applying $Hom_{K(X)}(\mathscr{X}, -)$ and deducing a morphism of long exact sequences (note that $K(\mathfrak{Qco}(X)) \longrightarrow K(X)$ is fully faithful, so we can replace $K(X)$ by $K(\mathfrak{Qco}(X))$ and then commute the products in the top row outside the Hom)

$$\begin{array}{ccccccc} \cdots \to \prod Hom_{K(\mathfrak{Qco}(X))}(\mathscr{X}, \Sigma^{-1}\mathscr{I}_n) & \to & Hom_{K(\mathfrak{Qco}(X))}(\mathscr{X}, \underleftarrow{holim}^{qc}\mathscr{I}_n) & \to & \prod Hom_{K(\mathfrak{Qco}(X))}(\mathscr{X}, \mathscr{I}_n) \to \cdots \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow \prod Hom_{K(X)}(\mathscr{X}, \Sigma^{-1}\mathscr{J}_n) & \longrightarrow & Hom_{K(X)}(\mathscr{X}, \underleftarrow{holim}\,\mathscr{J}_n) & \longrightarrow & \prod Hom_{K(X)}(\mathscr{X}, \mathscr{J}_n) \longrightarrow \cdots \end{array}$$

All the complexes in the second position of these Hom sets are hoinjective, so we can replace $K(-)$ with $\mathfrak{D}(-)$ throughout. If we can show that (19) is a bijection for $\mathscr{Y}$ bounded below then we would deduce from the Five Lemma that the canonical map

$$Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \underleftarrow{holim}^{qc}\mathscr{I}_n) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{X}, \underleftarrow{holim}\,\mathscr{J}_n)$$

is a bijection, from which it follows that (19) is a bijection for our original $\mathscr{Y}$. This completes the reduction step, so we can now assume that $\mathscr{Y}$ is bounded below.

In the first variable all the reductions are very easy, because we can use hocolimits which are much better behaved than holimits. Write $\mathscr{X}$ as the direct limit $\varinjlim \mathscr{X}_{\leq n}$ of its truncations. By (DTC,Proposition 65) we have a triangle in $\mathfrak{Dqcoh}(X)$

$$\bigoplus_{n\geq 0} \mathscr{X}_{\leq n} \xrightarrow{1-\nu} \bigoplus_{n\geq 0} \mathscr{X}_{\leq n} \longrightarrow \mathscr{X} \longrightarrow \Sigma \bigoplus_{n\geq 0} \mathscr{X}_{\leq n}$$

Applying $Hom_{\mathfrak{D}(X)}(-, \mathscr{Y})$ and using the Five Lemma we reduce to the case where $\mathscr{X}$ is bounded above. Now there is a rapid series of reductions for $\mathscr{X}$:

- Since $\mathfrak{Qco}(X) \cong R\mathbf{Mod}$ we can use (DTC,Proposition 69) to reduce to the case where $\mathscr{X}$ is a bounded above complex of *free* sheaves of modules (i.e. a coproduct of copies of $\mathcal{O}_X$).
- Writing $\mathscr{X}$ as the direct limit of its brutal trunctions $\mathscr{X} = \varinjlim_{n\leq 0} {}^b\mathscr{X}_{\geq n}$ we reduce to the case where $\mathscr{X}$ is a *bounded* complex of free sheaves.
- Such a complex can be built up successively from single free sheaves by (DTC,Remark 32), so finally we can assume $\mathscr{X} = \mathcal{O}_X$.

That is, we have reduced to showing that for a bounded below complex $\mathscr{Y}$ in $\mathfrak{Qco}(X)$ the map

$$Hom_{\mathfrak{Dqcoh}(X)}(\mathcal{O}_X, \mathscr{Y}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathcal{O}_X, \mathscr{Y}) \tag{23}$$

is a bijection. The final reduction is to replace $\mathscr{Y}$ by a single quasi-coherent sheaf. For any $n \in \mathbb{Z}$ we have a triangle in $\mathfrak{Dqcoh}(X)$ (DTC,Lemma 27)

$$c_n H^n(\mathscr{Y}) \longrightarrow \mathscr{Y}_{\geq n} \longrightarrow \mathscr{Y}_{\geq (n+1)} \longrightarrow \Sigma c_n H^n(\mathscr{Y})$$

Assuming that (23) is a bijection for $\mathscr{Y}$ a quasi-coherent *sheaf*, it follows from induction and applying $Hom_{\mathfrak{D}(X)}(\mathcal{O}_X, -)$ to this triangle that (23) is a bijection for any bounded below $\mathscr{Y}$ (here we use the fact that $Hom_{\mathfrak{D}(X)}(\mathcal{O}_X, \mathscr{Y}) = 0$ if $\mathscr{Y}$ is nonzero only in positive degrees). So finally we reduce to showing that

$$Hom_{\mathfrak{Dqcoh}(X)}(\mathcal{O}_X, \Sigma^n\mathscr{F}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathcal{O}_X, \Sigma^n\mathscr{F})$$

is a bijection for a quasi-coherent sheaf $\mathscr{F}$ and $n \in \mathbb{Z}$. If $n < 0$ both groups are zero. If $n = 0$ then both sides are $\Gamma(X, \mathscr{F})$ (DTC,Proposition 28). If $n > 0$ then the left hand side is $Ext^n(\mathcal{O}_X, \mathscr{F}) = 0$ (DTC2,Lemma 28) and the right hand side is $H^n(X, \mathscr{F})$ (DCOS,Theorem 12) which is zero by Serre's theorem (COS,Theorem 14). □

Proposition 41. *Let $X = Spec(R)$ be an affine scheme. Then the canonical triangulated functor $F : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ induces a triequivalence*

$$\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}_{qc}(X)$$

Proof. We already know that $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful, and we are claiming that every complex in $\mathfrak{Mod}(X)$ with quasi-coherent cohomology is isomorphic in $\mathfrak{D}(X)$ to a complex of quasi-coherent sheaves. That is, we claim $\mathfrak{D}_{qc}(X) = \mathscr{Q}$ where $\mathscr{Q}$ is the essential image of F. The proof is divided into several steps.

Step 1: Every bounded complex in $\mathfrak{D}_{qc}(X)$ is in $\mathscr{Q}$. The proof is by induction on the number n of nonzero terms in the complex. If $n \leq 1$ this is trivial. Given a bounded complex $\mathscr{Y}$ in $\mathfrak{D}_{qc}(X)$ we have a triangle

$$\mathscr{Y}_{\leq k} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Y}_{\geq(k+1)} \longrightarrow \Sigma\mathscr{Y}_{\leq k}$$

where we choose k so that $\mathscr{Y}_{\leq k}, \mathscr{Y}_{\geq(k+1)}$ have strictly fewer nonzero terms. By the inductive hypothesis $\mathscr{Y}_{\leq k}, \mathscr{Y}_{\geq(k+1)}$ are in $\mathscr{Q}$, and since F is full there is a morphism α in $\mathfrak{Dqcoh}(X)$ corresponding to $\mathscr{Y}_{\geq(k+1)} \longrightarrow \Sigma\mathscr{Y}_{\leq k}$. Completing α to a triangle in $\mathfrak{Dqcoh}(X)$ and mapping to $\mathfrak{D}(X)$, we deduce that $\mathscr{Y}$ is in the essential image of F.

Step 2: Every bounded below complex in $\mathfrak{D}_{qc}(X)$ is in $\mathscr{Q}$. If $\mathscr{Y}$ is a bounded below complex in $\mathfrak{D}_{qc}(X)$ then $\mathscr{Y} = \underrightarrow{holim}_{n\geq 0}\mathscr{Y}_{\leq n}$. That is, there is a triangle in $\mathfrak{D}(X)$ (DTC,Proposition 65)

$$\bigoplus \mathscr{Y}_{\leq n} \longrightarrow \bigoplus \mathscr{Y}_{\leq n} \longrightarrow \mathscr{Y} \longrightarrow \Sigma \bigoplus \mathscr{Y}_{\leq n}$$

But the $\mathscr{Y}_{\leq n}$ are all in the essential image of F, so we can form this hocolimit already in $\mathfrak{Dqcoh}(X)$. The functor F preserves coproducts, so mapping this hocolimit in $\mathfrak{Dqcoh}(X)$ into $\mathfrak{D}(X)$ we deduce that $\mathscr{Y}$ is in the essential image of F.

Step 3: Every complex in $\mathfrak{D}_{qc}(X)$ is in $\mathscr{Q}$. The general idea is to write a complex $\mathscr{Y}$ in $\mathfrak{D}_{qc}(X)$ as a holimit of bounded below complexes, therefore reducing to Step 2. But as usual, the devil is in the details.

For each $n \leq 0$ we can by Step 2 find an isomorphism $\mathscr{Y}_{\geq n} \longrightarrow \mathscr{I}_n$ in $\mathfrak{D}(X)$ where $\mathscr{I}_n$ is a complex of injectives in $\mathfrak{Qco}(X)$ with $\mathscr{I}_n^i = 0$ for $i < n$. Since the functor $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful and the $\mathscr{I}_n$ hoinjective in $K(\mathfrak{Qco}(X))$, we deduce a canonical sequence in $K(\mathfrak{Qco}(X))$

$$\cdots \longrightarrow \mathscr{I}_{-3} \longrightarrow \mathscr{I}_{-2} \longrightarrow \mathscr{I}_{-1} \longrightarrow \mathscr{I}_0 \tag{24}$$

which fits into a commutative diagram in $\mathfrak{D}(X)$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathscr{Y}_{\geq n} & \longrightarrow \cdots \longrightarrow & \mathscr{Y}_{\geq -2} & \longrightarrow & \mathscr{Y}_{\geq -1} & \longrightarrow & \mathscr{Y}_{\geq 0} \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & \mathscr{I}_n & \longrightarrow \cdots \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_0 \end{array} \tag{25}$$

Notice that the vertical isomorphisms in $\mathfrak{D}(X)$ don't necessarily lift to quasi-isomorphisms in $K(X)$, because while the $\mathscr{I}_n$ are hoinjective in $K(\mathfrak{Qco}(X))$, they may not be hoinjective in $K(X)$.

Considering (24) as a sequence in $K(X)$ we can inductively resolve it to produce a commutative diagram in $K(X)$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathscr{I}_n & \longrightarrow \cdots \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & \mathscr{J}_n & \longrightarrow \cdots \longrightarrow & \mathscr{J}_{-2} & \longrightarrow & \mathscr{J}_{-1} & \longrightarrow & \mathscr{J}_0 \end{array} \tag{26}$$

with vertical quasi-isomorphisms and the $\mathscr{J}_n$ bounded below complexes of injectives in $\mathfrak{Mod}(X)$. The composite $\mathscr{Y}_{\geq n} \longrightarrow \mathscr{I}_n \longrightarrow \mathscr{J}_n$ in $\mathfrak{D}(X)$ must lift uniquely to $K(X)$ because $\mathscr{J}_n$ is hoinjective. In other words, the composite of the two diagrams (25) and (26) is the image in $\mathfrak{D}(X)$ of a resolution of the truncations of $\mathscr{Y}$ in $K(X)$ as described by Proposition 104. In particular $\mathscr{Y} \longrightarrow \underleftarrow{holim}\, \mathscr{J}_n$ is a quasi-isomorphism.

Take the triangle defining $\underleftarrow{holim}^{qc}\mathscr{I}_n$ and map it into $K(X)$. The morphism of sequences (26) induces a morphism of triangles

$$\begin{array}{ccccccc} \underleftarrow{holim}^{qc}\mathscr{I}_n & \longrightarrow & \prod^{qc}\mathscr{I}_n & \longrightarrow & \prod^{qc}\mathscr{I}_n & \longrightarrow & \Sigma\underleftarrow{holim}^{qc}\mathscr{I}_n \\ \downarrow{\scriptstyle\vartheta} & & \downarrow & & \downarrow & & \downarrow \\ \underleftarrow{holim}\,\mathscr{I}_n & \longrightarrow & \prod\mathscr{I}_n & \longrightarrow & \prod\mathscr{I}_n & \longrightarrow & \Sigma\underleftarrow{holim}\,\mathscr{I}_n \end{array} \tag{27}$$

and we claim that ϑ is a quasi-isomorphism. Then we would have $\mathscr{Y}$ isomorphic in $\mathfrak{D}(X)$ to the quasi-coherent complex $\underleftarrow{holim}^{qc}\mathscr{I}_n$, so the proof would be complete.

We dealt successfully with a very similar situation in Proposition 40. The difference is that there our complex $\mathscr{Y}$ was already assumed to be in $\mathfrak{Qco}(X)$. We can reuse the argument, but there is a trick: we have to find a complex of quasi-coherent sheaves $\mathscr{Y}'$ so that the sequence (24) can be reinterpreted as a resolution in $\mathfrak{Qco}(X)$ of the truncations of $\mathscr{Y}'$. The obvious candidate is $\mathscr{Y}' = \underleftarrow{holim}^{qc}\mathscr{I}_n$.

From commutativity of (25) in $\mathfrak{D}(X)$ we deduce that $H^i(\mathscr{I}_{n-1}) \longrightarrow H^i(\mathscr{I}_n)$ is an isomorphism for $i \geq n$. This puts us in the situation of (DTC,Lemma 78) with $\mathcal{A} = \mathfrak{Qco}(X)$ and the sequence (24), from which we conclude that $H^i(\underleftarrow{holim}^{qc}\mathscr{I}_n) \longrightarrow H^i(\mathscr{I}_n)$ is an isomorphism for $i \geq n$. This yields a canonical quasi-isomorphism $\mathscr{Y}'_{\geq n} \longrightarrow \mathscr{I}_n$ in $K(\mathfrak{Qco}(X))$. There is a commutative diagram in $K(\mathfrak{Qco}(X))$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathscr{Y}'_{\geq n} & \longrightarrow & \cdots & \longrightarrow & \mathscr{Y}'_{\geq -2} & \longrightarrow & \mathscr{Y}'_{\geq -1} & \longrightarrow & \mathscr{Y}'_{\geq 0} \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathscr{I}_n & \longrightarrow & \cdots & \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_0 \end{array} \tag{28}$$

with vertical quasi-isomorphisms. This realises (24) as the resolution in $\mathfrak{Qco}(X)$ of the truncations of $\mathscr{Y}'$. Now we are in precisely the situation studied in the proof of Proposition 40 and in the same way we deduce that $\vartheta : \underleftarrow{holim}^{qc}\mathscr{I}_n \longrightarrow \underleftarrow{holim}\,\mathscr{I}_n$ is a quasi-isomorphism, which completes the proof. □

Theorem 42 (Bökstedt-Neeman). *Let X be a quasi-compact semi-separated scheme. The canonical triangulated functor $U : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful and induces a triequivalence*

$$\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}_{qc}(X)$$

Proof. First we show that U is fully faithful. To make the exposition clearer, for the duration of this proof we say that a complex $\mathscr{Y}$ in $\mathfrak{Qco}(X)$ is *swift* if for every complex $\mathscr{X}$ in $\mathfrak{Qco}(X)$ the canonical map

$$Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathscr{Y}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathscr{Y}) \tag{29}$$

is a bijection. We want to show that every complex is swift, which we do in two steps.

Step 1. For any complex $\mathscr{F}$ in $\mathfrak{Qco}(X)$, finite semi-separating cover $\mathfrak{U}$ and $p \geq 0$ the derived Čech complex $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is swift. It is clear that a finite coproduct in $\mathfrak{Dqcoh}(X)$ of swift complexes is swift, so it suffices to show that for $f : W \longrightarrow X$ the inclusion of an affine open subset, the complex $\mathbb{R}_q f_*(\mathscr{F}|_W)$ is swift. We have a bijection

$$\begin{aligned} Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathbb{R}_q f_*(\mathscr{F}|_W)) &\cong Hom_{\mathfrak{Dqcoh}(W)}(\mathscr{X}|_W, \mathscr{F}|_W) \\ &\cong Hom_{\mathfrak{D}(W)}(\mathscr{X}|_W, \mathscr{F}|_W) \\ &\cong Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathbb{R}f_*(\mathscr{F}|_W)) \\ &\cong Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathbb{R}_q f_*(\mathscr{F}|_W)) \end{aligned}$$

where we have used the following facts: (i) Restriction is left adjoint to $\mathbb{R}_q f_*$ for quasi-coherent sheaves, and to $\mathbb{R}f_*$ for arbitrary sheaves. (ii) In the affine case every complex is swift by Proposition 40. (iii) For morphisms of quasi-compact semi-separated schemes $\mathbb{R}f_*$ and $\mathbb{R}_q f_*$ agree by

Theorem 31. To check that this bijection is actually (29), let a morphism $\alpha : \mathscr{X} \longrightarrow \mathbb{R}_q f_*(\mathscr{F}|_W)$ in $\mathfrak{Dqcoh}(X)$ be given. We check that $\alpha, U(\alpha)$ end up at the same element of $Hom_{\mathfrak{D}(W)}(\mathscr{X}|_W, \mathscr{F}|_W)$. This amounts to showing that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}_q f_*(\mathscr{F}|_W)|_W & \xrightarrow{\mu|_W} & \mathbb{R} f_*(\mathscr{F}|_W)|_W \\ & {\scriptstyle U(\varepsilon^\diamondsuit)}\searrow \quad \swarrow{\scriptstyle \varepsilon^\diamondsuit} & \\ & \mathscr{F}|_W & \end{array}$$

We observed in the proof of Lemma 4 that $\varepsilon^\diamondsuit = (\zeta|_W)^{-1}$, so commutativity of the above diagram follows from the defining property of μ given in Theorem 31. This shows that $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is swift.

Step 2. Every complex is swift. Any translation of a swift complex is swift, and if we have a triangle in $\mathfrak{Dqcoh}(X)$

$$\mathscr{Y}' \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Y}'' \longrightarrow \Sigma\mathscr{Y}'$$

with $\mathscr{Y}$ and $\mathscr{Y}''$ swift, a simple argument using long exact sequences and the Five Lemma shows that $\mathscr{Y}'$ is swift. We know that the derived Čech complexes are swift, so the result now follows by climbing up the sequence of Čech triangles given in Proposition 33.

To be precise, let an arbitrary complex $\mathscr{Y}$ in $\mathfrak{Qco}(X)$ be given and let $\mathfrak{U}$ be a finite semi-separating cover of X. Since we know the result for affines, we may as well assume it contains $d > 1$ elements. By Remark 15 we have a sequence of triangles in $\mathfrak{Dqcoh}(X)$

$$\begin{gathered} \mathscr{Y} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{Y}) \longrightarrow \mathscr{D}^1(\mathfrak{U}, \mathscr{Y}) \longrightarrow \Sigma\mathscr{Y} \\ \mathscr{D}^1(\mathfrak{U}, \mathscr{Y}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{Y}) \longrightarrow \mathscr{D}^2(\mathfrak{U}, \mathscr{Y}) \longrightarrow \Sigma\mathscr{D}^1(\mathfrak{U}, \mathscr{Y}) \\ \vdots \\ \mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{Y}) \longrightarrow \mathscr{C}^{d-2}(\mathfrak{U}, \mathscr{Y}) \longrightarrow \mathscr{C}^{d-1}(\mathfrak{U}, \mathscr{Y}) \longrightarrow \Sigma\mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{Y}) \end{gathered}$$

In the last triangle the second and third objects are swift by Step 1, so it follows that $\mathscr{D}^{d-2}(\mathfrak{U}, \mathscr{Y})$ is swift. Proceeding in this way, we end up showing that $\mathscr{Y}$ is swift which completes the proof that U is fully faithful.

