Derived Categories Part I

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In this first part of our notes on derived categories we aim to give the definition and elementary properties of derived categories as efficiently as possible. The high point is the proof that the unbounded derived category of a grothendieck abelian category has enough hoinjectives.

Contents

1 Introduction

These notes are meant to complement our notes on Triangulated Categories, but the reader will always be warned at the beginning of each section exactly how much about triangulated categories they are expected to know. Until Section 3 the reader is expected to know nothing.

All notation and conventions are from our notes on Derived Functors. In particular we assume that every abelian category comes with canonical structures which allow us to define the cohomology of cochain complexes in an unambiguous way. If we write *complex* we mean *cochain complex*, and we write $\mathbf{C}(\mathcal{A})$ for the abelian category of all complexes in $\mathcal{A}$. As usual we write $A = 0$ to indicate that A is a zero object (not necessarily the canonical one). We use the terms *preadditive category* and *additive category* as defined in (AC,Section 2). See also the definition of an *acyclic complex* in (DF,Definition 5), which disagrees with some references. We say a complex is *exact* if all its cohomology objects are zero.

None of this material is new. The results on homotopy limits and colimits are from [BN93]. The original reference for hoinjective complexes is [Spa88], and all the major results leading to the proof of the existence of hoinjective resolutions for a grothendieck abelian category are from [ATJLSS00].

1.1 History and Motivation

Since they first appeared in Verdier's thesis [Ver96], derived categories have experienced an explosion of applications in most fields of algebra and even mathematical physics. It is not difficult to find articles describing the power of the technology. We assume here that the reader is ready to learn the material in depth.

There are now several good places to learn about derived categories. The original reference is [Ver96] and historically many people learnt the material from [Har68]. There is a short chapter in Weibel's book [Wei94] while the recent book of Gelfand & Manin [GM03] is more comprehensive. An excellent careful presentation aimed at applications in algebraic geometry can be found in Lipman's widely read notes [Lip].

Due to technical limitations, early work in field focused on the bounded derived category. The work of Neeman on Grothendieck duality [Nee96] showed that the natural setting for many questions is actually the *unbounded* derived category. Even if one is only interested in bounded complexes, it is often more convenient to allow unbounded complexes in one's arguments. Then one can deploy powerful tools borrowed from algebraic topology, such as the homotopy colimit. We refer the reader to the introduction of [Nee06] for a survey of the uses of the infinite techniques.

The reader can find many good introductions to the bounded derived category. But despite the great success of the method, it is much harder to find an introduction to the unbounded derived category. This problem is confounded by the fact that in the recent literature it is common to find references to [Nee96] in discussions of Grothendieck duality. In these notes we aim to give a complete, careful treatment of the unbounded derived category. This continues in our notes on Derived Categories of Sheaves (DCOS) and Derived Categories of Quasi-coherent Sheaves (DCOQS), where we finally reach a proof of Neeman's version of Grothendieck duality, with each step hopefully accessible to the graduate student just starting in the area.

The derived category is a special case of the verdier quotient studied in our notes on Triangulated Categories (TRC), which is to say, in Neeman's excellent book on the subject [Nee01]. The main technical problem one encounters with unbounded complexes is in the construction of resolutions: given a bounded below complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

It is not difficult (see Proposition 75) to construct inductively an injective resolution, by which we mean a quasi-isomorphism $X \longrightarrow I$ with I a bounded below complex of injectives. This is much harder if X is unbounded, but the necessary technology was introduced by Spaltenstein [Spa88] and later independently by several other authors. Our approach to the question of "existence of injective resolutions" follows a more recent paper [ATJLSS00] which gives a different proof. This is the main technical difficulty of the theory and it occupies much of these notes.

2 Homotopy Categories

Definition 1. Let $\mathcal{A}$ be an abelian category. Then $K(\mathcal{A})$ is the category whose objects are complexes in $\mathcal{A}$ and whose morphisms are homotopy equivalence classes of morphisms of complexes. That is, we begin with the abelian category $\mathbf{C}(\mathcal{A})$ and use the homotopy equivalence relation (DF,Lemma 8) to divide the morphism sets up into equivalence classes.

A complex X is said to be *bounded below* if there exists $N \in \mathbb{Z}$ such that $X^n = 0$ for all $n \leq N$ and *bounded above* if there exists $N \in \mathbb{Z}$ such that $X^n = 0$ for all $n \geq N$. A complex is *bounded* if it is both bounded above and bounded below. We define three full subcategories of $K(\mathcal{A})$:

$$\begin{aligned} K^+(\mathcal{A}) &: \text{ bounded below complexes} \\ K^-(\mathcal{A}) &: \text{ bounded above complexes} \\ K^b(\mathcal{A}) &: \text{ bounded complexes} \end{aligned}$$

There is an obvious functor $\mathbf{C}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ which is the identity on objects and sends morphisms of complexes to their equivalence class.

Lemma 1. *Let $\mathcal{A}$ be an abelian category. Then $K(\mathcal{A})$ is an additive category and the functor $\mathbf{C}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ is additive.*

Proof. Let $f, g : X \longrightarrow Y$ be morphisms of complexes and suppose that $f \simeq 0$ and $g \simeq 0$. It is not hard to check that $f + g \simeq 0$. Since $f \simeq 0$ implies $-f \simeq 0$ the equivalence class of $0 : X \longrightarrow Y$ under the relation $\simeq$ is actually a subgroup of $Hom_{\mathcal{A}}(X, Y)$. Therefore the set of equivalence classes is actually a quotient of the group $Hom_{\mathcal{A}}(X, Y)$, so it has a canonical additive structure defined by choosing representatives and adding them. This addition is clearly bilinear, so $K(\mathcal{A})$ is a preadditive category. The functor $\mathbf{C}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ is clearly additive, and by a standard argument (AC,Proposition 28) it follows that $K(\mathcal{A})$ has binary coproducts and therefore all finite products and coproducts. □

Remark 1. Let $\mathcal{A}$ be an abelian category. The categories $K^+(\mathcal{A}), K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ all inherit a natural structure as additive categories from $K(\mathcal{A})$.

Definition 2. Let $\mathcal{A}$ be an abelian category, X a complex in $\mathcal{A}$ and $n \in \mathbb{Z}$. We define another complex in $\mathcal{A}$ by "shifting" n places to the left (writing cochain complexes with indices ascending to the right)

$$X[n]^p = X^{p+n} \quad \partial^p_{X[n]} = (-1)^n \partial^{p+n}_X$$

If $f : X \longrightarrow Y$ is a morphism of complexes then $f[n]^p = f^{p+n}$ defines a morphism of complexes $f[n] : X[n] \longrightarrow Y[n]$. This defines the additive functor $(-)[n] : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{A})$. It is clear that $(-)[0]$ is the identity functor, and if $n, m \in \mathbb{Z}$ then $(-)[n] \circ (-)[m] = (-)[n + m]$. In particular $X[n][-n] = X$ for any complex X and for any $n \in \mathbb{Z}$ the functor $(-)[n]$ is an additive automorphism of $\mathbf{C}(\mathcal{A})$.

If $f, g : X \longrightarrow Y$ are homotopic morphisms of complexes then so are $f[n], g[n]$ so this construction also defines an additive automorphism $(-)[n] : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$.

Definition 3. Let $\mathcal{A}$ be an abelian category, $u : X \longrightarrow Y$ a morphism of complexes. The *mapping cone* Z of u is defined for $n \in \mathbb{Z}$ by $Z^n = X^{n+1} \oplus Y^n$ with the differential

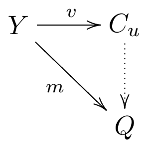
$$\partial^n_Z : X^{n+1} \oplus Y^n \longrightarrow X^{n+2} \oplus Y^{n+1}$$
$$\partial^n_Z = \begin{pmatrix} -\partial^{n+1}_X & 0 \\ u^{n+1} & \partial^n_Y \end{pmatrix}$$

There are canonical morphisms of complexes $v : Y \longrightarrow Z$ and $w : Z \longrightarrow X[1]$ defined for $n \in \mathbb{Z}$ to be the injection $v^n : Y^n \longrightarrow X^{n+1} \oplus Y^n$ and projection $w^n : X^{n+1} \oplus Y^n \longrightarrow X^{n+1}$ respectively. So given a morphism of complexes u we have produced the morphisms in the following diagram

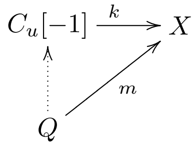
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \tag{1}$$

Shifting we also have a morphism of complexes $k = w[-1] : Z[-1] \longrightarrow X$. We will often write C_u instead of Z to indicate the dependence on the morphism u. The next result gives some reason why you should care about this construction.

Proposition 2. *Let $\mathcal{A}$ be an abelian category, $u : X \longrightarrow Y$ a morphism of complexes and $v : Y \longrightarrow C_u$, $k : C_u[-1] \longrightarrow X$ the canonical morphisms of complexes. Then* *(i) A morphism of complexes $m : Y \longrightarrow Q$ factors through v if and only if $mu \simeq 0$.*



(ii) A morphism of complexes $m : Q \longrightarrow X$ factors through k if and only if $um \simeq 0$.



There are also canonical *homotopies $vu \simeq 0$ and $uk \simeq 0$.*

Proof. (i) Let Q be any complex. Let us study what it means to define a morphism of complexes $f : C_u \longrightarrow Q$. A collection of morphisms $f^n : X^{n+1} \oplus Y^n \longrightarrow Q^n$ for each n, say with components $\Sigma^{n+1} : X^{n+1} \longrightarrow Q^n$ and $g^n : Y^n \longrightarrow Q^n$, define a morphism of complexes $C_u \longrightarrow Q$ if and only if the following equations are satisfied for $n \in \mathbb{Z}$

$$(gu)^n = \partial_Q^{n-1}\Sigma^n + \Sigma^{n+1}\partial_X^n$$
$$g^{n+1}\partial_Y^n = \partial_Q^n g^n$$

So a morphism of complexes $f : C_u \longrightarrow Q$ consists of a morphism of complexes $g : Y \longrightarrow Q$ and a homotopy $0 \simeq gu$, and it is clear that g is the composite fv. It is now easy to check that a given morphism of complexes $m : Y \longrightarrow Q$ factors through v if and only if $0 \simeq mu$, and in fact there is a bijection between the factorisations $C_u \longrightarrow Q$ and homotopies $0 \simeq mu$.

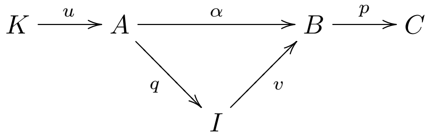
(ii) Let Q be any complex. A collection of morphisms $f^n : Q^n \longrightarrow X^n \oplus Y^{n-1}$ with components $g^n : Q^n \longrightarrow X^n$ and $\Sigma^n : Q^n \longrightarrow Y^{n-1}$ defines a morphism of complexes $f : Q \longrightarrow C_u[-1]$ if and only if $g : Q \longrightarrow X$ is a morphism of complexes and Σ is a homotopy $ug \simeq 0$, and moreover we can recover g as the composite kf. It is now easy to check that a given morphism of complexes $m : Q \longrightarrow X$ factors through k if and only if $um \simeq 0$. In fact there is a bijection between the factorisations $Q \longrightarrow C_u[-1]$ and homotopies $um \simeq 0$.

For the last statement, observe that $v : Y \longrightarrow C_u$ factors through itself via the identity, and this factorisation corresponds to a canonical homotopy $\Sigma : 0 \longrightarrow vu$ given by $\Sigma^n = \binom{1}{0}$. Similarly the factorisation of k through itself corresponds to a canonical homotopy $\Theta : uk \longrightarrow 0$ given by $\Theta^n = (0\ \ 1)$. $\square$

Corollary 3. *Let $\mathcal{A}$ be an abelian category and $f, g : X \longrightarrow Y$ morphisms of complexes. Then $f \simeq g$ if and only if $f - g$ factors through the mapping cone $X \longrightarrow C_1$ of the identity $1 : X \longrightarrow X$.*

Remark 2. So in some sense $v : Y \longrightarrow C_u$ is a *homotopy cokernel* and $k : C_u[-1] \longrightarrow X$ a *homotopy kernel* of $u : X \longrightarrow Y$. The remarkable thing is that unlike the actual cokernel and cokernel (whose constructions are quite different) the homotopy kernel and cokernel arise from the same object C_u, the mapping cone. In terms of homotopy theoretic information about the abelian category $\mathbf{C}(\mathcal{A})$ the triangle (1) encodes all the important information about the morphism u.

Remark 3. Let $\mathcal{A}$ be an abelian category, $\alpha : A \longrightarrow B$ a morphism with cokernel $p : B \longrightarrow C$ and kernel $u : K \longrightarrow A$ and epi-mono factorisation $q : A \longrightarrow I$ followed by $v : I \longrightarrow B$, as in the diagram



Then by the axioms of an abelian category, p is the cokernel of v and u is the kernel of q. It follows that v is the kernel of p and q the cokernel of u. The morphism $v : I \longrightarrow B$ is the *image* of α and the morphism $q : A \longrightarrow I$ is the *coimage*. To define a homotopy theoretic version of the image and coimage, we need to take the homotopy cokernel of the homotopy kernel, and the homotopy kernel of the homotopy cokernel.

Definition 4 (Homotopy Coimage). Let $\mathcal{A}$ be an abelian category, $u : X \longrightarrow Y$ a morphism of complexes with $v : Y \longrightarrow C_u$ and $k : C_u[-1] \longrightarrow X$ canonical. Then k is the homotopy kernel of u and we call the complex $\widetilde{C}_u = C_k$ the *mapping cylinder* of u. By definition of the mapping cone there are canonical morphisms of complexes $\widetilde{u} : X \longrightarrow \widetilde{C}_u$ and $\widetilde{v} : \widetilde{C}_u \longrightarrow C_u$. We call $\widetilde{u}$ the *homotopy coimage* of u. Let us examine the complex $\widetilde{C}_u$ more closely. For $n \in \mathbb{Z}$ we have

$$\widetilde{C}_u^n = C_u^n \oplus X^n = X^{n+1} \oplus Y^n \oplus X^n$$

The differential is defined by

$$\partial^n : X^{n+1} \oplus Y^n \oplus X^n \longrightarrow X^{n+2} \oplus Y^{n+1} \oplus X^{n+1}$$

$$\partial^n = \begin{pmatrix} -\partial_X^{n+1} & 0 & 0 \\ u^{n+1} & \partial_Y^n & 0 \\ 1 & 0 & \partial_X^n \end{pmatrix}$$

For $n \in \mathbb{Z}$ the morphism $\widetilde{u}^n : X^n \longrightarrow X^{n+1} \oplus Y^n \oplus X^n$ is the injection and $\widetilde{v}^n : X^{n+1} \oplus Y^n \oplus X^n \longrightarrow X^{n+1} \oplus Y^n$ is the projection onto the first two coordinates. There is also a morphism of complexes $\varphi : \widetilde{C}_u \longrightarrow Y$ defined by

$$\varphi^n : X^{n+1} \oplus Y^n \oplus X^n \longrightarrow Y^n$$

$$\varphi^n = \begin{pmatrix} 0 & -1 & u^n \end{pmatrix}$$

It is clear that the following diagram commutes

$$\begin{array}{ccccccc} C_u[-1] & \xrightarrow{k} & X & \xrightarrow{\quad u \quad} & Y & \xrightarrow{v} & C_u \\ & & & \searrow^{\widetilde{u}} \quad \nearrow_{\varphi} & & & \\ & & & \widetilde{C}_u & & & \end{array}$$

We know from Proposition 2 that $\widetilde{u}k \simeq 0$, but it is not in general true that $v\varphi \simeq 0$. This seems confusing only because in an abelian category, coimages and images coincide, but *homotopy* coimages and images do not necessarily agree. Note that the following sequence is trivially exact

$$0 \longrightarrow X \xrightarrow{\widetilde{u}} \widetilde{C}_u \xrightarrow{\widetilde{v}} C_u \longrightarrow 0 \tag{2}$$

Definition 5 (Homotopy Image). Let $\mathcal{A}$ be an abelian category, $u : X \longrightarrow Y$ a morphism of complexes with $v : Y \longrightarrow C_u$ and $k : C_u[-1] \longrightarrow X$ canonical. Then v is the homotopy cokernel and we consider the complex $\widehat{C}_u = C_v[-1]$. By definition there are canonical morphisms of complexes $\widehat{u} : \widehat{C}_u \longrightarrow Y$ and $\widehat{v} : C_u[-1] \longrightarrow \widehat{C}_u$ and we call $\widehat{u}$ the *homotopy image* of u. Let us examine the complex $\widehat{C}_u$ more closely. For $n \in \mathbb{Z}$ we have

$$\widehat{C}_u^n = Y^n \oplus C_u^{n-1} = Y^n \oplus X^n \oplus Y^{n-1}$$

The differential is defined by

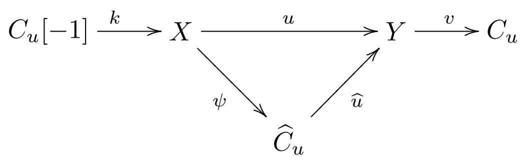
$$\partial^n : Y^n \oplus X^n \oplus Y^{n-1} \longrightarrow Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

$$\partial^n = \begin{pmatrix} \partial_Y^n & 0 & 0 \\ 0 & \partial_X^n & 0 \\ -1 & -u^n & -\partial_Y^{n-1} \end{pmatrix}$$

For $n \in \mathbb{Z}$ the morphism $\widehat{u}^n : Y^n \oplus X^n \oplus Y^{n-1} \longrightarrow Y^n$ is the projection and $\widehat{v}^n : X^n \oplus Y^{n-1} \longrightarrow Y^n \oplus X^n \oplus Y^{n-1}$ is the injection into the second two coordinates. There is also a morphism of complexes $\psi : X \longrightarrow \widehat{C}_u$ defined by

$$\psi^n : X^n \longrightarrow Y^n \oplus X^n \oplus Y^{n-1}$$
$$\psi^n = \begin{pmatrix} u^n \\ -1 \\ 0 \end{pmatrix}$$

It is clear that the following diagram commutes



We know that $v\widehat{u} \simeq 0$, but it is not in general true that $\psi k \simeq 0$. We have the following trivial short exact sequence

$$0 \longrightarrow C_u[-1] \xrightarrow{\widehat{v}} \widehat{C}_u \xrightarrow{\widehat{u}} Y \longrightarrow 0 \tag{3}$$

Lemma 4. *Let $\mathcal{A}$ be an abelian category and $u : X \longrightarrow Y$ a morphism of complexes. Then*

(i) The morphism of complexes $\varphi : \widetilde{C}_u \longrightarrow Y$ is a homotopy equivalence.

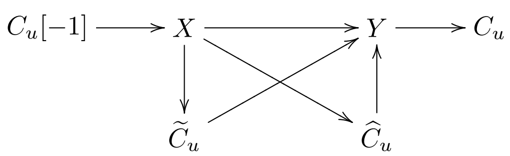
(ii) The morphism of complexes $\psi : X \longrightarrow \widehat{C}_u$ is a homotopy equivalence.

Proof. (i) Define a morphism of complexes $\kappa : Y \longrightarrow \widetilde{C}_u$ to be the additive inverse of the injection into the middle factor $\kappa^n : Y^n \longrightarrow X^{n+1} \oplus Y^n \oplus X^n$. Then $\varphi\kappa = 1$ and if we define a morphism $\Sigma^n : \widetilde{C}^n_u \longrightarrow \widetilde{C}^{n-1}_u$ by the matrix

$$\Sigma^n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is not difficult to check that this is a homotopy of $\kappa\varphi$ with 1. The statement (ii) is proved similarly. □

Remark 4. In other words, given a morphism of complexes $u : X \longrightarrow Y$ the homotopy kernel, cokernel, image and coimage fit into the following commutative diagram



where the homotopy image and coimage are both homotopy equivalences (these are the diagonal morphisms in the diagram).

Lemma 5. *Let $\mathcal{A}$ be an abelian category and X a complex. Let C_X denote the mapping cone of the identity $1_X : X \longrightarrow X$. Then the identity morphism $1_{C_X} : C_X \longrightarrow C_X$ is homotopic to the zero morphism.*

Proof. One checks that the morphisms $\Sigma^n = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) : C^n_X \longrightarrow C^{n-1}_X$ are a homotopy of the identity with zero. □

Remark 5. Let $\mathcal{A}$ be an abelian category and suppose we have a short exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{q} Q \longrightarrow 0$$

Let $\delta^n : H^n(Q) \longrightarrow H^{n+1}(X)$ be the canonical connecting morphism. The same sequence with q replaced by $-q$ is also a short exact sequence of complexes, with its own connecting morphism $\omega^n : H^n(Q) \longrightarrow H^{n+1}(X)$. It is straightforward to check that in fact $\omega^n = -\delta^n$. The same statement holds if we leave q fixed and replace u by $-u$.

Lemma 6. *Let $\mathcal{A}$ be an abelian category, and suppose we have an exact sequence of complexes*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{q} Q \longrightarrow 0$$

Then there is a canonical commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\widetilde{u}} & \widetilde{C}_u & \xrightarrow{-\widetilde{v}} & C_u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & X & \xrightarrow[u]{} & Y & \xrightarrow[q]{} & Q & \longrightarrow & 0 \end{array} \tag{4}$$

And therefore a morphism of the long exact sequences of cohomology

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^n(X) & \longrightarrow & H^n(\widetilde{C}_u) & \longrightarrow & H^n(C_u) & \xrightarrow{\omega^n} & H^{n+1}(X) & \longrightarrow & \cdots \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & H^n(X) & \longrightarrow & H^n(Y) & \longrightarrow & H^n(Q) & \longrightarrow & H^{n+1}(X) & \longrightarrow & \cdots \end{array} \tag{5}$$

Proof. Since $qu = 0$ the zero morphisms give a homotopy $qu \simeq 0$ and therefore by Proposition 2 there is an associated factorisation $f : C_u \longrightarrow Q$ of q through $v : Y \longrightarrow C_u$. To be precise, $f^n = (0 \quad q^n)$. All other morphisms are canonical, and it is not hard to check commutativity and exactness of the first row. We know from Lemma 4 that φ is a homotopy equivalence, so $H^n(\widetilde{C}_u) \longrightarrow H^n(Y)$ is an isomorphism for every $n \in \mathbb{Z}$. The commutative diagram (4) induces the morphism of long exact sequences (5) (DF,Proposition 30), and it follows from the 5-Lemma that $H^n(C_u) \longrightarrow H^n(Q)$ is an isomorphism for all $n \in \mathbb{Z}$. Note that the connecting morphism in the first row $\omega^n : H^n(C_u) \longrightarrow H^{n+1}(X)$ is the additive inverse of the connecting morphism of (2) (since we have used $-\widetilde{v}$). ☐

Lemma 7. *Let $\mathcal{A}$ be an abelian category, and suppose we have an exact sequence of complexes*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{q} Q \longrightarrow 0$$

Then there is a canonical commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{q} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & C_q[-1] & \xrightarrow[-\widehat{v}]{} & \widehat{C}_q & \xrightarrow[\widehat{q}]{} & Q & \longrightarrow & 0 \end{array}$$

And therefore a morphism of the long exact sequences of cohomology

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^n(X) & \longrightarrow & H^n(Y) & \longrightarrow & H^n(Q) & \longrightarrow & H^{n+1}(X) & \longrightarrow & \cdots \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \cdots & \longrightarrow & H^n(C_q[-1]) & \longrightarrow & H^n(\widehat{C}_q) & \longrightarrow & H^n(Q) & \xrightarrow[\omega^n]{} & H^{n+1}(C_q[-1]) & \longrightarrow & \cdots \end{array} \tag{6}$$

Proof. Since $qu = 0$ the zero morphisms give a homotopy $qu \simeq 0$ and therefore by Proposition 2 there is an associated factorisation $g : X \longrightarrow C_q[-1]$ of u through the homotopy kernel of q. To be precise, $g^n = \binom{u^n}{0}$. As before, it follows from Lemma 4 and the 5-Lemma that $H^n(X) \longrightarrow H^n(C_q[-1])$ is an isomorphism for all $n \in \mathbb{Z}$ and that we have the desired isomorphism of long exact sequences. Note that the connecting morphism in the second row ω^n is the additive inverse of the connecting morphism of (3) (since we have used $-\widehat{v}$). $\square$

Remark 6. Intuitively, Lemma 6 and Lemma 7 say something very important about short exact sequences of complexes. Let $\mathcal{A}$ be an abelian category and suppose we have an exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{q} Q \longrightarrow 0$$

Then $qu = 0$ means that there is a canonical homotopy $qu \simeq 0$, and this yields a factorisation $f : C_u \longrightarrow Q$ of q through the homotopy cokernel of u, and a factorisation $g : X \longrightarrow C_q[-1]$ of u through the homotopy kernel of q. From Lemma 6 and Lemma 7 we deduce that both morphisms f, g are *quasi-isomorphisms of complexes* (that is, they induce an isomorphism on cohomology in every degree).

Let $\mathcal{A}$ be an abelian category. So far we have defined homotopy kernels, cokernels and coimages. The usual kernels, cokernels and coimages are natural with respect to commutative diagrams, in the obvious sense. The homotopy versions are natural with respect to diagrams which are commutative *up to homotopy*. Suppose we have a diagram of complexes

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[u']{} & Y' \end{array} \tag{7}$$

together with a homotopy $\Sigma : fu \longrightarrow u'g$. Let $v : Y \longrightarrow C_u, v' : Y' \longrightarrow C_{u'}$ be the homotopy cokernels and $\Theta : 0 \longrightarrow v'u'$ the canonical homotopy $\Theta^n = \binom{1}{0}$. Then together Σ, Θ give a homotopy $\Lambda : 0 \longrightarrow v'fu$ defined by $\Lambda^n = \Theta^n g^n - v'^{n-1}\Sigma^n = \binom{g^n}{-\Sigma^n}$. By Proposition 2$(i)$ this homotopy corresponds to a morphism of complexes $h : C_u \longrightarrow C_{u'}$ making the right hand square in the following diagram commute

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & C_u \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \vdots{\scriptstyle h} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & C_{u'} \end{array} \tag{8}$$

$$h^n = \begin{pmatrix} g^{n+1} & 0 \\ -\Sigma^{n+1} & f^n \end{pmatrix} \tag{9}$$

Now let $k : C_u[-1] \longrightarrow X, k' : C_{u'}[-1] \longrightarrow X'$ be the homotopy kernels and $\Theta : uk \longrightarrow 0$ the canonical homotopy $\Theta^n = (0 \quad 1)$. The homotopy $\Sigma : fu \longrightarrow u'g$ remains fixed. Then $\Lambda^n = f^{n-1}\Theta^n - \Sigma^n k^n = (-\Sigma^n, \quad f^{n-1})$ is a homotopy $\Lambda : u'gk \longrightarrow 0$. By Proposition 2$(ii)$ this homotopy corresponds to a morphism of complexes $j : C_u[-1] \longrightarrow C_{u'}[-1]$ making the left hand square in the following diagram commute

$$\begin{array}{ccccc} C_u[-1] & \xrightarrow{k} & X & \xrightarrow{u} & Y \\ {\scriptstyle j}\vdots & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ C_{u'}[-1] & \xrightarrow[k']{} & X' & \xrightarrow[u']{} & Y \end{array} \tag{10}$$

$$j^n = \begin{pmatrix} g^n & 0 \\ -\Sigma^n & f^{n-1} \end{pmatrix} \tag{11}$$

Observe that $j = h[-1]$. In particular we obtain the morphisms $h : C_u \longrightarrow C_{u'}$ and $j : C_u[-1] \longrightarrow C_{u'}[-1]$ if the square (7) actually commutes, in which case $\Sigma^n = 0$ defines a homotopy $fu \longrightarrow u'g$. In particular we can apply the construction of the morphism on the homotopy cokernels to the left hand square of (10) (which actually commutes) to obtain a morphism of complexes $l : \widetilde{C}_u \longrightarrow \widetilde{C}_{u'}$. In summary

Definition 6. Let $\mathcal{A}$ be an abelian category, and suppose we have a diagram of morphisms of complexes (7) (not necessarily commutative) and a homotopy $\Sigma : fu \longrightarrow u'g$. Then there are canonical morphisms of complexes

$$\begin{array}{rl} h : C_u \longrightarrow C_{u'} & h^n = \begin{pmatrix} g^{n+1} & 0 \\ -\Sigma^{n+1} & f^n \end{pmatrix} \\ j : C_u[-1] \longrightarrow C_{u'}[-1] & j^n = \begin{pmatrix} g^n & 0 \\ -\Sigma^n & f^{n-1} \end{pmatrix} \\ l : \widetilde{C}_u \longrightarrow \widetilde{C}_{u'} & l^n = \begin{pmatrix} g^{n+1} & 0 & 0 \\ -\Sigma^{n+1} & f^n & 0 \\ 0 & 0 & g^n \end{pmatrix} \end{array}$$

and it is easy to check that the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\widetilde{u}} & \widetilde{C}_u & \xrightarrow{\widetilde{v}} & C_u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X' & \xrightarrow[\widetilde{u'}]{} & \widetilde{C}_{u'} & \xrightarrow[\widetilde{v'}]{} & C_{u'} & \longrightarrow & 0 \end{array} \tag{12}$$

In terms of cohomology, in every sense that matters we have replaced the diagram (7), which only commutes up to homotopy, with the left hand square in the above diagram which *does* commute.

Definition 7. Let $\mathcal{A}$ be an abelian category $f, g : X \to Y$ morphisms of complexes and $\Sigma : f \to g$ a homotopy. If $t : Q \longrightarrow X$ is another morphism of complexes then the morphisms $\Sigma^n t^n : Q^n \longrightarrow Y^{n-1}$ define a homotopy $\Sigma t : ft \longrightarrow gt$. Similarly if $t : Y \longrightarrow Q$ is a morphism of complexes the morphisms $t^{n-1}\Sigma^n : X^n \longrightarrow Q^{n-1}$ define a homotopy $t\Sigma : tf \longrightarrow tg$.

Definition 8. Let $\mathcal{A}$ be an abelian category $f, g : X \longrightarrow Y$ morphisms of complexes and $\Sigma, \Theta : f \longrightarrow g$ homotopies. A *homotopy of homotopies* (or 2*-homotopy*) $\vartheta : \Sigma \longrightarrow \Theta$ is a collection of morphisms $\vartheta^n : X^n \longrightarrow Y^{n-2}$ with the property that for all $n \in \mathbb{Z}$

$$\Theta^n - \Sigma^n = \partial_Y^{n-2}\vartheta^n - \vartheta^{n+1}\partial_X^n$$

In this case we say that Σ, Θ are *homotopic* and write $\Sigma \simeq \Theta$. Observe that if $\vartheta : \Sigma \longrightarrow \Theta$ is a homotopy then $-\vartheta$ is a homotopy $\Theta \longrightarrow \Sigma$ so there is no ambiguity in saying that Σ, Θ are homotopic. We denote the set of all 2-homotopies $\Sigma \longrightarrow \Theta$ by $Hom(\Sigma, \Theta)$.

Lemma 8. *The* 2*-homotopy relation* $\simeq$ *is an equivalence relation.*

Proof. That is, given an abelian category $\mathcal{A}$ and morphisms $f, g : X \longrightarrow Y$ of complexes the relation of 2-homotopy is an equivalence relation on the set $Hom(f, g)$ of homotopies. It is clearly reflexive and symmetric. For transitivity, suppose that we have 2-homotopies $\vartheta : \Sigma \longrightarrow \Theta$ and $\rho : \Theta \longrightarrow \Lambda$. Then the morphisms $\vartheta^n + \rho^n$ define a 2-homotopy $\Sigma \longrightarrow \Lambda$, as required. □

Theorem 9. *Let* $\mathcal{A}$ *be an abelian category, and suppose we have a diagram of complexes with exact rows (not necessarily commutative)*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{q} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} & & \\ 0 & \longrightarrow & X' & \xrightarrow[u']{} & Y' & \xrightarrow[q']{} & Q' & \longrightarrow & 0 \end{array} \tag{13}$$

Suppose that there exist homotopies $\Sigma : fu \longrightarrow u'g$ and $\Theta : q'f \longrightarrow eq$ with the property that the induced homotopies $q'\Sigma, \Theta u : q'fu \longrightarrow 0$ are homotopic. Then we have a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^n(X) & \longrightarrow & H^n(Y) & \longrightarrow & H^n(Q) & \xrightarrow{\delta} & H^{n+1}(X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^n(X') & \longrightarrow & H^n(Y') & \longrightarrow & H^n(Q') & \xrightarrow[\delta]{} & H^{n+1}(X') & \longrightarrow & \cdots \end{array}$$

Proof. Since $fu \simeq u'g$ and $q'f \simeq eq$ the only square whose commutativity is not obvious is the one involving the connecting morphisms. From the left hand square of complexes and the homotopy Σ we obtain a commutative diagram with exact rows (12), which remains commutative if we replace $\widetilde{v}$ and $\widetilde{v'}$ with $-\widetilde{v}, -\widetilde{v'}$ respectively. We therefore obtain a morphism of the corresponding long exact sequences of cohomology (for the modified short exact sequences). Applying Lemma 6 to both rows of (13) we obtain a commutative diagram

$$\begin{array}{ccccccc} H^n(X) & \longrightarrow & H^n(Y) & \longrightarrow & H^n(Q) & \longrightarrow & H^{n+1}(X) \\ \| & & \Uparrow & & \Uparrow & & \| \\ H^n(X) & \longrightarrow & H^n(\widetilde{C}_u) & \longrightarrow & H^n(C_u) & \longrightarrow & H^{n+1}(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(X') & \longrightarrow & H^n(\widetilde{C}_{u'}) & \longrightarrow & H^n(C_{u'}) & \longrightarrow & H^{n+1}(X') \\ \| & & \Downarrow & & \Downarrow & & \| \\ H^n(X') & \longrightarrow & H^n(Y') & \longrightarrow & H^n(Q') & \longrightarrow & H^{n+1}(X') \end{array}$$

So to complete the proof it suffices to show that the following diagram of complexes commutes up to homotopy

$$\begin{array}{ccc} C_u & \xrightarrow{\kappa} & Q \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle e} \\ C_{u'} & \xrightarrow[\kappa']{} & Q' \end{array}$$

where $\kappa^n = (0 \ q^n), \kappa'^n = (0 \ q'^n)$ and h is as given in Definition 6. Let $\vartheta : \Theta u \longrightarrow q'\Sigma$ be a homotopy of homotopies. For $n \in \mathbb{Z}$ we define a morphism $\Lambda^n = (\vartheta^{n+1} \ \Theta^n) : C_u^n \longrightarrow Q'^{n-1}$. It is not hard to check that for $n \in \mathbb{Z}$ we have

$$(e\kappa)^n - (\kappa' h)^n = \partial_{Q'}^{n-1}\Lambda^n + \Lambda^{n+1}\partial_{C_u}^n$$

Therefore $\Lambda : \kappa' h \longrightarrow e\kappa$ is a homotopy and the proof is complete. □

It may seem like the hypothesis of Theorem 9 are too restrictive to be of any use. In fact, nothing could be further from the truth.

Proposition 10. *Let $\mathcal{A}$ be an abelian category, C, D positive cochain complexes with C acyclic and D injective. If $\varphi, \psi : C \longrightarrow D$ are morphisms of cochain complexes and $\Sigma, \Theta : \psi \longrightarrow \varphi$ are homotopies, there is a homotopy $\vartheta : \Sigma \longrightarrow \Theta$.*

Proof. Let $\varphi, \psi : C \longrightarrow D$ be morphisms of cochain complexes. These morphisms are homotopic if and only if they induce the same morphism on cohomology $H^0(C) \longrightarrow H^0(D)$ (DF,Theorem 19). We claim that (up to homotopy) there is really only one homotopy $\psi \longrightarrow \varphi$.

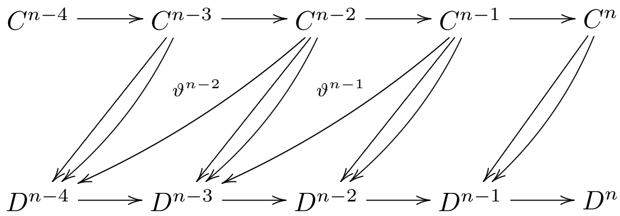
Let $\Sigma, \Theta : \psi \longrightarrow \varphi$ be homotopies. We have to construct morphisms $\vartheta^n : X^n \longrightarrow Y^{n-2}$ with the following property for all $i \in \mathbb{Z}$

$$\Theta^i - \Sigma^i = \partial_D^{i-2}\vartheta^i - \vartheta^{i+1}\partial_C^i \tag{14}$$

If we set $\vartheta^n = 0$ for all $n < 2$, this condition is trivially satisfied for $i < 1$. By assumption we have the following equations for $n \in \mathbb{Z}$

$$\varphi^n - \psi^n = \partial_D^{n-1}\Sigma^n + \Sigma^{n+1}\partial_C^n = \partial_D^{n-1}\Theta^n + \Theta^{n+1}\partial_C^n$$

In particular we have $(\Sigma^1 - \Theta^1)\partial_C^0 = 0$. Since C is acyclic, the morphism $\Sigma^1 - \Theta^1 : C^1 \longrightarrow D^0$ factors through $Im(\partial_C^1)$ and therefore by injectivity of D^0 we can lift this factorisation to a morphism $\vartheta^2 : C^2 \longrightarrow D^0$ with the property that $\vartheta^2\partial_C^1 = \Sigma^1 - \Theta^1$. So we have constructed morphisms ϑ^n for $n < 3$ satisfying (14) for $i < 2$. We proceed by recursion: suppose for some $n > 2$ we have constructed $\vartheta^0, \vartheta^1, \cdots, \vartheta^{n-1}$ satisfying (14) for $i < n-1$. Consider the following diagram



Set $\kappa = \partial_D^{n-3}\vartheta^{n-1} + \Sigma^{n-1} - \Theta^{n-1}$ and observe that

$$\kappa\partial_C^{n-2} = \partial_D^{n-3}\vartheta^{n-1}\partial_C^{n-2} + \Sigma^{n-1}\partial_C^{n-2} - \Theta^{n-1}\partial_C^{n-2} = \partial_D^{n-3}\partial_D^{n-4}\vartheta^{n-2} = 0$$

Therefore since C is acyclic κ factors through $Im(\partial_C^{n-1})$ and by injectivity of D this can be lifted to a morphism $\vartheta^n : C^n \longrightarrow D^{n-2}$ satisfying (14) for $i = n-1$. Proceeding recursively, we have defined the required homotopy $\vartheta : \Sigma \longrightarrow \Theta$. $\square$

3 Derived Categories

The reader is expected to know the contents of our Triangulated Categories notes, up to and including (TRC,Section 2). Throughout this section $\mathcal{A}$ is an abelian category, and all complexes are objects of $\mathbf{C}(\mathcal{A})$. We let Σ denote the additive automorphism $(-)[1] : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$ defined above, and agree that all candidate triangles (TRC,Definition 2) are with respect to Σ. It follows from Proposition 2 that for every morphism of complexes $u : X \longrightarrow Y$ the image of the diagram (1) in $K(\mathcal{A})$ is a candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X \tag{15}$$

We say a candidate triangle in $K(\mathcal{A})$ is *distinguished* if it is isomorphic (as a candidate triangle *in* $K(\mathcal{A})$) to a candidate triangle (15) arising from a morphism of complexes $u : X \longrightarrow Y$ in $\mathbf{C}(\mathcal{A})$. It is sometimes technically convenient to replace the object Y in (15) by the mapping cylinder of u, which is what we accomplish in the next result.

Lemma 11. *For any morphism of complexes $u : X \longrightarrow Y$ the following candidate triangle in $K(\mathcal{A})$ is distinguished*

$$X \xrightarrow{\widetilde{u}} \widetilde{C}_u \xrightarrow{-\widetilde{v}} C_u \xrightarrow{w} \Sigma X \tag{16}$$

and every distinguished triangle in $K(\mathcal{A})$ is isomorphic to a triangle of this form.

Proof. Here the morphisms $\widetilde{u}, \widetilde{v}$ are as given in Definition 4, and $w : C_u \longrightarrow \Sigma X$ is the usual morphism out of the mapping cone. Although trivially $\widetilde{v}\widetilde{u} = 0$ it is not immediately clear that $w\widetilde{v} = 0$. In any case, consider the following diagram in $K(\mathcal{A})$

$$\begin{array}{ccccccc} X & \xrightarrow{\widetilde{u}} & \widetilde{C}_u & \xrightarrow{-\widetilde{v}} & C_u & \xrightarrow{w} & \Sigma X \\ {\scriptstyle 1}\Downarrow & & \downarrow{\scriptstyle \varphi} & & {\scriptstyle 1}\Downarrow & & {\scriptstyle 1}\Downarrow \\ X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & C_u & \xrightarrow[w]{} & \Sigma X \end{array}$$

where $\varphi^n = (0 \;\; -1 \;\; u^n)$, also discussed in Definition 4. We have $\varphi\widetilde{u} = u$ and the matrix $\left(\begin{smallmatrix}0&0&1\\0&0&0\end{smallmatrix}\right)$ gives a homotopy $v\varphi \simeq -\widetilde{v}$. Therefore this diagram commutes in $K(\mathcal{A})$, where it is an isomorphism of candidate triangles since by Lemma 4 the morphism φ is a homotopy equivalence. This shows simultaneously that (16) is a distinguished triangle in $K(\mathcal{A})$, and that every distinguished triangle in $K(\mathcal{A})$ is isomorphic to a triangle of this form. ☐

Theorem 12. *The additive category $K(\mathcal{A})$ together with the additive automorphism Σ and the class of distinguished triangles defined above is a triangulated category.*

Proof. **TR0**. It is clear that any candidate triangle isomorphic to a distinguished triangle is distinguished. If X is a complex, then we have to show that the following candidate triangle is distinguished

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

By Lemma 5 the following diagram is an isomorphism of candidate triangles in $K(\mathcal{A})$

$$\begin{array}{ccccccc} X & \xrightarrow{1} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ X & \xrightarrow[1]{} & X & \xrightarrow[v]{} & C_1 & \xrightarrow[w]{} & \Sigma X \end{array}$$

Since the bottom row is distinguished, so is the top row, which completes the proof of TR0.

TR1. Since any morphism in $K(\mathcal{A})$ can be lifted to $\mathbf{C}(\mathcal{A})$, where it has a mapping cone, it is clear that any morphism in $K(\mathcal{A})$ can be extended to a distinguished triangle.

TR2. We have to show that the twists of a distinguished triangle are distinguished. By Lemma 11 it suffices to prove this for distinguished triangles in $K(\mathcal{A})$ of the following form

$$X \xrightarrow{\widetilde{u}} \widetilde{C}_u \xrightarrow{-\widetilde{v}} C_u \xrightarrow{w} \Sigma X$$

That is, we have to show that the following candidate triangles in $K(\mathcal{A})$ are distinguished

$$\widetilde{C}_u \xrightarrow{\widetilde{v}} C_u \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma\widetilde{u}} \Sigma\widetilde{C}_u$$

$$\Sigma^{-1}C_u \xrightarrow{-\Sigma^{-1}w} X \xrightarrow{-\widetilde{u}} \widetilde{C}_u \xrightarrow{\widetilde{v}} C_u$$

By definition $\widetilde{C}_u$ is the mapping cone of $\Sigma^{-1}w$, so it is easy to check that the second candidate triangle is distinguished. To show that the first candidate triangle is distinguished, we show that it is isomorphic in $K(\mathcal{A})$ to the canonical triangle induced by $\widetilde{v}$. That is, we will give a commutative diagram with vertical isomorphisms in $K(\mathcal{A})$

$$\begin{array}{ccccccc} \widetilde{C}_u & \xrightarrow{\widetilde{v}} & C_u & \xrightarrow{-w} & \Sigma X & \xrightarrow{-\Sigma\widetilde{u}} & \Sigma\widetilde{C}_u \\ {\scriptstyle 1}\Downarrow & & {\scriptstyle 1}\Downarrow & & {\scriptstyle -\Delta}\Downarrow & & {\scriptstyle 1}\Downarrow \\ \widetilde{C}_u & \xrightarrow[\widetilde{v}]{} & C_u & \xrightarrow[x]{} & C_{\widetilde{v}} & \xrightarrow[y]{} & \Sigma\widetilde{C}_u \end{array} \tag{17}$$

We define the morphism $\Delta : \Sigma X \longrightarrow C_{\widetilde{v}}$ as follows

$$\Delta^n = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} : X^{n+1} \longrightarrow X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus X^{n+1} \oplus Y^n$$

One checks that this is a morphism of complexes. Similarly we define a morphism of complexes $\square : C_{\widetilde{v}} \longrightarrow \Sigma X$ by $\square^n = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \end{pmatrix}$. Trivially $\square\Delta = 1$ and if we define

$$H^n = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : C^n_{\widetilde{v}} \longrightarrow C^{n-1}_{\widetilde{v}}$$

then H is a homotopy $\Delta\square \simeq 1$. Therefore in $K(\mathcal{A})$ the morphism Δ is an isomorphism with inverse $\square$. It is not difficult to check that $y\Delta = \Sigma\widetilde{u}$ and $\square x = w$, which shows that (17) is an isomorphism of candidate triangles in $K(\mathcal{A})$, as claimed.

TR4'. Suppose we have a diagram in $\mathbf{C}(\mathcal{A})$

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[u']{} & Y' \end{array}$$

which commutes up to some homotopy $\Phi : gu \longrightarrow u'f$. Then by Definition 6 we have a canonical morphism of complexes $h : C_u \longrightarrow C_{u'}$ making the following diagram *commute*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & C_u & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & C_{u'} & \xrightarrow[w']{} & \Sigma X' \end{array}$$

$$h^n = \begin{pmatrix} f^{n+1} & 0 \\ -\Phi^{n+1} & g^n \end{pmatrix}$$

The axiom TR3 is an immediate consequence, so $K(\mathcal{A})$ is a pretriangulated category. To show that it is triangulated, it would be enough to show that the image of the following sequence in $K(\mathcal{A})$ is a distinguished triangle

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} C_u \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus C_{u'} \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma(Y \oplus X') \tag{18}$$

The first thing we observe is that

$$\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & v'u' \end{pmatrix} \simeq 0 \tag{19}$$

since $v'u' \simeq 0$ by the canonical homotopy $\Theta^n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If we write $\alpha : Y \oplus X' \longrightarrow C_u \oplus Y'$ for the first morphism in (18) then the specific homotopy (19) induces by Proposition 2 a morphism of complexes $\tau : C_\alpha \longrightarrow \Sigma X \oplus C_{u'}$ making the middle square in the following diagram commute

$$\begin{array}{ccccccc} Y \oplus X' & \xrightarrow{\alpha} & C_u \oplus Y' & \longrightarrow & C_\alpha & \longrightarrow & \Sigma(Y \oplus X') \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle 1} \\ Y \oplus X' & \xrightarrow[\alpha]{} & C_u \oplus Y' & \longrightarrow & \Sigma X \oplus C_{u'} & \longrightarrow & \Sigma(Y \oplus X') \end{array}$$

$$\tau^n = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & f^{n+1} & 0 & 0 \\ 0 & 0 & -\Phi^{n+1} & g^n & 1 \end{pmatrix}$$

One also checks that the right hand square commutes, but only up to homotopy. Since the top row of this diagram is a distinguished triangle in $K(\mathcal{A})$, to show that (18) is distinguished in $K(\mathcal{A})$ it only remains to show that τ is a homotopy equivalence. We define the inverse $\sigma : \Sigma X \oplus C_{u'} \longrightarrow C_\alpha$ on components:

(i) Consider the morphism $\left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right) : X \longrightarrow Y \oplus X'$. We have a canonical homotopy

$$\alpha \begin{pmatrix} u \\ -f \end{pmatrix} = \begin{pmatrix} -vu \\ gu - u'f \end{pmatrix} \simeq 0$$

and therefore by Proposition 2 a canonical factorisation $\sigma_1' : X \longrightarrow \Sigma^{-1} C_\alpha$. Applying Σ and alternating the sign gives our first component $\sigma_1 = -\Sigma\sigma_1' : \Sigma X \longrightarrow C_\alpha$.

(ii) It is easy to see that the composite of $u' : X' \longrightarrow Y'$ with the injection $Y' \longrightarrow C_\alpha$ is canonically homotopic to zero. The canonical factorisation gives our second component $\sigma_2 : C_{u'} \longrightarrow C_\alpha$.

Putting these morphisms together we have a morphism of complexes $\sigma : \Sigma X \oplus C_{u'} \longrightarrow C_\alpha$ defined in terms of matrices by

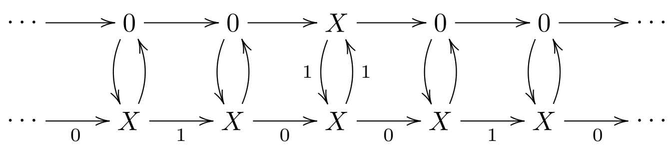
$$\sigma^n = \begin{pmatrix} -u^{n+1} & 0 & 0 \\ f^{n+1} & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ -\Phi^{n+1} & 0 & 1 \end{pmatrix}$$

One checks that $\tau\sigma = 1$, and the following matrix defines a homotopy $1 \simeq \sigma\tau$

$$\Psi^n = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : C_\alpha^n \longrightarrow C_\alpha^{n-1}$$

This finishes the proof that τ is a homotopy equivalence, which implies that (18) is a distinguished triangle in $K(\mathcal{A})$. Using these facts, it is now straightforward to check that TR4' holds for $K(\mathcal{A})$, which is consequently a triangulated category. □

Remark 7. Our convention is that triangulated subcategories must be replete (TRC,Definition 16). So in general the full additive subcategories $K^+(\mathcal{A}), K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ of $K(\mathcal{A})$ are *not* triangulated subcategories. Here is a counter-example: let X be any nonzero object of $\mathcal{A}$, and consider the following complexes in $\mathbf{C}(\mathcal{A})$



Here the top row is concentrated in degree zero. In the bottom row all the objects are X, and away from degree zero the differentials alternative between the identity and zero. We have two canonical morphisms of complexes $f : T \longrightarrow B, g : B \longrightarrow T$, where T denotes the top row and B the bottom. Clearly $gf = 1$ and $\Sigma^n = 1$ defines a homotopy $fg \simeq 1$. Therefore in $K(\mathcal{A})$ we have an isomorphism $T \cong B$, where T is bounded and B is not. For the definition of the bounded derived categories, see Section 3.3.

Definition 9. For any $n \in \mathbb{Z}$ the additive cohomology functor $H^n : \mathbf{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ induces an additive functor $K(\mathcal{A}) \longrightarrow \mathcal{A}$ which we also denote by H^n. If we set $H = H^0$ then it is easy to check that $H^n = H \circ \Sigma^n$ for any $n \in \mathbb{Z}$.

Proposition 13. *For any $n \in \mathbb{Z}$ the additive functor $H^n : K(\mathcal{A}) \longrightarrow \mathcal{A}$ is homological. For any distinguished triangle in $K(\mathcal{A})$*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

we have a long exact sequence in $\mathcal{A}$

$$\cdots \longrightarrow H^{n-1}(Z) \xrightarrow{H^{n-1}(w)} H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) \longrightarrow \cdots \tag{20}$$

Proof. Since $H^n = H \circ \Sigma^n$ it follows from (TRC,Remark 7) that we need only show that H is homological. By Lemma 11 we may assume our triangle is of the form

$$X \xrightarrow{\widetilde{u}} \widetilde{C}_u \xrightarrow{-\widetilde{v}} C_u \xrightarrow{w} \Sigma X$$

for some morphism of complexes $u : X \longrightarrow Y$. But then we have an exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{\widetilde{u}} \widetilde{C}_u \xrightarrow{\widetilde{v}} C_u \longrightarrow 0$$

and the corresponding long exact cohomology sequence includes $H(X) \longrightarrow H(\widetilde{C}_u) \longrightarrow H(C_u)$, which is therefore exact. This shows that H is a homological functor. ☐

Definition 10. A morphism of complexes $u : X \longrightarrow Y$ is a *quasi-isomorphism* if the morphism $H^n(u) : H^n(X) \longrightarrow H^n(Y)$ is an isomorphism in $\mathcal{A}$ for every $n \in \mathbb{Z}$. Since this property is stable under homotopy equivalence, it makes sense to say that a morphism in $K(\mathcal{A})$ is a quasi-isomorphism. If X is an exact complex, then the zero morphisms $0 \longrightarrow X, X \longrightarrow 0$ are clearly quasi-isomorphisms.