It remains to show that every complex $\mathscr{Y}$ in $\mathfrak{Mod}(X)$ with quasi-coherent cohomology is in the essential image of U. Let $n(X)$ denote the smallest number of affine open subsets that you can use to form a semi-separating cover of X. We proceed by induction on $n(X)$, with the case $n(X) = 1$ having already been established in Proposition 41. Assume that $n(X) > 1$ with $X = X_1 \cup \cdots \cup X_n$ and set $U = X_1, V = X_2 \cup \cdots \cup X_n$. This is a semi-separating open cover for V, which is therefore quasi-compact semi-separated with $n(V) < n$. Set $Z = X \setminus U$ and let $j : U \longrightarrow X, i : V \longrightarrow X$ be the inclusions. Consider the local cohomology triangle in $\mathfrak{D}(X)$ of Lemma 23

$$\mathbb{R}\Gamma_Z(\mathscr{Y}) \longrightarrow \mathscr{Y} \longrightarrow \mathbb{R}j_*(\mathscr{Y}|_U) \xrightarrow{\tau} \Sigma\mathbb{R}\Gamma_Z(\mathscr{F}) \tag{30}$$

Restricting to U it is clear that $\mathbb{R}\Gamma_Z(\mathscr{Y})|_U = 0$. From the Mayer-Vietoris triangle for $\mathbb{R}\Gamma_Z(\mathscr{Y})$ (DCOS,Lemma 21) we conclude that the canonical morphism

$$\mathbb{R}\Gamma_Z(\mathscr{Y}) \longrightarrow \mathbb{R}i_*(\mathbb{R}\Gamma_Z(\mathscr{Y})|_V) \tag{31}$$

is an isomorphism in $\mathfrak{D}(X)$. Now $\mathscr{Y}|_U$ has quasi-coherent cohomology and U is affine, so there is an isomorphism in $\mathfrak{D}(U)$ of $\mathscr{Y}|_U$ with a complex of quasi-coherent sheaves. Both i, j are morphisms of quasi-compact semi-separated schemes, so from Theorem 31 we deduce that $\mathbb{R}i_*, \mathbb{R}j_*$ send quasi-coherent complexes into complexes in the essential image of $U : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$. In particular $\mathbb{R}j_*(\mathscr{Y}|_U)$ is in the essential image of U.

From the triangle (30) we infer that $\mathbb{R}\Gamma_Z(\mathscr{Y})$ has quasi-coherent cohomology, so the same argument together with the inductive hypothesis on V and the isomorphism (31) shows that $\Sigma\mathbb{R}\Gamma_Z(\mathscr{F})$ is in the essential image of U. Since U is fully faithful, τ is (up to isomorphism) the image in $\mathfrak{D}(U)$ of a morphism τ' in $\mathfrak{Dqcoh}(U)$. If we extend τ' to a triangle in $\mathfrak{Dqcoh}(U)$, this triangle must be isomorphic in $\mathfrak{D}(U)$ to (30) from which we deduce that $\mathscr{Y}$ is in the essential image of U, completing the proof. $\square$

Corollary 43. *If $f : X \longrightarrow Y$ is a morphism of schemes then $\mathbb{L}f^*(\mathfrak{D}_{qc}(Y)) \subseteq \mathfrak{D}_{qc}(X)$.*

Proof. That is, we claim the triangulated functor $\mathbb{L}f^* : \mathfrak{D}(Y) \longrightarrow \mathfrak{D}(X)$ sends complexes with quasi-coherent cohomology to complexes with quasi-coherent cohomology. The derived inverse image is local (DCOS,Lemma 87) so we can reduce to Y affine. Let $\mathscr{Y}$ be a complex of sheaves of modules on Y with quasi-coherent cohomology. In light of Proposition 41 we may as well assume that $\mathscr{Y}$ is actually a complex of quasi-coherent sheaves. An affine scheme has enough quasi-coherent hoflats, so it follows from (DCOQS,Proposition 11) that $\mathbb{L}f^*(\mathscr{Y})$ has quasi-coherent cohomology. □

Corollary 44. *If X is a scheme and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules with quasi-coherent cohomology, then $\mathscr{X} \underline{\otimes} \mathscr{Y}$ also has quasi-coherent cohomology.*

Proof. Here $\mathscr{X} \underline{\otimes} \mathscr{Y}$ denotes the derived tensor product on $\mathfrak{D}(X)$ (DCOS,Definition 13). The derived tensor is local (DCOS,Lemma 55) so we may as well assume X is affine. By Proposition 41 it is then enough to consider the case where $\mathscr{X}, \mathscr{Y}$ are complexes of quasi-coherent sheaves. An affine scheme has enough quasi-coherent hoflats, so it follows from (DCOQS,Lemma 18) that $\mathscr{X} \underline{\otimes} \mathscr{Y}$ has quasi-coherent cohomology. For an alternative proof see [Lip] (2.5.8). □

Corollary 45. *Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes. The restricted functor $\mathbb{R}f_* : \mathfrak{D}_{qc}(X) \to \mathfrak{D}(Y)$ preserves coproducts.*

Proof. The triangulated functor $\mathbb{R}f_* : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(Y)$ composed with $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ preserves coproducts by Corollary 38. By Theorem 42 this second functor factors as an equivalence $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}_{qc}(X)$ followed by the inclusion $k : \mathfrak{D}_{qc}(X) \longrightarrow \mathfrak{D}(X)$, so it is clear that $\mathbb{R}f_* \circ k$ also preserves coproducts. □

Proposition 46. *Let X be a quasi-compact semi-separated scheme. For complexes $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ of quasi-coherent sheaves on X there is a canonical isomorphism of abelian groups natural in all three variables*

$$Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X} \underline{\otimes}_q \mathscr{Y}, \mathscr{Z}) \longrightarrow Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathbb{R}_q\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z}))$$

In particular we have an adjoint pair

$$\mathfrak{Dqcoh}(X) \underset{-\underline{\otimes}_q \mathscr{Y}}{\overset{\mathbb{R}_q\mathscr{H}om^\bullet(\mathscr{Y},-)}{\rightleftarrows}} \mathfrak{Dqcoh}(X) \qquad - \underline{\otimes}_q \mathscr{Y} \dashv \mathbb{R}_q\mathscr{H}om^\bullet(\mathscr{Y}, -)$$

Proof. To avoid confusion we write $- \underline{\otimes}_q -$ for the derived tensor product on $\mathfrak{Dqcoh}(X)$ defined in Definition 8. This exists because X has enough quasi-coherent hoflats by Lemma 8. The derived sheaf Hom on $\mathfrak{Dqcoh}(X)$ is the one given in Definition 2. The functor $i : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful by Theorem 42 so using the adjunction on $\mathfrak{D}(X)$ we have a canonical isomorphism of abelian groups natural in all variables

$$\begin{aligned} Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathbb{R}Q\mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) &\cong Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{Y}, \mathscr{Z})) \\ &\cong Hom_{\mathfrak{D}(X)}(\mathscr{X} \underline{\otimes} \mathscr{Y}, \mathscr{Z}) \\ &\cong Hom_{\mathfrak{D}(X)}(\mathscr{X} \underline{\otimes}_q \mathscr{Y}, \mathscr{Z}) \\ &\cong Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X} \underline{\otimes}_q \mathscr{Y}, \mathscr{Z}) \end{aligned}$$

as required. □

Corollary 47. *Let X be a quasi-compact semi-separated scheme and $\mathscr{X}$ a complex of quasi-coherent sheaves. The triangulated functors $\mathscr{X} \underline{\otimes}_q -$ and $- \underline{\otimes}_q \mathscr{X}$ preserve coproducts.*

5.1 Quasi-coherent Hypercohomology

Recall the definition of *hypercohomology* from (DCOS,Definition 19). There is a version for quasi-coherent sheaves, just as there is a quasi-coherent version of sheaf cohomology (HDIS,Section 6). For any reasonable scheme, the two types of hypercohomology agree.

Definition 15. Let X be a concentrated scheme, $\Gamma_{qc}(X,-) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ the global sections functor, and $\mathbb{H}_q(X,-)$ a right derived functor. We call $\mathbb{H}_q(X,\mathscr{F})$ the *quasi-coherent hypercohomology* of a complex $\mathscr{F}$ of quasi-coherent sheaves, and write $\mathbb{H}^m_q(X,\mathscr{F})$ for $H^m\mathbb{H}(X,\mathscr{F})$.

Lemma 48. *Let X be a quasi-compact semi-separated scheme. Any complex of dilute sheaves on X is acyclic for the additive functor $\Gamma_{qc}(X,-) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$.*

Proof. Just copy the proof of Proposition 28, but instead of applying (HDIS,Proposition 33) use (HDIS,Lemma 32). □

To actually calculate sheaf cohomology one uses the Čech cohomology and (COS,Theorem 35). We can now give the analagous result for quasi-coherent hypercohomology.

Proposition 49. *Let X be a quasi-compact semi-separated scheme and $\mathscr{F}$ a complex of quasi-coherent sheaves on X. For a finite semi-separating cover $\mathfrak{U}$ of X there is a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$ natural in $\mathscr{F}$*

$$\mathbb{H}_q(X,\mathscr{F}) \longrightarrow \Gamma(X,\mathscr{C}_{tot}(\mathfrak{U},\mathscr{F}))$$

Proof. By definition we have a canonical quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{C}_{tot}(\mathfrak{U},\mathscr{F})$, and the complex $\mathscr{C}_{tot}(\mathfrak{U},\mathscr{F})$ is acyclic for $\Gamma_{qc}(X,-) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ by Lemma 48. We have therefore a canonical isomorphism in $\mathfrak{D}(\mathbf{Ab})$ natural in $\mathscr{F}$

$$\mathbb{H}_q(X,\mathscr{F}) \cong \mathbb{H}_q(X,\mathscr{C}_{tot}(\mathfrak{U},\mathscr{F})) \cong \Gamma(X,\mathscr{C}_{tot}(\mathfrak{U},\mathscr{F}))$$

using the isomorphism of (DTC2,Remark 2). □

On a concentrated scheme X the sheaf cohomology functors $H^i(X,-)$ preserve coproducts (COS,Theorem 26). The next result should be interpreted as generalising this statement to quasi-coherent hypercohomology.

Corollary 50. *Let X be a quasi-compact semi-separated scheme. The triangulated functor*

$$\mathbb{H}_q(X,-) : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

preserves coproducts.

Proof. To be precise, we have an additive functor $\Gamma_{qc}(X,-) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ and we claim that any right derived functor $\mathbb{H}_q(X,-) = \mathbb{R}\Gamma_{qc}(X,-) : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$ preserves coproducts. The unique morphism $f : X \longrightarrow Spec(\mathbb{Z})$ has a coproduct-preserving derived functor $\mathbb{R}_q f_*$ by Proposition 37. Composing with the canonical equivalence $\mathfrak{Dqcoh}(Spec(\mathbb{Z})) \cong \mathfrak{D}(\mathbf{Ab})$ yields the desired result. □

By (HDIS,Lemma 32) if you take a reasonable scheme X then there is a uniform bound on the number of nonzero cohomology groups of any quasi-coherent sheaf. This is also true of the quasi-coherent hypercohomology, and moreover it is true of the *usual* hypercohomology restricted to complexes with quasi-coherent cohomology.

Proposition 51. *Let X be a concentrated scheme. The restricted hypercohomology functor*

$$\mathbb{H}(X,-) : \mathfrak{D}_{qc}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

is bounded. If X is a quasi-compact semi-separated scheme the quasi-coherent hypercohomology functor

$$\mathbb{H}_q(X,-) : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

is also bounded.

Proof. To be precise, let $(\mathbb{H}(X,-),\zeta)$ be an arbitrary right derived functor of the additive functor $\Gamma(X,-) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$ and $(\mathbb{H}_q(X,-),\omega)$ be an arbitrary right derived functor of $\Gamma_{qc}(X,-) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$. It follows immediately from Proposition 30 and the discussion given in the proof of Corollary 50 that if X is quasi-compact semi-separated then $\mathbb{H}_q(X,-)$ is bounded. It remains to show that the functor $\mathbb{H}(X,-)$ composed with the inclusion $\mathfrak{D}_{qc}(X) \longrightarrow \mathfrak{D}(X)$ is bounded whenever X is concentrated.

This is not a special case of Theorem 21, but as one might expect a small modification of the proof gives the desired result. To be precise, let $\mathscr{F}$ be a complex of sheaves of modules on X with quasi-coherent cohomology. Then with the notation used in the proof of Theorem 21 we have an isomorphism $\Gamma(X,\mathscr{C}_n) \cong \Sigma^{-n}\mathbb{R}\Gamma(X, H^n(\mathscr{F}))$ in $\mathfrak{D}(\mathbf{Ab})$ for $n < 0$. Since X is concentrated we can by (HDIS,Lemma 32) find an integer $d \geq 0$ such that $H^i(X,\mathscr{G}) = 0$ for every quasi-coherent sheaf $\mathscr{G}$ and $i > d$. Then for $m > n + d$ we have

$$H^m\Gamma(X,\mathscr{C}_n) \cong H^{m-n}(X, H^n(\mathscr{F})) = 0$$

and by the now standard argument we deduce for $m \geq n + d$ an isomorphism

$$H^m\Gamma(X, \underleftarrow{holim}\mathscr{I}_n) \longrightarrow H^m\Gamma(X,\mathscr{I}_n)$$

Since the complexes $\underleftarrow{holim}\mathscr{I}_n$ and $\mathscr{I}_n$ are hoinjective, this is actually an isomorphism

$$H^m(\mathbb{H}(X, \underleftarrow{holim}\mathscr{I}_n)) \longrightarrow H^m(\mathbb{H}(X,\mathscr{I}_n))$$

Copying the proof of Theorem 21 we end up with an isomorphism for arbitrary $n \leq 0$ and $m \geq n+d$

$$H^m(\mathbb{H}(X,\mathscr{F})) \longrightarrow H^m(\mathbb{H}(X,\mathscr{F}_{\geq n}))$$

from which it follows that $\mathbb{H}(X,-)$ sends $\mathfrak{D}_{qc}(X)^{\leq -1}$ into $\mathfrak{D}(\mathbf{Ab})^{\leq (d-1)}$, completing the proof that $\mathbb{H}(X,-)$ is bounded on $\mathfrak{D}_{qc}(X)$. ∎

On a quasi-compact semi-separated scheme the quasi-coherent sheaf cohomology $H^i_{qc}(X,-)$ agrees with the usual sheaf cohomology $H^i(X,-)$ (HDIS,Corollary 41). Again this generalises to quasi-coherent hypercohomology.

Proposition 52. *Let X be a quasi-compact semi-separated scheme. The diagram of triangulated functors*

$$\begin{array}{ccc} \mathfrak{Dqcoh}(X) & \xrightarrow{U} & \mathfrak{D}(X) \\ & {\scriptstyle \mathbb{H}_q(X,-)} \searrow \quad \swarrow {\scriptstyle \mathbb{H}(X,-)} & \\ & \mathfrak{D}(\mathbf{Ab}) & \end{array}$$

commutes up to canonical trinatural equivalence.

Proof. We have a commutative diagram of additive functors

$$\begin{array}{ccc} \mathfrak{Qco}(X) & \xrightarrow{U} & \mathfrak{Mod}(X) \\ & {\scriptstyle \Gamma_{qc}(X,-)} \searrow \quad \swarrow {\scriptstyle \Gamma(X,-)} & \\ & \mathbf{Ab} & \end{array}$$

Let $(\mathbb{H}(X,-),\zeta)$ and $(\mathbb{H}_q(X,-),\omega)$ be arbitrary right derived functors of $\Gamma(X,-)$ and $\Gamma_{qc}(X,-)$ respectively. The trinatural transformation $\zeta K(U)$ induces a unique trinatural transformation $\mu : \mathbb{H}_q(X,-) \longrightarrow \mathbb{H}(X,-)U$ unique making the following diagram commute

$$\begin{array}{ccc} & QK(\Gamma_{qc}(X,-)) & \\ & {\scriptstyle \omega} \swarrow \quad \searrow {\scriptstyle \zeta K(U)} & \\ \mathbb{H}_q(X,-) \circ Q & \xrightarrow[\mu Q]{} & \mathbb{H}(X,-) \circ U \circ Q \end{array}$$

and we claim that μ is a trinatural equivalence. The triangulated functors $\mathbb{H}_q(X,-)$ and $\mathbb{H}(X,-)\circ U$ are bounded by Proposition 51. Therefore by (DTC2,Proposition 38) it is enough to show that

$$\mu : \mathbb{H}_q(X,\mathscr{F}) \longrightarrow \mathbb{H}(X,\mathscr{F})$$

is an isomorphism in $\mathfrak{D}(\mathbf{Ab})$ for any quasi-coherent *sheaf* $\mathscr{F}$ on X. Let $\mathfrak{U}$ be a finite semi-separating cover of X and consider the Čech complex

$$\mathscr{C}(\mathfrak{U},\mathscr{F}) : 0 \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \longrightarrow \cdots$$

This is a complex of dilute sheaves, so $\mathscr{C}(\mathfrak{U},\mathscr{F})$ is acyclic for $\Gamma_{qc}(X,-)$ by Lemma 48. Each $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ is acyclic for $\Gamma(X,-)$ by the definition of a dilute sheaf, so the complex $\mathscr{C}(\mathfrak{U},\mathscr{F})$ is acyclic for $\Gamma(X,-)$ by (DTC2,Corollary 43). It follows easily from these observations that $\mu : \mathbb{H}_q(X,\mathscr{F}) \longrightarrow \mathbb{H}(X,\mathscr{F})$ is an isomorphism, as required. □

Remark 17. Let A be a commutative ring and X a quasi-compact semi-separated scheme over $Spec(A)$. Then $\mathfrak{Dqcoh}(X)$ and $\mathfrak{D}(X)$ are A-linear triangulated categories (DTC,Remark 11) and we have additive functors

$$\Gamma_q(X,-) : \mathfrak{Qco}(X) \longrightarrow A\mathbf{Mod}$$
$$\Gamma(X,-) : \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$$

Take arbitrary right derived functors $\mathbb{H}_q(X,-) : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(A)$ and $\mathbb{H}(X,-) : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(A)$. It is easily verified that these agree up to canonical trinatural equivalence with the hypercohomologies defined with values in $\mathbf{Ab}$ (see (DCOS,Remark 35)). It is therefore a consequence of Proposition 51 that that $\mathbb{H}_q(X,-)$ and $\mathbb{H}(X,-)$ are bounded triangulated functors (to be precise, the latter functor is bounded on $\mathfrak{D}_{qc}(X)$). One can check exactly as in Proposition 52 that the diagram

$$\begin{array}{ccc} \mathfrak{Dqcoh}(X) & \xrightarrow{\quad U \quad} & \mathfrak{D}(X) \\ & {\scriptstyle \mathbb{H}_q(X,-)} \searrow \quad \swarrow {\scriptstyle \mathbb{H}(X,-)} & \\ & \mathfrak{D}(A) & \end{array}$$

commutes up to canonical trinatural equivalence.