Corollary 14. *A morphism of complexes $u : X \longrightarrow Y$ is a quasi-isomorphism if and only if the mapping cone C_u is exact.*

Proof. By definition the following sequence is a triangle in $K(\mathcal{A})$

$$X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X$$

so by Proposition 13 we have a long exact cohomology sequence

$$\cdots \longrightarrow H^{n-1}(C_u) \longrightarrow H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(C_u) \longrightarrow H^{n+1}(X) \longrightarrow \cdots$$

It follows that $H^n(C_u) = 0$ for every $n \in \mathbb{Z}$ if and only if $H^n(u)$ is an isomorphism for every $n \in \mathbb{Z}$, which is what we wanted to show. ☐

Corollary 15. *The exact complexes form a thick triangulated subcategory $\mathcal{Z}$ of $K(\mathcal{A})$. The corresponding class of morphisms $Mor_{\mathcal{Z}}$ is the class of all quasi-isomorphisms in $K(\mathcal{A})$.*

Proof. Let $\mathcal{Z}$ be the full subcategory of $K(\mathcal{A})$ consisting of the exact complexes. This class of objects is closed under isomorphisms and finite coproducts in $K(\mathcal{A})$, so it is a replete additive category. It is also closed under the functor Σ and its inverse. If we have a distinguished triangle in $K(\mathcal{A})$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

with $X, Y \in \mathcal{Z}$, then we deduce from the long exact cohomology sequence (20) that $H^n(Z) = 0$ for every $n \in \mathbb{Z}$. That is, Z is exact. This shows that $\mathcal{Z}$ is a triangulated subcategory of $K(\mathcal{A})$, and thickness is straightforward to check. The second claim follows from Corollary 14. ☐

Remark 8. In a moment we will be forced to pay attention to some annoying set-theoretic details, in the guise of "portly categories". The careful reader will want to consult (TRC,Remark 37) and (FCT,Section 4) for the relevant background. In our Triangulated Categories notes we are very careful to write "portly subcategory" and "portly triangulated subcategory" throughout, but here we work under the convention that whenever we write "subcategory" we really mean "portly subcategory". Of course if the ambient portly category is actually a category, "subcategory" and "portly subcategory" are the same thing.

Definition 11. Let $\mathcal{A}$ be an abelian category. The *derived category* of $\mathcal{A}$ is the verdier quotient $K(\mathcal{A})/\mathcal{Z}$, which is a portly triangulated category. We use the notation $\mathfrak{D}(\mathcal{A})$ for this portly triangulated category. There is a canonical triangulated functor $F : K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$.

Lemma 16. *For any complex X we have $F(X) = 0$ if and only if X is exact. If $u : X \longrightarrow Y$ is a morphism of complexes then $F(u)$ is an isomorphism if and only if u is a quasi-isomorphism.*

Proof. By (TRC,Theorem 68) the kernel of F is precisely the thick closure of $\mathcal{Z}$, which we have already shown is $\mathcal{Z}$ itself. The second claim follows from (TRC,Proposition 64) and thickness of $\mathcal{Z}$. □

Proposition 17. *Let $\mathcal{A}$ be an abelian category with derived category $\mathfrak{D}(\mathcal{A})$. If $G : K(\mathcal{A}) \longrightarrow \mathcal{S}$ is a triangulated functor into a portly triangulated category which sends quasi-isomorphisms to isomorphisms, there is a* unique *triangulated functor $H : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{S}$ making the following diagram commute*

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{F} & \mathfrak{D}(\mathcal{A}) \\ & {\scriptstyle G}\searrow & \downarrow{\scriptstyle H} \\ & & \mathcal{S} \end{array}$$

Proof. Let $G : K(\mathcal{A}) \longrightarrow \mathcal{S}$ be such a triangulated functor. If X is an exact complex then $0 \longrightarrow X$ is a quasi-isomorphism in $K(\mathcal{A})$, which is sent to an isomorphism in $\mathcal{S}$. Therefore $\mathcal{Z} \subseteq Ker(G)$. By (TRC,Theorem 68) there is a *unique* triangulated functor $H : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{S}$ making the above diagram commute, so the proof is complete. □

Definition 12. For each $n \in \mathbb{Z}$ the homological functor $H^n : K(\mathcal{A}) \longrightarrow \mathcal{A}$ sends quasi-isomorphisms to isomorphisms, and therefore by (TRC,Proposition 54), (TRC,Remark 43) there is a unique functor $H^n : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{A}$ making the following diagram commute

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{F} & \mathfrak{D}(\mathcal{A}) \\ & {\scriptstyle H^n}\searrow & \downarrow{\scriptstyle H^n} \\ & & \mathcal{A} \end{array}$$

The functor $H^n : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{A}$ is easily checked to be homological.

Lemma 18. *A morphism $\gamma : X \longrightarrow Y$ in $\mathfrak{D}(\mathcal{A})$ is an isomorphism if and only if $H^n(\gamma)$ is an isomorphism in $\mathcal{A}$ for every $n \in \mathbb{Z}$.*

Proof. The condition is clearly necessary. Now suppose that $H^n(\gamma)$ is an isomorphism for every $n \in \mathbb{Z}$. By (TRC,Remark 41) we can write $\gamma = F(g)F(f)^{-1}$ for some morphisms of complexes fitting into a diagram of the form

$$\begin{array}{ccccc} & & W & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} & \\ X & & & & Y \end{array}$$

Since $H^n(\gamma) = H^n(g)H^n(f)^{-1}$ we infer that $H^n(g)$ is an isomorphism for every $n \in \mathbb{Z}$. That is, g is a quasi-isomorphism. By Lemma 16 the image $F(g)$ of g in $\mathfrak{D}(\mathcal{A})$ is an isomorphism, and hence so is $\gamma = F(g)F(f)^{-1}$. □

Lemma 19. *Let $f, g : X \longrightarrow Y$ be morphisms of complexes. Then the following statements are equivalent*

(i) $F(f) = F(g)$.

(ii) There exists a quasi-isomorphism $\alpha : W \longrightarrow X$ such that $f\alpha, g\alpha$ are homotopic.

(iii) In $K(\mathcal{A})$ the morphism $f - g$ factors through an exact complex.

In particular $F(f) = 0$ if and only if $f\alpha \simeq 0$ for some quasi-isomorphism α.

Proof. This is a special case of (TRC,Lemma 55). $\square$

Remark 9. In fact for two morphisms $f, g : X \longrightarrow Y$ of complexes the following implications are all strict (see the example on p.39 of RD)

$$f = g \Rightarrow f \simeq g \Rightarrow F(f) = F(g) \Rightarrow H^n(f) = H^n(g) \text{ for all } n \in \mathbb{Z}$$

For actually proving statements about the derived category, triangles involving the mapping cone and cylinder of a morphism are very convenient. But intuitively the more natural triangles are those arising from short exact sequences of complexes. We are very familiar with the fact that such a short exact sequence gives rise to "connecting morphisms" on cohomology (DF,Theorem 29). The formalism of the derived category allows us to lift these connecting morphisms on cohomology to an *actual morphism* in the derived category.

Proposition 20. *Suppose we are given a short exact sequence of complexes*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{q} Q \longrightarrow 0$$

Then there is a canonical morphism $z : Q \longrightarrow \Sigma X$ in $\mathfrak{D}(\mathcal{A})$, called the connecting morphism, *making the following diagram into a triangle*

$$X \xrightarrow{u} Y \xrightarrow{q} Q \xrightarrow{-z} \Sigma X \tag{21}$$

Proof. Combining Lemma 11 and Lemma 6 we have a commutative diagram in $\mathfrak{D}(\mathcal{A})$ with the first row a triangle

$$\begin{array}{ccccccc} X & \xrightarrow{\widetilde{u}} & \widetilde{C}_u & \xrightarrow{-\widetilde{v}} & C_u & \xrightarrow{w} & \Sigma X \\ {\scriptstyle 1}\Downarrow & & {\scriptstyle \varphi}\Downarrow & & {\scriptstyle f}\Downarrow & & \\ X & \xrightarrow[u]{} & Y & \xrightarrow[q]{} & Q & & \end{array}$$

Let $F : K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ be the quotient and define z to be the morphism $-F(w)F(f)^{-1} : Q \longrightarrow \Sigma X$ of $\mathfrak{D}(\mathcal{A})$ then it is clear that (21) is a triangle in $\mathfrak{D}(\mathcal{A})$ (in the notation of (TRC,Section 2) we have $z = -[f, w]$). The reason for the sign will become apparent in a moment. $\square$

Lemma 21. *The connecting morphism is natural in the exact sequence. That is, suppose we are given a commutative diagram of complexes with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{q} & Q & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \\ 0 & \longrightarrow & X' & \xrightarrow[u']{} & Y' & \xrightarrow[q']{} & Q' & \longrightarrow & 0 \end{array} \tag{22}$$

Then the following diagram commutes in $\mathfrak{D}(\mathcal{A})$

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{q} & Q & \xrightarrow{-z} & \Sigma X \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & {\scriptstyle \Sigma\alpha}\downarrow \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[q']{} & Q' & \xrightarrow[-z']{} & \Sigma X' \end{array}$$

where z, z' are the canonical connecting morphisms.

Proof. Let $f : C_u \longrightarrow Q$ and $f' : C_{u'} \longrightarrow Q'$ be the morphisms involved in the definition of the connecting morphisms. That is, $f^n = (0\ q^n)$ and $(f')^n = (0\ (q')^n)$. Let $w : C_u \longrightarrow \Sigma X$ and $w' : C_{u'} \longrightarrow \Sigma X'$ be the canonical morphisms out of the mapping cone, so that $z = -F(w)F(f)^{-1}$ and $z' = -F(w')F(f')^{-1}$. The left hand commutative square in (22) induces a morphism of complexes $h : C_u \longrightarrow C_{u'}$ as defined in (9) above. This makes the following diagram of complexes commute

$$\begin{array}{ccccc} Q & \xleftarrow{f} & C_u & \xrightarrow{w} & \Sigma X \\ {\scriptstyle\gamma}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle\Sigma\alpha} \\ Q' & \xleftarrow[f']{} & C_{u'} & \xrightarrow[w']{} & \Sigma X' \end{array}$$

from which we deduce that $\Sigma\alpha \circ (-z) = (-z') \circ \gamma$ in $\mathfrak{D}(\mathcal{A})$. The commutativity of the rest of the diagram is trivial, so this completes the proof. □

Lemma 22. *Suppose we are given a short exact sequence of complexes*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{q} Q \longrightarrow 0 \tag{23}$$

Let $z : Q \longrightarrow \Sigma X$ be the canonical connecting morphism in $\mathfrak{D}(\mathcal{A})$ of Proposition 20 and let $\delta^n : H^n(Q) \longrightarrow H^{n+1}(X)$ be the canonical connecting morphism of (23) in the sense of (DF, Theorem 29). Then $H^n(z) = \delta^n$.

Proof. By definition $z = -[f, w]$ and so from commutativity of the diagram (5) of Lemma 6 we deduce that it is enough to show that for the exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{\widetilde{u}} \widetilde{C}_u \xrightarrow{\widetilde{v}} C_u \longrightarrow 0$$

the canonical connecting morphism $H^n(C_u) \longrightarrow H^{n+1}(X)$ of (DF,Theorem 29) is equal to $H^n(w)$ where $w : C_u \longrightarrow \Sigma X$ is the morphism of complexes given as part of the definition of the mapping cone. The connecting morphism $H^n(C_u) \longrightarrow H^{n+1}(X)$ is defined by diagram chasing using the following diagram of (DF,Theorem 26)

$$\begin{array}{ccccccc} & H^n(X) & \longrightarrow & H^n(\widetilde{C}_u) & \longrightarrow & H^n(C_u) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & Coker\partial^{n-1} & \longrightarrow & Coker\partial^{n-1} & \longrightarrow & Coker\partial^{n-1} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & Ker\partial^{n+1} & \longrightarrow & Ker\partial^{n+1} & \longrightarrow & Ker\partial^{n+1} & \\ & \downarrow & & \downarrow & & \downarrow & \\ & H^{n+1}(X) & \longrightarrow & H^{n+1}(\widetilde{C}_u) & \longrightarrow & H^{n+1}(C_u) & \end{array} \tag{24}$$

The uniqueness part of (DCAC,Theorem 13) and the proof of (DCAC,Lemma 14) mean that to show $H^n(w)$ is equal to the connecting morphism, we have to show that for some small, full, abelian subcategory $\mathcal{C}$ of $\mathcal{A}$ containing (24) and some exact embedding $T : \mathcal{C} \longrightarrow \mathbf{Ab}$ the morphism of abelian groups $TH^n(w)$ is the canonical connecting morphism of the image under T of the diagram (24). Since we know explicitly the objects and differentials of the complexes $\widetilde{C}_u, C_u$ this is technical but not difficult to check. □

Remark 10. The connecting morphism in $\mathfrak{D}(\mathcal{A})$ reduces to the usual connecting morphisms when you apply cohomology, but amazingly enough the connecting morphism in the derived category contains *even more* information. For example, given $X \in \mathcal{A}$ let $c(X)$ denote the complex concentrated in degree zero with $c(X)^0 = X$. Suppose we have a short exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and therefore also a short exact sequence $0 \longrightarrow c(A) \longrightarrow c(B) \longrightarrow c(C) \longrightarrow 0$. By the above there is a canonical connecting morphism $z : c(C) \longrightarrow \Sigma c(A)$ in $\mathfrak{D}(\mathcal{A})$. It is clear that $H^n(z) = 0$ for every $n \in \mathbb{Z}$ so one might expect that $z = 0$. But in fact the morphism z is very useful. See for example (DTC2,Remark 5).

Remark 11. Let k be a commutative ring and $\mathcal{A}$ a k-linear abelian category (AC,Definition 35). The abelian category $\mathbf{C}(\mathcal{A})$ is also k-linear, with action $(r \cdot \psi)^i = r \cdot \psi^i$. With the same action $K(\mathcal{A})$ is a k-linear triangulated category in the sense of (TRC,Definition 32). Therefore the verdier quotient $\mathfrak{D}(\mathcal{A}) = K(\mathcal{A})/\mathcal{Z}$ is canonically k-linear and the functor $K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ is k-linear. For any $n \in \mathbb{Z}$ the cohomology functors $H^n(-) : K(\mathcal{A}) \longrightarrow \mathcal{A}$ and $H^n(-) : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{A}$ are k-linear.

3.1 Extending Functors

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. This induces a canonical additive functor $\mathbf{C}(F) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$ (which we sometimes also denote by F) with the property that if $\varphi, \psi : C \longrightarrow D$ are homotopic morphisms of complexes, then so are $\mathbf{C}(F)(\varphi), \mathbf{C}(F)(\psi)$ (DF,Lemma 7). That is, there is a unique additive functor $K(F)$ making the following diagram commute

$$\begin{array}{ccc} \mathbf{C}(\mathcal{A}) & \xrightarrow{\mathbf{C}(F)} & \mathbf{C}(\mathcal{B}) \\ \downarrow & & \downarrow \\ K(\mathcal{A}) & \xrightarrow[K(F)]{} & K(\mathcal{B}) \end{array}$$

There is an equality of functors $\Sigma K(F) = K(F)\Sigma$ and it is easy to see that given a morphism $u : X \longrightarrow Y$ of complexes in $\mathcal{A}$ there is a canonical isomorphism of complexes $C_{F(u)} \longrightarrow F(C_u)$ in $\mathcal{B}$ making the following diagram commute

$$\begin{array}{ccccccc} F(X) & \xrightarrow{F(u)} & F(Y) & \longrightarrow & C_{F(u)} & \longrightarrow & \Sigma F(X) \\ {\scriptstyle 1}\Downarrow & & {\scriptstyle 1}\Downarrow & & \Downarrow & & {\scriptstyle 1}\Downarrow \\ F(X) & \xrightarrow[F(u)]{} & F(Y) & \longrightarrow & F(C_u) & \longrightarrow & F(\Sigma X) \end{array}$$

Therefore $K(F)$ is a triangulated functor $K(\mathcal{A}) \longrightarrow K(\mathcal{B})$. It is clear that $K(1) = 1$ and $K(GF) = K(G)K(F)$ for another additive functor $G : \mathcal{B} \longrightarrow \mathcal{C}$ between abelian categories. In particular isomorphic abelian categories have isomorphic homotopy categories.

Lemma 23. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor between abelian categories. There is a unique triangulated functor $\mathfrak{D}(F) : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ making the following diagram commute*

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}) \\ \downarrow & & \downarrow \\ \mathfrak{D}(\mathcal{A}) & \xrightarrow[\mathfrak{D}(F)]{} & \mathfrak{D}(\mathcal{B}) \end{array}$$

Proof. We need only observe that the composite $K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow \mathfrak{D}(\mathcal{B})$ sends exact complexes to zero, which is trivial. □

Remark 12. It is clear that if $F : \mathcal{A} \longrightarrow \mathcal{B}, G : \mathcal{B} \longrightarrow \mathcal{C}$ are exact functors between abelian categories then $\mathfrak{D}(GF) = \mathfrak{D}(G)\mathfrak{D}(F)$. Clearly $\mathfrak{D}(1) = 1$. It follows that isomorphic abelian categories have isomorphic derived categories.

Remark 13. Let $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ be additive functors between abelian categories, and let $\theta : F \longrightarrow G$ be a natural transformation. Then there are corresponding natural transformations

$$\mathbf{C}(\theta) : \mathbf{C}(F) \longrightarrow \mathbf{C}(G) \quad \mathbf{C}(\theta)^i_X = \theta_{X^i}$$
$$K(\theta) : K(F) \longrightarrow K(G) \quad K(\theta)_X = [\mathbf{C}(\theta)_X]$$

where $K(\theta)$ is actually a trinatural transformation. If F, G are exact and $Q : K(\mathcal{B}) \longrightarrow \mathfrak{D}(\mathcal{B})$ canonical, then we have a trinatural transformation

$$\mathfrak{D}(\theta) : \mathfrak{D}(F) \longrightarrow \mathfrak{D}(G) \quad \mathfrak{D}(\theta)_X = Q(K(\theta)_X)$$

In particular if F, G are naturally equivalent then $\mathbf{C}(F), \mathbf{C}(G)$ are naturally equivalent and $K(F), K(G)$ are trinaturally equivalent. If F, G are in addition exact, then $\mathfrak{D}(F), \mathfrak{D}(G)$ are trinaturally equivalent. An immediate consequence of this is that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an equivalence of abelian categories, then $\mathbf{C}(F)$ is an equivalence and $K(F), \mathfrak{D}(F)$ are triequivalences.

Lemma 24. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and suppose we have additive functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ with G left adjoint to F. Then $\mathbf{C}(G)$ is left adjoint to $\mathbf{C}(F)$.*

Proof. Let G be left adjoint to F with unit $\eta : 1 \longrightarrow FG$ and counit $\varepsilon : GF \longrightarrow 1$. The natural transformations $\mathbf{C}(\eta) : 1 \longrightarrow \mathbf{C}(F)\mathbf{C}(G)$ and $\mathbf{C}(\varepsilon) : \mathbf{C}(G)\mathbf{C}(F) \longrightarrow 1$ are easily checked to be the unit and counit of an adjunction $\mathbf{C}(G) \dashv \mathbf{C}(F)$, which completes the proof. □

Lemma 25. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and suppose we have additive functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ with G left adjoint to F. Then $K(G)$ is left triadjoint to $K(F)$.*

Proof. See (TRC,Section 2.1) for the definition of *triadjoints*. Let G be left adjoint to F with unit $\eta : 1 \longrightarrow FG$ and counit $\varepsilon : GF \longrightarrow 1$ and suppose we have two morphisms of complexes $u, v : X \longrightarrow FY$ and a homotopy $\Sigma : v \longrightarrow u$. For each $n \in \mathbb{Z}$ the morphism $\Sigma^n : X^n \longrightarrow FY^{n-1}$ corresponds via the adjunction to a morphism $\Lambda^n : GX^n \longrightarrow Y^{n-1}$. If we let $\hat{u}, \hat{v} : GX \longrightarrow Y$ denote the morphisms of complexes associated to u, v by the adjunction, then one checks that $\Lambda : \hat{v} \longrightarrow \hat{u}$ is a homotopy. Conversely if $\hat{u} \simeq \hat{v}$ then $u \simeq v$, so the bijection

$$Hom_{\mathbf{C}(\mathcal{B})}(X, FY) \cong Hom_{\mathbf{C}(\mathcal{A})}(GX, Y)$$

defined by the adjunction $\mathbf{C}(G) \dashv \mathbf{C}(F)$ induces a bijection

$$Hom_{K(\mathcal{B})}(X, FY) \cong Hom_{K(\mathcal{A})}(GX, Y)$$

which is clearly natural in both variables. This gives an adjunction $K(G) \dashv K(F)$ whose unit is the trinatural transformation $K(\eta) : 1 \longrightarrow K(F)K(G)$ and whose unit is $K(\varepsilon) : K(G)K(F) \longrightarrow 1$. It is therefore a triadjunction, so the proof is complete. □

3.2 Truncations and Hearts

In this section we take a preliminary look at ways of embedding the abelian category $\mathcal{A}$ into its derived category. Throughout this section $\mathcal{A}$ is an abelian category.

Definition 13. A complex X is *concentrated in degree* n for some $n \in \mathbb{Z}$ if we have $X^i = 0$ for $i \neq n$ (note that we do not require $X^n \neq 0$). We say X has *cohomology concentrated in degree* n if $H^i(X) = 0$ for $i \neq n$. We say X is *cohomologically bounded above* if there exists $n \in \mathbb{Z}$ such that $H^i(X) = 0$ for $i > n$ and similarly define *cohomologically bounded below* and *cohomologically bounded*.

Definition 14 (Truncation from above). Let W be a complex and $n \in \mathbb{Z}$. We define $W_{\leq n}$ to be the complex with $W^i_{\leq n} = W^i$ for $i < n$, $W^n_{\leq n} = Ker\partial^n_W$ and $W^i_{\leq n} = 0$ for $i > n$. Graphically,

$$\cdots \longrightarrow W^{n-2} \xrightarrow{\partial_W^{n-2}} W^{n-1} \longrightarrow Ker\partial^n_W \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

There is an obvious monomorphism $v : W_{\leq n} \longrightarrow W$ and $H^i(v)$ is an isomorphism for $i \leq n$. As subobjects of W it is clear that $W_{\leq n} \leq W_{\leq(n+1)}$ and in fact the inclusions $\{W_{\leq n} \longrightarrow W\}_{n\geq 0}$ are a direct limit in $\mathbf{C}(\mathcal{A})$. That is, $W = \varinjlim_{n\geq 0} W_{\leq n}$. It is also clear that the complex $W_{\leq n}$ is functorial in W, and that v is natural. The morphism v has a universal property: for any complex X with $X^i = 0$ for $i > n$ composition with v defines isomorphisms

$$Hom_{\mathbf{C}(\mathcal{A})}(X, W_{\leq n}) \longrightarrow Hom_{\mathbf{C}(\mathcal{A})}(X, W)$$
$$Hom_{K(\mathcal{A})}(X, W_{\leq n}) \longrightarrow Hom_{K(\mathcal{A})}(X, W)$$

In other words, a morphism from X to W factors uniquely through $W_{\leq n}$ in both $\mathbf{C}(\mathcal{A})$ and $K(\mathcal{A})$.

Definition 15 (Truncation from below). Let W be a complex and $n \in \mathbb{Z}$. We define $W_{\geq n}$ to be the complex with $W^i_{\geq n} = W^i$ for $i > n$, $W^n_{\geq n} = Coker\partial^{n-1}_W$ and $W^i_{\geq n} = 0$ for $i < n$. Graphically

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Coker\partial_W^{n-1} \longrightarrow W^{n+1} \xrightarrow{\partial_W^{n+1}} W^{n+2} \longrightarrow \cdots$$

There is an obvious epimorphism $q : W \longrightarrow W_{\geq n}$ and $H^i(q)$ is an isomorphism for $i \geq n$. There is a canonical epimorphism $W_{\geq n} \longrightarrow W_{\geq(n+1)}$ and in fact the quotients $\{W \longrightarrow W_{\geq n}\}_{n\leq 0}$ are an inverse limit in $\mathbf{C}(\mathcal{A})$. That is, $W = \varprojlim_{n\leq 0} W_{\geq n}$. It is clear that the complex $W_{\geq n}$ is functorial in W, and that q is natural. The morphism q has a universal property: for any complex X with $X^i = 0$ for $i < n$ composition with q defines isomorphisms

$$Hom_{\mathbf{C}(\mathcal{A})}(W_{\geq n}, X) \longrightarrow Hom_{\mathbf{C}(\mathcal{A})}(W, X)$$
$$Hom_{K(\mathcal{A})}(W_{\geq n}, X) \longrightarrow Hom_{K(\mathcal{A})}(W, X)$$

In other words, a morphism from W to X factors uniquely through $W_{\geq n}$ in both $\mathbf{C}(\mathcal{A})$ and $K(\mathcal{A})$.

This allows us to write any complex as a direct limit of bounded above complexes, or alternatively as an inverse limit of bounded below complexes. Although the truncations given above behave well on cohomology, they have the disadvantage that not all of their objects occur in the original complex. Next we introduce the *brutal truncations* which ruin the cohomology, but are made up of objects from the original complex.