Corollary 53. *Let X be a quasi-compact semi-separated scheme. The restricted functor*

$$\mathbb{H}(X,-) : \mathfrak{D}_{qc}(X) \longrightarrow \mathfrak{D}(\mathbf{Ab})$$

preserves coproducts.

Proof. Here $\mathbb{H}(X,-)$ denotes an arbitrary right derived functor of $\Gamma(X,-) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$. Given a coproduct in $\mathfrak{D}_{qc}(X)$ we can by Theorem 42 assume all the objects are actually in $\mathfrak{Dqcoh}(X)$, in which case $\mathbb{H}(X,-)$ preserves the coproduct by Proposition 52 and Corollary 50. □

Here is the derived version of (HDIS,Theorem 30), one of the most important properties of the "old-fashioned" derived direct image.

Proposition 54. *Let $f : X \longrightarrow Y$ be a morphism of schemes where X is concentrated and $Y = SpecA$ is affine. Then for any complex $\mathscr{F}$ of quasi-coherent sheaves of modules on X there is a canonical isomorphism in $\mathfrak{Dqcoh}(Y)$ natural in $\mathscr{F}$*

$$\gamma : \mathbb{H}_q(X,\mathscr{F})^{\sim} \longrightarrow \mathbb{R}_q f_*(\mathscr{F})$$

Proof. We have three additive functors

$$f_* : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$$
$$\Gamma_{qc}(X,-) : \mathfrak{Qco}(X) \longrightarrow A\mathbf{Mod}$$
$$\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Qco}(Y)$$

where the last is actually an equivalence. In particular it is exact, so it extends to the derived category and $\widetilde{-}\circ\mathbb{H}_q(X,-)$ is a right derived functor of $\widetilde{-}\circ\Gamma(X,-)$, which is naturally equivalent to $f_*(-)$. We deduce a canonical trinatural equivalence $\widetilde{-}\circ\mathbb{H}_q(X,-) \longrightarrow \mathbb{R}_q f_*(-)$, as required.. $\square$

The reader will observe that Proposition 54 is completely trivial, so it seems like we should be able to take cohomology of both sides and obtain a very easy proof of (HDIS,Theorem 30). Unfortunately this is an illusion, because of the possible distinction between the right derived functors of $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ and $f_* : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$ (see [TT90] or our HDIS notes). Somewhere one has to prove something.

Corollary 55. *Let $f : X \longrightarrow Spec(k)$ be a concentrated scheme over a field k. Then for any complex $\mathscr{F}$ of quasi-coherent sheaves of modules on X there is a canonical isomorphism in $\mathfrak{Dqcoh}(k)$ natural in $\mathscr{F}$ between $\mathbb{R}_q f_*(\mathscr{F})$ and the following complex*

$$\cdots \xrightarrow{0} \mathbb{H}_q^{n-1}(X,\mathscr{F})^{\sim} \xrightarrow{0} \mathbb{H}_q^{n}(X,\mathscr{F})^{\sim} \xrightarrow{0} \mathbb{H}_q^{n+1}(X,\mathscr{F})^{\sim} \xrightarrow{0} \cdots$$

Proof. From Proposition 54 we have the isomorphism $\mathbb{R}_q f_*(\mathscr{F}) \cong \mathbb{H}_q(X,\mathscr{F})^{\sim}$. Since k is a field the abelian category k**Mod** is semisimple (DTC,Definition 23), so complexes in $\mathfrak{Dqcoh}(k)$ are canonically isomorphic to complexes with zero differentials and the cohomology of the complex in each position (DTC,Proposition 39). This isomorphism is natural, so the proof is complete. $\square$

6 Perfect Complexes

Definition 16. Let $(X,\mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules on X. Then the following conditions are equivalent:

(i) $\mathscr{F}$ is locally finitely free (MRS,Definition 14).

(ii) $\mathscr{F}$ is locally free of finite type (MOS,Lemma 56).

In the literature such sheaves are more commonly known as *vector bundles*. A vector bundle on a scheme is clearly a coherent sheaf. If X is a noetherian scheme then $\mathscr{F}$ is a vector bundle if and only if it is a locally free coherent sheaf (MOS,Corollary 28) (MOS,Lemma 34).

Lemma 56. *Let X be a scheme and $\mathscr{F}$ a locally finitely presented sheaf of modules on X. Then the following conditions are equivalent:*

(i) $\mathscr{F}$ is a vector bundle.

(ii) For every $x \in X$ the $\mathcal{O}_{X,x}$-module $\mathscr{F}_x$ is free (equivalently, projective or flat).

(iii) For every $x \in X$ there is an open affine neighborhood $x \in U$ with $\mathscr{F}|_U$ projective in $\mathfrak{Qco}(U)$.

Proof. $(i) \Rightarrow (ii)$ is trivial. See (MAT,Proposition 24) for the proof that free $\Leftrightarrow$ projective $\Leftrightarrow$ flat over a local ring. $(ii) \Rightarrow (i)$ follows from (MRS,Corollary 90). Clearly on an affine scheme a free sheaf is projective in $\mathfrak{Qco}(X)$, so $(i) \Rightarrow (iii)$. For $(iii) \Rightarrow (ii)$ suppose that $x \in X$ is given with open affine neighborhood U and $\mathscr{F}|_U$ projective in $\mathfrak{Qco}(U)$. Then $\mathscr{F}|_U$ is a direct summand in $\mathfrak{Qco}(U)$ of some coproduct of copies of $\mathcal{O}_X|_U$, from which we deduce that $\mathscr{F}_x$ is projective. $\square$

Remark 18. If you have a bounded complex of vector bundles on a neighborhood of a point, then you can shrink your open neighborhood until all the objects of the complex are simultaneously free of finite rank. So replacing the occurrence of "vector bundles" in $(i),(ii)$ above with "free sheaves of finite rank" we have two new conditions $(i)',(ii)'$ which are equivalent to $(i),(ii)$ respectively.

In the theory of sheaves of modules on a scheme, vector bundles play an important role. As an example, see the basic results of (MRS,Section 1.12) where we observe that various canonical isomorphisms connecting constructions on sheaves are isomorphisms for vector bundles. The *perfect complexes* are the analogous objects in the derived category. Our definition follows the one given in SGA6 I §4 and [TT90] Definition 2.2.10. We make no attempt to develop the theory in utmost generality, for which we would need pseudo-coherence, tor-dimension etc. For these details the reader is referred to SGA6 I.

Definition 17. Let X be a scheme and $\mathscr{X}$ a complex of sheaves of modules on X. We say that $\mathscr{X}$ is a *strict perfect complex* if it is a bounded complex of vector bundles, and a *perfect complex* if it is locally isomorphic in the derived category to a strict perfect complex. That is, $\mathscr{X}$ is a perfect complex if for every $x \in X$ there is an open neighborhood $x \in U$ and an isomorphism $\mathscr{P} \cong \mathscr{X}|_U$ in $\mathfrak{D}(U)$ where $\mathscr{P}$ is a bounded complex of vector bundles on U. This property is stable under isomorphism in $\mathfrak{D}(X)$, and it is clear that a perfect complex has quasi-coherent cohomology.

If $U \subseteq X$ is open and $\mathscr{X}$ a perfect complex on X, then $\mathscr{X}|_U$ is a perfect complex on U. Conversely if $\mathscr{X}$ is a complex of sheaves of modules which is perfect on a neighborhood of every point, then $\mathscr{X}$ is perfect.

Lemma 57. *Given a scheme X and a complex $\mathscr{X}$ of quasi-coherent sheaves the following conditions are equivalent:*

(i) *$\mathscr{X}$ is perfect.*

(ii) *For every $x \in X$ there is an open neighborhood $x \in U$ and a quasi-isomorphism $\mathscr{P} \longrightarrow \mathscr{X}|_U$ where $\mathscr{P}$ is a bounded complex of vector bundles on U.*

(iii) *For every $x \in X$ there is an open neighborhood $x \in U$ and an isomorphism $\mathscr{P} \cong \mathscr{X}|_U$ in $\mathfrak{Dqcoh}(U)$ where $\mathscr{P}$ is a bounded complex of vector bundles on U.*

Proof. Clearly $(ii) \Rightarrow (iii), (iii) \Rightarrow (i)$. For $(iii) \Rightarrow (ii)$ let $x \in X$ be given. Since $\mathscr{P}$ is bounded we can, by shrinking the neighborhood U if necessary, assume that $\mathscr{P}$ is a bounded complex of free sheaves of finite rank and that U is affine. Such a sheaf is projective in $\mathfrak{Qco}(U)$, so $\mathscr{P}$ is hoprojective as an object of $K(\mathfrak{Qco}(U))$. Therefore any isomorphism $\mathscr{P} \cong \mathscr{X}|_U$ must lift to a quasi-isomorphism of complexes $\mathscr{P} \longrightarrow \mathscr{X}|_U$. $(i) \Rightarrow (iii)$ Let $x \in X$ be given and find an open *affine* neighborhood $x \in U$ together with an isomorphism $\mathscr{P} \cong \mathscr{X}|_U$ in $\mathfrak{D}(U)$ where $\mathscr{P}$ is a bounded complex of vector bundles (in particular, a complex of quasi-coherent sheaves). By Theorem 42 the canonical functor $\mathfrak{Dqcoh}(U) \longrightarrow \mathfrak{D}(U)$ is fully faithful, so this isomorphism must come from an isomorphism in $\mathfrak{Dqcoh}(U)$. □

Remark 19. The equivalence of the conditions $(ii), (iii)$ is true more generally, but the proof is complicated (see SGA I). In some sense we only really care about perfect complexes of *quasi-coherent* sheaves, because for a quasi-compact semi-separated scheme X any perfect complex has quasi-coherent cohomology, and is therefore by Theorem 42 isomorphic in $\mathfrak{D}(X)$ to a quasi-coherent complex (which is trivially perfect). However, it is technically convenient to allow perfect complexes of arbitrary sheaves of modules.

Proposition 58. *Let X be a scheme and $\mathscr{X}$ a perfect complex on X. The triangulated functor*

$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, -) : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$

preserves coproducts and sends $\mathfrak{D}_{qc}(X)$ into $\mathfrak{D}_{qc}(X)$.

Proof. The derived sheaf Hom is local (DCOS,Lemma 24), and we can check locally if a cocone is a coproduct (DCOS,Lemma 2), so in proving both claims we can reduce to the case where $\mathscr{X}$ is a bounded complex of free sheaves of finite rank.

Let $\mathcal{S}$ be the full subcategory of $\mathfrak{D}(X)$ consisting of those complexes $\mathscr{F}$ for which the triangulated functor $\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, -)$ preserves coproducts. This is clearly replete, and it is closed under Σ^{-1} by (DTC2,Remark 8). Suppose we have a triangle in $\mathfrak{D}(X)$

$$\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow \Sigma\mathscr{F}$$

with $\mathscr{F}, \mathscr{G} \in \mathcal{S}$ and let a coproduct $\{\mathscr{A}_\lambda \longrightarrow \mathscr{A}\}_{\lambda \in \Lambda}$ in $\mathfrak{D}(X)$ be given. Applying the triangulated functors $\mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(-, \mathscr{A}_\lambda)$ and $\mathbb{R}_{\mathcal{I}}\mathscr{H}om^\bullet(-, \mathscr{A})$ to the following triangle of $\mathfrak{D}(X)^{\mathrm{op}}$

$$\Sigma^{-1}\mathscr{H} \longleftarrow \mathscr{F} \longleftarrow \mathscr{G} \longleftarrow \mathscr{H}$$

and taking coproducts we deduce a morphism of triangles in $\mathfrak{D}(X)$

$$\begin{array}{ccccccc}
\Sigma \oplus_\lambda \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}, \mathscr{A}_\lambda) & \leftarrow & \oplus_\lambda \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{A}_\lambda) & \leftarrow & \oplus_\lambda \mathbb{R}\mathscr{H}om^\bullet(\mathscr{G}, \mathscr{A}_\lambda) & \leftarrow & \oplus_\lambda \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}, \mathscr{A}_\lambda) \\
\downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow \alpha \\
\Sigma \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}, \mathscr{A}) & \longleftarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{A}) & \longleftarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{G}, \mathscr{A}) & \longleftarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}, \mathscr{A})
\end{array}$$

By assumption β, γ are isomorphisms, so α is an isomorphism. It follows that $\mathscr{H} \in \mathcal{S}$, which by (TRC,Lemma 33) is enough to show that $\mathcal{S}$ is a triangulated subcategory of $\mathfrak{D}(X)$.

Next we claim that the sheaf $\mathcal{O}_X$ belongs to $\mathcal{S}$. This follows from the natural isomorphism

$$\mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X, \mathscr{Y}) \cong \mathscr{Y} \tag{32}$$

of (DCOS,Lemma 25). But if $\mathcal{S}$ contains $\mathcal{O}_X$ then it contains any bounded complex built out of finite coproducts of copies of $\mathcal{O}_X$ in $\mathfrak{Mod}(X)$ (DTC,Lemma 79). That is, it contains any bounded complex of free sheaves of finite rank. Since we have already reduced to this case, this proves that $\mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, -)$ preserves coproducts for any perfect $\mathscr{X}$.

For the second claim, fix a complex $\mathscr{A} \in \mathfrak{D}_{qc}(X)$ and let $\mathcal{S}$ be the full subcategory of $\mathfrak{D}(X)$ consisting of those complexes $\mathscr{F}$ for which $\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{A})$ has quasi-coherent cohomology. One checks that this is a triangulated subcategory of $\mathfrak{D}(X)$, which contains $\mathcal{O}_X$ by virtue of (32). It therefore contains any bounded complex of free sheaves of finite rank, which is what we needed to show. $\square$

Corollary 59. *Let X be a quasi-compact semi-separated scheme. If $\mathscr{X}$ is a perfect complex of quasi-coherent sheaves on X then it is compact as an object of $\mathfrak{Dqcoh}(X)$.*

Proof. See (AC,Definition 18) for the definition of a *compact* object in a category. Let $\mathscr{X}$ be a perfect complex on X and $\{\mathscr{F}_\lambda\}_{\lambda \in \Lambda}$ a family of objects in $\mathfrak{Dqcoh}(X)$. For each $\lambda \in \Lambda$ the complex $\mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathscr{F}_\lambda)$ has quasi-coherent cohomology by Proposition 58, so we have an isomorphism in $\mathfrak{D}(\mathbf{Ab})$

$$\begin{aligned}
\mathbb{R}Hom^\bullet(\mathscr{X}, \oplus_\lambda \mathscr{F}_\lambda) &\cong \mathbb{H}(X, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \oplus_\lambda \mathscr{F}_\lambda)) \\
&\cong \mathbb{H}(X, \oplus_\lambda \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathscr{F}_\lambda)) \\
&\cong \oplus_\lambda \mathbb{H}(X, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, \mathscr{F}_\lambda)) \\
&\cong \oplus_\lambda \mathbb{R}Hom^\bullet(\mathscr{X}, \mathscr{F}_\lambda)
\end{aligned} \tag{33}$$

using (DCOS,Proposition 75), Proposition 58 and Corollary 53. In other words, the composite triangulated functor

$$\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X) \xrightarrow{\mathbb{R}Hom^\bullet(\mathscr{X}, -)} \mathfrak{D}(\mathbf{Ab})$$

preserves coproducts. Taking H^0 of both sides of (33) and using (DTC2,Lemma 26) we have an isomorphism of abelian groups

$$Hom_{\mathfrak{D}(X)}(\mathscr{X}, \oplus_\lambda \mathscr{F}_\lambda) \cong \oplus_\lambda Hom_{\mathfrak{D}(X)}(\mathscr{X}, \mathscr{F}_\lambda)$$

and since by Theorem 42 the canonical functor $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful, this yields an isomorphism $Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \oplus_\lambda \mathscr{F}_\lambda) \cong \oplus_\lambda Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, \mathscr{F}_\lambda)$ from which it follows that $\mathscr{X}$ is compact in $\mathfrak{Dqcoh}(X)$, as required. $\square$

Remark 20. From the above results we see that a perfect complex $\mathscr{X}$ of quasi-coherent sheaves is compact in every way you can imagine: the functors $\mathbb{R}\mathscr{H}om^\bullet(\mathscr{X}, -), \mathbb{R}Hom^\bullet(\mathscr{X}, -)$ and $Hom_{\mathfrak{Dqcoh}(X)}(\mathscr{X}, -)$ all preserve coproducts.

Proposition 60. *Let X be a quasi-compact semi-separated scheme with an ample family $\{\mathscr{L}_\alpha\}_{\alpha \in \Lambda}$ of invertible sheaves. Then $\mathfrak{Dqcoh}(X)$ is compactly generated by the set*

$$\mathcal{C} = \{\Sigma^m \mathscr{L}_\alpha^{\otimes n} \,|\, \alpha \in \Lambda, m, n \in \mathbb{Z}\}$$

Proof. See (AMF,Definition 2) for the definition of an *ample family*. An invertible sheaf is a vector bundle, so the sheaves $\mathscr{L}_\alpha^{\otimes n}$ are perfect as complexes and therefore compact in $\mathfrak{Dqcoh}(X)$ by Corollary 59. To see that $\mathcal{C}$ is a family of compact generators in the sense of (TRC3,Definition 9) suppose we are given a complex $\mathscr{X}$ of quasi-coherent sheaves with

$$Hom_{\mathfrak{Dqcoh}(X)}(\Sigma^m \mathscr{L}_\alpha^{\otimes n}, \mathscr{X}) = 0$$

for every $\alpha \in \Lambda, m, n \in \mathbb{Z}$. The category $\mathfrak{Qco}(X)$ has enough hoinjectives, so we may as well assume that $\mathscr{X}$ is hoinjective in $K(\mathfrak{Qco}(X))$, and consequently that

$$Hom_{K(\mathfrak{Qco}(X))}(\Sigma^m \mathscr{L}_\alpha^{\otimes n}, \mathscr{X}) = 0$$

for every $\alpha \in \Lambda, m, n \in \mathbb{Z}$. It follows from (DTC,Lemma 31) that every morphism of sheaves of modules $\mathscr{L}_\alpha^{\otimes n} \longrightarrow Ker\partial_{\mathscr{X}}^{-m}$ factors through $\mathscr{X}^{-m-1} \longrightarrow Ker\partial_{\mathscr{X}}^{-m}$. Since the sheaves $\mathscr{L}_\alpha^{\otimes n}$ generate $\mathfrak{Qco}(X)$ (AMF,Lemma 8) it follows that the epimorphism $Ker\partial_{\mathscr{X}}^{-m} \longrightarrow H^{-m}(\mathscr{X})$ is the zero morphism, and therefore trivially $H^{-m}(\mathscr{X}) = 0$. Since m is arbitrary we conclude that $\mathscr{X}$ is exact and hence zero in $\mathfrak{Dqcoh}(X)$. Hence $\mathcal{C}$ compactly generates $\mathfrak{Dqcoh}(X)$, as claimed. To be pedantic, you can actually omit the non-negative tensor powers; that is, the compact objects $\{\Sigma^m \mathscr{L}_\alpha^{\otimes n}\}_{\alpha \in \Lambda, m \in \mathbb{Z}, n<0}$ generate $\mathfrak{Dqcoh}(X)$. □

Remark 21. Neeman has shown that $\mathfrak{Dqcoh}(X)$ is compactly generated even without the existence of an ample family of invertible sheaves [Nee96]. The careful reader can therefore delete this hypothesis from any results in these notes that rely on Proposition 60. We do not include the more general result here because we are mainly interested in quasi-projective varieties, which automatically admit an ample family.

Proposition 61. *Let X be a scheme and $\mathcal{P}$ the full subcategory of $\mathfrak{Dqcoh}(X)$ consisting of the perfect complexes. Then $\mathcal{P}$ is a thick triangulated subcategory of $\mathfrak{Dqcoh}(X)$.*

Proof. We use the criterion of (TRC,Lemma 33). It is easy to check that $\mathcal{P}$ is replete and closed under Σ^{-1}. Suppose that we have a triangle in $\mathfrak{Dqcoh}(X)$

$$\mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow \Sigma\mathscr{X}$$

with $\mathscr{X}, \mathscr{Y}$ perfect. Given $x \in X$ let $x \in U$ be an affine open neighborhood small enough that $\mathscr{X}|_U, \mathscr{Y}|_U$ are isomorphic in $\mathfrak{Dqcoh}(U)$ to bounded complexes of free sheaves of finite rank $\mathscr{P}, \mathscr{Q}$ respectively. Then we have a triangle in $\mathfrak{Dqcoh}(U)$

$$\mathscr{P} \longrightarrow \mathscr{Q} \longrightarrow \mathscr{Z}|_U \longrightarrow \Sigma\mathscr{P}$$

The morphism $\mathscr{P} \longrightarrow \mathscr{Q}$ must lift to $K(\mathfrak{Qco}(U))$ because $\mathscr{P}$ is hoprojective, and therefore $\mathscr{Z}|_U$ is isomorphic in $\mathfrak{Dqcoh}(U)$ to the mapping cone on a morphism of bounded complexes of free sheaves of finite rank. Such a mapping cone is clearly itself a bounded complex of free sheaves of finite rank, so $\mathscr{Z}$ is perfect and $\mathcal{P}$ triangulated.

To see that $\mathcal{P}$ is thick, suppose we have $\mathscr{X} = \mathscr{Y} \oplus \mathscr{Z}$ in $\mathfrak{Dqcoh}(X)$ with $\mathscr{X}$ perfect. Given $x \in X$ let U be an affine neighborhood of x so small that $\mathscr{X}$ is isomorphic in $\mathfrak{Dqcoh}(U)$ to a bounded complex of free sheaves of finite rank. Passing to $\mathfrak{D}(R)$ where $U \cong Spec(R)$ we have $X = Y \oplus Z$ where X is a bounded complex of finitely generated free R-modules. From (DTC2,Proposition 50) we deduce that Y is isomorphic in $\mathfrak{D}(R)$ to a bounded complex of finitely presented projective R-modules (since a finitely generated projective R-module is finitely presented).

Returning to $\mathfrak{Dqcoh}(U)$ we see that $\mathscr{Y}|_U$ is isomorphic in $\mathfrak{Dqcoh}(U)$ to a bounded complex $\mathscr{Q}$ of finitely presented sheaves projective in $\mathfrak{Qco}(U)$. Therefore $\mathscr{Q}_x$ is a bounded complex of free $\mathcal{O}_{X,x}$-modules of finite rank (for finitely generated modules over a local ring free $\Leftrightarrow$ projective). Using (MRS,Corollary 90) we can find an open neighborhood $x \in V \subseteq U$ such that $\mathscr{Q}|_V$ is a bounded complex of free sheaves of finite rank, which proves that $\mathscr{Y}$ is perfect and completes the proof. □

Corollary 62. *If X is a scheme the perfect complexes form a triangulated subcategory of $\mathfrak{D}(X)$.*

Proof. First of all assume that X is affine. Let $\mathcal{P}, \mathcal{Q}$ denote the full subcategories of $\mathfrak{Dqcoh}(X), \mathfrak{D}(X)$ respectively consisting of the perfect complexes. By Theorem 42 the canonical functor U : $\mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{D}(X)$ is fully faithful, and as we observed in Remark 19 the subcategory $\mathcal{Q}$ is the essential image of $\mathcal{P}$ under U. Since $\mathcal{P}$ is triangulated, it follows that $\mathcal{Q}$ is as well. In other words, the claim is true for affine schemes.

Now for general X let $\mathcal{Q}$ be the full subcategory of perfect complexes in $\mathfrak{D}(X)$. This is clearly replete and closed under Σ^{-1}. If we have a triangle in $\mathfrak{D}(X)$

$$\mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow \Sigma\mathscr{X}$$

with $\mathscr{X}, \mathscr{Y}$ perfect, then restricting to affine open neighborhoods and using the previous paragraph we deduce that $\mathscr{Z}$ is perfect on a neighborhood of every point, and therefore perfect. Hence $\mathcal{Q}$ is a triangulated subcategory of $\mathfrak{D}(X)$. □

Theorem 63. *Let X be a quasi-compact semi-separated scheme with an ample family of invertible sheaves. Then $\mathfrak{Dqcoh}(X)$ is compactly generated and an object of $\mathfrak{Dqcoh}(X)$ is compact if and only if it is a perfect complex.*

Proof. Let $\{\mathscr{L}_\alpha\}_{\alpha\in\Lambda}$ be an ample family of invertible sheaves and set $\mathcal{T} = \mathfrak{Dqcoh}(X)$. If $\mathcal{T}^c$ denotes the triangulated subcategory of compact objects and $\mathcal{P}$ the subcategory of perfect objects, then we know from Corollary 59 that $\mathcal{P} \subseteq \mathcal{T}^c$. By Proposition 60 the triangulated category $\mathfrak{Dqcoh}(X)$ is compactly generated by complexes $\Sigma^m \mathscr{L}_\alpha^{\otimes n}$ and therefore by (TRC3,Lemma 17) the subcategory $\mathcal{T}^c$ is the smallest thick triangulated subcategory of $\mathfrak{Dqcoh}(X)$ containing these compact generators. By Proposition 61 the subcategory $\mathcal{P}$ is thick, and since each of the generators is perfect we must have $\mathcal{T}^c \subseteq \mathcal{P}$. Therefore $\mathcal{T}^c = \mathcal{P}$ as required. □

7 Projection Formula and Friends

Lemma 64 (Projection Formula). *Let $f : X \longrightarrow Y$ be a morphism of concentrated schemes and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules on X, Y respectively. There is a canonical morphism of complexes of sheaves of modules on Y trinatural in both variables*

$$\pi : f_*(\mathscr{X}) \otimes \mathscr{Y} \longrightarrow f_*(\mathscr{X} \otimes f^*(\mathscr{Y}))$$

which is an isomorphism if for every $j \in \mathbb{Z}$ the sheaf $\mathscr{Y}^j$ is locally finitely free.

Proof. Since the schemes X, Y are concentrated the additive functor $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ preserves coproducts (HDIS,Proposition 37). This is essential for the proof, which is why we work over schemes instead of arbitrary ringed spaces. Using the projection morphism for sheaves (MRS,Lemma 80) we have for $n \in \mathbb{Z}$ a canonical morphism of sheaves of modules

$$\begin{aligned}\pi^n : (f_*(\mathscr{X}) \otimes \mathscr{Y})^n &= \oplus_{i+j=n} f_*(\mathscr{X}^i) \otimes \mathscr{Y}^j \\ &\longrightarrow \oplus_{i+j=n} f_*(\mathscr{X}^i \otimes f^*(\mathscr{Y}^j)) \\ &\cong f_*\left(\oplus_{i+j=n} \mathscr{X}^i \otimes f^*(\mathscr{Y}^j)\right) \\ &= f_*(\mathscr{X} \otimes f^*(\mathscr{Y}))^n\end{aligned}$$

and together these define the required morphism of complexes π, which is clearly natural in both variables. If every $\mathscr{Y}^j$ is locally finitely free then it follows from (MRS,Lemma 80) that π is an isomorphism. When we say that π is *trinatural* in both variables we mean that the following diagrams commute

$$\begin{array}{ccc} f_*(\Sigma\mathscr{X})\otimes\mathscr{Y} & \xrightarrow{\pi} & f_*((\Sigma\mathscr{X})\otimes f^*(\mathscr{Y})) \\ \Downarrow & & \Downarrow \\ \Sigma(f_*(\mathscr{X})\otimes\mathscr{Y}) & \xrightarrow[\Sigma\pi]{} & \Sigma f_*(\mathscr{X}\otimes f^*(\mathscr{Y})) \end{array}$$

$$\begin{array}{ccc} f_*(\mathscr{X})\otimes(\Sigma\mathscr{Y}) & \xrightarrow{\pi} & f_*(\mathscr{X}\otimes f^*(\Sigma\mathscr{Y})) \\ \Downarrow & & \Downarrow \\ \Sigma(f_*\mathscr{X}\otimes\mathscr{Y}) & \xrightarrow[\Sigma\pi]{} & \Sigma f_*(\mathscr{X}\otimes f^*\mathscr{Y}) \end{array}$$

which is easily checked. □

Remark 22. With the notation of Lemma 64 let $V \subseteq Y$ be a quasi-compact open subset and $g : U \longrightarrow V$ the induced morphism of schemes, where $U = f^{-1}V$. Observe that U, V are both concentrated. Then one checks that the projection morphism is local, by which we mean that the following diagram commutes

$$\begin{array}{ccc} (f_*(\mathscr{X})\otimes\mathscr{Y})|_V & \xrightarrow{\pi|_V} & f_*(\mathscr{X}\otimes f^*(\mathscr{Y}))|_V \\ \Downarrow & & \Downarrow \\ g_*(\mathscr{X}|_U)\otimes\mathscr{Y}|_V & \xrightarrow[\pi]{} & g_*(\mathscr{X}|_U\otimes g^*(\mathscr{Y}|_V)) \end{array}$$

Commutativity of this diagram follows immediately from (MRS,Remark 16).

Next we generalise the projection formula to the derived category, following [Nee96] Proposition 5.3. In the ordinary projection formula (MRS,Lemma 80) it is crucial that the sheaf $\mathscr{E}$ be locally finitely free. By passing to the derived category we can prove that the projection morphism is an isomorphism for essentially every pair of complexes, which is quite surprising.

Proposition 65 (Derived Projection Formula). *Let $f : X \longrightarrow Y$ be a morphism of concentrated schemes and $\mathscr{X}, \mathscr{Y}$ complexes of sheaves of modules on X, Y respectively. There is a canonical morphism in $\mathfrak{D}(Y)$ trinatural in both variables*

$$\varpi : \mathbb{R}f_*(\mathscr{X}) \underline{\otimes} \mathscr{Y} \longrightarrow \mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y}))$$

If X, Y are quasi-compact semi-separated schemes and $\mathscr{X}, \mathscr{Y}$ have quasi-coherent cohomology, this is an isomorphism.

Proof. First assume that $\mathscr{X}$ is hoinjective and $\mathscr{Y}$ hoflat. In that case we using Lemma 64 a canonical morphism in $\mathfrak{D}(Y)$

$$\begin{aligned} \mathbb{R}f_*(\mathscr{X}) \underline{\otimes} \mathscr{Y} \cong f_*(\mathscr{X}) \underline{\otimes} \mathscr{Y} &\cong f_*(\mathscr{X}) \otimes \mathscr{Y} \\ &\longrightarrow f_*(\mathscr{X} \otimes f^*(\mathscr{Y})) \longrightarrow \mathbb{R}f_*(\mathscr{X} \otimes f^*(\mathscr{Y})) \\ &\cong \mathbb{R}f_*(\mathscr{X} \underline{\otimes} f^*(\mathscr{Y})) \cong \mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y})) \end{aligned}$$

where we have used the fact that $f^*(\mathscr{Y})$ is also hoflat (DCOS,Lemma 52). Given arbitrary complexes $\mathscr{X}, \mathscr{Y}$ we can find isomorphic complexes $\mathscr{X}', \mathscr{Y}'$ which are respectively hoinjective and hoflat, and define $\varpi_{\mathscr{X},\mathscr{Y}}$ to be the composite

$$\mathbb{R}f_*(\mathscr{X}) \underline{\otimes} \mathscr{Y} \Longrightarrow \mathbb{R}f_*(\mathscr{X}') \underline{\otimes} \mathscr{Y}' \xrightarrow{\varpi} \mathbb{R}f_*(\mathscr{X}' \underline{\otimes} \mathbb{L}f^*(\mathscr{Y}')) \Longrightarrow \mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y}))$$

which does not depend on the choice of $\mathscr{X}', \mathscr{Y}'$ and is therefore canonical. It is straightforward to check that ϖ is natural in both variables, with respect to morphisms of the derived categories.

When we say that ϖ is *trinatural* in both variables we mean that the following diagrams commute

$$
\begin{array}{ccc}
\mathbb{R}f_*(\mathscr{X}) \underline{\otimes} (\Sigma \mathscr{Y}) & \xrightarrow{\ \varpi\ } & \mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\Sigma \mathscr{Y})) \\
\Big\Downarrow & & \Big\Downarrow \\
 & & \mathbb{R}f_*(\mathscr{X} \underline{\otimes} \Sigma \mathbb{L}f^*(\mathscr{Y})) \\
 & & \Big\Downarrow \\
 & & \mathbb{R}f_*\Sigma(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y})) \\
 & & \Big\Downarrow \\
\Sigma(\mathbb{R}f_*(\mathscr{X}) \underline{\otimes} \mathscr{Y}) & \xrightarrow[\Sigma\varpi]{} & \Sigma\mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y}))
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{R}f_*(\Sigma\mathscr{X}) \underline{\otimes} \mathscr{Y} & \xrightarrow{\ \varpi\ } & \mathbb{R}f_*((\Sigma\mathscr{X}) \underline{\otimes} \mathbb{L}f^*(\mathscr{Y})) \\
\Big\Downarrow & & \Big\Downarrow \\
(\Sigma\mathbb{R}f_*(\mathscr{X})) \underline{\otimes} \mathscr{Y} & & \mathbb{R}f_*\Sigma(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y})) \\
\Big\Downarrow & & \Big\Downarrow \\
\Sigma(\mathbb{R}f_*(\mathscr{X}) \underline{\otimes} \mathscr{Y}) & \xrightarrow[\Sigma\varpi]{} & \Sigma\mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y}))
\end{array}
$$

which is easily checked. Now suppose that X, Y are quasi-compact semi-separated schemes and fix an arbitrary complex $\mathscr{X}$ of quasi-coherent sheaves. We can then consider ϖ as a trinatural transformation between two triangulated functors $\mathfrak{Dqcoh}(Y) \longrightarrow \mathfrak{D}(Y)$

$$\varpi : \mathbb{R}f_*(\mathscr{X}) \underline{\otimes} (-) \longrightarrow \mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(-))$$

The derived tensor product preserves coproducts (DCOS,Corollary 74), as does $\mathbb{L}f^*$ by virtue of having a right adjoint (DCOS,Proposition 86). The restricted functor $\mathbb{R}f_* : \mathfrak{D}_{qc}(X) \longrightarrow \mathfrak{D}(Y)$ preserves coproducts by Corollary 42. These facts together with Corollary 43 and Corollary 44 show that ϖ is a trinatural transformation of coproduct preserving triangulated functors $\mathfrak{Dqcoh}(Y) \longrightarrow \mathfrak{D}(Y)$. We claim that this is a trinatural equivalence.

The derived projection morphism ϖ is local (see Remark 23 for precisely what we mean by this statement) so in verifying the claim we can reduce to the case where Y is affine.

Let $\mathcal{S}$ be the full subcategory of $\mathfrak{Dqcoh}(Y)$ consisting of those complexes $\mathscr{Y}$ for which $\varpi_{\mathscr{Y}}$ is an isomorphism. Since both functors preserve coproducts this is a localising subcategory of $\mathfrak{Dqcoh}(Y)$ (TRC,Remark 30). The scheme Y is affine so $\mathcal{O}_Y$ is ample and Proposition 60 implies that $\mathfrak{Dqcoh}(Y)$ is compactly generated by the shifts $\{\Sigma^m \mathcal{O}_Y\}_{m \in \mathbb{Z}}$. It is clear that $\mathcal{O}_Y \in \mathcal{S}$ so all of these generators must belong to $\mathcal{S}$, and therefore $\mathcal{S} = \mathfrak{Dqcoh}(Y)$ by (TRC3,Corollary 9). This shows that ϖ is a trinatural equivalence for affine Y, and therefore also for arbitrary quasi-compact semi-separated Y.

To complete the proof we need only refer to Theorem 42 which tells us that complexes with quasi-coherent cohomology on X, Y are actually isomorphic to complexes of quasi-coherent sheaves. $\square$

Remark 23. Let $f : X \longrightarrow Y$ be a morphism of concentrated schemes, $V \subseteq Y$ a quasi-compact open subset and let $g : U \longrightarrow V$ be the induced morphism of schemes where $U = f^{-1}V$. The derived projection morphism ϖ is *local*, in the sense that the following diagram commutes

$$\begin{array}{ccc}
\big(\mathbb{R}f_*(\mathscr{X}) \underline{\otimes} \mathscr{Y}\big)|_V & \xrightarrow{\varpi|_V} & \mathbb{R}f_*(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y}))|_V \\
\Downarrow & & \Downarrow \\
\mathbb{R}f_*(\mathscr{X})|_V \underline{\otimes} \mathscr{Y}|_V & & \mathbb{R}g_*\big(\big(\mathscr{X} \underline{\otimes} \mathbb{L}f^*(\mathscr{Y})\big)|_U\big) \\
\Downarrow & & \Downarrow \\
 & & \mathbb{R}g_*(\mathscr{X}|_U \underline{\otimes} \mathbb{L}f^*(\mathscr{Y})|_U) \\
 & & \Downarrow \\
\mathbb{R}g_*(\mathscr{X}|_U) \underline{\otimes} \mathscr{Y}|_V & \xrightarrow[\varpi]{} & \mathbb{R}g_*(\mathscr{X}|_U \underline{\otimes} \mathbb{L}g^*(\mathscr{Y}|_V))
\end{array}$$

One checks commutativity of this diagram using Remark 22 and various other compatibilities verified earlier in these notes.