Definition 16 (Brutal truncation from above). Let W be a complex and $n \in \mathbb{Z}$. We define ${}_bW_{\leq n}$ to be the complex with ${}_bW^i_{\leq n} = W^i$ for $i \leq n$ and ${}_bW^i_{\leq n} = 0$ for $i > n$ with the obvious differentials. Graphically,

$$\cdots \longrightarrow W^{n-1} \xrightarrow{\partial_W^{n-1}} W^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

There is an obvious epimorphism of complexes $W \longrightarrow {}_bW_{\leq n}$ and ${}_bW_{\leq(n+1)} \longrightarrow {}_bW_{\leq n}$ and in fact the quotients $\{W \longrightarrow {}_bW_{\leq n}\}_{n\leq 0}$ are an inverse limit in $\mathbf{C}(\mathcal{A})$. That is, $W = \varprojlim_{n\geq 0} {}_bW_{\geq n}$.

Definition 17 (Brutal truncation from below). Let W be a complex and $n \in \mathbb{Z}$. We define ${}_bW_{\geq n}$ to be the complex with ${}_bW^i_{\geq n} = W^i$ for $i \geq n$ and ${}_bW^i_{\geq n} = 0$ for $i < n$ with the obvious differentials. Graphically,

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow W^n \xrightarrow{\partial_W^n} W^{n+1} \longrightarrow \cdots$$

There is an obvious monomorphism of complexes ${}_bW_{\geq n} \longrightarrow W$. As subobjects of W it is clear that ${}_bW_{\geq n} \leq {}_bW_{\geq(n-1)}$ and in fact the inclusions $\{{}_bW_{\geq n} \longrightarrow W\}_{n\leq 0}$ are a direct limit in $\mathbf{C}(\mathcal{A})$. That is, $W = \varinjlim_{n\leq 0} {}_bW_{\geq n}$.

Together these four truncations allow us to write any complex as a direct or inverse limit of bounded above complexes, and also as a direct or inverse limit of bounded below complexes. If we truncate brutally below and normally above, we can write any complex as a direct limit of bounded complexes.

Definition 18 (Double truncation). Let W be a complex and $m, n \in \mathbb{Z}$ with $n < m$. We define $W_{[n,m]}$ to be the following complex

$$\cdots \longrightarrow 0 \longrightarrow W^n \xrightarrow{\partial_W^n} W^{n+1} \longrightarrow \cdots \longrightarrow W^{m-1} \longrightarrow Ker\partial_W^m \longrightarrow 0 \longrightarrow \cdots$$

There are obvious monomorphisms $z : W_{[n,m]} \longrightarrow W$ and $W_{[n,m]} \longrightarrow W_{[n',m']}$ for any "larger" interval, that is, $n' \leq n$ and $m' \geq m$. In particular we have a direct system

$$W_{[0,1]} \longrightarrow W_{[-1,2]} \longrightarrow W_{[-2,3]} \longrightarrow \cdots \longrightarrow W_{[-n,n+1]} \longrightarrow \cdots$$

and the canonical morphisms $W_{[-n,n+1]} \longrightarrow W$ are a direct limit in $\mathbf{C}(\mathcal{A})$. That is, we have $W = \varinjlim_{n \geq 0} W_{[-n,n+1]}$.

The morphisms relating the original complex to its truncations have certain universal properties. We give one example.

Remark 14. Let W be a complex and X a complex with $X^i = 0$ for $i < n$. Then any morphism of complexes $X \longrightarrow W$ factors *uniquely* through the canonical morphism ${}_bW_{\geq(n-1)} \longrightarrow W$. That is, there is an isomorphism $Hom_{\mathbf{C}(\mathcal{A})}(X, {}_bW_{\geq(n-1)}) \longrightarrow Hom_{\mathbf{C}(\mathcal{A})}(X, W)$. This identifies null-homotopic morphisms, so we deduce an isomorphism of abelian groups

$$Hom_{K(\mathcal{A})}(X, {}_bW_{\geq(n-1)}) \longrightarrow Hom_{K(\mathcal{A})}(X, W)$$

Note that we truncate at $n-1$ rather than n so that the null-homotopic morphisms agree. The reader will observe that everything we have said remains true when we replace ${}_bW_{\geq(n-1)}$ by ${}_bW_{\geq(n-k)}$ for any integer $k \geq 1$.

Definition 19. For each $n \in \mathbb{Z}$ there is a canonical full additive embedding $\mathcal{A} \longrightarrow \mathbf{C}(\mathcal{A})$ sending A to the complex whose only nonzero object is A in degree n. One checks that composing with the functor $\mathbf{C}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ gives another full additive embedding. Finally, composing this functor with $K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ gives an additive functor $c_n : \mathcal{A} \longrightarrow \mathfrak{D}(\mathcal{A})$ which is distinct on objects. Given an object $A \in \mathcal{A}$, we will sometimes denote the complex $c_n(A)$ by $A[-n]$ (the sign is needed since the functor $(-)[n]$ on $\mathbf{C}(\mathcal{A})$ shifts n places to the *left*). Often we will not distinguish between the object A and complex $c_0(A)$.

There are several ways we can recover the cohomology objects of a complex from its truncations. We can either truncate twice at the same position and be left with a single object (which happens to the cohomology at that position of the complex), or we can take the homotopy cokernel (resp. kernel) of the morphism $X_{\leq(n-1)} \longrightarrow X_{\leq n}$ (resp. $X_{\geq n} \longrightarrow X_{\geq(n+1)}$).

Lemma 26. *Let W be a complex and $n \in \mathbb{Z}$. Then there is a canonical isomorphism of complexes $c_n(H^n(W)) \longrightarrow (W_{\geq n})_{\leq n}$ natural in W.*

Proof. The complex $(W_{\geq n})_{\leq n}$ is concentrated in degree n, where it is the kernel of the induced morphism $Coker\partial_W^{n-1} \longrightarrow W^{n+1}$. By (DF,Lemma 25) this is canonically naturally isomorphic to $H^n(W)$. So one we embed $H^n(W)$ in degree n, we have the desired isomorphism. □

Lemma 27. *Let X be a complex and $n \in \mathbb{Z}$. There are canonical triangles in $\mathfrak{D}(\mathcal{A})$ natural in X*

$$X_{\leq(n-1)} \longrightarrow X_{\leq n} \longrightarrow c_nH^n(X) \longrightarrow \Sigma X_{\leq(n-1)}$$
$$c_nH^n(X) \longrightarrow X_{\geq n} \longrightarrow X_{\geq(n+1)} \longrightarrow \Sigma c_nH^n(X)$$

Proof. We begin with the first triangle. Let $X_{\leq n} \longrightarrow C$ be the canonical cokernel of the monomorphism $X_{\leq(n-1)} \longrightarrow X_{\leq n}$. We have a canonical morphism of complexes $C \longrightarrow c_nH^n(X)$ depicted in the following diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & X^{n-1}/Ker\partial_X^{n-1} & \longrightarrow & Ker\partial_X^n & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^n(X) & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

which is clearly a quasi-isomorphism. From the short exact sequence

$$0 \longrightarrow X_{\leq(n-1)} \longrightarrow X_{\leq n} \longrightarrow C \longrightarrow 0$$

we deduce a canonical triangle in $\mathfrak{D}(\mathcal{A})$

$$X_{\leq(n-1)} \longrightarrow X_{\leq n} \longrightarrow C \longrightarrow \Sigma X_{\leq(n-1)}$$

the quasi-isomorphism $C \longrightarrow c_n H^n(X)$ becomes an isomorphism in $\mathfrak{D}(\mathcal{A})$, so we obtain the desired triangle by replacing C by $c_n H^n(X)$. Naturality with respect to morphisms of complexes in X is easily checked. For the second triangle we deduce a canonical quasi-isomorphism of complexes $c_n H^n(X) \longrightarrow K$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & H^n(X) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & Coker\partial_X^{n-1} & \longrightarrow & Im\partial^n & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

where the complex K in the bottom row is a kernel of $X_{\geq n} \longrightarrow X_{\geq(n+1)}$. So we can define the second triangle in the same way as the first, and once again naturality is easily checked. □

Proposition 28. *For every $n \in \mathbb{Z}$ the additive functor $c_n : \mathcal{A} \longrightarrow \mathfrak{D}(\mathcal{A})$ is a full embedding.*

Proof. We claim that $H^n c_n = 1$, where $H^n : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{A}$ is the additive functor given in Definition 12. It suffices to observe that the embedding $\mathcal{A} \longrightarrow \mathbf{C}(\mathcal{A})$ followed by $H^n : \mathbf{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ is the identity, by our agreed conventions on the choices of canonical kernels and cokernels. In particular c_n is faithful and distinct on objects, so it only remains to show that c_n is full.

Let $A, B \in \mathcal{A}$ be given, and suppose we have a morphism $\gamma : c_n(A) \longrightarrow c_n(B)$ in $\mathfrak{D}(\mathcal{A})$, which we can write as $\gamma = F(g)F(f)^{-1}$ for some morphisms of complexes $f : W \longrightarrow c_n(A), g : W \longrightarrow c_n(B)$ with f a quasi-isomorphism. In particular we must have $H^i(W) = 0$ for $i \neq n$.

Let $W_{\leq n}$ be the truncated complex, as defined above. By composing the projection $Ker\partial_W^n \longrightarrow H^n(W)$ with the respective morphisms $H^n(f), H^n(g)$ we obtain morphisms of complexes $W_{\leq n} \longrightarrow c_n(A), W_{\leq n} \longrightarrow c_n(B)$. This first morphism is clearly a quasi-isomorphism, and the following diagram commutes

$$\begin{array}{ccccc}
 & \overset{f}{\swarrow} & W & \overset{g}{\searrow} & \\
 & & \uparrow & & \\
c_n(A) & \longleftarrow & W_{\leq n} & \longrightarrow & c_n(B) \\
 & & \downarrow & & \\
 & \underset{1}{\nwarrow} & c_n(A) & \underset{c_n H^n(\gamma)}{\nearrow} &
\end{array}$$

so by definition of the category $\mathfrak{D}(\mathcal{A}) = K(\mathcal{A})/\mathcal{Z}$, the morphism γ is equal to the image under F of the embedding of $H^n(\gamma) : A \longrightarrow B$ in $K(\mathcal{A})$. That is, $\gamma = c_n H^n(\gamma)$. This shows that c_n is full, and completes the proof. □

Remark 15. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor between abelian categories then we have a triangulated functor $K(F) : K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$ and there is an *equality* of functors $F = H^0 K(F) c_0$.

Lemma 29. *Let W be a complex with cohomology concentrated in degree n for some $n \in \mathbb{Z}$. Then there is a canonical isomorphism $W \cong c_n H^n(W)$ in $\mathfrak{D}(\mathcal{A})$.*

Proof. If W has cohomology concentrated in degree n, then the inclusion of the truncation $v : W_{\leq n} \longrightarrow W$ is a quasi-isomorphism, and therefore an isomorphism in $\mathfrak{D}(\mathcal{A})$. On the other hand, since the cohomology of $W_{\leq n}$ also vanishes for $i < n$ the canonical morphism $W_{\leq n} \longrightarrow c_n H^n(W)$ is a quasi-isomorphism as well. The desired isomorphism in $\mathfrak{D}(\mathcal{A})$ is the composite $W \cong W_{\leq n} \cong c_n H^n(W)$. □

Lemma 30. *The essential image of the functor $c_n : \mathcal{A} \longrightarrow \mathfrak{D}(\mathcal{A})$ is the class of all complexes whose cohomology is concentrated in degree n.*

Proof. By the *essential image* of the functor c_n we mean the class of objects of $\mathfrak{D}(\mathcal{A})$ which are isomorphic to $c_n(A)$ for some $A \in \mathcal{A}$. It is clear that every $c_n(A)$ has cohomology only in degree n, and consequently the same is true for any object in the image of c_n. Conversely, suppose X has cohomology concentrated in degree n. Then in $\mathfrak{D}(\mathcal{A})$ there is an isomorphism $X \cong c_n H^n(X)$, which completes the proof. □

Lemma 31. *Given an object $A \in \mathcal{A}$ and a complex X in $\mathcal{A}$ there is a canonical isomorphism of abelian groups natural in both variables*

$$Hom_{\mathbf{C}(\mathcal{A})}(c_i(A), X) \longrightarrow Hom_{\mathcal{A}}(A, Ker\partial_X^i)$$

which identifies null-homotopic morphisms $c_i(A) \longrightarrow X$ with morphisms $A \longrightarrow Ker\partial_X^i$ factoring through the canonical morphism $X^{i-1} \longrightarrow Ker\partial_X^i$. Dually we have a canonical isomorphism of abelian groups natural in both variables

$$Hom_{\mathbf{C}(\mathcal{A})}(X, c_i(A)) \longrightarrow Hom_{\mathcal{A}}(Coker\partial_X^{i-1}, A)$$

which identifies null-homotopic morphisms $X \longrightarrow c_i(A)$ with the morphisms $Coker\partial_X^{i-1} \longrightarrow A$ factoring through the canonical morphism $Coker\partial_X^{i-1} \longrightarrow X^{i+1}$.

Proof. A morphism of complexes $c_i(A) \longrightarrow X$ is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots \end{array}$$

which is determined by the morphism $A \longrightarrow X^i$, which must clearly factor through $Ker\partial_X^i$. Sending the morphism of complexes to this factorisation defines our isomorphism of abelian groups $Hom(c_i(A), X) \longrightarrow Hom(A, Ker\partial_X^i)$, which clearly identifies null-homotopic morphisms with those factoring through X^{i-1}.

The second claim is proved in the same way. Any morphism of complexes $X \longrightarrow c_i(A)$ is just a morphism $X^i \longrightarrow A$ which must factor through $Coker\partial_X^{i-1}$. Sending the morphism to this factorisation defines the desired isomorphism. □

Definition 20. Given $n \in \mathbb{Z}$ let $\mathfrak{D}(\mathcal{A})^{\geq n}$ (resp. $\mathfrak{D}(\mathcal{A})^{\leq n}$) denote the full subcategory of $\mathfrak{D}(\mathcal{A})$ consisting of complexes X with $H^i(X) = 0$ for $i < n$ (resp. $H^i(X) = 0$ for $i > n$). These are both replete additive subcategories of $\mathfrak{D}(\mathcal{A})$. The intersection $\mathfrak{D}(\mathcal{A})^{\geq n} \cap \mathfrak{D}(\mathcal{A})^{\leq n}$ is the full subcategory of all complexes whose cohomology is concentrated in degree n, so it follows from Lemma 30 that this is a portly abelian category equivalent to $\mathcal{A}$.

Clearly a complex W belongs to $\mathfrak{D}(\mathcal{A})^{\geq n}$ if and only if the morphism of complexes $W \longrightarrow W_{\geq n}$ is an isomorphism in $\mathfrak{D}(\mathcal{A})$, and belongs to $\mathfrak{D}(\mathcal{A})^{\leq n}$ if and only if $W_{\leq n} \longrightarrow W$ is an isomorphism in $\mathfrak{D}(\mathcal{A})$.

Lemma 32. *Given $m, n \in \mathbb{Z}$ with $m < n$ and complexes $W \in \mathfrak{D}(\mathcal{A})^{\leq m}, Q \in \mathfrak{D}(\mathcal{A})^{\geq n}$ we have $Hom_{\mathfrak{D}(\mathcal{A})}(W, Q) = 0$.*

Proof. We may as well assume that $W^i = 0$ for $i > m$ and $Q^i = 0$ for $i < n$. A morphism $W \longrightarrow Q$ in $\mathfrak{D}(\mathcal{A})$ can be represented by a diagram of morphisms of complexes

$$\begin{array}{ccccc} & & T & & \\ & \swarrow & & \searrow & \\ W & & & & Q \end{array}$$

where $T \longrightarrow W$ is a quasi-isomorphism, so $H^i(T) = 0$ for $i > m$. The canonical morphism of complexes $T_{\leq m} \longrightarrow T$ is therefore a quasi-isomorphism. Since it is clear that the composite $T_{\leq m} \longrightarrow T \longrightarrow Q$ is zero, we deuce that $T \longrightarrow Q$ is zero in $\mathfrak{D}(\mathcal{A})$ and therefore so is $W \longrightarrow Q$. □

Proposition 33. *Let X be a complex and $n \in \mathbb{Z}$. Then there is a canonical triangle in $\mathfrak{D}(\mathcal{A})$ natural in X*

$$X_{\leq n} \xrightarrow{v} X \xrightarrow{q} X_{\geq(n+1)} \longrightarrow \Sigma X_{\leq n}$$

Proof. We have canonical morphisms of complexes $v : X_{\leq n} \longrightarrow X$ and $q : X \longrightarrow X_{\geq(n+1)}$ and we claim there exists a morphism $t : X_{\geq(n+1)} \longrightarrow \Sigma X_{\leq n}$ of $\mathfrak{D}(\mathcal{A})$ fitting these into a triangle. If such a morphism exists it must be unique by Lemma 32 and (TRC,Remark 16). Form the following exact sequence of complexes

$$0 \longrightarrow X_{\leq n} \xrightarrow{v} X \xrightarrow{t} Q \longrightarrow 0$$

and then using Proposition 20 a triangle in $\mathfrak{D}(\mathcal{A})$

$$X_{\leq n} \xrightarrow{v} X \xrightarrow{t} Q \longrightarrow \Sigma X_{\leq n}$$

There is an induced morphism of complexes $a : Q \longrightarrow W_{\geq(n+1)}$ unique such that $at = q$. One checks that a is a quasi-isomorphism, so that we can replace Q by $W_{\geq(n+1)}$ in the above triangle and obtain the desired result. $\square$

There are many questions suggested by these results. Are there abelian categories embedded in $\mathfrak{D}(\mathcal{A})$ which are not equivalent to $\mathcal{A}$? In fact an important recent discovery in homological algebra is that two different abelian categories can have equivalent derived categories. This leads to the notions of *t-structures* and *hearts* which are developed in our notes on Hearts of Triangulated Categories (HRT).

3.3 Bounded Derived Categories

Proposition 34. *Let $\mathcal{C}$ be a full, replete, additive subcategory of $\mathbf{C}(\mathcal{A})$ which is closed under translation and mapping cones. Then the full subcategory of $K(\mathcal{A})$ whose objects are the complexes in $\mathcal{C}$ is a fragile triangulated subcategory $K^{\mathcal{C}}(\mathcal{A}) \longrightarrow K(\mathcal{A})$.*

Proof. See (TRC,Definition 20) for the definition of a fragile triangulated subcategory. Let $\mathcal{C}$ be a category with the stated properties. When we say $\mathcal{C}$ is closed under translation, we mean that whenever a complex X is in $\mathcal{C}$, so is the translation $X[n]$ for any $n \in \mathbb{Z}$ (as given in Definition 2). Let $K^{\mathcal{C}}(\mathcal{A})$ denote the full subcategory of $K(\mathcal{A})$ formed by the objects of $\mathcal{C}$. This is certainly an additive category (although it is not necessarily replete in $K(\mathcal{A})$, by Remark 7), and the additive automorphism Σ of $K(\mathcal{A})$ restricts to an additive automorphism $\Sigma : K^{\mathcal{C}}(\mathcal{A}) \longrightarrow K^{\mathcal{C}}(\mathcal{A})$. Given any morphism $u : X \longrightarrow Y$ of complexes in $\mathcal{C}$, we have the following candidate triangle in $K^{\mathcal{C}}(\mathcal{A})$

$$X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X \tag{25}$$

We say a candidate triangle in $K^{\mathcal{C}}(\mathcal{A})$ is *distinguished* if it is isomorphic (as a candidate triangle *in* $K^{\mathcal{C}}(\mathcal{A})$) to a candidate triangle (25) arising from a morphism $u : X \longrightarrow Y$ in $\mathcal{C}$. If a candidate triangle in $K^{\mathcal{C}}(\mathcal{A})$ is distinguished, then it is certainly distinguished as a candidate triangle in $K(\mathcal{A})$. We must now show that the additive category $K^{\mathcal{C}}(\mathcal{A})$, together with the additive automorphism Σ and class of distinguished triangles just defined, is a triangulated category. One does this by carefully copying the proof of Theorem 12 (observe that since $\mathcal{C}$ is closed under mapping cones and translation it is also closed under mapping cylinders, so we can still use Lemma 11). The inclusion $K^{\mathcal{C}}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ is now obviously a fragile triangulated subcategory. In particular we deduce that the distinguished triangles of $K^{\mathcal{C}}(\mathcal{A})$ are just the triangles of $K(\mathcal{A})$ whose objects happen to lie in $\mathcal{C}$. $\square$

Corollary 35. *The additive categories $K^{+}(\mathcal{A}), K^{-}(\mathcal{A})$ and $K^{b}(\mathcal{A})$ become triangulated categories in a canonical way, and they are all fragile triangulated subcategories of $K(\mathcal{A})$. Moreover $K^{b}(\mathcal{A})$ is a fragile triangulated subcategory of both $K^{+}(\mathcal{A})$ and $K^{-}(\mathcal{A})$.*

Proof. The full subcategories $\mathcal{C}^+, \mathcal{C}^-, \mathcal{C}^b$ of $\mathbf{C}(\mathcal{A})$ formed by the bounded above, bounded below and bounded complexes respectively are all clearly full, replete, additive subcategories closed under translation and mapping cones. It follows from Proposition 34 that $K^+(\mathcal{A}), K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ are all triangulated categories, in which the translation functors are the restriction of Σ on $K(\mathcal{A})$, and the distinguished triangles are those arising from mapping cones in $\mathbf{C}(\mathcal{A})$. ☐

Throughout the remainder of this section whenever we write $K^*(\mathcal{A})$ or $\mathfrak{D}^*(\mathcal{A})$ we mean that the given statement holds with $*$ replaced by $+, -$ or b.

Lemma 36. *The exact complexes in $K^*(\mathcal{A})$ form a thick triangulated subcategory $\mathcal{Z}^*$, and the corresponding class of morphisms $Mor_{\mathcal{Z}^*}$ is the class of all quasi-isomorphisms in $K^*(\mathcal{A})$.*

Proof. As in Corollary 15 one checks that $\mathcal{Z}^*$ is a thick triangulated subcategory of $K^*(\mathcal{A})$, and the second claim is also easily checked. ☐

Definition 21. Let $\mathcal{A}$ be an abelian category. Then we define the portly triangulated categories $\mathfrak{D}^+(\mathcal{A}), \mathfrak{D}^-(\mathcal{A})$ and $\mathfrak{D}^b(\mathcal{A})$ as the verdier quotients $K^*(\mathcal{A})/\mathcal{Z}^*$. We call $\mathfrak{D}^b(\mathcal{A})$ the *bounded derived category* of $\mathcal{A}$. There are canonical triangulated functors $F : K^*(\mathcal{A}) \longrightarrow \mathfrak{D}^*(\mathcal{A})$.

Lemma 37. *There is a canonical commutative diagram of triangulated functors, in which each functor is a full embedding*

$$\begin{array}{ccccc} & & \mathfrak{D}^+(\mathcal{A}) & & \\ & \nearrow & & \searrow & \\ \mathfrak{D}^b(\mathcal{A}) & & \longrightarrow & & \mathfrak{D}(\mathcal{A}) \\ & \searrow & & \nearrow & \\ & & \mathfrak{D}^-(\mathcal{A}) & & \end{array}$$

All these functors are defined on morphisms by $[f, g] \mapsto [f, g]$, and in each case the embedding is the unique triangulated functor making the following diagram commute

$$\begin{array}{ccc} K^*(\mathcal{A}) & \longrightarrow & K(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathfrak{D}^*(\mathcal{A}) & \longrightarrow & \mathfrak{D}(\mathcal{A}) \end{array}$$

Proof. The triangulated category $K^*(\mathcal{A})$ is a fragile triangulated subcategory of $K(\mathcal{A})$ and $\mathcal{Z}_* = \mathcal{Z} \cap K^*(\mathcal{A})$ is a thick triangulated subcategory of $K^*(\mathcal{A})$. To show that the induced triangulated functor $K^*(\mathcal{A})/\mathcal{Z}_* \longrightarrow K(\mathcal{A})/\mathcal{Z}$ is a full embedding, it suffices by (TRC,Proposition 70) to check one of the conditions $(a), (b)$ given there. We treat each case separately:

Case $* = +$ Suppose we are given a quasi-isomorphism of complexes $t : Y \longrightarrow X$ with Y bounded below. Say $Y^i = 0$ for $i \le N$. Then if $q : X \longrightarrow X_{\geq N}$ is canonical, it is easy to check that qt is a quasi-isomorphism, so the condition (b) of (TRC,Proposition 70) is satisfied.

Case $* = -$ Suppose we are given a quasi-isomorphism of complexes $s : X \longrightarrow Y$ with Y bounded above. Say $Y^i = 0$ for $i \geq N$. Then if $v : X_{\leq N} \longrightarrow X$ is the inclusion, it is again easy to check that sv is a quasi-isomorphism, so the condition (a) of (TRC,Proposition 70) is satisfied.

This shows that there are canonical full embeddings of triangulated categories $\mathfrak{D}^+(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ and $\mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$. Of course $K^b(\mathcal{A})$ is a fragile triangulated subcategory of both $K^+(\mathcal{A}), K^-(\mathcal{A})$ and one checks using the same arguments that there are canonical full embeddings of triangulated categories $\mathfrak{D}^b(\mathcal{A}) \longrightarrow \mathfrak{D}^+(\mathcal{A})$ and $\mathfrak{D}^b(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})$ which are the unique triangulated functors making the following diagrams commute

$$\begin{array}{ccc} K^b(\mathcal{A}) & \longrightarrow & K^+(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathfrak{D}^b(\mathcal{A}) & \longrightarrow & \mathfrak{D}^+(\mathcal{A}) \end{array} \qquad \begin{array}{ccc} K^b(\mathcal{A}) & \longrightarrow & K^-(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathfrak{D}^b(\mathcal{A}) & \longrightarrow & \mathfrak{D}^-(\mathcal{A}) \end{array}$$

Composing we have two full triangulated embeddings $\mathfrak{D}^b(\mathcal{A}) \longrightarrow \mathfrak{D}^{\pm}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$. But by the universal property of the derived category these must agree, and the proof is complete. $\square$

Lemma 38. *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a full replete subcategory which is abelian. Then*

(i) $\mathbf{C}(\mathcal{B})$ *is a full, replete subcategory of* $\mathbf{C}(\mathcal{A})$*, which is closed under translation and mapping cones. If $\mathcal{B}$ is an abelian subcategory of $\mathcal{A}$ then* $\mathbf{C}(\mathcal{B})$ *is an abelian subcategory of* $\mathbf{C}(\mathcal{A})$*.*

(ii) $K(\mathcal{B})$ *is a fragile triangulated subcategory of* $K(\mathcal{A})$*.*

(iii) If $\mathcal{B}$ is an abelian subcategory of $\mathcal{A}$ then there is a canonical triangulated functor $\mathfrak{D}(\mathcal{B}) \longrightarrow \mathfrak{D}(\mathcal{A})$*.*

Proof. $(i), (ii)$ are easily checked. Observe that as subcategories of $K(\mathcal{A})$ we actually have $K(\mathcal{B}) = K^{\mathcal{B}}(\mathcal{A})$ in the notation of Proposition 34, and the induced triangulated structure on $K^{\mathcal{B}}(\mathcal{A})$ agrees with the canonical structure on $K(\mathcal{B})$. (iii) If we denote by $\mathcal{Z}_{\mathcal{A}}, \mathcal{Z}_{\mathcal{B}}$ the categories of exact complexes in $K(\mathcal{A}), K(\mathcal{B})$ respectively then it is clear that $\mathcal{Z}_{\mathcal{A}} \cap K(\mathcal{B}) = \mathcal{Z}_{\mathcal{B}}$. So the triangulated functor $K(\mathcal{B}) \longrightarrow K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ certainly sends objects of $\mathcal{Z}_{\mathcal{B}}$ to zero, and we obtain a triangulated functor $\mathfrak{D}(\mathcal{B}) \longrightarrow \mathfrak{D}(\mathcal{A})$. In special cases, where we can verify one of the conditions $(a), (b)$ of (TRC,Proposition 70), this functor is a full embedding. Observe that this functor is just the one induced by the exact functor $\mathcal{B} \longrightarrow \mathcal{A}$ as in Lemma 23. $\square$

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. There is a unique triangulated functor $K^*(F)$ making the following diagram commute

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^*(\mathcal{A}) & \xrightarrow[K^*(F)]{} & K^*(\mathcal{B}) \end{array}$$

It is clear that $K^*(1) = 1$ and $K^*(GF) = K^*(G)K^*(F)$ for another additive functor $G : \mathcal{B} \longrightarrow \mathcal{C}$ between abelian categories. If F is exact then there is a unique triangulated functor $\mathfrak{D}^*(F) : \mathfrak{D}^*(\mathcal{A}) \longrightarrow \mathfrak{D}^*(\mathcal{B})$ making the following diagram commute

$$\begin{array}{ccc} K^*(\mathcal{A}) & \xrightarrow{K^*(F)} & K^*(\mathcal{B}) \\ \downarrow & & \downarrow \\ \mathfrak{D}^*(\mathcal{A}) & \xrightarrow[\mathfrak{D}^*(F)]{} & \mathfrak{D}^*(\mathcal{B}) \end{array}$$

Clearly $\mathfrak{D}^*(1) = 1$ and if $G : \mathcal{B} \longrightarrow \mathcal{C}$ is another exact functor between abelian categories then $\mathfrak{D}^*(GF) = \mathfrak{D}^*(G)\mathfrak{D}^*(F)$.

3.4 Plump Subcategories

Definition 22. Let $\mathcal{A}$ be an abelian category. A full replete subcategory $\mathcal{C} \subseteq \mathcal{A}$ is called a *plump subcategory* if it contains all the zero objects of $\mathcal{A}$ and if for any exact sequence in $\mathcal{A}$

$$X_1 \longrightarrow X_2 \longrightarrow X \longrightarrow X_3 \longrightarrow X_4$$

with $X_1, X_2, X_3, X_4 \in \mathcal{C}$, we have $X \in \mathcal{C}$ as well. It is clear that $\mathcal{C}$ is an abelian subcategory of $\mathcal{A}$. We denote by $K_{\mathcal{C}}(\mathcal{A})$ and $\mathfrak{D}_{\mathcal{C}}(\mathcal{A})$ the full subcategories of $K(\mathcal{A}), \mathfrak{D}(\mathcal{A})$ respectively consisting of complexes whose cohomology objects all belong to $\mathcal{C}$. These are both triangulated subcategories. If $\mathcal{A}$ has exact coproducts and if $\mathcal{C}$ is closed under coproducts in $\mathcal{A}$, then both subcategories are localising. If X is a complex in $\mathcal{A}$ whose cohomology objects all belong to $\mathcal{C}$, then the same is true for $X_{\le n}, X_{\ge n}$ for any $n \in \mathbb{Z}$.

3.5 Remarks on Duality

Throughout this section let $\mathcal{A}$ be an abelian category. We show that $\mathfrak{D}(\mathcal{A})^{\mathrm{op}} = \mathfrak{D}(\mathcal{A}^{\mathrm{op}})$, which allows us to prove statements about derived categories using duality arguments. We define a covariant isomorphism of categories

$$\begin{gathered} F : \mathbf{C}(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathbf{C}(\mathcal{A}^{\mathrm{op}}) \\ F(X)^j = X^{-j}, \quad \partial^j_{F(X)} = -\partial_X^{-j-1} \\ F(\psi)^j = \psi_{-j} \end{gathered}$$

There is also an *equality* of complexes $F(\Sigma^{-1}X) = \Sigma F(X)$. Observe that for a complex X and $i \in \mathbb{Z}$ there is a canonical isomorphism $H^i(X) \cong H^{-i}(F(X))$ in $\mathcal{A}^{\mathrm{op}}$. Let $\psi : X \longrightarrow Y$ be a morphism of complexes in $\mathcal{A}$ (that is, a morphism of the category $\mathbf{C}(\mathcal{A})$). There is a canonical isomorphism in the category $\mathbf{C}(\mathcal{A}^{\mathrm{op}})$

$$\begin{gathered} \Phi : C_{F(\psi)} \longrightarrow \Sigma F(C_\psi) \\ \Phi^j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : Y^{-j-1} \oplus X^{-j} \longrightarrow X^{-j} \oplus Y^{-j-1} \end{gathered}$$

Let $v : Y \longrightarrow C_\psi$ and $k : \Sigma^{-1} C_\psi \longrightarrow X$ be the homotopy cokernel and kernel respectively. We also have the corresponding morphisms for $F(\psi)$, which we denote $V : F(X) \longrightarrow C_{F(\psi)}$ and $K : \Sigma^{-1} C_{F(\psi)} \longrightarrow F(Y)$. The following diagram commutes in $\mathbf{C}(\mathcal{A}^{\mathrm{op}})$

$$\begin{array}{ccccccc} F(C_\psi) & \xrightarrow{F(v)} & F(Y) & \xrightarrow{F(\psi)} & F(X) & \xrightarrow{V} & C_{F(\psi)} \\ {\scriptstyle \Sigma^{-1}\Phi}\Uparrow & \nearrow{\scriptstyle K} & & & & \searrow{\scriptstyle F(k)} & \Downarrow{\scriptstyle \Phi} \\ \Sigma^{-1} C_{F(\psi)} & & & & & & \Sigma F(C_\psi) \end{array} \tag{26}$$

The isomorphism F extends to an isomorphism of categories $K(\mathcal{A})^{\mathrm{op}} \longrightarrow K(\mathcal{A}^{\mathrm{op}})$ making the following diagram commute

$$\begin{array}{ccc} \mathbf{C}(\mathcal{A})^{\mathrm{op}} & \xrightarrow{F} & \mathbf{C}(\mathcal{A}^{\mathrm{op}}) \\ \downarrow & & \downarrow \\ K(\mathcal{A})^{\mathrm{op}} & \xrightarrow[F]{} & K(\mathcal{A}^{\mathrm{op}}) \end{array}$$

Let $\phi : F\Sigma^{-1} \longrightarrow \Sigma F$ be the natural equivalence defined by $\phi_X = -1$. Using (26) one checks that the pair (F, ϕ) is a triangulated functor, and therefore an isomorphism of triangulated categories. It preserves exact complexes, so it induces a unique triangulated functor making the following diagram commute

$$\begin{array}{ccc} K(\mathcal{A})^{\mathrm{op}} & \xrightarrow{F} & K(\mathcal{A}^{\mathrm{op}}) \\ \downarrow & & \downarrow \\ \mathfrak{D}(\mathcal{A})^{\mathrm{op}} & \xrightarrow[F]{} & \mathfrak{D}(\mathcal{A}^{\mathrm{op}}) \end{array}$$

It is clear that F is an isomorphism of triangulated categories, so we have our canonical isomorphism of triangulated categories $\mathfrak{D}(\mathcal{A})^{\mathrm{op}} \cong \mathfrak{D}(\mathcal{A}^{\mathrm{op}})$.

3.6 Derived Categories of Rings

The philosophy of the derived category is that by taking cohomology one throws away information. One should deal throughout with the original complexes. However, in this section we will see how in some cases objects in the derived category are completely determined by their cohomology. The reader should see (DIM,Definition 3) for the definition of the *global dimension* of an abelian category with enough injectives.

Definition 23. Let $\mathcal{A}$ be an abelian category with enough injectives. We say that $\mathcal{A}$ is *semisimple* if $gl.dim.\mathcal{A} \leq 0$, or equivalently if $Ext^1(-,-)$ is zero. We say that $\mathcal{A}$ is *hereditary* if $gl.dim.\mathcal{A} \leq 1$, or equivalently if $Ext^2(-,-)$ is zero.

Example 1. The global dimension of any field is zero, so any field is semisimple. The global dimension of any commutative principal ideal domain which is *not* a field is one, so $gl.dim(\mathbb{Z}) = 1$. In particular the category of abelian groups is hereditary.

Proposition 39. *Let $\mathcal{A}$ be a semisimple abelian category. Given a complex X in $\mathcal{A}$ let $H^\bullet(X)$ denote the following complex*

$$\cdots \xrightarrow{0} H^{n-1}(X) \xrightarrow{0} H^n(X) \xrightarrow{0} H^{n+1}(X) \xrightarrow{0} \cdots$$

There is a canonical isomorphism $H^\bullet(X) \longrightarrow X$ in $\mathfrak{D}(\mathcal{A})$ natural in X. If $\mathcal{A}$ has exact products and exact coproducts then the functor $X \mapsto (H^i(X))_{i\in\mathbb{Z}}$ defines an equivalence $\mathfrak{D}(\mathcal{A}) \longrightarrow \prod_{\mathbb{Z}} \mathcal{A}$.

Proof. Consider the short exact sequence

$$0 \longrightarrow Im(\partial^{n-1}) \longrightarrow Ker(\partial^n) \longrightarrow H^n(X) \longrightarrow 0$$

Since $\mathcal{A}$ is semisimple this must split. Let $g^n : H^n(X) \longrightarrow Ker(\partial^n)$ be such a splitting for each $n \in \mathbb{Z}$. Then we have the following morphism of complexes

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{0} & H^{n-1}(X) & \xrightarrow{0} & H^n(X) & \xrightarrow{0} & H^{n+1}(X) & \xrightarrow{0} & \cdots \\ & & \downarrow{\scriptstyle g^{n-1}} & & \downarrow{\scriptstyle g^n} & & \downarrow{\scriptstyle g^{n+1}} & & \\ \cdots & \xrightarrow{0} & Ker(\partial^{n-1}) & \xrightarrow{0} & Ker(\partial^n) & \xrightarrow{0} & Ker(\partial^{n+1}) & \xrightarrow{0} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \end{array}$$

which is clearly a quasi-isomorphism. We claim that this morphism of complexes $H^\bullet(X) \longrightarrow X$ is independent (up to homotopy) of the choice of splittings g^n. Let h^n be another choice of splittings, so that $g^n - h^n$ vanishes when composed with $Ker(\partial^n) \longrightarrow H^n(X)$. It therefore factors through $Im(\partial^{n-1}) \longrightarrow Ker(\partial^n)$, and composing this factorisation with an arbitrary splitting of the epimorphism $X^{n-1} \longrightarrow Im(\partial^{n-1})$ we have for each $n \in \mathbb{Z}$ a morphism $\Sigma^n : H^n(X) \longrightarrow X^{n-1}$. This is clearly a homotopy, as required. One checks that this isomorphism $H^\bullet(X) \longrightarrow X$ in $\mathfrak{D}(\mathcal{A})$ is natural in X, with respect to morphisms of $\mathfrak{D}(\mathcal{A})$

Now suppose that $\mathcal{A}$ has exact products and coproducts, so that products and coproducts in $\mathfrak{D}(\mathcal{A})$ exist and can be calculated on the level of complexes by Proposition 44 and Remark 18. Then $H^\bullet(X)$ is both a product $\prod_{i\in\mathbb{Z}} c_i H^i(X)$ and a coproduct $\bigoplus_{i\in\mathbb{Z}} c_i H^i(X)$ in $\mathfrak{D}(\mathcal{A})$, so there is a bijection between morphisms $H^\bullet(X) \longrightarrow H^\bullet(Y)$ and the abelian group

$$\prod_{i,j\in\mathbb{Z}} Ext^{j-i}_{\mathcal{A}}(H^i(X), H^j(Y))$$

Since $\mathcal{A}$ is semisimple this reduces to $\prod_{n\in\mathbb{Z}} Hom_{\mathcal{A}}(H^n(X), H^n(Y))$, which proves the last statement. $\square$

More generally we have

Proposition 40. *Let $\mathcal{A}$ be a hereditary abelian category with exact coproducts. For any complex X in $\mathcal{A}$ there is an isomorphism $H^\bullet(X) \longrightarrow X$ in $\mathfrak{D}(\mathcal{A})$.*

Proof. For each $n \in \mathbb{Z}$ we have a short exact sequence

$$0 \longrightarrow Ker(\partial^{n-1}) \longrightarrow X^{n-1} \longrightarrow Im(\partial^{n-1}) \longrightarrow 0$$

whose long exact Ext sequence for $H^n(X)$ in the first variable contains

$$\cdots \longrightarrow Ext^1_{\mathcal{A}}(H^n(X), X^{n-1}) \longrightarrow Ext^1_{\mathcal{A}}(H^n(X), Im(\partial^{n-1})) \longrightarrow Ext^2_{\mathcal{A}}(H^n(X), Ker(\partial^{n-1})) \longrightarrow \cdots$$

Since $Ext^2_{\mathcal{A}}(-,-) = 0$ the map $Ext^1_{\mathcal{A}}(H^n(X), X^{n-1}) \longrightarrow Ext^1_{\mathcal{A}}(H^n(X), Im(\partial^{n-1}))$ is surjective. Using the Yoneda characterisation of Ext groups, we deduce a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & H^n(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & Im(\partial^{n-1}) & \longrightarrow & Ker(\partial^n) & \longrightarrow & H^n(X) & \longrightarrow & 0
\end{array}$$

where $E^n \in \mathcal{A}$ and the left hand square is a pushout. We have a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^n(X) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots
\end{array}$$

in which the vertical morphisms induce an isomorphism on cohomology in degree n. Let Y be the complex obtained by taking the coproduct of the middle row above over all $n \in \mathbb{Z}$. Since $\mathcal{A}$ has exact coproducts the induced morphisms $Y \longrightarrow X$ and $Y \longrightarrow H^\bullet(X)$ are both quasi-isomorphisms. In $\mathfrak{D}(\mathcal{A})$ this yields an isomorphism $H^\bullet(X) \longrightarrow X$. $\square$

Remark 16. Observe that **Ab** is hereditary with exact coproducts, so in $\mathfrak{D}(\mathbf{Ab})$ every complex X is isomorphic to the complex $\cdots \longrightarrow H^n(X) \longrightarrow H^{n+1}(X) \longrightarrow \cdots$ with zero differentials.

4 Homotopy Resolutions

The reader is expected to know the contents of our notes on Triangulated Categories (TRC) notes, up to and including (TRC,Section 4). Throughout this section $\mathcal{A}$ denotes an abelian category.

In classical homological algebra the most fundamental concept is that of a *resolution* of an object. In the theory of derived categories an analogous role is played by resolutions of complexes. Resolutions of bounded complexes can be handled in much the same way as resolutions of objects (morally this is because of Remark 32), but the situation for unbounded complexes is different. We proceed by generalising a defining property of injective and projective resolutions to define *hoinjective* and *hoprojective* complexes, first introduced in [Spa88]. It then remains to show that resolutions by these special complexes exist. In the current section we achieve this goal for several important special cases, which will allow us to study the general case in Section 7.

Proposition 41. *If $\mathcal{A}$ is a grothendieck abelian category then so is $\mathbf{C}(\mathcal{A})$.*

Proof. We already know that $\mathbf{C}(\mathcal{A})$ is cocomplete abelian (DF,Lemma 65). Using the fact that colimits in $\mathbf{C}(\mathcal{A})$ can be computed pointwise, one checks easily that direct limits in $\mathbf{C}(\mathcal{A})$ are exact. So it only remains to show that $\mathbf{C}(\mathcal{A})$ has a generating family.

Let U be a generator for $\mathcal{A}$, and for $i \in \mathbb{Z}$ let $d_i(U)$ denote the following complex

$$\cdots \longrightarrow 0 \longrightarrow U \xrightarrow{1} U \longrightarrow 0 \longrightarrow \cdots \tag{27}$$

where the first U occurs in position i. Given a nonzero morphism of complexes $\psi : S \longrightarrow T$ there is some $i \in \mathbb{Z}$ with $\psi^i \neq 0$. Since U is a generator there is a morphism $x : U \longrightarrow S^i$ with $\psi^i x \neq 0$. Define a morphism of complexes $\phi : d_i(U) \longrightarrow S$ by $\phi^i = x$ and $\phi^{i+1} = \partial_S^i x$. Then $\psi\phi \neq 0$, so the complexes $\{d_i(U)\}_{i\in\mathbb{Z}}$ form a generating family for $\mathbf{C}(\mathcal{A})$. ☐

Remark 17. In fact the proof of Proposition 41 is a little more general than the statement. If $\mathcal{A}$ is an abelian category with generator U then the complexes $\{d_i(U)\}_{i\in\mathbb{Z}}$ form a generating family for $\mathbf{C}(\mathcal{A})$. If moreover U is projective, then each $d_i(U)$ is projective, so this is a generating family of projectives. In particular if $\mathcal{A}$ is cocomplete with a projective generator, then $\mathbf{C}(\mathcal{A})$ also has a projective generator.

Lemma 42. *If $\mathcal{A}$ is cocomplete, then $K(\mathcal{A})$ is a triangulated category with coproducts.*

Proof. If $\mathcal{A}$ is cocomplete, so is the abelian category $\mathbf{C}(\mathcal{A})$ (DF,Lemma 65). One checks that the canonical functor $\mathbf{C}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ preserves coproducts, which completes the proof. ☐

Lemma 43. *If $\mathcal{A}$ is complete, then $K(\mathcal{A})$ is a triangulated category with products.*

Proof. If $\mathcal{A}$ is complete, so is the abelian category $\mathbf{C}(\mathcal{A})$ (DF,Lemma 65). One checks that the canonical functor $\mathbf{C}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ preserves products, which completes the proof. ☐

Proposition 44. *If $\mathcal{A}$ has exact coproducts, then $\mathfrak{D}(\mathcal{A})$ is a portly triangulated category with coproducts, and the functor $K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ preserves coproducts.*

Proof. See (AC,Definition 45) for the definition of an abelian category with exact coproducts. In particular we require that $\mathcal{A}$ be cocomplete, so $K(\mathcal{A})$ is a triangulated category with coproducts. The triangulated subcategory $\mathcal{Z}$ of exact complexes is localising, since an arbitrary coproduct in $\mathbf{C}(\mathcal{A})$ of exact complexes is exact (DF,Lemma 66). The result now follows from (TRC,Lemma 91). ☐

Example 2. In particular if $\mathcal{A}$ is a grothendieck abelian category, then $\mathcal{A}$ has exact coproducts (AC,Lemma 57) and consequently $\mathfrak{D}(\mathcal{A})$ has coproducts. We have also shown that the exact complexes $\mathcal{Z}$ form a thick localising subcategory of $K(\mathcal{A})$.

Remark 18. Using the duality of Section 3.5 we infer from Proposition 44 that if $\mathcal{A}$ has exact products, then $\mathfrak{D}(\mathcal{A})$ has products and the functor $K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ preserves products.

Lemma 45. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between cocomplete abelian categories. If F preserves coproducts then so does the induced functor $K(F) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$.*

Proof. The induced functor on complexes $\mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$ preserves coproducts (DF,Lemma 65), so this follows immediately from Lemma 42 and its proof. ☐

Lemma 46. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor between grothendieck abelian categories. If F preserves coproducts then so does the induced functor $\mathfrak{D}(F) : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{B})$.*

Proof. The induced functor on complexes $\mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$ preserves coproducts (DF,Lemma 65) and by Proposition 44 coproducts in the derived category can be calculated on the level of complexes, so it is clear that $\mathfrak{D}(F)$ preserves all coproducts. □

The exact complexes $\mathcal{Z}$ give a triangulated subcategory of $K(\mathcal{A})$ so we can define the thick colocalising subcategory $\mathcal{Z}^{\perp}$ of $\mathcal{Z}$-local objects of $K(\mathcal{A})$ and dually the thick localising subcategory ${}^{\perp}\mathcal{Z}$ of $\mathcal{Z}$-colocal objects (TRC,Definition 37) (TRC,Lemma 90). These classes of objects are so important that we give them a special name.

Definition 24. The $\mathcal{Z}$-local objects of $K(\mathcal{A})$ are called *homotopy injective* (or *hoinjective*) complexes and the $\mathcal{Z}$-colocal objects are called *homotopy projective* (or *hoprojective*) complexes. The hoinjective complexes form a thick colocalising subcategory of $K(\mathcal{A})$ which we denote by $K(I)$, and the hoprojective complexes form a thick localising subcategory which we denote by $K(P)$. That is, arbitrary products of hoinjectives in $K(\mathcal{A})$ are hoinjective, and arbitrary coproducts of hoprojectives in $K(\mathcal{A})$ are hoprojective.

Remark 19. Just to be perfectly clear, a complex I is hoinjective if and only if every morphism of complexes $X \longrightarrow I$ from an exact complex is null-homotopic. Dually a complex P is hoprojective if and only if every morphism of complexes $P \longrightarrow X$ into an exact complex is null-homotopic.

Remark 20. In the literature there are many different names for the complexes which we call here hoinjective. In [BN93] they are called "special complexes of injectives" which we find exhausting to write too often, and in [Spa88] [ATJLSS00] they are called "K-injective", which is better but in the author's opinon looks ugly when he wants to write "with I K-injective". It appears that "K-injective" has become the standard notation in the literature.

Proposition 47. *Any bounded below complex of injectives in $\mathcal{A}$ is hoinjective.*

Proof. When we say that I is injective, we mean that I^n is injective for every $n \in \mathbb{Z}$. Suppose that $I^i = 0$ for $i < N$ and let $f : X \longrightarrow I$ be a morphism of complexes with X exact. We have to show that f is null-homotopic. To construct a homotopy $\Sigma : 0 \longrightarrow f$, we begin at the obvious place

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & X^{N-2} & \longrightarrow & X^{N-1} & \longrightarrow & X^{N} & \longrightarrow & X^{N+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f^N} & & \downarrow{\scriptstyle f^{N+1}} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I^{N} & \longrightarrow & I^{N+1} & \longrightarrow & \cdots \end{array}$$

Since $f^N \partial_X^{N-1} = 0$ we can factor f^N through $Im\partial_X^N$ and then lift using injectivity of I^N to a morphism $\Sigma^{N+1} : X^{N+1} \longrightarrow I^N$ with $\Sigma^{N+1}\partial_X^N = f^N$. One constructs the morphisms Σ^i for $i > N+1$ in the usual way, defining the necessary homotopy $f \simeq 0$. □

Proposition 48. *Any bounded above complex of projectives in $\mathcal{A}$ is hoprojective.*

Lemma 49. *An object $A \in \mathcal{A}$ is injective in $\mathcal{A}$ (resp. projective) if and only if it is hoinjective (resp. hoprojective) considered as a complex concentrated in degree zero.*

Corollary 50. *Suppose we are given complexes X, Y in $\mathcal{A}$ with either X a hoprojective complex or Y a hoinjective complex. Then the canonical morphism of abelian groups*

$$Hom_{K(\mathcal{A})}(X, Y) \longrightarrow Hom_{\mathfrak{D}(\mathcal{A})}(X, Y)$$

is an isomorphism.

Proof. This is a special case of (TRC,Proposition 92). □

In fact this property characterises the hoinjective and hoprojective complexes. The next result also proves that our hoinjective complexes agree with the K-injective complexes of [Spa88] (also called q-injective in [Lip]).

Proposition 51. *Given a complex I the following conditions are equivalent* *(i) I is hoinjective.*

(ii) For any complex X the map $Hom_{K(\mathcal{A})}(X, I) \longrightarrow Hom_{\mathfrak{D}(\mathcal{A})}(X, I)$ is an isomorphism.

(iii) For any diagram of morphisms of complexes

$$X \xleftarrow{\ s\ } W \xrightarrow{\ f\ } I$$

with s a quasi-isomorphism, there is a morphism of complexes $g : X \longrightarrow I$ such that $gs \simeq f$.

(iv) Every quasi-isomorphism of complexes $I \longrightarrow Y$ is a coretraction in $K(\mathcal{A})$.

Proof. The implication $(i) \Rightarrow (ii)$ is Corollary 50 and $(ii) \Rightarrow (iii), (iii) \Rightarrow (iv)$ are trivial. See (TRC,Proposition 12) for a list of conditions on a morphism in $K(\mathcal{A})$ which are equivalent to being a coretraction. Suppose that (iv) is satisfied and let $f : C \longrightarrow I$ be a morphism of complexes with C exact. We can extend this to a triangle in $K(\mathcal{A})$

$$C \longrightarrow I \longrightarrow Z \longrightarrow \Sigma C$$

The morphism $I \longrightarrow Z$ is therefore a quasi-isomorphism, which by (iv) is a coretraction and so by (TRC,Proposition 12) we can conclude that $C \longrightarrow I$ is zero, as required. □

Lemma 52. *A morphism of hoinjective complexes $\varphi : I \longrightarrow J$ is a homotopy equivalence if and only if it is a quasi-isomorphism.*

Proposition 53. *Given a complex P the following conditions are equivalent*

(i) P is hoprojective.