It is worthwhile writing down the analogue of Proposition 65 in the world of quasi-coherent sheaves.

Proposition 66. *Let $f : X \longrightarrow Y$ be a morphism of quasi-compact semi-separated schemes and $\mathscr{X}, \mathscr{Y}$ complexes of quasi-coherent sheaves on X, Y respectively. There is a canonical isomorphism in $\mathfrak{Dqcoh}(Y)$ trinatural in both variables*

$$\varpi : \mathbb{R}_q f_*(\mathscr{X}) \underline{\otimes}_q \mathscr{Y} \longrightarrow \mathbb{R}_q f_*(\mathscr{X} \underline{\otimes}_q \mathbb{L}_q f^*(\mathscr{Y}))$$

Proof. The schemes X, Y have enough quasi-coherent hoflats Proposition 16 so $\mathfrak{Dqcoh}(X)$ and $\mathfrak{Dqcoh}(Y)$ acquire canonical derived tensor products $- \underline{\otimes}_q -$ as in Definition 8, and the additive functor $f^* : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Qco}(X)$ has a left derived functor $\mathbb{L}_q f^* : \mathfrak{Dqcoh}(Y) \longrightarrow \mathfrak{Dqcoh}(X)$. One proves existence and trinaturality of ϖ as in Proposition 65.

The morphism ϖ is local, in the following sense: let $V \subseteq Y$ be an open subset whose inclusion is affine. Then both V and $U = f^{-1}V$ are quasi-compact semi-separated and ϖ is well-defined for the morphism $g : U \longrightarrow V$ and complexes $\mathscr{X}|_U, \mathscr{Y}|_V$, and we obtain a commutative diagram of the form given in Remark 23, *mutatis mutandis*. The rest of the proof proceeds exactly as in (DCOQS,Proposition 65). ◻

Most of what follows can be found in SGA I §7. However, since the publication of SGA we have learned how to work more effectively with unbounded complexes, so the development here can avoid some of the boundedness hypotheses of SGA.

Proposition 67 (Derived Double Dual). *Given a scheme X and a complex $\mathscr{X}$ of sheaves of modules there is a canonical morphism in $\mathfrak{D}(X)$ trinatural in $\mathscr{X}$*

$$\tau' : \mathscr{X} \longrightarrow (\mathscr{X}^\vee)^\vee$$

which is an isomorphism if $\mathscr{X}$ is perfect.

Proof. See (DCOS,Definition 16) for the definition of the derived dual complex $\mathscr{X}^\vee$ and for the existence of a canonical morphism $\tau' : \mathscr{X} \longrightarrow (\mathscr{X}^\vee)^\vee$ see (DCOS,Lemma 77). The double derived dual is a triangulated functor $(-^\vee)^\vee : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$ and τ' is a trinatural transformation $1 \longrightarrow (-^\vee)^\vee$, so the full subcategory of objects $\mathscr{X} \in \mathfrak{D}(X)$ for which τ' is an isomorphism is a triangulated subcategory $\mathcal{S}$ of $\mathfrak{D}(X)$. First we claim that $\mathcal{O}_X \in \mathcal{S}$.

From (DCOS,Lemma 77) we have a commutative diagram in $\mathfrak{D}(X)$

$$\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{\tau} & \mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathcal{O}_X, \mathcal{O}_X), \mathcal{O}_X) \\
{\scriptstyle \tau'}\downarrow & & \downarrow \\
\mathbb{R}\mathscr{H}om^\bullet(\mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X, \mathcal{O}_X), \mathcal{O}_X) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathcal{O}_X, \mathcal{O}_X), \mathcal{O}_X)
\end{array} \qquad (34)$$

We observed in the proof of (DCOS,Lemma 25) that the canonical morphism $\mathscr{H}om^\bullet(\mathcal{O}_X,\mathcal{O}_X) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X,\mathcal{O}_X)$ is an isomorphism, from which we deduce that the right and bottom sides of (34) are isomorphisms. To show that $\mathcal{O}_X \in \mathcal{S}$ it therefore suffices to show that τ is an isomorphism of complexes. We know τ explicitly from the proof of (DCOS,Lemma 77), so this is easily checked.

Having shown that $\mathcal{O}_X \in \mathcal{S}$ we also know that any bounded complex of free sheaves of finite rank belongs to $\mathcal{S}$ (DTC,Lemma 79). We can check locally whether τ' is an isomorphism (DCOS,Remark 23), and locally a perfect complex is isomorphic to a bounded complex of free sheaves of finite rank, so the proof is complete. □

Lemma 68. *Let X be a scheme and $\mathscr{X},\mathscr{Y}$ complexes of sheaves of modules with $\mathscr{X}$ strictly perfect. The canonical morphism $\zeta : \mathscr{H}om^\bullet(\mathscr{X},\mathscr{Y}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{X},\mathscr{Y})$ is an isomorphism.*

Proof. Fix the object $\mathscr{Y}$ and consider this morphism as a trinatural transformation of triangulated functors $K(X)^{\text{op}} \longrightarrow \mathfrak{D}(X)$. The full subcategory of complexes $\mathscr{X}$ for which ζ is an isomorphism is therefore a triangulated subcategory $\mathcal{T} \subseteq K(X)$. We observed in the proof of (DCOS,Lemma 25) that $\mathcal{O}_X \in \mathcal{T}$, and therefore $\mathcal{T}$ contains any bounded complex of free sheaves of finite rank. We can check locally whether ζ is an isomorphism (DCOS,Lemma 24), and locally a strictly perfect complex is equal to a bounded complex of free sheaves of finite rank, so the proof is complete. □

Lemma 69. *If X is a scheme and $\mathscr{X}$ a perfect complex then $\mathscr{X}^\vee$ is a perfect complex.*

Proof. The question is local, so it suffices to show that for X affine and $\mathscr{X}$ a bounded complex of free sheaves of finite rank, $\mathscr{X}^\vee$ is perfect. Let $\mathcal{S}$ be the full subcategory of $\mathfrak{D}(X)$ consisting of those complexes $\mathscr{X}$ for which $\mathscr{X}^\vee$ is perfect. This is the inverse image under the triangulated functor $(-)^\vee$ of the triangulated subcategory of perfect complexes (see Corollary 62), so $\mathcal{S}$ is a triangulated subcategory of $\mathfrak{D}(X)$. The dual $\mathcal{O}_X^\vee$ is perfect, because $\mathcal{O}_X^\vee = \mathbb{R}\mathscr{H}om^\bullet(\mathcal{O}_X,\mathcal{O}_X) \cong \mathcal{O}_X$. Therefore $\mathcal{S}$ contains $\mathcal{O}_X$, and hence any bounded complex of free sheaves of finite rank, as required. □

Lemma 70. *If X is a scheme and $\mathscr{X},\mathscr{Y}$ are perfect complexes then $\mathscr{X} \underline{\otimes} \mathscr{Y}$ is perfect.*

Proof. The question is local, so we may as well assume that $\mathscr{X},\mathscr{Y}$ are bounded complexes of free sheaves of finite rank. Such complexes are hoflat, so we need to show that $\mathscr{X} \otimes \mathscr{Y}$ is perfect. But this is clearly a bounded complex of free sheaves of finite rank, so the proof is complete. □

Lemma 71. *Given a scheme X and complexes $\mathscr{E},\mathscr{F},\mathscr{G}$ of sheaves of modules there is a canonical morphism in $\mathfrak{D}(X)$ natural in all three variables*

$$\xi' : \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F}) \underline{\otimes} \mathscr{G} \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{F} \underline{\otimes} \mathscr{G})$$

which is an isomorphism if $\mathscr{E}$ or $\mathscr{G}$ is perfect.

Proof. For the existence and properties of the morphism ξ' see (DCOS,Lemma 78). Since ξ' is trinatural in $\mathscr{E}$ and $\mathscr{G}$ and local, we can reduce by the now standard argument (see the proof of Proposition 67) to the two cases (i) $\mathscr{E} = \mathcal{O}_X$ and (ii) $\mathscr{G} = \mathcal{O}_X$. We deal with each case separately.

(i) By naturality we can assume that $\mathscr{F}$ is hoinjective and $\mathscr{G}$ hoflat. Then by virtue of Lemma 68 every vertical morphism in the compatibility diagram for ξ,ξ' of (DCOS,Lemma 78) is an isomorphism in $\mathfrak{D}(X)$, and therefore we have reduced to showing that

$$\xi : \mathscr{H}om^\bullet(\mathcal{O}_X,\mathscr{F}) \otimes \mathscr{G} \longrightarrow \mathscr{H}om^\bullet(\mathcal{O}_X,\mathscr{F} \otimes \mathscr{G})$$

is an isomorphism of complexes for arbitrary complexes $\mathscr{F},\mathscr{G}$. The left hand side is canonically isomorphic to $((-1)^{\bullet+1}\mathscr{F}) \otimes \mathscr{G}$ and the right to $(-1)^{\bullet+1}(\mathscr{F} \otimes \mathscr{G})$, and one checks that ξ is none other than the isomorphism α of (DCOS,Remark 7).

(ii) By naturality we can assume that $\mathscr{F}$ is hoinjective, and therefore as in (i) we reduce to showing that the canonical morphism

$$\xi : \mathscr{H}om^\bullet(\mathscr{E},\mathscr{F}) \otimes \mathcal{O}_X \longrightarrow \mathscr{H}om^\bullet(\mathscr{E},\mathscr{F} \otimes \mathcal{O}_X)$$

is an isomorphism of complexes, which is again straightforward. □

Lemma 72. *Given a scheme X and complexes $\mathscr{E},\mathscr{G}$ of sheaves of modules there is a canonical morphism in $\mathfrak{D}(X)$ trinatural in both variables*

$$\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G} \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G})$$

which is an isomorphism if $\mathscr{E}$ or $\mathscr{G}$ is perfect.

Proof. This is the special case of Lemma 71 where $\mathscr{F} = \mathcal{O}_X$, together with the canonical isomorphism $\mathcal{O}_X \underline{\underline{\otimes}} \mathscr{G} \longrightarrow \mathscr{G}$. Observe that this morphism is local with respect to open subsets. □

Lemma 73. *Given a scheme X and complexes $\mathscr{E},\mathscr{F},\mathscr{G}$ of sheaves of modules there are canonical morphisms in $\mathfrak{D}(X)$ and $\mathfrak{D}(\mathbf{Ab})$ respectively natural in all three variables*

$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G})$$
$$\mathbb{R}Hom^\bullet(\mathscr{F},\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}) \longrightarrow \mathbb{R}Hom^\bullet(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G})$$

which are isomorphisms if $\mathscr{E}$ or $\mathscr{G}$ is perfect. Taking cohomology we have a canonical morphism of abelian groups natural in all three variables

$$Hom_{\mathfrak{D}(X)}(\mathscr{F},\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G})$$

which is an isomorphism if $\mathscr{E}$ or $\mathscr{G}$ is perfect.

Proof. The canonical morphism is the following composite, using the adjunction isomorphism of (DCOS,Proposition 69) and the canonical morphism of Lemma 72

$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F},\mathbb{R}\mathscr{H}om^\bullet(\mathscr{E},\mathscr{G})) \cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G})$$

By construction this is natural in all three variables and an isomorphism for $\mathscr{E}$ or $\mathscr{G}$ perfect. Similarly one defines the morphism for $\mathbb{R}Hom^\bullet(-,-)$. Observe that if X is a scheme over an affine scheme $Spec(A)$ then this is a morphism in $\mathfrak{D}(A)$ in the spirit of Remark 1. Consequently the third map is a morphism of A-modules. □

Remark 24. With the notation of Lemma 73 we can describe the morphism

$$\Phi : Hom_{\mathfrak{D}(X)}(\mathscr{F},\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\mathscr{G})$$

explicitly as follows. Given $\alpha : \mathscr{F} \longrightarrow \mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}$ in $\mathfrak{D}(X)$ it is a consequence of (DCOS,Lemma 84) that $\Phi(\alpha)$ is the following composite

$$\begin{array}{ccccccc}
\mathscr{F} \underline{\underline{\otimes}} \mathscr{E} & \xrightarrow{\alpha \underline{\underline{\otimes}} 1} & (\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}) \underline{\underline{\otimes}} \mathscr{E} & \Longrightarrow & \mathscr{E}^\vee \underline{\underline{\otimes}} (\mathscr{G} \underline{\underline{\otimes}} \mathscr{E}) & \Longrightarrow & \mathscr{E}^\vee \underline{\underline{\otimes}} (\mathscr{E} \underline{\underline{\otimes}} \mathscr{G}) \\
 & & & & & & \Downarrow \\
 & & \mathscr{G} & \Longleftarrow & \mathcal{O}_X \underline{\underline{\otimes}} \mathscr{G} & \xleftarrow[\varepsilon \underline{\underline{\otimes}} 1]{} & (\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{E}) \underline{\underline{\otimes}} \mathscr{G}
\end{array}$$

Using this observation it is straightforward to check that the following triadjunction diagram of (TRC,Theorem 42) commutes

$$\begin{array}{ccc}
Hom_{\mathfrak{D}(X)}(\Sigma\mathscr{F},\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G}) & \longrightarrow & Hom_{\mathfrak{D}(X)}((\Sigma\mathscr{F}) \underline{\underline{\otimes}} \mathscr{E},\mathscr{G}) \\
\downarrow & & \downarrow \\
Hom_{\mathfrak{D}(X)}(\mathscr{F},\Sigma^{-1}(\mathscr{E}^\vee \underline{\underline{\otimes}} \mathscr{G})) & & Hom_{\mathfrak{D}(X)}(\Sigma(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E}),\mathscr{G}) \\
\downarrow & & \downarrow \\
Hom_{\mathfrak{D}(X)}(\mathscr{F},\mathscr{E}^\vee \underline{\underline{\otimes}} (\Sigma^{-1}\mathscr{G})) & \longrightarrow & Hom_{\mathfrak{D}(X)}(\mathscr{F} \underline{\underline{\otimes}} \mathscr{E},\Sigma^{-1}\mathscr{G})
\end{array}$$

Lemma 74. *Given a scheme X and a perfect complex $\mathscr{E}$ of sheaves of modules, there is a canonical triadjunction*

$$\mathfrak{D}(X) \underset{-\underline{\otimes}\mathscr{E}}{\overset{\mathscr{E}^\vee\underline{\otimes}-}{\rightleftarrows}} \mathfrak{D}(X) \qquad - \underline{\otimes}\, \mathscr{E} \longrightarrow\!\!\!\!\!\shortmid\ \mathscr{E}^\vee \underline{\otimes} -$$

Proof. The third isomorphism of Lemma 73 is natural in $\mathscr{F}$ and $\mathscr{G}$, so it defines an adjunction between $- \underline{\otimes}\, \mathscr{E}$ and $\mathscr{E}^\vee \underline{\otimes} -$, which is a triadjunction by virtue of Remark 24. □

Lemma 75. *Given a scheme X and complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules there are canonical isomorphisms in $\mathfrak{D}(X), \mathfrak{D}(\mathbf{Ab})$ and $\mathbf{Ab}$ respectively, natural in both variables*

$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}^\vee) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}^\vee)$$
$$\mathbb{R}Hom^\bullet(\mathscr{F}, \mathscr{E}^\vee) \longrightarrow \mathbb{R}Hom^\bullet(\mathscr{E}, \mathscr{F}^\vee)$$
$$Hom_{\mathfrak{D}(X)}(\mathscr{F}, \mathscr{E}^\vee) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{E}, \mathscr{F}^\vee)$$

Proof. Note that we do not require either of $\mathscr{E}, \mathscr{F}$ to be perfect. Using (DCOS,Proposition 69) we have a canonical isomorphism in $\mathfrak{D}(X)$ natural in both variables

$$\begin{aligned}\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}^\vee) &= \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathcal{O}_X)) \cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\otimes}\, \mathscr{E}, \mathcal{O}_X)\\ &\cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E} \underline{\otimes}\, \mathscr{F}, \mathcal{O}_X) \cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathcal{O}_X)) = \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}^\vee)\end{aligned}$$

The proofs for $\mathbb{R}Hom^\bullet(-,-)$ and $Hom_{\mathfrak{D}(X)}(-,-)$ are identical, using (DCOS,Corollary 70) in the second case. □

Lemma 76. *Given a scheme X and complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules there are canonical morphisms in $\mathfrak{D}(X), \mathfrak{D}(\mathbf{Ab})$ and $\mathbf{Ab}$ respectively, natural in both variables*

$$\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}^\vee, \mathscr{F}^\vee)$$
$$\mathbb{R}Hom^\bullet(\mathscr{F}, \mathscr{E}) \longrightarrow \mathbb{R}Hom^\bullet(\mathscr{E}^\vee, \mathscr{F}^\vee)$$
$$Hom_{\mathfrak{D}(X)}(\mathscr{F}, \mathscr{E}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{E}^\vee, \mathscr{F}^\vee)$$

which are isomorphisms if $\mathscr{E}$ is perfect.

Proof. Using Lemma 75 and Proposition 67 we have a canonical morphism in $\mathfrak{D}(X)$ natural in both variables $\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, (\mathscr{E}^\vee)^\vee) \cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}^\vee, \mathscr{F}^\vee)$ that is an isomorphism if $\mathscr{E}$ is perfect. The proof for $\mathbb{R}Hom^\bullet(-,-)$ and $Hom_{\mathfrak{D}(X)}(-,-)$ is similar. Observe that given complexes $\mathscr{E}, \mathscr{F}$ the map

$$Hom_{\mathfrak{D}(X)}(\mathscr{F}, \mathscr{E}) \longrightarrow Hom_{\mathfrak{D}(X)}(\mathscr{E}^\vee, \mathscr{F}^\vee)$$

is actually the map determined by the contravariant triangulated functor $(-)^\vee$. □

Lemma 77. *Given a scheme X and complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules there is a canonical morphism in $\mathfrak{D}(X)$ natural in both variables*

$$\mathscr{E}^\vee \underline{\otimes}\, \mathscr{F}^\vee \longrightarrow (\mathscr{E} \underline{\otimes}\, \mathscr{F})^\vee$$

which is an isomorphism if $\mathscr{E}$ or $\mathscr{F}$ is perfect.