(ii) For any complex X the map $Hom_{K(\mathcal{A})}(P, X) \longrightarrow Hom_{\mathfrak{D}(\mathcal{A})}(P, X)$ is an isomorphism.

(iii) For any diagram of morphisms of complexes

$$P \xrightarrow{\ f\ } W \xleftarrow{\ s\ } X$$

with s a quasi-isomorphism, there is a morphism of complexes $g : P \longrightarrow X$ such that $sg \simeq f$.

(iv) Every quasi-isomorphism of complexes $Y \longrightarrow P$ is a retraction in $K(\mathcal{A})$.

Lemma 54. *A morphism of hoprojective complexes $\varphi : P \longrightarrow Q$ is a homotopy equivalence if and only if it is a quasi-isomorphism.*

Let $\mathcal{T}$ be a triangulated category, $S \subseteq \mathcal{T}$ a nonempty class of objects. We define $\langle S \rangle$ to be the smallest localising subcategory of $\mathcal{T}$ containing the objects of S. That is, it is intersection of every such subcategory. Given a family of objects $\{E_\lambda\}_{\lambda \in \Lambda}$ we denote $\langle \{E_\lambda\}_\lambda \rangle$ by $\langle E_\lambda \rangle_{\lambda \in \Lambda}$. Categories of this form are discussed in our Triangulated Categories Part II (TRC2) notes, but we do not require any of these results. Similarly one defines the smallest colocalising subcategory of $\mathcal{T}$ containing S.

Lemma 55. *Let $\mathcal{T}$ be a triangulated category and S a nonempty class of objects of $\mathcal{T}$. An object B is $\langle S \rangle$-local if and only if $Hom_{\mathcal{T}}(\Sigma^j Y, B) = 0$ for every $Y \in S$ and $j \in \mathbb{Z}$.*

Proof. The condition is clearly necessary. Suppose now that $Hom(\Sigma^j Y, B) = 0$ for every $j \in \mathbb{Z}$ and $Y \in S$. Denote by $\mathcal{S}$ the full subcategory of $\mathcal{T}$ whose objects X are such that $Hom(\Sigma^j X, B) = 0$ for every $j \in \mathbb{Z}$. Clearly $\mathcal{S}$ is closed under Σ^{-1} and an argument involving long exact sequences shows that it is also closed under mapping cones. It is therefore a triangulated subcategory of $\mathcal{T}$. If we are given a nonempty family of objects $\{X_i\}_{i\in I}$ of $\mathcal{S}$ and a coproduct $\bigoplus_{i\in I} X_i$ in $\mathcal{T}$ then

$$Hom\left(\Sigma^j \bigoplus_{i\in I} X_i, B\right) \cong \prod_{i\in I} Hom(\Sigma^j X_i, B) = 0$$

so $\mathcal{S}$ is localising. Since it contains S we must have $\langle S\rangle \subseteq \mathcal{S}$ which completes the proof. □

Lemma 56. *Let $\mathcal{T}$ be a triangulated category and S a nonempty class of objects of $\mathcal{T}$. Let $\mathcal{L}$ be the smallest colocalising subcategory of $\mathcal{T}$ containing the objects of S. Then an object B is $\mathcal{L}$-colocal if and only if $Hom_{\mathcal{T}}(B, \Sigma^j Y) = 0$ for every $Y \in S$ and $j \in \mathbb{Z}$.*

Proposition 57. *Let $\mathcal{A}$ be an abelian category. For any object $A \in \mathcal{A}$ and complex X in $\mathcal{A}$ there is a canonical morphism of abelian groups natural in both variables*

$$\zeta : Hom_{K(\mathcal{A})}(c_i(A), X) \longrightarrow Hom_{\mathcal{A}}(A, H^i(X))$$

which is an isomorphism if A is projective. Dually there is a canonical morphism of abelian groups natural in both variables

$$\omega : Hom_{K(\mathcal{A})}(X, c_i(A)) \longrightarrow Hom_{\mathcal{A}}(H^i(X), A)$$

which is an isomorphism if A is injective.

Proof. We have an exact sequence of abelian groups

$$0 \longrightarrow Hom_{\mathcal{A}}(A, Im\partial_X^{i-1}) \longrightarrow Hom_{\mathcal{A}}(A, Ker\partial_X^i) \xrightarrow{\alpha} Hom_{\mathcal{A}}(A, H^i(X)) \tag{28}$$

Therefore the isomorphism of Lemma 31 sends null-homotopic morphisms into the kernel of α. We deduce a unique morphism of abelian groups ζ making the following diagram commute

$$\begin{array}{ccc} Hom_{\mathbf{C}(\mathcal{A})}(c_i(A), X) & \Longrightarrow & Hom_{\mathcal{A}}(A, Ker\partial_X^i) \\ \downarrow & & \downarrow \alpha \\ Hom_{K(\mathcal{A})}(c_i(A), X) & \xrightarrow[\zeta]{} & Hom_{\mathcal{A}}(A, H^i(X)) \end{array} \tag{29}$$

which is clearly natural in A and X. If A is projective then the last morphism of (28) is an epimorphism, and moreover the image of the first morphism is *precisely* the set of morphisms $A \longrightarrow Ker\partial_X^i$ factoring through the morphism $X^{i-1} \longrightarrow Ker\partial_X^i$. It follows that ζ is an isomorphism, as required.

For the second statement, consider the following commutative diagram containing the second isomorphism of Lemma 31

$$\begin{array}{ccc} Hom_{\mathbf{C}(\mathcal{A})}(X, c_i(A)) & \Longrightarrow & Hom_{\mathcal{A}}(Coker\partial_X^{i-1}, A) \\ \downarrow & & \downarrow \\ Hom_{K(\mathcal{A})}(X, c_i(A)) & \xrightarrow[\omega]{} & Hom_{\mathcal{A}}(H^i(X), A) \end{array} \tag{30}$$

where we have an induced morphism ω because the right hand morphism sends the image of null-homotopic morphisms to zero. This morphism is clearly natural in A and X. If A is injective then we have an exact sequence

$$Hom_{\mathcal{A}}(Ker\partial_X^{i+1}, A) \longrightarrow Hom_{\mathcal{A}}(Coker\partial_X^{i-1}, A) \longrightarrow Hom_{\mathcal{A}}(H^i(X), A) \longrightarrow 0$$

and an epimorphism $Hom_{\mathcal{A}}(X^{i+1}, A) \longrightarrow Hom_{\mathcal{A}}(Ker\partial_X^{i+1}, A)$, so the right hand morphism in (30) is an epimorphism whose kernel is the image of the top morphism. We deduce that ω is an isomorphism, as claimed. □

Remark 21. If P is projective then $c_i(P)$ is hoprojective, so we deduce a canonical isomorphism of abelian groups natural in both variables $Hom_{\mathfrak{D}(\mathcal{A})}(c_i(P), X) \longrightarrow Hom_{\mathcal{A}}(P, H^i(X))$. Dually if I injective then we have a canonical natural isomorphism $Hom_{\mathfrak{D}(\mathcal{A})}(X, c_i(I)) \longrightarrow Hom_{\mathcal{A}}(H^i(X), I)$.

Example 3. Let R be a ring (not necessarily commutative) and set $\mathcal{A} = \mathbf{Mod}R$. It follows from Proposition 57 that for every $i \in \mathbb{Z}$ and complex X of R-modules there is a canonical isomorphism of abelian groups $Hom_{\mathfrak{D}(R)}(c_i(R), X) \longrightarrow H^i(X)$ natural in X.

Lemma 58. *Let $\mathcal{A}$ be an abelian category with exact coproducts. If an object A is compact in $\mathcal{A}$, then it is also compact in $K(\mathcal{A})$.*

Proof. If a complex is compact in $\mathbf{C}(\mathcal{A})$ then it is also compact in $K(\mathcal{A})$ (provided $\mathcal{A}$ is cocomplete), so it suffices to show that A is compact in $\mathbf{C}(\mathcal{A})$. Then by Lemma 31 for any family of complexes $\{X_i\}_{i\in I}$ we have

$$\begin{aligned} Hom_{\mathbf{C}(\mathcal{A})}(A, \oplus_i X_i) &\cong Hom_{\mathcal{A}}(A, \oplus_i Ker\partial_i^0) \\ &\cong \bigoplus_i Hom_{\mathcal{A}}(A, Ker\partial_i^0) \\ &\cong \bigoplus_i Hom_{\mathbf{C}(\mathcal{A})}(A, X_i) \end{aligned}$$

since by hypothesis coproducts preserve kernels. This shows that A is compact as a complex, as claimed. □

Lemma 59. *Let $\mathcal{A}$ be an abelian category with generating family $\{U_\lambda\}_{\lambda\in\Lambda}$ and let $\mathcal{L} = \langle U_\lambda \rangle_{\lambda\in\Lambda}$ be the smallest localising subcategory of $K(\mathcal{A})$ containing these generators. Then $\mathcal{L}^\perp \subseteq \mathcal{Z}$, with equality if all the U_λ are projective.*

Proof. We consider each U_λ as a complex in degree zero in the usual way. Let F be a complex in $\mathcal{A}$ that belongs to $\mathcal{L}^\perp$. By Lemma 31 for any $i \in \mathbb{Z}$ every morphism $U_\lambda \longrightarrow Ker\partial_F^i$ factors through F^{i-1}. That is, F is exact and therefore belongs to $\mathcal{Z}$. Now assume that all the U_λ are projective and let X be an exact complex. It is easy to see that every morphism $U_\lambda \longrightarrow Ker\partial_X^i$ must factor through X^{i-1}, so $Hom_{K(\mathcal{A})}(c_i(U_\lambda), X) = 0$. Since $c_i(U_\lambda) = \Sigma^{-i} U_\lambda$ it follows from Lemma 55 that $X \in \mathcal{L}^\perp$, as required. □

Lemma 60. *Let $\mathcal{A}$ be an abelian category with cogenerating family $\{V_\lambda\}_{\lambda\in\Lambda}$ and let $\mathcal{L}$ be the smallest colocalising subcategory of $K(\mathcal{A})$ containing these cogenerators. Then ${}^\perp\mathcal{L} \subseteq \mathcal{Z}$, with equality if all the V_λ are injective.*

Definition 25. Let $\mathcal{A}$ be an abelian category and X a complex in $\mathcal{A}$. A *homotopy projective* (or *hoprojective*) *resolution* of X is morphism of complexes $\vartheta : P \longrightarrow X$ with P a hoprojective complex, which fits into a triangle in $K(\mathcal{A})$

$$P \longrightarrow X \longrightarrow Z \longrightarrow \Sigma P$$

with Z exact. Equivalently, ϑ is a quasi-isomorphism. A *homotopy injective* (or *hoinjective*) *resolution* of X is a morphism of complexes $\rho : X \longrightarrow I$ with I a hoinjective complex, which fits into a triangle in $K(\mathcal{A})$

$$Z \longrightarrow X \longrightarrow I \longrightarrow \Sigma Z$$

with Z exact. Equivalently, ρ is a quasi-isomorphism.

Definition 26. Let $\mathcal{A}$ be an abelian category. If every complex in $\mathcal{A}$ has a hoprojective resolution then we say that $\mathcal{A}$ *has enough hoprojectives*. If every complex in $\mathcal{A}$ has a hoinjective resolution then we say that $\mathcal{A}$ *has enough hoinjectives*.

Remark 22. Let $\mathcal{A}$ be an abelian category and X a complex in $\mathcal{A}$. It follows from Corollary 50 that a hoprojective resolution exists for X if and only if X is isomorphic in $\mathfrak{D}(\mathcal{A})$ to an object of $K(P)$. Similarly a hoinjective resolution exists for X if and only if X is isomorphic in $\mathfrak{D}(\mathcal{A})$ to an object of $K(I)$.

Example 4. The concept of a hoprojective resolution of a complex is an elegant generalisation of the usual projective resolution of an object. Let X be an object of our abelian category $\mathcal{A}$ and suppose we have a projective resolution

$$\cdots \longrightarrow P_{-2} \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

which to be precise consists of an acyclic (that is, exact at $i < 0$) projective complex P with $P^i = 0$ for $i > 0$, and a morphism $j : P_0 \longrightarrow X$ making the above sequence exact in $\mathcal{A}$. Another way of encapsulating this condition is to say that j defines a quasi-isomorphism from P to the complex X concentrated in degree zero.

Lemma 61. *Let $\mathcal{A}$ be an abelian category with enough hoinjectives or hoprojectives. Then the portly triangulated category $\mathfrak{D}(\mathcal{A})$ has small morphism conglomerates.*

Proof. For definiteness assume that $\mathcal{A}$ has enough hoinjectives. Given complexes X, Y in $\mathcal{A}$, let $\nu : Y \longrightarrow I$ be a hoinjective resolution of Y. Then we have by Corollary 50 a bijection

$$Hom_{\mathfrak{D}(\mathcal{A})}(X, Y) \cong Hom_{\mathfrak{D}(\mathcal{A})}(X, I) \cong Hom_{K(\mathcal{A})}(X, I)$$

and this latter conglomerate is certainly small, completing the proof. □

It is standard that a functor between abelian categories which has an exact left adjoint preserves injectives. In the same way, we show that such a functor also preserves hoinjectives.

Lemma 62. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories that has an exact left adjoint. Then $K(F) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ preserves hoinjectives.*

Proof. Let G be an exact left adjoint of F. By 25 the functor $K(G)$ is left adjoint to $K(F)$, so for an exact complex Z in $\mathcal{B}$ and hoinjective complex I in $\mathcal{A}$ we have

$$Hom_{K(\mathcal{B})}(Z, FI) \cong Hom_{K(\mathcal{A})}(GZ, I) = 0$$

since G is exact. This proves that FI is hoinjective, as required. □

4.1 Homotopy Limits and Colimits

It is common in homological algebra to prove a statement by giving a proof for finite subobjects, and then passing to the direct limit. The same idea is useful in the derived category, where we give a proof for bounded complexes and then pass to various direct limits to prove the result for unbounded complexes. Realising this intuitive idea on the level of direct limits of complexes leads to unnecessary technical complications [Spa88], and it became clear from the work of Bökstedt and Neeman [BN93] that the correct tool is the *homotopy colimit*. This is analogous to how the homotopy kernel and cokernel replace the ordinary kernel and cokernel. However, there is still something to be learnt from the interplay between the homotopy colimit and ordinary direct limit.

To motivate the definition of the homotopy colimit, we give a construction of the usual direct limit as a cokernel. Then by replacing the cokernel by a homotopy cokernel, we will obtain the homotopy colimit.

Remark 23. The set $\mathbb{N} = \{0, 1, 2, \ldots\}$ is a directed set in the canonical way (with minimum 0). Let $\mathcal{A}$ be a cocomplete abelian category and suppose we are given a direct system $\{G_n, \mu_{nm}\}_{n \in \mathbb{N}}$ over this directed set. This is just a sequence of objects and morphisms in $\mathcal{A}$ (writing μ_n for $\mu_{n(n+1)}$)

$$G_0 \xrightarrow{\mu_0} G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \longrightarrow \cdots \tag{31}$$

Let $\nu : \bigoplus_{n \in \mathbb{N}} G_n \longrightarrow \bigoplus_{n \in \mathbb{N}} G_n$ be the morphism induced out of the first coproduct by the morphisms $\mu_n : G_n \longrightarrow G_{n+1}$. That is, $\nu u_n = u_{n+1} \mu_n$ where u_n is the injection of G_n into the coproduct. Given a cokernel $\bigoplus_{n \in \mathbb{N}} G_n \longrightarrow C$ of the morphism $1 - \nu$ it is clear that the composites $G_n \longrightarrow \bigoplus_{n \in \mathbb{N}} G_n \longrightarrow C$ are a colimit for direct system $\{G_n, \mu_{nm}\}_{n \in \mathbb{N}}$.

Lemma 63. *Let $\mathcal{A}$ be a cocomplete abelian category and suppose we have a sequence (31). Then*

(i) If $\mathcal{A}$ is grothendieck abelian then $1-\nu$ is a monomorphism.

(ii) If every μ_n is a coretraction then $1-\nu$ is a coretraction.

(iii) If (31) is eventually constant, then $1-\nu$ is a coretraction.

Proof. (i) In any grothendieck abelian category the canonical morphism $\bigoplus_{i\in I} X_i \longrightarrow \prod_{i\in I} X_i$ is a monomorphism (see Mitchell III Corollary 1.3). Suppose we are given the direct system $\{G_n, \mu_{nm}\}_{n\in\mathbb{N}}$ and a morphism $\alpha : Y \longrightarrow \bigoplus_{n\in\mathbb{N}} G_n$ with $(1-\nu)\alpha = 0$. Denote by u_n, p_n the injections and projections for the coproduct $\bigoplus_{n\in\mathbb{N}} G_n$. It suffices to show that the morphism $\alpha_n = p_n\alpha$ is zero for every $n \geq 0$. But from $\alpha = \nu\alpha$ we deduce that $\alpha_n = p_n\nu\alpha$ for every $n \geq 0$. Since

$$p_n\nu\alpha = \begin{cases} \mu_{n-1}\alpha_{n-1} & n > 0 \\ 0 & n = 0 \end{cases}$$

we have $\alpha_0 = 0$ and $\alpha_n = \mu_{n-1}\alpha_{n-1}$ for $n > 0$. This implies $\alpha_n = 0$ for $n > 0$, as required. (ii) is straightforward to check. (iii) When we say that (31) is eventually constant, we mean that there exists $N \geq 0$ such that μ_j is an isomorphism for all $j \geq N$. We need to construct a morphism κ with components $\kappa_n : G_n \longrightarrow \oplus_{n\in\mathbb{N}} G_n$ satisfying the equations

$$\kappa_n - \kappa_{n+1}\mu_n = u_n \quad n \geq 0$$

First set $\kappa_N = u_N$ and define the morphisms $\kappa_{N-1}, \kappa_{N-2}, \ldots, \kappa_0$ by $\kappa_j = u_j + \kappa_{j+1}\mu_j$. For $j > N$ we define recursively $\kappa_j = (\kappa_{j-1} - u_{j-1})\mu_{j-1}^{-1}$. It is now clear that $\kappa(1-\nu) = 1$ as required.

More generally if (31) is an arbitrary coproduct of sequences for which individually the morphism $1-\nu$ is a coretraction, the morphism $1-\nu$ for the coproduct sequence is also a coretraction. □

Now we upgrade these statements to complexes. First we introduce some notation.

Definition 27. Let $\mathcal{A}$ be an abelian category and $f : X \longrightarrow Y$ a morphism of complexes. We say that f is a *fibration* or a *retraction in each degree* if $f^i : X^i \longrightarrow Y^i$ is a retraction in $\mathcal{A}$ for every $i \in \mathbb{Z}$. Dually we say that f is a *cofibration* or a *coretraction in each degree* if $f^i : X^i \longrightarrow Y^i$ is a coretraction for every $i \in \mathbb{Z}$. Clearly a fibration is an epimorphism and a cofibration is a monomorphism.

Definition 28. Let $\mathcal{A}$ be a cocomplete abelian category, and suppose we have a sequence of morphisms of complexes in $\mathcal{A}$

$$G_0 \xrightarrow{\mu_0} G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \longrightarrow \cdots \tag{32}$$

Then $\mathbf{C}(\mathcal{A})$ is a cocomplete abelian category, so as in Remark 23 we can define a morphism $\nu : \bigoplus_{n\in\mathbb{N}} G_n \longrightarrow \bigoplus_{n\in\mathbb{N}} G_n$. The *homotopy colimit* of (32) is the mapping cone $\bigoplus_{n\in\mathbb{N}} G_n \longrightarrow C_{1-\nu}$, and we denote the object $C_{1-\nu}$ by $\underrightarrow{holim} G_n$. It follows that we have a triangle in $K(\mathcal{A})$

$$\bigoplus_{n\in\mathbb{N}} G_n \xrightarrow{1-\nu} \bigoplus_{n\in\mathbb{N}} G_n \longrightarrow \underrightarrow{holim} G_n \longrightarrow \Sigma \bigoplus_{n\in\mathbb{N}} G_n$$

so that $\underrightarrow{holim} G_n$ is a homotopy colimit in the more general sense of (TRC,Definition 34). Given a morphism of direct systems of complexes

$$\begin{array}{ccccccccc} G_0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle\psi_0} & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\psi_3} & & \\ H_0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & \cdots \end{array}$$

We have a commutative diagram of complexes

$$\begin{array}{ccccc} \bigoplus_{n\in\mathbb{N}} G_n & \xrightarrow{1-\nu} & \bigoplus_{n\in\mathbb{N}} G_n & \longrightarrow & \underrightarrow{holim} G_n \\ {\scriptstyle \oplus\psi^n}\downarrow & & \downarrow{\scriptstyle \oplus\psi_n} & & \vdots\, {\scriptstyle \underrightarrow{holim}\psi_n} \\ \bigoplus_{n\in\mathbb{N}} H_n & \xrightarrow[1-\nu]{} & \bigoplus_{n\in\mathbb{N}} H_n & \xrightarrow[v]{} & \underrightarrow{holim} H_n \end{array}$$

Since the composite $v\circ\oplus\psi_n\circ(1-\nu)$ is canonically homotopic to zero, there is a canonical morphism $\underrightarrow{holim}\psi_n$ making the diagram commute.

Remark 24. Suppose we are in the situation of Definition 28. The canonical morphism of complexes $\bigoplus_{n\in\mathbb{N}} G_n \longrightarrow \varinjlim G_n$ composes with $1-\nu$ to give zero, so from the basic property of the homotopy cokernel we deduce a canonical morphism of complexes $f : \underrightarrow{holim} G_n \longrightarrow \varinjlim G_n$ making the following diagram commute

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{N}} G_n & \longrightarrow & \underrightarrow{holim} G_n \\ & \searrow & \downarrow{\scriptstyle f} \\ & & \varinjlim G_n \end{array}$$

As we will show in a moment, $1-\nu$ is often a monomorphism. In this case it follows from Remark 6 that f is a quasi-isomorphism.

Remark 25. We have now defined the canonical hocolimit of a sequence of complexes. One defines hocolimits in an arbitrary triangulated category analogously (TRC,Section 3) and our canonical hocolimit is a hocolimit in $K(\mathcal{A})$ in this broader sense. In many applications an arbitrary hocolimit will do, but sometimes (for example in Remark 24) the canonical one is necessary. Hocolimits only agree in $K(\mathcal{A})$ up to noncanonical isomorphism, so occasionally this distinction is significant.

Lemma 64. *Let $\mathcal{A}$ be a cocomplete abelian category and suppose we have a sequence of* complexes *of the form* (32)*. Then*

(i) If $\mathcal{A}$ is grothendieck abelian then $1-\nu$ is a monomorphism.

(ii) If every μ_n is a cofibration then $1-\nu$ is a cofibration, and in particular it is a monomorphism.

(iii) If (32) *is eventually constant in each degree, then $1-\nu$ is a cofibration.*

Proof. (i) If $\mathcal{A}$ is grothendieck abelian then so is $\mathbf{C}(\mathcal{A})$ by Proposition 41, so this follows from Lemma 63 with $\mathbf{C}(\mathcal{A})$ in the place of $\mathcal{A}$. (ii) Applying Lemma 63 in each degree we deduce that $1-\nu : \bigoplus_{n\in\mathbb{N}} G_n \longrightarrow \bigoplus_{n\in\mathbb{N}} G_n$ is a coretraction in each degree. (iii) When we say that (32) is eventually constant in each degree, we mean that for each $i \in \mathbb{Z}$ there is an integer $N_i \geq 0$ (that can vary with i) such that μ^i_j is an isomorphism for all $j \geq N_i$. In that case it follows from Lemma 63(iii) that $1-\nu$ is a coretraction. □

Proposition 65. *Let $\mathcal{A}$ be a cocomplete abelian category, $\{G_n, \mu_{nm}\}_{n\in\mathbb{N}}$ a direct system in $\mathbf{C}(\mathcal{A})$ and suppose any of the conditions of Lemma 64 are satisfied. Then there is a canonical triangle in $\mathfrak{D}(\mathcal{A})$*

$$\bigoplus_{n\in\mathbb{N}} G_n \xrightarrow{1-\nu} \bigoplus_{n\in\mathbb{N}} G_n \longrightarrow \varinjlim_{n\in\mathbb{N}} G_n \longrightarrow \Sigma(\bigoplus_{n\in\mathbb{N}} G_n)$$

It follows that if $\mathcal{L}$ is a localising subcategory of $\mathfrak{D}(\mathcal{A})$ with $G_n \in \mathcal{L}$ for every $n \geq 0$, then we have $\varinjlim_{n\in\mathbb{N}} G_n \in \mathcal{L}$.

Proof. As before we consider $\mathbb{N}$ as a directed set in the obvious way. By hypothesis we have a short exact sequence in $\mathbf{C}(\mathcal{A})$

$$0 \longrightarrow \bigoplus_{n\in\mathbb{N}} G_n \xrightarrow{1-\nu} \bigoplus_{n\in\mathbb{N}} G_n \longrightarrow \varinjlim_{n\in\mathbb{N}} G_n \longrightarrow 0$$

By Proposition 20 this leads to a triangle in $\mathfrak{D}(\mathcal{A})$ of the required form. For what we mean by a localising subcategory of $\mathfrak{D}(\mathcal{A})$ (which is after all only a portly triangulated category) see (TRC,Section 7). By Proposition 44 the functor $\mathbf{C}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ preserves coproducts, so we can use the above triangle and the fact that $\mathcal{L}$ is localising to conclude that $\varinjlim_{n\in\mathbb{N}} G_n \in \mathcal{L}$. ☐

Next we study the dual notion of a homotopy limit. As before, we motivate it by giving an alternative construction of the usual inverse limit and replacing kernels by homotopy kernels.