Proof. Using Lemma 72 and Lemma 73 with $\mathscr{G} = \mathscr{F}^\vee$ and $\mathscr{G} = \mathcal{O}_X$ respectively, we have a canonical morphism in $\mathfrak{D}(X)$

$$\begin{aligned}\mathscr{E}^\vee \underline{\otimes}\, \mathscr{F}^\vee &\longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}^\vee) \cong \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \mathscr{F}^\vee \underline{\otimes}\, \mathcal{O}_X)\\ &\longrightarrow \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E} \underline{\otimes}\, \mathscr{F}, \mathcal{O}_X) = (\mathscr{E} \underline{\otimes}\, \mathscr{F})^\vee\end{aligned}$$

If either of $\mathscr{E}, \mathscr{F}$ is perfect then this is an isomorphism, using Lemma 69. □

Definition 18. Let X be a scheme. We denote by $\mathfrak{Dper}(X)$ the full subcategory of $\mathfrak{D}(X)$ consisting of the perfect complexes. By Corollary 62 this is a triangulated subcategory.

Proposition 78. *Given a scheme X the dualising functor $(-)^\vee : \mathfrak{D}(X)^{op} \longrightarrow \mathfrak{D}(X)$ restricts to a triangulated functor $(-)^\vee : \mathfrak{Dper}(X)^{op} \longrightarrow \mathfrak{Dper}(X)$ and moreover this functor is an equivalence.*

Proof. By Lemma 69 the dual of a perfect complex is perfect, so $(-)^\vee$ certainly restricts to a triangulated functor $D : \mathfrak{Dper}(X)^{\mathrm{op}} \longrightarrow \mathfrak{Dper}(X)$, whose opposite functor is a triangulated functor $D^{\mathrm{op}} : \mathfrak{Dper}(X) \longrightarrow \mathfrak{Dper}(X)^{\mathrm{op}}$. From Proposition 67 we have canonical trinatural equivalences

$$1 \longrightarrow D^{\mathrm{op}} \circ D, \quad 1 \longrightarrow D \circ D^{\mathrm{op}}$$

so D is an equivalence, as claimed. □

Proposition 79. *Given a scheme X, a point $x \in X$ and complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules there is a canonical morphism in $\mathfrak{D}(\mathcal{O}_{X,x})$ natural in both variables*

$$\mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E}, \mathscr{F})_x \longrightarrow \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x)$$

which is an isomorphism if $\mathscr{E}$ is perfect.

Proof. For the existence and properties of this morphism see (DCOS,Lemma 107). The morphism is trinatural in $\mathscr{E}$ and local by (DCOS,Remark 34), so by the standard argument (see the proof of Proposition 67) we can assume $\mathscr{E} = \mathcal{O}_X$. We have a commutative diagram in $\mathfrak{D}(\mathcal{O}_{X,x})$

$$\begin{array}{ccc} \mathscr{H}om^\bullet_X(\mathcal{O}_X, \mathscr{F})_x & \longrightarrow & Hom^\bullet_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathscr{F}_x) \\ \downarrow & & \downarrow \\ \mathbb{R}\mathscr{H}om^\bullet_X(\mathcal{O}_X, \mathscr{F})_x & \longrightarrow & \mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathscr{F}_x) \end{array}$$

in which the vertical morphisms are clearly isomorphisms, so we have reduced to checking that the top is an isomorphism of complexes. But the domain and codomain are both canonically isomorphic to $(-1)^{\bullet+1}\mathscr{F}_x$ and one checks the determined morphism is just the identity, so the proof is complete. This result should be compared with (MRS,Proposition 89) for sheaves. □

Let us establish some notation for the next few results. Let X be a scheme, $x \in X$ a point and $x \in U$ an open neighborhood. We have a canonical isomorphism of rings $\kappa : (\mathcal{O}_X|_U)_x \longrightarrow \mathcal{O}_{X,x}$ which induces an isomorphism of triangulated categories, fitting into a diagram that commutes up to canonical trinatural equivalence

$$\begin{array}{ccc} \mathfrak{D}(X) & \longrightarrow & \mathfrak{D}(\mathcal{O}_{X,x}) \\ {\scriptstyle (-)|_U}\downarrow & & \Downarrow{\scriptstyle \kappa_*} \\ \mathfrak{D}(U) & \longrightarrow & \mathfrak{D}((\mathcal{O}_X|_U)_x) \end{array}$$

For clarity we typically pretend that κ_* is the identity when making statements, but we are usually more careful in the proofs. If we are given complexes $\mathscr{E}, \mathscr{F}$ of sheaves of modules on X and a morphism $\phi : \mathscr{E}|_U \longrightarrow \mathscr{F}|_U$ in $\mathfrak{D}(U)$ then by abuse of notation we write ϕ_x for both the morphism $(\mathscr{E}|_U)_x \longrightarrow (\mathscr{F}|_U)_x$ in $\mathfrak{D}((\mathcal{O}_X|_U)_x)$ and the composite

$$\mathscr{E}_x \Longrightarrow (\mathscr{E}|_U)_x \xrightarrow{\kappa_*^{-1}(\phi_x)} (\mathscr{F}|_U)_x \Longrightarrow \mathscr{F}_x$$

Corollary 80. *Let X be a scheme, $x \in X$ a point and $\mathscr{E}, \mathscr{F}$ complexes of sheaves of modules. There is a canonical morphism of $\mathcal{O}_{X,x}$-modules natural in both variables*

$$\varinjlim_{x \in U} Hom_{\mathfrak{D}(U)}(\mathscr{E}|_U, \mathscr{F}|_U) \longrightarrow Hom_{\mathfrak{D}(\mathcal{O}_{X,x})}(\mathscr{E}_x, \mathscr{F}_x) \tag{35}$$

which is an isomorphism if $\mathscr{E}$ is perfect.

Proof. For an open set $U \subseteq X$ the abelian group $Hom_{\mathfrak{D}(U)}(\mathscr{E}|_U, \mathscr{F}|_U)$ is canonically a $\Gamma(U, \mathcal{O}_X)$-module, and similarly $Hom_{\mathfrak{D}(\mathcal{O}_{X,x})}(\mathscr{E}_x, \mathscr{F}_x)$ is canonically an $\mathcal{O}_{X,x}$-module (DCOS,Remark 3). Therefore the left hand side of (35) becomes a $\mathcal{O}_{X,x}$-module in the usual way. For each open neighborhood $x \in U$ we have a canonical morphism of abelian groups natural in both variables

$$Hom_{\mathfrak{D}(U)}(\mathscr{E}|_U, \mathscr{F}|_U) \longrightarrow Hom_{\mathfrak{D}((\mathcal{O}_X|_U)_x)}((\mathscr{E}|_U)_x, (\mathscr{F}|_U)_x) \cong Hom_{\mathfrak{D}(\mathcal{O}_{X,x})}(\mathscr{E}_x, \mathscr{F}_x)$$

which sends the action of $\Gamma(U, \mathcal{O}_X)$ to the action of $\mathcal{O}_{X,x}$. One checks that this map is compatible with the direct system on the left, so we have an induced morphism of $\mathcal{O}_{X,x}$-modules (35) natural in both variables.

It remains to show that this map is an isomorphism if $\mathscr{E}$ is perfect. In that case we have using Proposition 79, (DCOS,Proposition 29) and (DTC2,Lemma 26) an isomorphism of $\mathcal{O}_{X,x}$-modules

$$\begin{aligned} \mathbb{D}\mathscr{H}om(\mathscr{E}, \mathscr{F})_x &\cong H^0(\mathbb{R}\mathscr{H}om^\bullet_X(\mathscr{E}, \mathscr{F})_x) \\ &\cong H^0(\mathbb{R}Hom^\bullet_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x)) \\ &\cong Hom_{\mathfrak{D}(\mathcal{O}_{X,x})}(\mathscr{E}_x, \mathscr{F}_x) \end{aligned} \tag{36}$$

Here $\mathbb{D}\mathscr{H}om(\mathscr{E}, \mathscr{F})$ is the sheafification of the presheaf of $\mathcal{O}_X$-modules defined by

$$\Gamma(U, D\mathscr{H}om(\mathscr{E}, \mathscr{F})) = Hom_{\mathfrak{D}(U)}(\mathscr{E}|_U, \mathscr{F}|_U)$$

so there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules

$$\varinjlim_{x \in U} Hom_{\mathfrak{D}(U)}(\mathscr{E}|_U, \mathscr{F}|_U) \cong \mathbb{D}\mathscr{H}om(\mathscr{E}, \mathscr{F})_x \cong Hom_{\mathfrak{D}(\mathcal{O}_{X,x})}(\mathscr{E}_x, \mathscr{F}_x)$$

To complete the proof we have to show that this isomorphism agrees with the map (35) that we have already defined. In checking this we can assume $\mathscr{F}$ hoinjective. The proof is straightforward and involves checking commutativity of various diagrams, which we leave to the reader. □

Corollary 81. *Let X be a scheme, $x \in X$ a point and $\mathscr{E}, \mathscr{F}$ complexes of sheaves of modules with $\mathscr{E}$ perfect. Then*

(a) Given open neighborhoods $x \in U, V$ and morphisms $\phi : \mathscr{E}|_U \longrightarrow \mathscr{F}|_U$ and $\psi : \mathscr{E}|_V \longrightarrow \mathscr{F}|_V$ in $\mathfrak{D}(U), \mathfrak{D}(V)$ respectively, we have $\phi_x = \psi_x$ in $\mathfrak{D}(\mathcal{O}_{X,x})$ if and only if $\phi|_W = \psi|_W$ for some open neighborhood $x \in W \subseteq U \cap V$.

(b) Given a morphism $t : \mathscr{E}_x \longrightarrow \mathscr{F}_x$ in $\mathfrak{D}(\mathcal{O}_{X,x})$ there exists an open neighborhood $x \in U$ and a morphism $\phi : \mathscr{E}|_U \longrightarrow \mathscr{F}|_U$ in $\mathfrak{D}(U)$ with $\phi_x = t$.

Proof. Both statements are immediate from Corollary 80. □

Corollary 82. *Let X be a scheme, $x \in X$ a point and $\mathscr{E}, \mathscr{F}$ perfect complexes of sheaves of modules. If $\mathscr{E}_x \cong \mathscr{F}_x$ in $\mathfrak{D}(\mathcal{O}_{X,x})$ then $\mathscr{E}|_U \cong \mathscr{F}|_U$ in $\mathfrak{D}(U)$ for some open neighborhood $x \in U$.*

Proof. More precisely, if we are given an isomorphism $t : \mathscr{E}_x \longrightarrow \mathscr{F}_x$ in $\mathfrak{D}(\mathcal{O}_{X,x})$ with inverse s then there is by Corollary 81 an open neighborhood $x \in U$ and morphisms $\phi : \mathscr{E}|_U \longrightarrow \mathscr{F}|_U$ and $\psi : \mathscr{F}|_U \longrightarrow \mathscr{E}|_U$ in $\mathfrak{D}(U)$ with $\phi_x = t, \psi_x = s$ and $\phi\psi = 1, \psi\phi = 1$ in $\mathfrak{D}(U)$. □

Lemma 83. *Let $f : X \longrightarrow Y$ be a morphism of schemes. For complexes of sheaves of modules $\mathscr{E}, \mathscr{F}$ on Y there is a canonical morphism in $\mathfrak{D}(X)$ natural in both variables*

$$\mathbb{L}f^*\mathbb{R}\mathscr{H}om^\bullet_Y(\mathscr{E}, \mathscr{F}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathscr{E}, \mathbb{L}f^*\mathscr{F})$$

which is an isomorphism if $\mathscr{E}$ is perfect.

Proof. See (DCOS,Lemma 99) for the definition of this morphism, which is local and trinatural in $\mathscr{E}$, so by the usual argument we may assume $\mathscr{E} = \mathcal{O}_X$. The first step is to verify that the following diagram commutes in $\mathfrak{D}(Y)$ for any complex $\mathscr{F}$ of sheaves of modules on Y

$$\begin{array}{ccc}
\mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathcal{O}_Y, \mathscr{F}) & \xrightarrow{\aleph} & \mathbb{R}\mathscr{H}om^\bullet_Y(\mathcal{O}_Y, \mathbb{R}f_*\mathscr{F}) \\
\downarrow & & \\
\mathbb{R}f_*\mathbb{R}\mathscr{H}om^\bullet_X(\mathcal{O}_X, \mathscr{F}) & & \downarrow \\
\downarrow & & \\
\mathbb{R}f_*(-1)^{\bullet+1}\mathscr{F} & \longrightarrow & (-1)^{\bullet+1}\mathbb{R}f_*\mathscr{F}
\end{array}$$

where $\aleph$ is defined in (DCOS,Proposition 98) and all other morphisms are canonical. One also checks that the following diagram commutes in $\mathfrak{D}(Y)$

$$\begin{array}{ccc}
(-1)^{\bullet+1}\mathscr{F} & \longrightarrow & (-1)^{\bullet+1}\mathbb{R}f_*\mathbb{L}f^*\mathscr{F} \\
 & \searrow & \downarrow \\
 & & \mathbb{R}f_*(-1)^{\bullet+1}\mathbb{L}f^*\mathscr{F} \\
 & & \downarrow \\
 & & \mathbb{R}f_*\mathbb{L}f^*(-1)^{\bullet+1}\mathscr{F}
\end{array}$$

Using these facts one checks that the composite

$$\begin{aligned}
\mathbb{L}f^*(-1)^{\bullet+1}\mathscr{F} &\cong \mathbb{L}f^*\mathbb{R}\mathscr{H}om^\bullet_Y(\mathcal{O}_Y, \mathscr{F}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathcal{O}_Y, \mathbb{L}f^*\mathscr{F}) \\
&\cong \mathbb{R}\mathscr{H}om^\bullet_X(\mathcal{O}_X, \mathbb{L}f^*\mathscr{F}) \cong (-1)^{\bullet+1}\mathbb{L}f^*\mathscr{F} \cong \mathbb{L}f^*(-1)^{\bullet+1}\mathscr{F}
\end{aligned} \tag{37}$$

corresponds under the adjunction to the unit $(-1)^{\bullet+1}\mathscr{F} \longrightarrow \mathbb{R}f_*\mathbb{L}f^*(-1)^{\bullet+1}\mathscr{F}$. It follows that (37) is the identity, and therefore $\mathbb{L}f^*\mathbb{R}\mathscr{H}om^\bullet_Y(\mathcal{O}_Y, \mathscr{F}) \longrightarrow \mathbb{R}\mathscr{H}om^\bullet_X(\mathbb{L}f^*\mathcal{O}_Y, \mathbb{L}f^*\mathscr{F})$ is an isomorphism as required. ☐

Lemma 84. *Let $f : X \longrightarrow Y$ be a morphism of schemes. For a complex $\mathscr{E}$ of sheaves of modules on Y there is a canonical natural morphism in $\mathfrak{D}(X)$*

$$\mathbb{L}f^*(\mathscr{E}^\vee) \longrightarrow (\mathbb{L}f^*\mathscr{E})^\vee$$

which is an isomorphism if $\mathscr{E}$ is perfect.

Proof. This is the special case $\mathscr{F} = \mathcal{O}_Y$ of Lemma 83. ☐

Lemma 85. *Let $f : X \longrightarrow Y$ be a morphism of schemes and $\mathscr{E}$ a perfect complex on Y. Then $\mathbb{L}f^*\mathscr{E}$ is a perfect complex on X.*

Proof. By Corollary 62 the perfect complexes form a triangulated subcategory of $\mathfrak{D}(Y)$, so the full subcategory of $\mathfrak{D}(Y)$ consisting of those perfect $\mathscr{E}$ for which $\mathbb{L}f^*\mathscr{E}$ is perfect is also a triangulated subcategory of $\mathfrak{D}(Y)$. The question is local, so we can assume $\mathscr{E}$ is a bounded complex of free sheaves of finite rank. But by what we have just said it is then enough to prove the claim for $\mathscr{E} = \mathcal{O}_Y$ which is trivial since $\mathbb{L}f^*\mathcal{O}_Y \cong \mathcal{O}_X$. ☐

7.1 Commutative Diagrams

Continuing in the spirit of (DCOS,Section 5.3) we take the time to record here some commutative diagrams relating the various constructions of the previous section.

Lemma 86. *Let X be a scheme and $\mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}$ complexes of sheaves of modules on X. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}^\vee \underline{\otimes} \mathscr{G}) \underline{\otimes} \mathscr{H} & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, (\mathscr{E}^\vee \underline{\otimes} \mathscr{G}) \underline{\otimes} \mathscr{H}) \\
\Big\downarrow & & \Big\Downarrow \\
 & & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{E}^\vee \underline{\otimes} (\mathscr{G} \underline{\otimes} \mathscr{H})) \\
 & & \Big\downarrow \\
\mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\otimes} \mathscr{E}, \mathscr{G}) \underline{\otimes} \mathscr{H} & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{F} \underline{\otimes} \mathscr{E}, \mathscr{G} \underline{\otimes} \mathscr{H})
\end{array}$$

Proof. To be clear, the morphisms are the ones defined in Lemma 71 and Lemma 73. Commutativity of this diagram follows from (DCOS,Lemma 79) and (DCOS,Lemma 81). □

Lemma 87. *Let X be a scheme and $\omega, \mathscr{E}, \mathscr{G}$ complexes of sheaves of modules on X with ω perfect. The following diagram commutes in $\mathfrak{D}(X)$*

$$\begin{array}{ccc}
\mathscr{E}^\vee \underline{\otimes} (\omega^\vee \underline{\otimes} \mathscr{G}) & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E}, \omega^\vee \underline{\otimes} \mathscr{G}) \\
\Big\Downarrow & & \\
(\mathscr{E}^\vee \underline{\otimes} \omega^\vee) \underline{\otimes} \mathscr{G} & & \Big\Downarrow \\
\Big\Downarrow & & \\
(\mathscr{E} \underline{\otimes} \omega)^\vee \underline{\otimes} \mathscr{G} & \longrightarrow & \mathbb{R}\mathscr{H}om^\bullet(\mathscr{E} \underline{\otimes} \omega, \mathscr{G})
\end{array}$$

Proof. To be clear, the morphisms are the ones defined in Lemma 72, Lemma 73 and Lemma 77. Commutativity of this diagram follows from (DCOS,Lemma 79) and other minor facts. □

8 Invertible Complexes

The *Picard group* of a scheme X is the group of units in $\mathfrak{Mod}(X)$ under the tensor product, and this invariant is closely connected with the properties of divisors on X. A result of Balmer tells roughly that X can be reconstructed from the triangulated category of perfect complexes $\mathfrak{Dper}(X)$ together with its derived tensor product, so it should be possible to extract $Pic(X)$ directly from $\mathfrak{Dper}(X)$. The contents of this section have also been published recently by Balmer.

Lemma 88. *Let k be a field and X, Y complexes of k-modules with X cohomologically bounded above. If $X \underline{\otimes}_k Y \cong k$ in $\mathfrak{D}(k)$ there exists an integer $i \in \mathbb{Z}$ and isomorphisms $X \cong \Sigma^i k, Y \cong \Sigma^{-i} k$ in $\mathfrak{D}(k)$.*

Proof. By (DTC,Proposition 39) we may as well assume that the differentials in the complexes X, Y are all zero and that X is actually bounded above. Every module over a field is free, therefore flat, so X is hoflat and we have an isomorphism $X \otimes_k Y \cong k$ in $\mathfrak{D}(k)$. The differentials of $X \otimes_k Y$ are all zero, so we deduce isomorphisms of k-modules

$$(X \otimes_k Y)^0 \cong k$$
$$(X \otimes_k Y)^n = 0 \quad n \neq 0$$

From which we deduce that there exists an integer $i \in \mathbb{Z}$ with $X^i \otimes_k Y^{-i} \cong k$ as k-modules and $X^s \otimes_k Y^t = 0$ whenever $s + t \neq 0$. It is then clear that $X^i \cong k, Y^{-i} \cong k$ as k-modules, and that these are the only nonzero terms in the complexes X, Y. In other words, we have $X \cong \Sigma^{-i} k$ and $Y \cong \Sigma^i k$, as required. □

Let $(A, \mathfrak{m}, k)$ be a local ring. We say that a morphism of A-modules $u : M \longrightarrow N$ is *minimal* if $u \otimes k : M \otimes k \longrightarrow N \otimes k$ is an isomorphism (MAT,Definition 26). If M is a finitely generated A-module then there exists a minimal morphism $u : F \longrightarrow M$ with F a free A-module of finite rank $rank_A F = rank_k(M \otimes k)$ and u surjective (MAT,Lemma 140).

Lemma 89. *Let $(A, \mathfrak{m}, k)$ be a local ring and $\varphi : M \longrightarrow N$ a morphism of free A-modules of finite rank. Then*

(i) If $\varphi \otimes k : M \otimes k \longrightarrow N \otimes k$ is injective then so is φ. Moreover φ is a coretraction.

(ii) If $\varphi \otimes k : M \otimes k \longrightarrow N \otimes k$ is an isomorphism then so is φ.

Proof. (i) Set $m = rank_A(M)$ and $n = rank_A(N)$ and fix bases of both modules. These map to a basis of k-modules in $M \otimes k, N \otimes k$ so we have $m = rank_k(M \otimes k)$ and $n = rank_k(N \otimes k)$. Hence injectivity of $\varphi \otimes k$ is only possible if $m \leq n$. The case $m = 0$ is trivial, so assume $m > 0$. We begin with the case $m = n$, so that φ is represented by some square matrix $A = (a_{ij})$ and $\varphi \otimes k$ by the matrix $\overline{A} = (a_{ij} + \mathfrak{m})$. The morphism $\varphi \otimes k$ is injective if and only if it is an isomorphism, which is if and only if $det \overline{A} \neq 0$. But $det \overline{A} = \overline{det(A)}$ so we must have $det(A) \notin \mathfrak{m}$. Since A is a local ring, this means that $det(A)$ is a unit and φ is therefore also an isomorphism.

Now assume $m < n$. The idea is to patch M until the ranks are equal. Consider the morphism $\varphi \otimes k : M \otimes k \longrightarrow N \otimes k$ as the inclusion of a submodule. Since k is a field this is a direct summand whose complement is of rank $n-m$ with basis $\overline{x_1}, \ldots, \overline{x_{n-m}}$ for some $x_i \in N$ (in the usual fashion we confuse $N \otimes k$ and $N/\mathfrak{m}N$). Let $T = A^{\oplus(n-m)} \longrightarrow N$ be determined by the x_i. The induced morphism $T \oplus M \longrightarrow N$ maps under $- \otimes k$ to an isomorphism by construction, and so by the first part of the proof $T \oplus M \longrightarrow N$ is also an isomorphism. Its second component $\varphi : M \longrightarrow N$ is therefore a coretraction, as required. (ii) If $\varphi \otimes k$ is an isomorphism then certainly $m = n$ so the claim follows from the first part of (i). ☐

Remark 25. Let $(A, \mathfrak{m}, k)$ be a local ring and $M \longrightarrow N, M' \longrightarrow N$ two morphisms of free A-modules of finite rank. It follows from Lemma 89(ii) that this pair of morphisms is a coproduct in A**Mod** if and only if the image under $- \otimes k$ is a coproduct in k**Mod**.

The following consequence of Lemma 89 seems a little surprising at first glance, but it is reasonable because over a local ring freeness is very cheap (MAT,Proposition 24).

Lemma 90. *Let $(A, \mathfrak{m}, k)$ be a local ring and $\varphi : M \longrightarrow N$ a morphism of finitely generated A-modules with N free. If $\varphi \otimes k : M \otimes k \longrightarrow N \otimes k$ is injective then M is free and φ is injective.*

Proof. Let $v : F \longrightarrow M$ be a surjective morphism of A-modules with F free of finite rank, such that $v \otimes k$ is an isomorphism (MAT,Lemma 140). The composite $uv : F \longrightarrow N$ is a morphism of free A-modules of finite rank with $uv \otimes k$ injective, so from Lemma 89(i) we deduce that uv is injective. Hence v is an isomorphism, from which we deduce the desired conclusion. Observe that if φ were a coretraction (one easy way to know that $\varphi \otimes k$ is injective) then it follows immediately from (MAT,Proposition 24) that M is free. ☐

We can now prove a version of Nakayama's lemma for complexes.

Lemma 91. *Let $(A, \mathfrak{m}, k)$ be a local ring and M a bounded complex of free A-modules of finite rank. If $M \otimes k$ is exact, then M is also exact.*

Proof. The proof is by induction on the number $n \geq 0$ of nonzero terms in the complex M. The case $n = 0$ is trivial, and if $n = 1$ then M is a single free A-module of finite rank in some degree, and exactness of $M \otimes k$ means that $M/\mathfrak{m}M = 0$. By Nakayama's lemma we conclude that $M = 0$, as required. If $n = 2$ then either the terms are separated, in which case the claim follows again by Nakayama's lemma, or they are adjacent in which case the claim is Lemma 89(ii).

Now assume that $n > 2$. To simplify our notation we may as well assume that $M^i = 0$ for $i < 0$ and that $M^0 \neq 0$. Note that the truncation $M_{\geq 1}$ of (DTC,Definition 15) works by replacing M^1 by a cokernel, and $- \otimes k$ preserves cokernels, so we have $(M \otimes k)_{\geq 1} \cong M_{\geq 1} \otimes k$ as complexes of k-modules. If $M \otimes k$ is exact then the same is true of $(M \otimes k)_{\geq 1}$ (since they have the same cohomology except at zero) and therefore also of $M_{\geq 1} \otimes k$. The complex $M_{\geq 1}$ begins with

$$0 \longrightarrow Coker\partial_M^0 \longrightarrow M^2 \longrightarrow \cdots$$

The cokernel is finitely generated, and we know that $Coker\partial_M^0 \otimes k \longrightarrow M^2 \otimes k$ is injective, so it follows from Lemma 90 that $Coker\partial_M^0$ is a free A-module of finite rank. Then $M_{\geq 1}$ is a bounded complex of free A-modules of finite rank with $M_{\geq 1} \otimes k$ exact, so by the inductive hypothesis $M_{\geq 1}$ is exact. In other words, $H^n(M) = 0$ for $n > 0$. To show that M is exact we need only show that $M^0 \longrightarrow M^1$ is injective. But we know that $M^0 \otimes k \longrightarrow M^1 \otimes k$ is injective, so this follows from Lemma 89(i). □

Remark 26. In this generalisation of Nakayama's lemma it really is necessary that the modules be free (at least, the statement is not true if the modules are only finitely generated). The obvious counterexample is the complex of A-modules consisting of the morphism $A \longrightarrow A/\mathfrak{m}$. Tensored with k this is an isomorphism (hence an exact complex) but the original complex is not exact.

We can abstract one of the key points of the proof of Lemma 91 as follows.

Lemma 92. *Let $(A, \mathfrak{m}, k)$ be a local ring and $\varphi : M \longrightarrow N$ a morphism of free A-modules of finite rank. If $\varphi \otimes k$ is injective then $Coker(\varphi)$ is a free A-module of finite rank.*

Proof. If $\varphi \otimes k$ is injective then by Lemma 89 the morphism φ is a coretraction, and if we choose a splitting $\rho : N \longrightarrow M$ then we have canonically $N = M \oplus Coker(\varphi)$. As the retract of a free A-module of finite rank, $Coker(\varphi)$ has the necessary properties (MAT,Proposition 24). □

Proposition 93. *Let $(A, \mathfrak{m}, k)$ be a local ring and M a bounded complex of free A-modules of finite rank. If $M \otimes k \cong \Sigma^i k$ in $\mathfrak{D}(k)$ for some $i \in \mathbb{Z}$ then $M \cong \Sigma^i A$ in $\mathfrak{D}(A)$.*

Proof. Once again the proof is by induction on the number $n \geq 0$ of nonzero terms in the complex M. If $n = 0$ this is trivial. If $n = 1$ then M is a single free A-module in some degree, of rank $rank_A(M) = rank_k(M \otimes k)$. The isomorphism $M \otimes k \cong k$ means that $rank_k(M \otimes k) = 1$, so we deduce an isomorphism $M \cong A$ as required.

Now assume that $n > 1$. We may as well assume that $M^j = 0$ for $j < 0$ and that $M^0 \neq 0$. Let $i \in \mathbb{Z}$ be such that $M \otimes k \cong \Sigma^{-i} k$. It is clear that $i \geq 0$ and $M^i \neq 0$. We divide into two cases:

Case $i > 0$. In this case $M^0 \otimes k \longrightarrow M^1 \otimes k$ is injective, and therefore by Lemma 89(i) so is $M^0 \longrightarrow M^1$. It follows that $M \cong M_{\geq 1}$ in $\mathfrak{D}(A)$ and $M \otimes k \cong (M \otimes k)_{\geq 1} \cong M_{\geq 1} \otimes k$ in $\mathfrak{D}(k)$, so we can pass to $M_{\geq 1}$. By Lemma 92 this is a bounded complex of free A-modules of finite rank, so the claim is true for $M_{\geq 1}$ by the inductive hypothesis.

Case $i = 0$. In this case $M \otimes k \cong k$ in $\mathfrak{D}(k)$. We treat the case $n = 2$ separately for clarity, and to motivate the general argument. If the terms of M are separated then we quickly come to a contradiction, so M is of the form $\cdots \longrightarrow 0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0 \longrightarrow \cdots$ and $M \otimes k \cong k$ in $\mathfrak{D}(k)$ means that we have an exact sequence

$$0 \longrightarrow k \longrightarrow M^0 \otimes k \longrightarrow M^1 \otimes k \longrightarrow 0$$

Setting $m_0 = rank_A(M^0)$ and $m_1 = rank_A(M^1)$ the first thing we deduce from this exact sequence is $m_0 = m_1 + 1$. Consider $k \longrightarrow M^0 \otimes k$ as a submodule and choose a basis $\overline{x_1}, \ldots, \overline{x_{m_1}}$ for the complement, where $x_j \in M^0$ and let $T = A^{\oplus n} \longrightarrow M^0$ be determined by the x_j. Let $A \longrightarrow M^0$ lift $k \longrightarrow M^0 \otimes k$. The morphisms $T \longrightarrow M^0$ and $A \longrightarrow M^0$ map to a coproduct under $- \otimes k$ and are therefore a coproduct of A-modules by Remark 25. Clearly $T \otimes k \longrightarrow M^0 \otimes k \longrightarrow M^1 \otimes k$ is an isomorphism, so from Lemma 89 we infer that $T \longrightarrow M^1$ is an isomorphism. In particular $M^0 \longrightarrow M^1$ must be an epimorphism, and the exact sequence

$$0 \longrightarrow Ker(\partial^0) \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0$$

is split exact. The image under $- \otimes k$ is still split exact, so we have $Ker(\partial^0) \otimes k \cong k$ and therefore $Ker(\partial^0) \cong A$ (as a retract of a free module of finite rank $Ker(\partial^0)$ is finitely generated and projective, and therefore free since we are working over a local ring). It is now clear that $M \cong A$ in $\mathfrak{D}(A)$.

In the general case $n > 2$ one defines T in the same way by choosing a basis for the complement of $k \longrightarrow M^0 \otimes k$ and lifting (the basis no longer necessarily contains $rank_A(M^1)$ elements). We do not exclude the possibility of T being zero. We then use the Nakayama lemma for complexes (Lemma 91) to lift exactness of the complex

$$0 \longrightarrow T \otimes k \longrightarrow M^1 \otimes k \longrightarrow M^2 \otimes k \longrightarrow \cdots M^t \otimes k \longrightarrow 0$$

to exactness of

$$0 \longrightarrow T \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots \longrightarrow M^t \longrightarrow 0$$

Since $M^{t-1} \longrightarrow M^t$ is an epimorphism of free A-modules it is a retraction and we have a split exact sequence

$$0 \longrightarrow Ker(\partial^{t-1}) \longrightarrow M^{t-1} \longrightarrow M^t \longrightarrow 0$$

and therefore $Ker(\partial^{t-1}) \otimes k = Ker(\partial^{t-1} \otimes k)$. It follows that $M_{\le(t-1)} \otimes k \cong (M \otimes k)_{\le(t-1)}$ and as before $Ker(\partial^{t-1})$ is free of finite rank, so we have reduced to the shorter complex $M_{\le(t-1)}$ and can invoke the inductive hypothesis to see that $M_{\le(t-1)} \cong A$ in $\mathfrak{D}(A)$. Since $M \cong M_{\le(t-1)}$ in $\mathfrak{D}(A)$, the proof is complete. □

Proposition 94. *Let $(A, \mathfrak{m}, k)$ be a local ring and X, Y bounded complexes of free A-modules of finite rank. If $X \underline{\otimes}_A Y \cong A$ in $\mathfrak{D}(A)$ there is an integer $i \in \mathbb{Z}$ and isomorphisms $X \cong \Sigma^i A$ and $Y \cong \Sigma^{-i} A$ in $\mathfrak{D}(A)$.*

Proof. Since X, Y are hoflat we have an isomorphism in $\mathfrak{D}(A)$

$$X \otimes_A Y \cong X \underline{\otimes}_A Y \cong A$$

and since $X \otimes_A Y$ is itself a bounded complex of free A-modules (hence hoprojective) we deduce an isomorphism $X \otimes_A Y \cong A$ in $K(A)$. Tensoring both sides of this isomorphism with $- \otimes_A k : K(A) \longrightarrow K(k)$ we have an isomorphism in $K(k)$

$$X/\mathfrak{m}X \otimes_k Y/\mathfrak{m}Y \cong k$$

The complexes $X/\mathfrak{m}X, Y/\mathfrak{m}Y$ are bounded, therefore hoflat, so we have $X/\mathfrak{m}X \underline{\otimes}_k Y/\mathfrak{m}Y \cong k$ in $\mathfrak{D}(k)$. By Lemma 88 this yields isomorphisms $X/\mathfrak{m}X \cong \Sigma^i k$ and $Y/\mathfrak{m}Y \cong \Sigma^{-i} k$ in $\mathfrak{D}(k)$ for some $i \in \mathbb{Z}$. From Proposition 93 we deduce isomorphisms $X \cong \Sigma^i A$ and $Y \cong \Sigma^{-i} A$ in $\mathfrak{D}(A)$, as required. □

Remark 27. Let X be a scheme and $\mathscr{E}$ a perfect complex of sheaves of modules. Given $x \in X$ it is clear that $\mathscr{E}_x$ is isomorphic in $\mathfrak{D}(\mathcal{O}_{X,x})$ to a bounded complex of free $\mathcal{O}_{X,x}$-modules of finite rank.

Definition 19. Let X be a scheme. We say that a complex $\mathscr{E}$ of sheaves of modules is *invertible* if for every $x \in X$ there exists an open neighborhood $x \in U$ and an isomorphism $\mathscr{E}|_U \cong \Sigma^i \mathcal{O}_X|_U$ in $\mathfrak{D}(U)$ for some $i \in \mathbb{Z}$. Clearly an invertible complex is perfect. This property is stable under isomorphism and suspension. If $\mathscr{E}$ is invertible it is clear that $\mathscr{E}^\vee$ is also invertible, and if $\mathscr{E}, \mathscr{F}$ are invertible then so is $\mathscr{E} \underline{\otimes} \mathscr{F}$.

Lemma 95. *Let X be a scheme and $\mathscr{E}$ an invertible complex of sheaves of modules. The canonical morphism $\mathscr{E}^\vee \underline{\otimes} \mathscr{E} \longrightarrow \mathcal{O}_X$ is an isomorphism in $\mathfrak{D}(X)$.*

Proof. The canonical morphism $\varepsilon : \mathscr{E}^\vee \underline{\otimes} \mathscr{E} \longrightarrow \mathcal{O}_X$ is the counit of (DCOS,Section 5.1). This morphism is local and trinatural, so we may assume $\mathscr{E} = \mathcal{O}_X$ in which case the compatibility diagram of (DCOS,Section 5.1) means that we need only check that the canonical morphism of complexes $\mathscr{H}om^\bullet(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is an isomorphism. This is trivial, so the proof is complete. □

Lemma 96. *Let $f : X \longrightarrow Y$ be a morphism of schemes and $\mathscr{E}$ an invertible complex on Y. Then $\mathbb{L}f^*(\mathscr{E})$ is invertible on X.*

Proof. The question is local and trinatural in $\mathscr{E}$ so we can assume that $\mathscr{E} = \mathcal{O}_Y$ in which case $\mathbb{L}f^*(\mathscr{E}) \cong \mathcal{O}_X$ is certainly invertible. $\square$

Definition 20 (Derived Picard Group). Let X be a scheme. The *derived Picard group of X*, denoted $DPic(X)$, is the abelian group of isomorphism classes of invertible complexes in $\mathfrak{D}(X)$ under the derived tensor product. The underlying conglomerate of this group may not be a set or even a class.

Lemma 97. *Let X be a scheme and $\mathscr{E}$ an invertible complex of sheaves of modules. There are induced equivalences of triangulated categories*

$$- \underline{\otimes} \mathscr{E} : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$$
$$- \underline{\otimes} \mathscr{E} : \mathfrak{Dper}(X) \longrightarrow \mathfrak{Dper}(X)$$

Proof. Let $\mathscr{E}$ be an arbitrary invertible complex. Such a complex is perfect, and the triangulated functor $- \underline{\otimes} \mathscr{E} : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(X)$ preserves perfection by Lemma 70, so we have an induced triangulated functor

$$S(-) = - \underline{\otimes} \mathscr{E} : \mathfrak{Dper}(X) \longrightarrow \mathfrak{Dper}(X)$$

Similarly we have a triangulated functor $T(-) = - \underline{\otimes} \mathscr{E}^\vee : \mathfrak{Dper}(X) \longrightarrow \mathfrak{Dper}(X)$ and it is clear that there are canonical trinatural equivalences $ST \cong 1, TS \cong 1$ by virtue of Lemma 95. $\square$

Remark 28. Let X be a scheme and $\mathscr{E}$ an invertible complex of sheaves of modules. Given a point $x \in X$ there is a *unique* integer $i \in \mathbb{Z}$ such that $H^i(\mathscr{E})_x \neq 0$, and there is an isomorphism of $\mathcal{O}_{X,x}$-modules $H^i(\mathscr{E})_x \cong \mathcal{O}_{X,x}$.