Remark 26. The set $\mathbb{N}^{\text{op}} = \{0, 1, 2, \ldots\}$ is an inverse directed set in the canonical way (with maximum 0). Let $\mathcal{A}$ be a complete abelian category and suppose we are given an inverse system $\{P_n, \mu_{nm}\}_{n\in\mathbb{N}^{\text{op}}}$ over this inverse directed set. This is just a sequence of objects and morphisms in $\mathcal{A}$ (writing μ_n for $\mu_{n(n-1)}$)

$$\cdots \longrightarrow P_3 \xrightarrow{\mu_3} P_2 \xrightarrow{\mu_2} P_1 \xrightarrow{\mu_1} P_0 \tag{33}$$

Let $\nu : \prod_{n\in\mathbb{N}} P_n \longrightarrow \prod_{n\in\mathbb{N}} P_n$ be the morphism induced into the second product by the morphisms $\mu_{n+1} : P_{n+1} \longrightarrow P_n$. That is, $p_n\nu = \mu_{n+1}p_{n+1}$ where p_n is the projection onto P_n out of the product. Given a kernel $K \longrightarrow \prod_{n\in\mathbb{N}} P_n$ of the morphism $1-\nu$ it is clear that the composites $K \longrightarrow \prod_{n\in\mathbb{N}} P_n \longrightarrow P_n$ are a limit for the inverse system $\{P_n, \mu_{nm}\}_{n\in\mathbb{N}}$.

Lemma 66. *Let $\mathcal{A}$ be a complete abelian category and suppose we have a sequence (33). Then*

(i) If $\mathcal{A}$ has a family of projective generators and μ_n is an epimorphism for all sufficiently large n, then $1-\nu$ is an epimorphism.

(ii) If every μ_n is a retraction then $1-\nu$ is a retraction.

Proof. (i) It suffices to show that every morphism $x : U \longrightarrow \prod_{n\in\mathbb{N}} P_n$ factors through $1-\nu$, for every object U of the family of projective generators. Let such a morphism x be given, and write x_n for $p_n x$. We have to solve recursively the following equations for the unknowns $a_i : U \longrightarrow P_i$

$$\begin{aligned} a_0 - \mu_1 a_1 &= x_0 \\ a_1 - \mu_2 a_2 &= x_1 \\ a_2 - \mu_3 a_3 &= x_2 \\ &\vdots \end{aligned}$$

Suppose for the moment that every μ_n is an epimorphism. We begin by setting $a_0 = 0$ and choosing a morphism $a_1 : U \longrightarrow P_1$ with $\mu_1 a_1 = -x_0$. Then choose a morphism $a_2 : U \longrightarrow P_2$ with $\mu_2 a_2 = a_1 - x_1$ and so on. Using a simple Zorn's Lemma argument, it is clear that we can construct the morphisms a_i satisfying all of these equations. Together these induce a morphism $a : U \longrightarrow \prod_{n\in\mathbb{N}} P_n$ with $(1-\nu)a = x$, as required.

Now we proceed to the general case. Let $N \geq 1$ be such that μ_n is an epimorphism for all $n \geq N$. Writing a morphism $x : U \longrightarrow \prod_{n\in\mathbb{N}} P_n$ as a matrix of its components, it is clear that

$$(1-\nu)\begin{pmatrix} \mu_1\mu_2\cdots\mu_k x_k \\ \mu_2\cdots\mu_k x_k \\ \vdots \\ \mu_k x_k \\ x_k \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_k \\ 0 \\ \vdots \end{pmatrix}$$

so any matrices with single nonzero entries are in the image of $1-\nu$. We can therefore individually kill off a finite number of initial terms of any given matrix, and reduce to showing that the high order terms are in the image: this is possible because for large n, the μ_n are epimorphisms, so we can use the recursive construction above.

(ii) The argument is much the same if each μ_n is a retraction. ☐

Now we upgrade these statements to complexes.

Definition 29. Let $\mathcal{A}$ be a complete abelian category, and suppose we have a sequence of morphisms of complexes in $\mathcal{A}$

$$\cdots \longrightarrow P_3 \xrightarrow{\mu_3} P_2 \xrightarrow{\mu_2} P_1 \xrightarrow{\mu_1} P_0 \tag{34}$$

Then $\mathbf{C}(\mathcal{A})$ is a complete abelian category, so as in Remark 26 we can define a morphism $\nu : \prod_{n\in\mathbb{N}} P_n \longrightarrow \prod_{n\in\mathbb{N}} P_n$. The *homotopy limit* of (34) is the homotopy kernel of $1-\nu$, which we denote by $\underleftarrow{holim} P_n$. We have a triangle in $K(\mathcal{A})$

$$\underleftarrow{holim} P_n \longrightarrow \prod_{n\in\mathbb{N}} P_n \xrightarrow{1-\nu} \prod_{n\in\mathbb{N}} P_n \longrightarrow \Sigma \underleftarrow{holim} P_n$$

Given a morphism of inverse systems of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \\ & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} \\ \cdots & \longrightarrow & Q_3 & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 \end{array}$$

We have a commutative diagram of complexes

$$\begin{array}{ccccc} \underleftarrow{holim} P_n & \xrightarrow{k} & \prod_{n\in\mathbb{N}} P_n & \xrightarrow{1-\nu} & \prod_{n\in\mathbb{N}} P_n \\ {\scriptstyle \underleftarrow{holim}\psi_n}\downarrow & & \downarrow{\scriptstyle \prod \psi_n} & & \downarrow{\scriptstyle \prod \psi_n} \\ \underleftarrow{holim} Q_n & \longrightarrow & \prod_{n\in\mathbb{N}} Q_n & \xrightarrow[1-\nu]{} & \prod_{n\in\mathbb{N}} Q_n \end{array}$$

Since the composite $(1-\nu)\circ \prod \psi_n \circ k$ is canonically homotopic to zero, there is a canonical morphism $\underleftarrow{holim}\psi_n$ making the diagram commute.

Remark 27. Suppose we are in the situation of Definition 29. The canonical morphism $\varprojlim P_n \longrightarrow \prod_{n\in\mathbb{N}} P_n$ composes with $1-\nu$ to give zero, so from the basic property of the homotopy kernel we deduce a canonical morphism of complexes $f : \varprojlim P_n \longrightarrow \underleftarrow{holim} P_n$ making the following diagram commute

$$\begin{array}{ccc} \underleftarrow{holim} P_n & \longrightarrow & \prod_{n\in\mathbb{N}} P_n \\ {\scriptstyle f}\uparrow & \nearrow & \\ \varprojlim P_n & & \end{array}$$

The morphism $1-\nu$ is sometimes an epimorphism, in which case it follows from Remark 6 that f is a quasi-isomorphism.

Remark 28. We have now defined the canonical holimit of a sequence of complexes. One defines holimits in an arbitrary triangulated category analogously (TRC,Definition 36) and our canonical holimit is a hocolimit in $K(\mathcal{A})$ in this broader sense.

Lemma 67. *Let $\mathcal{A}$ be a complete abelian category and suppose we have a sequence of* complexes *of the form* (34)*. Then*

(i) If $\mathcal{A}$ has a projective generator and every μ_n is an epimorphism, then $1-\nu$ is an epimorphism.

(ii) If every μ_n is a fibration then $1-\nu$ is a fibration, and in particular it is an epimorphism.

Proof. (i) If $\mathcal{A}$ has a projective generator then $\mathbf{C}(\mathcal{A})$ has a generating family of projectives, so this follows from Lemma 66. (ii) Applying Lemma 66 in each degree we deduce that $1-\nu : \prod_{n\in\mathbb{N}} P_n \longrightarrow \prod_{n\in\mathbb{N}} P_n$ is also a retraction in each degree. $\square$

4.2 Existence of Hoprojective Resolutions

In this section we show that given a class of objects $\mathcal{P}$ of an abelian category $\mathcal{A}$ with every object of $\mathcal{A}$ a quotient of an element of $\mathcal{P}$, you can construct a resolution of any bounded above complex by objects of $\mathcal{P}$. Taking hocolimits you obtain a resolution of any complex by a complex belonging to the smallest localising subcategory of $K(\mathcal{A})$ containing the objects of $\mathcal{P}$.

The most obvious application is with $\mathcal{P}$ equal to the projectives, in which case the category of hoprojectives is localising and contains $\mathcal{P}$, so we will have constructed hoprojective resolutions for any complex in an abelian category with enough projectives. It will be convenient later to have developed the material in slightly more generality.

Definition 30. Let $\mathcal{A}$ be an abelian category. A class $\mathcal{P} \subseteq \mathcal{A}$ is said to be *smothering* if it satisfies the following conditions:

(i) $\mathcal{P}$ is closed under isomorphism and contains all the zero objects.

(ii) Every object X in $\mathcal{A}$ admits an epimorphism $P \longrightarrow X$ for some $P \in \mathcal{P}$.

(iii) If $P, Q \in \mathcal{P}$ then $P \oplus Q \in \mathcal{P}$.

Obviously if $\mathcal{A}$ has enough projectives then the class of projective objects is smothering. A class $\mathcal{I} \subseteq \mathcal{A}$ is said to be *cosmothering* if it satisfies $(i), (iii)$ and the condition (ii') Every object X in $\mathcal{A}$ admits a monomorphism $X \longrightarrow I$ for some $I \in \mathcal{I}$. If $\mathcal{A}$ has enough injectives then the class of injective objects is cosmothering.

Let $\mathcal{A}$ be an abelian category. Given a class of objects $\mathcal{T} \subseteq \mathcal{A}$ we say that a complex $X \in \mathbf{C}(\mathcal{A})$ is a complex *in* $\mathcal{T}$ if $X^i \in \mathcal{T}$ for all $i \in \mathbb{Z}$.

Remark 29. If $\mathcal{P}$ is a smothering class then condition (iii) of smothering means that given a morphism of complexes $f : X \longrightarrow Y$ in $\mathcal{P}$ the mapping cone C_f is also a complex in $\mathcal{P}$.

Throughout this section $\mathcal{A}$ denotes an abelian category. The next technical lemma is useful in showing that various morphisms we will construct are quasi-isomorphisms.

Lemma 68. *Suppose we are given a pullback diagram in* $\mathcal{A}$

$$\begin{array}{ccc} A' & \xrightarrow{\varphi'} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

and an epimorphism $\psi : P \longrightarrow A'$*. If* $K \longrightarrow P$ *and* $Q \longrightarrow A$ *are kernels of* $\varphi'\psi$ *and* φ *respectively, then the induced morphism* $K \longrightarrow Q$ *is an epimorphism.*

Proof. By the embedding theorem (DCAC,Theorem 1) (DCAC,Lemma 2) we can reduce to the case where $\mathcal{A} = \mathbf{Ab}$ and A' the canonical pullback, defined to be the set of all pairs $(b, a) \in B' \times A$ mapping to the same element of B. We can also assume K, Q are the canonical kernels. Suppose we are given $a \in A$ with $\varphi(a) = 0$. Then the pair $(0, a) \in B' \times A$ belongs to A', so we can lift it to an element $b \in P$ with $\psi(b) = (0, a)$. By construction $\varphi'\psi(b) = 0$ so $b \in Q$. It is clear that the morphism $K \longrightarrow Q$ maps b to a, so the proof is complete. $\square$

Next we show that a smothering class yields resolutions, and moreover these resolutions can be chosen in a functorial way.

Proposition 69. *Let* $\mathcal{P}$ *be a smothering class for* $\mathcal{A}$*. Then*

(a) Every bounded above complex X *admits a quasi-isomorphism* $P \longrightarrow X$ *with* P *a bounded above complex in* $\mathcal{P}$*.*

(b) If $f : X \longrightarrow Y$ is a morphism of bounded above complexes, $u : P \longrightarrow X$ a quasi-isomorphism with P a bounded above complex in $\mathcal{P}$, then there exists a commutative diagram of complexes

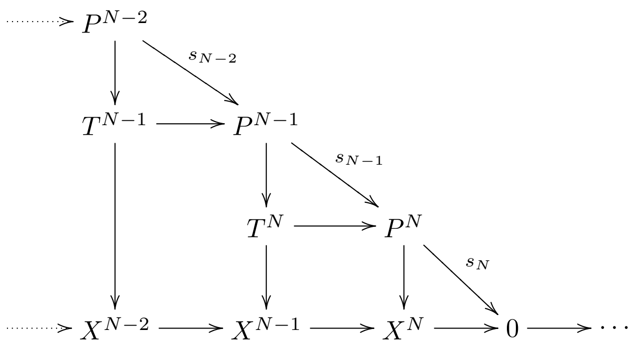
$$\begin{CD} P @>{m}>> Q \\ @V{u}VV @VV{v}V \\ X @>>{f}> Y \end{CD} \tag{35}$$

with $v : Q \longrightarrow Y$ a quasi-isomorphism and Q a bounded above complex in $\mathcal{P}$.

Proof. (a) Let X be a bounded above complex in $\mathcal{A}$, say $X^i = 0$ for all $i > N$. For $i > N$ we define $P^i = 0$. Choose any epimorphism $P^N \longrightarrow X^N$ with $P^N \in \mathcal{P}$. Suppose that for some $k < N$ we have constructed an object $P^i \in \mathcal{P}$ and morphisms $s_i : P^i \longrightarrow P^{i+1}, P^i \longrightarrow X^i$ for every $i > k$ (we have just done this trivially for $k = N - 1$). Form the following pullback

$$\begin{CD} T^k @>>> Ker(s_{k+1}) \\ @VVV @VVV \\ X^k @>>> X^{k+1} \end{CD} \tag{36}$$

and choose an epimorphism $P^k \longrightarrow T^k$ with $P^k \in \mathcal{P}$. Let $s_k : P^k \longrightarrow P^{k+1}$ be the composition $P^k \longrightarrow T^k \longrightarrow Ker(s_{k+1}) \longrightarrow P^{k+1}$ and define $P^k \longrightarrow X^k$ in the obvious way. The first few steps of this construction are outlined in the following diagram



We have recursively constructed a bounded above complex P in $\mathcal{P}$ together with a morphism of complexes $P \longrightarrow X$ (of course, as with any recursive construction involving non-canonical choices, we need a small Zorn's Lemma argument that the reader can easily provide). It remains to show that $P \longrightarrow X$ is a quasi-isomorphism. For $k < N$ the pullback (36) gives rise to a pullback

$$\begin{CD} Im(s_k) @>>> Ker(s_{k+1}) \\ @VVV @VVV \\ Im\partial_X^k @>>> Ker\partial_X^{k+1} \end{CD}$$

where the horizontal morphisms are monomorphisms. It follows from Lemma 68 that the vertical morphism on the right is an epimorphism, so by (AC,Lemma 35) the induced morphism on the cokernels $H^{k+1}(P) \longrightarrow H^{k+1}(X)$ must be an isomorphism.

(b) Take the homotopy kernel $C_{fu}[-1]$ of the composite $fu : P \longrightarrow X$ and find a bounded above complex T in $\mathcal{P}$ together with a quasi-isomorphism $T \longrightarrow C_{fu}[-1]$. Let g be the composite $g : T \longrightarrow C_{fu}[-1] \longrightarrow P$ and $m : P \longrightarrow Q = C_g$ the homotopy cokernel of g. We have the following diagram in $K(\mathcal{A})$ in which the rows are triangles

$$\begin{array}{ccccccc}
T & \xrightarrow{g} & P & \xrightarrow{m} & Q & \longrightarrow & \Sigma T \\
\downarrow & & \downarrow{\scriptstyle 1} & & \vdots{\scriptstyle v} & & \downarrow \\
\Sigma^{-1}C_{fu} & \longrightarrow & P & \xrightarrow[fu]{} & Y & \xrightarrow[-]{} & C_{fu}
\end{array}$$

which induces a morphism v making the diagram commute. The first two vertical morphisms are quasi-isomorphisms so we deduce from (TRC,Lemma 71) that v is a quasi-isomorphism as well. Since it is clear that Q is a bounded above complex in $\mathcal{P}$, the proof is complete. Observe that m is a coretraction in each degree, and in particular is a monomorphism. □

Corollary 70. *If $\mathcal{A}$ has enough projectives then every bounded above complex in $\mathcal{A}$ admits a quasi-isomorphism from a bounded above complex of projectives. In particular every bounded above complex has a hoprojective resolution.*

We are now ready to prove the existence of hoprojective resolutions for arbitrary complexes. Given a complex M, the idea is to take hoprojective resolutions for the bounded complexes $M_{\leq n}$ and then take the homotopy colimit to obtain a hoprojective resolution for the complex $M = \varinjlim_{n \geq 0} M_{\leq n}$.

Proposition 71. *Let $\mathcal{A}$ be a grothendieck abelian category, $\mathcal{P}$ a smothering class for $\mathcal{A}$ and $\langle \mathcal{P} \rangle$ the smallest localising subcategory of $K(X)$ containing every bounded above complex in $\mathcal{P}$. Then every complex X in $\mathcal{A}$ admits a quasi-isomorphism $P \longrightarrow X$ with $P \in \langle \mathcal{P} \rangle$.*

Proof. By Proposition 69 every bounded above complex admits a quasi-isomorphism from a bounded above complex in $\mathcal{P}$. Let M be any complex in $\mathcal{A}$ and for $n \geq 0$ let $M_{\leq n}$ denote the truncated complex of Definition 14. Since this complex is bounded above, we can find a bounded above complex P_n in $\mathcal{P}$ and a quasi-isomorphism $\vartheta_n : P_n \longrightarrow M_{\leq n}$. In fact proceeding inductively and using Proposition 69(b) at each stage we can choose these resolutions in such a way that we have a commutative diagram of complexes

$$\begin{array}{ccccccccccc}
P_0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & \cdots & \longrightarrow & P_n & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
M_{\leq 0} & \longrightarrow & M_{\leq 1} & \longrightarrow & M_{\leq 2} & \longrightarrow & \cdots & \longrightarrow & M_{\leq n} & \longrightarrow & \cdots
\end{array}$$

There is an induced morphism of the homotopy colimits $\underrightarrow{holim} P_n \longrightarrow \underrightarrow{holim} M_{\leq n}$ fitting into a morphism of triangles in $K(\mathcal{A})$

$$\begin{array}{ccccccc}
\bigoplus_{n\geq 0} P_n & \xrightarrow{1-\nu} & \bigoplus_{n\geq 0} P_n & \longrightarrow & \underrightarrow{holim} P_n & \longrightarrow & \Sigma \bigoplus_{n\geq 0} P_n \\
{\scriptstyle \bigoplus_n \vartheta_n}\downarrow & & {\scriptstyle \bigoplus_n \vartheta_n}\downarrow & & {\scriptstyle \underrightarrow{holim}\vartheta_n}\downarrow & & \downarrow \\
\bigoplus_{n\geq 0} M_{\leq n} & \xrightarrow[1-\nu]{} & \bigoplus_{n\geq 0} M_{\leq n} & \longrightarrow & \underrightarrow{holim} M_{\leq n} & \longrightarrow & \Sigma \bigoplus_{n\geq 0} M_n
\end{array} \qquad (37)$$

Looking at the long exact cohomology sequence we deduce that $\underrightarrow{holim}\vartheta_n$ is a quasi-isomorphism. Since $\mathcal{A}$ is grothendieck abelian the canonical morphism of complexes $\underrightarrow{holim} M_{\leq n} \longrightarrow \varinjlim M_{\leq n} = M$ is a quasi-isomorphism. Composing with $\underrightarrow{holim}\vartheta_n$ yields a quasi-isomorphism of complexes $P = \underrightarrow{holim} P_n \longrightarrow M$. It is clear that $P \in \langle \mathcal{P} \rangle$, so the proof is complete. □

Corollary 72. *If $\mathcal{A}$ is grothendieck abelian and has enough projectives then every complex in $\mathcal{A}$ has a hoprojective resolution.*

Example 5. If we take $\mathcal{A}$ to be a category of abelian groups **Ab** or of modules R**Mod**, **Mod**R over a ring R then $\mathcal{A}$ is grothendieck abelian with enough projectives, so complexes of abelian groups and modules have hoprojective resolutions.

4.3 Existence of Hoinjective Resolutions

In Proposition 71 we showed that a grothendieck abelian category admits hoprojective resolutions for its complexes provided it has enough projectives. The existence of hoinjective resolutions is a more subtle question, and in this section we only manage to prove it under a rather strong hypothesis. Throughout this section $\mathcal{A}$ denotes an abelian category. As in the previous section, we develop the preliminary material in the generality of a *cosmothering class*.

Proposition 73. *Let $\mathcal{I}$ be a cosmothering class for $\mathcal{A}$. Then*

(a) Every bounded below complex X admits a quasi-isomorphism $X \longrightarrow I$ with I a bounded below complex in $\mathcal{I}$.

(b) If $f : Y \longrightarrow X$ is a morphism of bounded below complexes, $u : X \longrightarrow I$ a quasi-isomorphism with I a bounded below complex in $\mathcal{I}$, then there exists a commutative diagram of complexes

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ J & \xrightarrow[m]{} & I \end{array}$$

with $v : Y \longrightarrow J$ a quasi-isomorphism and J a bounded below complex in $\mathcal{I}$.

Proof. (a) The argument is dual to the argument of Proposition 69(a). Let X be a bounded below complex in $\mathcal{A}$, say $X^i = 0$ for all $i < N$. For $i < N$ we define $I^i = 0$. Choose any monomorphism $X^N \longrightarrow I^N$ with $I^N \in \mathcal{I}$. Suppose that for some $k > N$ we have constructed an object $I^i \in \mathcal{I}$ and morphisms $s_{i-1} : I^{i-1} \longrightarrow I^i, X^i \longrightarrow I^i$ for every $i < k$ (we have just done this trivially for $k = N + 1$). Form the following pushout

$$\begin{array}{ccc} X^{k-1} & \longrightarrow & X^k \\ \downarrow & & \downarrow \\ Coker(s_{k-2}) & \longrightarrow & S^k \end{array} \tag{38}$$

and choose a monomorphism $S^k \longrightarrow I^k$ with $I^k \in \mathcal{I}$. Let $s_{k-1} : I^{k-1} \longrightarrow I^k$ be the composition $I^{k-1} \longrightarrow Coker(s_{k-2}) \longrightarrow S^k \longrightarrow I^k$ and define $X^k \longrightarrow I^k$ in the obvious way. We have recursively constructed a bounded below complex I in $\mathcal{I}$ together with a morphism of complexes $X \longrightarrow I$. It remains to show that $X \longrightarrow I$ is a quasi-isomorphism. For $k > N$ the pushout (38) gives rise to a pushout

$$\begin{array}{ccc} Coker(\partial_X^{k-2}) & \longrightarrow & Im(\partial_X^{k-1}) \\ \downarrow & & \downarrow \\ Coker(s_{k-2}) & \longrightarrow & Im(s_{k-1}) \end{array}$$

where the horizontal morphisms are epimorphisms. The dual of Lemma 68 implies that the vertical morphism on the left is a monomorphism. By (DF,Lemma 25) the induced morphism on the kernels is $H^{k-1}(X) \longrightarrow H^{k-1}(I)$. The dual of (AC,Lemma 35) implies that this morphism is an isomorphism.

(b) Take the homotopy cokernel C_{uf} of the composite $Y \longrightarrow I$ and find a quasi-isomorphism $C_{uf} \longrightarrow Q$ with Q a bounded below in $\mathcal{I}$. Denote the composite $I \longrightarrow Q$ by g, and take the homotopy kernel $m : J = C_g[-1] \longrightarrow I$ of g. We have the following diagram in $K(\mathcal{A})$ in which the rows are triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{uf} & I & \longrightarrow & C_{uf} & \longrightarrow & \Sigma Y \\ {\scriptstyle v}\vdots & & {\scriptstyle 1}\downarrow & & \downarrow & & \vdots{\scriptstyle \Sigma v} \\ C_g[-1] & \xrightarrow[m]{} & I & \xrightarrow[g]{} & Q & \xrightarrow[-]{} & C_g \end{array}$$

which induces a morphism v making the diagram commute. From (TRC,Lemma 71) we deduce that v is a quasi-isomorphism, and it is clear that J is a bounded below complex in $\mathcal{I}$, so the proof is complete. Observe that the morphism m is actually a retraction in each degree, and is in particular an epimorphism. □

Remark 30. With the notation of Proposition 73(b) suppose that there exists $N \in \mathbb{Z}$ with $X^i = 0, I^i = 0$ for $i < N$ and $Y^i = 0$ for $i < N-1$. This is often the case in applications. Going through the proof we observe that J can be found such that $J^i = 0$ for $i < N - 1$.

Corollary 74. *If $\mathcal{A}$ has enough injectives then every bounded below complex in $\mathcal{A}$ admits a quasi-isomorphism into a bounded below complex of injectives. In particular every bounded below complex has a hoinjective resolution.*

We are now ready to prove our first result about the existence of hoinjective resolutions for arbitrary complexes. We work under unnecessarily restrictive hypothesis (essentially, modules over a ring) because this is all we need to bootstrap ourselves up in Section 7 to the full generality of an arbitrary grothendieck abelian category. See [BN93] for a more complete treatment of these intermediate results.

Given a complex M, the idea is to take hoinjective resolutions for the bounded complexes $M_{\geq n}$ and then take the homotopy limit to obtain a hoinjective resolution for the complex $M = \varprojlim_{n \leq 0} M_{\geq n}$.