Lemma 98. *If X is an irreducible scheme a complex $\mathscr{E}$ of sheaves of modules is invertible if and only if it is isomorphic in $\mathfrak{D}(X)$ to $\Sigma^i \mathscr{L}$ for some invertible* sheaf *$\mathscr{L}$ of modules and $i \in \mathbb{Z}$.*

Proof. If $\mathscr{E}$ is invertible then we can associate to a point $x \in X$ the unique integer $i(x) \in \mathbb{Z}$ with $H^{i(x)}(\mathscr{E})_x \neq 0$, and moreover about every point $x \in X$ is an open neighborhood U with $i(x) = i(y)$ for $y \in U$. If X is irreducible then these open neighborhoods must all overlap, and there is a fixed integer $i \in \mathbb{Z}$ with $H^j(\mathscr{E}) = 0$ for $j \neq i$. Up to isomorphism in $\mathfrak{D}(X)$ we can therefore replace $\mathscr{E}$ by a sheaf $\mathscr{L}$ in degree i, and it is then clear that $\mathscr{L}$ is an invertible sheaf. $\square$

Proposition 99. *Let X be a scheme and $\mathscr{E}, \mathscr{F}$ perfect complexes of sheaves of modules with $\mathscr{E} \underline{\otimes} \mathscr{F} \cong \mathcal{O}_X$ in $\mathfrak{D}(X)$. Then $\mathscr{E}, \mathscr{F}$ are invertible.*

Proof. Given $x \in X$ we have after taking stalks an isomorphism in $\mathfrak{D}(\mathcal{O}_{X,x})$ (DCOS,Lemma 57)

$$\mathscr{E}_x \underline{\otimes} \mathscr{F}_x \cong (\mathscr{E} \underline{\otimes} \mathscr{F})_x \cong \mathcal{O}_{X,x}$$

Since $\mathscr{E}_x, \mathscr{F}_x$ are isomorphic in $\mathfrak{D}(\mathcal{O}_{X,x})$ to bounded complexes of free $\mathcal{O}_{X,x}$-modules of finite rank, we can by Proposition 94 find $i \in \mathbb{Z}$ and isomorphisms $\mathscr{E}_x \cong \Sigma^i \mathcal{O}_{X,x}$ and $\mathscr{F}_x \cong \Sigma^{-i} \mathcal{O}_{X,x}$ in $\mathfrak{D}(\mathcal{O}_{X,x})$. Using Corollary 82 we can find an open neighborhood $x \in U$ and isomorphisms $\mathscr{E}|_U \cong \Sigma^i \mathcal{O}_X|_U$ and $\mathscr{F}|_U \cong \Sigma^{-i} \mathcal{O}_X|_U$ in $\mathfrak{D}(U)$. In particular both $\mathscr{E}, \mathscr{F}$ are invertible. $\square$

Proposition 100. *Let X be a quasi-compact semi-separated scheme with an ample family of invertible sheaves and $\mathscr{E}, \mathscr{F} \in \mathfrak{D}_{qc}(X)$ complexes with $\mathscr{E} \underline{\otimes} \mathscr{F} \cong \mathcal{O}_X$ in $\mathfrak{D}(X)$. Then $\mathscr{E}, \mathscr{F}$ are invertible.*

Proof. By Lemma 18 we need not be specific about whether we are calculating the derived tensor product in $\mathfrak{D}(X)$ or $\mathfrak{Dqcoh}(X)$, and we can by Theorem 42 assume that $\mathscr{E}, \mathscr{F} \in \mathfrak{Dqcoh}(X)$ and $\mathscr{E} \underline{\otimes} \mathscr{F} \cong \mathcal{O}_X$ in $\mathfrak{Dqcoh}(X)$. It is enough by Proposition 99 to show that $\mathscr{E}, \mathscr{F}$ are perfect. But $\mathscr{E} \underline{\otimes} - : \mathfrak{Dqcoh}(X) \longrightarrow \mathfrak{Dqcoh}(X)$ is by hypothesis a triequivalence, and therefore sends compact objects to compact objects. By Theorem 63 the compacts are precisely the perfect complexes, so in particular $\mathscr{E} \cong \mathscr{E} \underline{\otimes} \mathcal{O}_X$ is perfect. By symmetry $\mathscr{F}$ is also perfect, and the proof is complete. $\square$

Corollary 101. *Let X be an irreducible quasi-compact semi-separated scheme with an ample family of invertible sheaves. If $\mathscr{E}, \mathscr{F} \in \mathfrak{D}_{qc}(X)$ are complexes with $\mathscr{E} \underline{\otimes} \mathscr{F} \cong \mathcal{O}_X$ in $\mathfrak{D}(X)$ then there exists an integer $i \in \mathbb{Z}$, an invertible sheaf of modules $\mathscr{L}$, and isomorphisms in $\mathfrak{D}(X)$*

$$\mathscr{E} \cong \Sigma^i \mathscr{L}, \quad \mathscr{F} \cong \Sigma^{-i} \mathscr{L}^\vee$$

Proof. By Proposition 100 both complexes are invertible, therefore perfect, and by Lemma 98 there exist integers $s, t \in \mathbb{Z}$ and invertible sheaves $\mathscr{L}, \mathscr{T}$ on X such that $\mathscr{E} \cong \Sigma^s \mathscr{L}$ and $\mathscr{F} \cong \Sigma^t \mathscr{T}$ in $\mathfrak{D}(X)$. As in the proof of Lemma 99 if we are given $x \in X$ we can find an open neighborhood $x \in U$ and $i \in \mathbb{Z}$ such that

$$\mathscr{E}|_U \cong \Sigma^i \mathcal{O}_X|_U, \quad \mathscr{F}|_U \cong \Sigma^{-i} \mathcal{O}_X|_U$$

in $\mathfrak{D}(U)$, from which we deduce that $t = -s$. The sheaves $\mathscr{L}, \mathscr{T}$ are flat, so we have an isomorphism in $\mathfrak{D}(X)$ of the form $(\Sigma^s \mathscr{L}) \otimes (\Sigma^{-s} \mathscr{T}) \cong \mathscr{E} \underline{\otimes} \mathscr{F} \cong \mathcal{O}_X$ which determines an isomorphism of sheaves of modules $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{T} \cong \mathcal{O}_X$. It is therefore clear that $\mathscr{T} \cong \mathscr{L}^\vee$ as sheaves of modules, completing the proof. □

Theorem 102. *Let X be an irreducible quasi-compact semi-separated scheme with an ample family of invertible sheaves. Given a complex $\mathscr{E} \in \mathfrak{D}_{qc}(X)$ the following are equivalent:*

(a) $\mathscr{E}$ is invertible.

(b) There exists an invertible sheaf $\mathscr{L}$ such that $\mathscr{E} \cong \Sigma^i \mathscr{L}$ in $\mathfrak{D}(X)$ for some $i \in \mathbb{Z}$.

(c) $\mathscr{E}$ is perfect and the canonical morphism $\mathscr{E}^\vee \underline{\otimes} \mathscr{E} \longrightarrow \mathcal{O}_X$ is an isomorphism in $\mathfrak{D}(X)$.

(d) There exists $\mathscr{F} \in \mathfrak{D}_{qc}(X)$ with $\mathscr{E} \underline{\otimes} \mathscr{F} \cong \mathcal{O}_X$ in $\mathfrak{D}(X)$.

Proof. $(a) \Leftrightarrow (b)$ is Lemma 98. $(a) \Rightarrow (c)$ is Lemma 95. $(c) \Rightarrow (d)$ If $\mathscr{E}$ is perfect then by Lemma 69 so is $\mathscr{E}^\vee$, which therefore has quasi-coherent cohomology. $(d) \Rightarrow (a)$ is Proposition 100. □

Corollary 103. *Let X be an irreducible quasi-compact semi-separated scheme. There is a canonical isomorphism of abelian groups*

$$Pic(X) \oplus \mathbb{Z} \longrightarrow DPic(X)$$
$$(\mathscr{L}, i) \mapsto \Sigma^{-i} \mathscr{L}$$

9 Appendix

Proposition 104. *Let X be a scheme and $\mathscr{F}$ a complex of sheaves of modules with quasi-coherent cohomology. Suppose we have a commutative diagram of complexes*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathscr{F}_{\geq n} & \longrightarrow & \cdots & \longrightarrow & \mathscr{F}_{\geq -2} & \longrightarrow & \mathscr{F}_{\geq -1} & \longrightarrow & \mathscr{F}_{\geq 0} \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathscr{I}_n & \longrightarrow & \cdots & \longrightarrow & \mathscr{I}_{-2} & \longrightarrow & \mathscr{I}_{-1} & \longrightarrow & \mathscr{I}_0 \end{array} \tag{38}$$

satisfying the following conditions

(i) Every vertical morphism is a quasi-isomorphism.

(ii) The $\mathscr{I}_n$ are complexes of injectives and $\mathscr{I}_n^i = 0$ for $i < n$.

(iii) Every morphism of complexes in the bottom row is a retraction in each degree.

Then the induced morphism $\vartheta : \mathscr{F} \longrightarrow \varprojlim_{n \leq 0} \mathscr{I}_n$ is a hoinjective resolution.

Proof. Set $\mathscr{I} = \varprojlim_{n \leq 0} \mathscr{I}_n$ and let $\mathscr{A}_n$ denote the kernel of $\mathscr{F}_{\geq n} \longrightarrow \mathscr{F}_{\geq (n+1)}$ and $\mathscr{C}_n$ the kernel of $\mathscr{I}_n \longrightarrow \mathscr{I}_{n+1}$. There is an induced morphism $\mathscr{A}_n \longrightarrow \mathscr{C}_n$ which one checks easily is a quasi-isomorphism. But by (DTC,Lemma 27) there is a canonical quasi-isomorphism $c_n H^n(\mathscr{F}) \longrightarrow \mathscr{A}_n$, so putting these together we have a quasi-isomorphism $c_n H^n(\mathscr{F}) \longrightarrow \mathscr{C}_n$. Our assumption (ii) means that $\mathscr{C}_n^i = 0$ for $i < n$ and from (iii) we infer that every sheaf $\mathscr{C}_n^i$ is injective, so $\Sigma^n \mathscr{C}_n$ is an injective resolution of $H^n(\mathscr{F})$ for every $n \leq -1$. Therefore for any affine open subset $U \subseteq X$ and $m > n$

$$H^m(\Gamma(U, \mathscr{C}_n)) \cong H^{m-n}(U, H^n(\mathscr{F})) = 0$$

since by assumption $H^n(\mathscr{F})$ is quasi-coherent, and therefore has vanishing higher cohomology on U by (COS,Theorem 14). This means that the sequence

$$\Gamma(U, \mathscr{C}_n^n) \longrightarrow \Gamma(U, \mathscr{C}_n^{n+1}) \longrightarrow \Gamma(U, \mathscr{C}_n^{n+2}) \longrightarrow \Gamma(U, \mathscr{C}_n^{n+3}) \longrightarrow \cdots$$

is exact. For each $i \in \mathbb{Z}$ we deduce morphisms of four inverse systems as in (DTC2,Lemma 30), with the induced kernel sequences being exact in every row if $i \geq 0$ and exact above row i otherwise (indexing the rows $0, -1, \ldots$ as they go up the page, and *above* meaning higher on the page)

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(U, \mathscr{I}_{-1}^{i-1}) & \longrightarrow & \Gamma(U, \mathscr{I}_{-1}^{i}) & \longrightarrow & \Gamma(U, \mathscr{I}_{-1}^{i+1}) & \longrightarrow & \Gamma(U, \mathscr{I}_{-1}^{i+2}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(U, \mathscr{I}_{0}^{i-1}) & \longrightarrow & \Gamma(U, \mathscr{I}_{0}^{i}) & \longrightarrow & \Gamma(U, \mathscr{I}_{0}^{i+1}) & \longrightarrow & \Gamma(U, \mathscr{I}_{0}^{i+2})
\end{array}$$

From (DTC2,Lemma 30) we infer that for any $j \leq 0$ and $i \geq j$ the canonical morphism

$$H^i(\Gamma(U, \mathscr{I})) \longrightarrow H^i(\Gamma(U, \mathscr{I}_j)) \tag{39}$$

is an isomorphism. Sheafifying and using (DCOS,Lemma 3) we see that $H^i(\mathscr{I}) \longrightarrow H^i(\mathscr{I}_j)$ is also an isomorphism. But we have commutative diagrams

$$\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\vartheta} & \mathscr{I} \\
{\scriptstyle q}\downarrow & & \downarrow \\
\mathscr{F}_{\geq j} & \longrightarrow & \mathscr{I}_j
\end{array}
\qquad
\begin{array}{ccc}
H^i(\mathscr{F}) & \xrightarrow{H^i(\vartheta)} & H^i(\mathscr{I}) \\
{\scriptstyle H^i(q)}\downarrow & & \downarrow \\
H^i(\mathscr{F}_{\geq j}) & \longrightarrow & H^i(\mathscr{I}_j)
\end{array} \tag{40}$$

where in the right hand diagram everything except the top morphism is an isomorphism, since by assumption $\mathscr{F}_{\geq n} \longrightarrow \mathscr{I}_n$ is a quasi-isomorphism. It follows that $H^i(\vartheta)$ is an isomorphism for $i \geq j$. But j is arbitrary, so ϑ must be a quasi-isomorphism. It follows from (DTC2,Proposition 32) that $\mathscr{I}$ is hoinjective, so the proof is complete. □

Remark 29. Let X be a scheme and $\mathscr{F}$ a complex of sheaves of modules. Then a diagram (38) satisfying the properties $(i), (ii), (iii)$ of Proposition 104 exists, by the inductive construction given in the proof of (DTC,Proposition 75). So provided $\mathscr{F}$ has quasi-coherent cohomology, the induced morphism $\mathscr{F} \longrightarrow \varprojlim \mathscr{I}_n$ will be a hoinjective resolution. Still assuming that $\mathscr{F}$ has quasi-coherent cohomology, observe that for $n \leq 0$ there is a unique morphism of complexes $\mathscr{I}_{\geq n} \longrightarrow \mathscr{I}_n$ making the following diagram commute

$$\begin{array}{ccccc}
 & & \mathscr{I} & & \\
 & \swarrow & & \searrow & \\
\mathscr{I}_{\geq n} & & \longrightarrow & & \mathscr{I}_n
\end{array}$$

We showed in the proof of Proposition 104 that $H^i(\mathscr{I}) \longrightarrow H^i(\mathscr{I}_n)$ is an isomorphism for $i \geq n$, so it is clear that $\mathscr{I}_{\geq n} \longrightarrow \mathscr{I}_n$ is a quasi-isomorphism. So we have a hoinjective resolution $\mathscr{I}$ of $\mathscr{F}$ whose truncations $\mathscr{I}_{\geq n}$ are (quasi-isomorphic to) hoinjective resolutions of the truncations $\mathscr{F}_{\geq n}$.

Corollary 105. *With the notation of Proposition 104 any holimit $\underleftarrow{holim}_{n\leq 0}\mathscr{I}_n$ is a hoinjective resolution of $\mathscr{F}$.*

Proof. By Proposition 104 the induced morphism $\mathscr{F} \longrightarrow \mathscr{I} = \varprojlim \mathscr{I}_n$ is a hoinjective resolution. Suppose we have a triangle in $K(X)$ defining a holimit

$$\underleftarrow{holim}\mathscr{I}_n \longrightarrow \prod_{n\leq 0}\mathscr{I}_n \xrightarrow{1-\nu} \prod_{n\leq 0}\mathscr{I}_n \longrightarrow \Sigma\underleftarrow{holim}\mathscr{I}_n$$

Note that we do *not* assume that this is the canonical holimit defined in (DTC,Definition 29). The canonical morphism $\varprojlim \mathscr{I}_n \longrightarrow \prod \mathscr{I}_n$ composes with $1-\nu$ to give zero, so there is a factorisation $f : \mathscr{I} \longrightarrow \underleftarrow{holim}\mathscr{I}_n$ in $K(X)$. The homotopy limit is certainly hoinjective, so to complete the proof we need only show that f is a quasi-isomorphism. It is therefore enough to show that $\Gamma(U, f)$ is a quasi-isomorphism of complexes of abelian groups for every affine open $U \subseteq X$ (DCOS,Lemma 5).

If we took the canonical holimit, and the canonical factorisation f of (DTC,Remark 27) then it would follow from the fact that all the morphisms $\mathscr{I}_{n+1} \longrightarrow \mathscr{I}_n$ are fibrations that $1-\nu$ is a fibration, and therefore f is a quasi-isomorphism (DTC,Remark 27) (DTC,Lemma 67). But it can be useful to know the result for an arbitrary holimit and arbitrary factorisation f in $K(X)$.

The inclusion of sheaves in presheaves means that $K(X)$ is a fragile triangulated subcategory of $K(Mod(X))$ (DTC,Lemma 38), and the inclusion preserves products so our homotopy limit is still valid on the level of presheaves. The functor $\Gamma(U,-)$ is exact on presheaves, so $\mathbb{R}\Gamma(U,-) = \Gamma(U,-)$ and we have a triangle in $\mathfrak{D}(\mathbf{Ab})$

$$\Gamma(U, \underleftarrow{holim}\mathscr{I}_n) \longrightarrow \prod_{n\leq 0}\Gamma(U, \mathscr{I}_n) \longrightarrow \prod_{n\leq 0}\Gamma(U,\mathscr{I}_n) \longrightarrow \Sigma\Gamma(U, \underleftarrow{holim}\mathscr{I}_n)$$

In other words, this means

$$\Gamma(U, \underleftarrow{holim}\mathscr{I}_n) = \underleftarrow{holim}\Gamma(U, \mathscr{I}_n)$$

We have already observed in (39) that the *presheaf* cohomology of the $\mathscr{I}_n$ stabilises (at least over open affines). That is, for open affine $U \subseteq X$ and fixed $i \in \mathbb{Z}$ the morphism

$$H^i(\Gamma(U, \mathscr{I})) \longrightarrow H^i(\Gamma(U, \mathscr{I}_n))$$

is an isomorphism for all sufficiently large negative $n \leq 0$. We are now in the situation of (DTC,Lemma 77), from which we deduce that $\Gamma(U, \mathscr{I}) \longrightarrow \Gamma(U, \underleftarrow{holim}\mathscr{I}_n)$ is a quasi-isomorphism. This is enough to show that $f : \mathscr{I} \longrightarrow \underleftarrow{holim}\mathscr{I}_n$ is a quasi-isomorphism, and complete the proof. ∎

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