Proposition 75. *Let $\mathcal{A}$ be an abelian category with exact products and a projective generator, $\mathcal{I}$ a cosmothering class for $\mathcal{A}$ and $\langle \mathcal{I} \rangle^{co}$ the smallest colocalising subcategory of $K(X)$ containing every bounded below complex in $\mathcal{I}$. Then every complex X in $\mathcal{A}$ admits a quasi-isomorphism $X \longrightarrow I$ with $I \in \langle \mathcal{I} \rangle^{co}$.*

Proof. See (AC,Definition 46) for what we mean by *has exact products*. By Proposition 73 every bounded below complex admits a quasi-isomorphism into a bounded below complex in $\mathcal{I}$. Let M be any complex in $\mathcal{A}$ and for $n \leq 0$ let $M_{\geq n}$ denote the truncated complex of Definition 15. Since this complex is bounded below, we can find a bounded below complex I_n in $\mathcal{I}$ and a quasi-isomorphism $\rho_n : M_{\geq n} \longrightarrow I_n$. In fact proceeding inductively and using Proposition 73(b) at each stage we can choose these resolutions in such a way that we have a commutative diagram of complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & M_{\geq n} & \longrightarrow & \cdots & \longrightarrow & M_{\geq -2} & \longrightarrow & M_{\geq -1} & \longrightarrow & M_{\geq 0} \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & I_n & \longrightarrow & \cdots & \longrightarrow & I_{-2} & \longrightarrow & I_{-1} & \longrightarrow & I_0 \end{array}$$

There is an induced morphism of the homotopy limits $\underleftarrow{holim} M_{\geq n} \longrightarrow \underleftarrow{holim} I_n$ fitting into a morphism of triangles in $K(\mathcal{A})$

$$\begin{array}{ccccccc} \underleftarrow{holim} M_{\geq n} & \longrightarrow & \prod_{n \leq 0} M_{\geq n} & \xrightarrow{1-\nu} & \prod_{n \leq 0} M_{\geq n} & \longrightarrow & \Sigma \underleftarrow{holim} M_{\geq n} \\ {\scriptstyle \underleftarrow{holim}\rho_n}\downarrow & & {\scriptstyle \prod_n \rho_n}\downarrow & & {\scriptstyle \prod_n \rho_n}\downarrow & & \downarrow \\ \underleftarrow{holim} I_n & \longrightarrow & \prod_{n \leq 0} I_n & \xrightarrow[1-\nu]{} & \prod_{n \leq 0} I_n & \longrightarrow & \Sigma \underleftarrow{holim} I_n \end{array}$$

Looking at the long exact cohomology sequence we deduce that $\underleftarrow{holim} \rho_n$ is a quasi-isomorphism (here we use the fact that $\mathcal{A}$ has exact products to see that arbitrary products of quasi-isomorphisms are quasi-isomorphisms). By Lemma 67(i) the canonical morphism of complexes $M = \varprojlim M_{\geq n} \longrightarrow \underleftarrow{holim} M_{\geq n}$ is a quasi-isomorphism. Composing with $\underleftarrow{holim} \rho_n$ yields a quasi-isomorphism of complexes $M \longrightarrow \underleftarrow{holim} I_n$. It is clear that $\underleftarrow{holim} I_n \in \langle \mathcal{I} \rangle^{co}$, so the proof is complete. □

Corollary 76. *If $\mathcal{A}$ is an abelian category with exact products, a projective generator and enough injectives then every complex in $\mathcal{A}$ has a hoinjective resolution.*

Remark 31. The reason why the going is harder for hoinjective resolutions (compare the hypothesis on Proposition 71 and Proposition 75) is that hoinjectives are not closed under coproducts in $K(\mathcal{A})$. If they were, we could replace $M_{\geq n}$ by the brutal truncation ${}_bM_{\geq n}$ and then replace the inverse limits by direct limits. See Proposition 92 for more details.

4.4 Remark on Homotopy Limits

In this short section we prove a useful technical lemma from [BN93]. It will not be used until our notes on Derived Categories of Sheaves (DCOS), so can be safely skipped on a first reading. Throughout this section let $\mathcal{A}$ be an abelian category with a family of projective generators and exact products. In this case the portly triangulated category $\mathfrak{D}(\mathcal{A})$ has products by Remark 18. Suppose we have a sequence of morphisms in $\mathfrak{D}(\mathcal{A})$

$$\cdots \longrightarrow X_3 \xrightarrow{\mu_3} X_2 \xrightarrow{\mu_2} X_1 \xrightarrow{\mu_1} X_0 \tag{39}$$

together with a complex X and morphisms $X \longrightarrow X_i$ in $\mathfrak{D}(\mathcal{A})$ compatible with the morphisms of the sequence. A homotopy limit of this sequence is by definition an object $\underleftarrow{holim} X_i$ fitting into a triangle in $\mathfrak{D}(\mathcal{A})$

$$\underleftarrow{holim} X_i \xrightarrow{\beta} \prod_{i\in\mathbb{N}} X_i \xrightarrow{1-\nu} \prod_{i\in\mathbb{N}} X_i \longrightarrow \Sigma \underleftarrow{holim} X_i \tag{40}$$

where ν is defined by $p_n\nu = \mu_{n+1}p_{n+1}$. The morphisms $X \longrightarrow X_i$ determine a morphism $\alpha : X \longrightarrow \prod_{i\in\mathbb{N}} X_i$ which gives zero when composed with $1-\nu$, so we deduce a (noncanonical) morphism $\gamma : X \longrightarrow \underleftarrow{holim} X_i$ with $\beta\gamma = \alpha$.

Lemma 77. *Suppose that for each $n \in \mathbb{Z}$ the morphism $H^n(X) \longrightarrow H^n(X_i)$ is an isomorphism for all sufficiently large $i \geq 0$. Then $\gamma : X \longrightarrow \underleftarrow{holim} X_i$ is an isomorphism in $\mathfrak{D}(\mathcal{A})$.*

Proof. We claim that the following sequence is exact for $n \in \mathbb{Z}$

$$0 \longrightarrow H^n(X) \xrightarrow{H^n(\alpha)} H^n(\prod_{i\in\mathbb{N}} X_i) \xrightarrow{H^n(1-\nu)} H^n(\prod_{i\in\mathbb{N}} X_i) \longrightarrow 0 \tag{41}$$

By assumption $\mathcal{A}$ has exact products, so cohomology commutes with products and $H^n(\alpha)$ is a morphism into the product $\prod_{i\in\mathbb{N}} H^n(X_i)$ with components $H^n(X) \longrightarrow H^n(X_i)$. Only finitely many of these are not isomorphisms, so it is clear that $H^n(\alpha)$ is a monomorphism. To see that the rest of the sequence is exact, consider the following inverse system in $\mathcal{A}$

$$\cdots \longrightarrow H^n(X_3) \longrightarrow H^n(X_2) \longrightarrow H^n(X_1) \longrightarrow H^n(X_0) \tag{42}$$

By assumption this eventually stabilises to $H^n(X)$, so it is clear that the morphisms $H^n(X) \longrightarrow H^n(X_i)$ are an inverse limit of this system in $\mathcal{A}$, and therefore that $H^n(\alpha)$ is the kernel of $H^n(1-\nu)$. Since the morphisms of the system (42) are eventually epimorphisms, it follows from Lemma 66 that $H^n(1-\nu)$ is an epimorphism (here we use the fact that $\mathcal{A}$ has projective generators). Therefore (41) is exact as claimed.

From the long exact cohomology sequence of (40), together with the fact that $H^n(1-\nu)$ is an epimorphism, we deduce another short exact sequence

$$0 \longrightarrow H^n(\underleftarrow{holim} X_i) \xrightarrow{H^n(\beta)} H^n(\prod_{i\in\mathbb{N}} X_i) \xrightarrow{H^n(1-\nu)} H^n(\prod_{i\in\mathbb{N}} X_i) \longrightarrow 0$$

so it is clear that $H^n(\gamma) : H^n(X) \longrightarrow H^n(\underleftarrow{holim} X_i)$ is an isomorphism for every $n \in \mathbb{Z}$. Thus γ is an isomorphism in $\mathfrak{D}(\mathcal{A})$, as claimed. □

If you take the inverse limit of a sequence which eventually stabilises, then the inverse limit must be this stable value. There is a similar general statement for homotopy limits and colimits which follows from the results of [Nee01] §1.7. Using the simple argument of Lemma 77 we can get something finer: if the cohomology in a certain degree stabilises, then the stable value is the cohomology of the holimit in that degree.

Lemma 78. *Suppose that for every $m \in \mathbb{Z}$ the sequence*

$$\cdots \longrightarrow H^m(X_3) \longrightarrow H^m(X_2) \longrightarrow H^m(X_1) \longrightarrow H^m(X_0)$$

eventually consists entirely of isomorphisms. Then for every $m \in \mathbb{Z}$ the morphism

$$H^m(\underleftarrow{holim} X_i) \longrightarrow H^m(X_i)$$

is an isomorphism for all sufficiently large $i \geq 0$.

Proof. Actually we prove something a little more specific. Fix some $m \in \mathbb{Z}$ and suppose that the following two sequences stabilise

$$\cdots \longrightarrow H^m(X_3) \longrightarrow H^m(X_2) \longrightarrow H^m(X_1) \longrightarrow H^m(X_0)$$
$$\cdots \longrightarrow H^{m-1}(X_3) \longrightarrow H^{m-1}(X_2) \longrightarrow H^{m-1}(X_1) \longrightarrow H^{m-1}(X_0)$$

To be precise, suppose that $N \geq 0$ is such that $H^m(X_{k+1}) \longrightarrow H^m(X_k)$ is an isomorphism for $k \geq N$. Then we show $H^m(\underleftarrow{holim} X_i) \longrightarrow H^m(X_k)$ is an isomorphism for $k \geq N$. In other words, we only need the cohomology sequence to stabilise in two degrees in order to deduce something.

In the situation described above, the argument given in the proof of Lemma 77 shows that $H^{m-1}(1-\nu)$ and $H^m(1-\nu)$ are epimorphisms. From the long exact cohomology sequence of the triangle defining $\underleftarrow{holim} X_i$, we infer a short exact sequence

$$0 \longrightarrow H^m(\underleftarrow{holim} X_i) \longrightarrow \prod_{i\in\mathbb{N}} H^m(X_i) \longrightarrow \prod_{i\in\mathbb{N}} H^m(X_i) \longrightarrow 0$$

and therefore a canonical isomorphism $H^m(\underleftarrow{holim} X_i) = \varprojlim H^m(X_i)$ by Remark 26. But $\varprojlim H^m(X_i)$ must be the stable value of the cohomology objects $H^m(X_i)$ for large $i \geq 0$, so we deduce that $H^m(\underleftarrow{holim} X_i) \longrightarrow H^m(X_k)$ is an isomorphism for $k \geq N$, as required. ☐

4.5 Building Bounded Complexes

In the previous sections we have begun to describe how to use homotopy colimits and direct limits to reduce problems to the case of bounded complexes. In this short section we explain how one can reduce from bounded complexes to single objects.

Remark 32. Let $\mathcal{A}$ be an abelian category and let X be a bounded complex of the form

$$\cdots \longrightarrow 0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0 \longrightarrow \cdots$$

where $n \geq 0$. Let $c_1(X^0)$ denote X^0 considered as a complex in degree 1, and let S be the complex obtained from X by replacing X^0 by zero. Then the differential $\partial^0 : c_1(X^0) \longrightarrow S$ is a morphism of complexes, as described by the following diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \partial^0} & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X^1 & \xrightarrow[\partial^1]{} & X^2 & \xrightarrow[\partial^2]{} & \cdots & \xrightarrow[\partial^{n-1}]{} & X^n & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

It is easy to check that the mapping cone of this morphism is canonically isomorphic to X. That is, we have a triangle in $K(\mathcal{A})$

$$c_1(X^0) \longrightarrow S \longrightarrow X \longrightarrow c_0(X^0)$$

This triangle is very useful, as it expresses X in terms of two strictly smaller bounded complexes.

Lemma 79. *Let $\mathcal{A}$ be an abelian category, $\mathcal{P} \subseteq \mathcal{A}$ a class of objects which contains the zero objects and $\mathcal{S}$ a triangulated subcategory of $K(\mathcal{A})$ (or $\mathfrak{D}(\mathcal{A})$) containing $c(P)$ for every $P \in \mathcal{P}$. Then $\mathcal{S}$ contains any bounded complex in $\mathcal{A}$ whose objects all belong to $\mathcal{P}$.*

Lemma 80. *Let $\mathcal{A}$ be a grothendieck abelian category and $S, T : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{Q}$ coproduct preserving triangulated functors. If $\psi : S \longrightarrow T$ is a trinatural transformation with $\psi_A : S(A) \longrightarrow T(A)$ an isomorphism for every $A \in \mathcal{A}$, then ψ is a natural equivalence.*

Proof. Let $\mathcal{T}$ be the full subcategory of $\mathfrak{D}(\mathcal{A})$ consisting of the complexes X such that ψ_X is an isomorphism. This is a localising subcategory of $\mathfrak{D}(\mathcal{A})$ (TRC,Remark 30). By assumption it contains all the objects of $\mathcal{A}$ and therefore by Lemma 79 it contains any bounded complex. Any complex X in $\mathcal{A}$ can be writen as a direct limit of bounded complexes by Definition 18, all of which must belong to $\mathcal{T}$, so it follows from Proposition 65 that $X \in \mathcal{T}$ as required. □

Although Remark 32 is useful, there is another way of decomposing a bounded complex that is more in keeping with the spirit of the derived category.

Remark 33. Let $\mathcal{A}$ be an abelian category and X a complex belonging to $\mathfrak{D}(\mathcal{A})^{\geq n}$. Then from Lemma 27 we have a triangle in $\mathfrak{D}(\mathcal{A})$

$$c_n H^n(X) \longrightarrow X \longrightarrow X_{\geq(n+1)} \longrightarrow \Sigma c_n H^n(X)$$

If X is bounded above, then so is $X_{\geq(n+1)}$ and we have once again placed X into a triangle with two strictly smaller bounded complexes.

We can now prove a "soft" version of the result following Remark 32.

Lemma 81. *Let $\mathcal{A}$ be an abelian category, $\mathcal{P} \subseteq \mathcal{A}$ a class of objects containing the zero objects and closed under isomorphism, and $\mathcal{S}$ a triangulated subcategory of $\mathfrak{D}(\mathcal{A})$ containing $c(P)$ for every $P \in \mathcal{P}$. Then $\mathcal{S}$ contains cohomologically bounded complex in $\mathcal{A}$ whose cohomology objects all belong to $\mathcal{P}$.*

Proof. See Definition 13 for what we mean by *cohomologically bounded*. Observe that any cohomologically bounded complex is isomorphic in $\mathfrak{D}(\mathcal{A})$ to a bounded complex. The proof is by induction on the number $n(X)$ of nonzero cohomology objects of the complex. If $n(X) = 1$ then $X \cong c_k H^k(X)$ for some $k \in \mathbb{Z}$ by Lemma 29, and therefore $X \in \mathcal{S}$. Suppose that $n(X) > 1$ and that k is such that $H^k(X) \neq 0$ and $H^i(X) = 0$ for all $i < k$. Then we have a triangle

$$c_k H^k(X) \longrightarrow X \longrightarrow X_{\geq(k+1)} \longrightarrow \Sigma c_k H^k(X)$$

where it is clear that $c_k H^k(X)$ and $X_{\geq(k+1)}$ belong to $\mathcal{S}$ by assumption and the inductive hypothesis. Therefore $X \in \mathcal{S}$ and the proof is complete. □

5 Homotopy Direct Limits

In Section 4 we defined the homotopy limit and colimit and gave their basic properties. In this section we study some more advanced properties needed in applications. We have already observed the following result

- A localising subcategory $\mathcal{L} \subseteq \mathfrak{D}(\mathcal{A})$ is closed under direct limits indexed by $\mathbb{N}$.

In this section we expand on this in two directions:

(a) In Section 5.1 we prove that a localising subcategory $\mathcal{L} \subseteq K(\mathcal{A})$ is closed under a certain special kind of direct limit indexed by $\mathbb{N}$. It will follow that if $\mathcal{L}$ contains a collection of objects from the abelian category, it contains any bounded above complex formed from these objects, generalising Lemma 79.

(b) In Section 5.2 we prove following [ATJLSS00] that a localising subcategory $\mathcal{L} \subseteq \mathfrak{D}(\mathcal{A})$ is closed under *arbitrary* direct limits, not just those indexed by $\mathbb{N}$. This is the key step in the proof of existence of hoinjective resolutions given in [ATJLSS00] for a grothendieck abelian category, which is also the major theorem of these notes.

Although (b) is crucial for the rest of these notes and for our notes on derived functors, the proof of (a) can be safely skipped on a first reading.

5.1 Split Direct Limits

We already know that in general there is a morphism of complexes connecting the homotopy colimit to the usual colimit. Under certain natural conditions this morphism is a quasi-isomorphism, which tells us that in the derived category the two constructions agree. The main technical result of this section shows that for a certain common type of direct system the two constructions agree already in the homotopy category. See Definition 27 for the definition of a *fibration* and *cofibration* of complexes.

The next two results should be compared with Proposition 20 and Lemma 21. The meaning of this analogy will be clarified in our notes on Stable Derived Categories (SDTC).

Proposition 82. *Let $\mathcal{A}$ be an abelian category and suppose we have a short exact sequence of complexes*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

If this sequence is split exact in each degree there is a canonical morphism $z : Z \longrightarrow \Sigma X$ in $K(\mathcal{A})$ making the following diagram into a triangle in $K(\mathcal{A})$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-z} \Sigma X$$

Proof. For each $n \in \mathbb{Z}$ the following sequence is split exact

$$0 \longrightarrow X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \longrightarrow 0$$

and we can choose splittings $a^n : Y^n \longrightarrow X^n, b^n : Z^n \longrightarrow Y^n$ in such a way that the tuple (a^n, f^n, g^n, b^n) makes Y^n into a biproduct $X^n \oplus Z^n$ in $\mathcal{A}$. One checks that the morphisms $Z^{n-1} \longrightarrow X^n$ defined by $h^n = a^n \partial^{n-1} b^{n-1}$ together define a morphism of complexes $h : \Sigma^{-1} Z \longrightarrow X$. With respect to the above biproduct structure the differential $\partial^n : Y^n \longrightarrow Y^{n+1}$ has the matrix

$$\partial^n = \begin{pmatrix} -\partial^{n+1}_{\Sigma^{-1}Z} & 0 \\ h^{n+1} & \partial^n_X \end{pmatrix}$$

That is, Y is canonically isomorphic as a complex to the mapping cone C_h and there is a triangle in $K(\mathcal{A})$

$$\Sigma^{-1} Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$$

Shifting we have a triangle in $K(\mathcal{A})$ of the desired form with $z = \Sigma h : Z \longrightarrow \Sigma X$. It remains to show that z is canonical, by which we mean that it does not depend on the choice of splittings (up to homotopy). By a "choice of splittings" we mean a choice of morphism $a^n : Y^n \longrightarrow X^n$ with $a^n f^n = 1$ for each $n \in \mathbb{Z}$. The morphism b^n is then uniquely determined by the requirements $g^n b^n = 1$ and $f^n a^n + b^n g^n = 1$.

Suppose that a'^n, b'^n is an alternative choice of splittings for each $n \in \mathbb{Z}$ and define $h'^n = a'^n \partial^{n-1} b'^{n-1}$. This is a morphism of complexes $h' : \Sigma^{-1} Z \longrightarrow X$ and it suffices to show that h, h' are homotopic. If we define $\Sigma^n : Z^{n-1} \longrightarrow X^{n-1}$ by $\Sigma^n = -a^{n-1} b'^{n-1}$ then

$$f^n(\Sigma^{n+1} \partial^n_{\Sigma^{-1}Z} + \partial^{n-1}_X \Sigma^n) = \partial^{n-1}_X (b^{n-1} - b'^{n-1}) + (b'^n - b^n) \partial^{n-1}_Z$$

where we use the trick of writing $f^n a^n = 1 - b^n g^n$. Applying this trick several times one checks that $f^n(h^n - h'^n)$ is equal to the same expression. Since f^n is a monomorphism we deduce

$$h^n - h'^n = \Sigma^{n+1} \partial^n_{\Sigma^{-1}Z} + \partial^{n-1}_X \Sigma^n$$

which is what we needed to show. □

Lemma 83. *Let $\mathcal{A}$ be an abelian category and suppose we are given a commutative diagram of complexes with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & 0 \end{array}$$

in which the rows are split exact in each degree. The following diagram commutes in $K(\mathcal{A})$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{-z} & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[-z']{} & \Sigma X' \end{array}$$

where z, z' are the canonical morphisms of Proposition 82.

Proof. Choose splittings $a^n : Y^n \longrightarrow X^n, b^n : Z^n \longrightarrow Y^n$ and $a'^n : Y'^n \longrightarrow X'^n, b'^n : Z'^n \longrightarrow Y'^n$ and define h, h' as in the proof of Proposition 82 so that $z = \Sigma h, z' = \Sigma h'$. It suffices to show that the diagram

$$\begin{array}{ccc} \Sigma^{-1}Z & \xrightarrow{h} & X \\ \downarrow{\scriptstyle \Sigma^{-1}\gamma} & & \downarrow{\scriptstyle \alpha} \\ \Sigma^{-1}Z' & \xrightarrow[h']{} & X' \end{array}$$

commutes up to homotopy. Define $\Sigma^n : (\Sigma^{-1}Z)^n \longrightarrow X'^{n-1}$ by $\Sigma^n = -a'^{n-1}\beta^{n-1}b^{n-1}$. One checks that $f'^n(\Sigma^{n+1}\partial^n_{\Sigma^{-1}Z} + \partial^{n-1}_{X'}\Sigma^n) = f'^n(h'^n\gamma^{n-1} - \alpha^n h^n)$. Since f'^n is a monomorphism we deduce that Σ is a homotopy of αh with $h'\Sigma^{-1}\gamma$, as claimed. □

The proof of Proposition 82 is independent of the one given in Proposition 20, but there is something to be gained by understanding it in the context of the original statement for $\mathfrak{D}(\mathcal{A})$. This also makes it clear that the two "canonical" morphisms $z : Z \longrightarrow \Sigma X$ agree in $\mathfrak{D}(\mathcal{A})$.

Remark 34. Let $\mathcal{A}$ be an abelian category and suppose we are given a short exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{g} Z \longrightarrow 0 \qquad (43)$$

As in the proof of Proposition 20 we have a canonical quasi-isomorphism of complexes $f : C_u \longrightarrow Z$. We claim that if (43) is split exact in each degree then f is actually already an isomorphism in $K(\mathcal{A})$ and moreover the canonical morphism $z : Z \longrightarrow \Sigma X$ of Proposition 82 is just the composite

$$Z \xrightarrow{f^{-1}} C_u \xrightarrow{-w} \Sigma X$$

in $K(\mathcal{A})$, where w is the canonical morphism of complexes. In particular this shows that if we map the connecting morphism $Z \longrightarrow \Sigma X$ of Proposition 82 into the derived category, it agrees there with the connecting morphism of Proposition 20.

Proof. Suppose that (43) is split exact in each degree and choose a specific splitting, with the notation a^n, b^n, h^n as in Proposition 82. Define a morphism

$$H^n = \begin{pmatrix} -h^{n+1} \\ b^n \end{pmatrix} : Z^n \longrightarrow C_u^n = X^{n+1} \oplus Y^n$$

It is straightforward to check that H is a morphism of complexes $Z \longrightarrow C_u$ with $fH = 1$. If we define morphisms

$$\Sigma^n = \begin{pmatrix} 0 & a^n \\ 0 & 0 \end{pmatrix} : X^{n+1} \oplus Y^n \longrightarrow X^n \oplus Y^{n-1}$$

then one checks that $1 - H^n f^n = \partial^{n-1}\Sigma^n + \Sigma^{n+1}\partial^n$. This shows that f is a homotopy equivalence with inverse H in $K(\mathcal{A})$, as required.

Now we are ready to show that for sequences of cofibrations, the direct limit is also a homotopy colimit in the homotopy category.

Proposition 84. *Let $\mathcal{A}$ be a cocomplete abelian category and*

$$X_1 \xrightarrow{\mu_1} X_2 \xrightarrow{\mu_2} X_3 \xrightarrow{\mu_3} \cdots$$

a sequence of morphisms in $\mathbf{C}(\mathcal{A})$ with each μ_i a cofibration. In $K(\mathcal{A})$ there is an isomorphism $\underrightarrow{holim} X_i \longrightarrow \varinjlim X_i$.

Proof. By definition an arbitrary homotopy colimit $\underrightarrow{holim} X_i$ in $K(\mathcal{A})$ fits into a triangle with the following morphism of complexes

$$\bigoplus_{i\geq 1} X_i \xrightarrow{1-\nu} \bigoplus_{i\geq 1} X_i \qquad (1-\nu)q_i = q_i - q_{i+1}\mu_i$$

By Lemma 64 this is a cofibration and we have an exact sequence of complexes split exact in each degree

$$0 \longrightarrow \bigoplus_{i\geq 1} X_i \xrightarrow{1-\nu} \bigoplus_{i\geq 1} X_i \longrightarrow \varinjlim X_i \longrightarrow 0$$

From Proposition 65 we can deduce an isomorphism $\underrightarrow{holim} X_i \cong \varinjlim X_i$ in $\mathfrak{D}(\mathcal{A})$, and we want to show that such an isomorphism already exists in $K(\mathcal{A})$. This is now an immediate consequence of Proposition 82. ☐

Proposition 85. *Let $\mathcal{A}$ be a complete abelian category and*

$$\cdots \xrightarrow{\mu_3} X_3 \xrightarrow{\mu_2} X_2 \xrightarrow{\mu_1} X_1$$

a sequence of morphisms in $\mathbf{C}(\mathcal{A})$ with each μ_i a fibration. In $K(\mathcal{A})$ there is an isomorphism $\underleftarrow{holim} X_i \longrightarrow \varprojlim X_i$.

Proof. By definition an arbitrary homotopy limit $\underleftarrow{holim} X_i$ in $K(\mathcal{A})$ fits into a triangle with the following morphism of complexes

$$\prod_{i\geq 1} X_i \xrightarrow{1-\nu} \prod_{i\geq 1} X_i \qquad p_n(1-\nu) = p_n - \mu_n p_{n+1}$$

By Lemma 67 this is a fibration and we have an exact sequence of complexes split exact in each degree

$$0 \longrightarrow \varprojlim X_i \longrightarrow \prod_{i\geq 1} X_i \xrightarrow{1-\nu} \prod_{i\geq 1} X_i \longrightarrow 0$$

The claim now follows from Proposition 82. ☐

Definition 31. Let $\mathcal{A}$ be an abelian category. An inverse system of complexes in $\mathcal{A}$

$$\cdots \longrightarrow X_4 \xrightarrow{\mu_3} X_3 \xrightarrow{\mu_2} X_2 \xrightarrow{\mu_1} X_1 \tag{44}$$

is a *split inverse system* if every μ_n is a fibration (in particular an epimorphism). An inverse limit of this system is called a *split inverse limit*. Dually, a direct system of complexes

$$X_1 \xrightarrow{\mu_1} X_2 \xrightarrow{\mu_2} X_3 \xrightarrow{\mu_3} \cdots \tag{45}$$

is a *split direct system* if every μ_n is a cofibration (in particular a monomorphism). A direct limit of this system is called a *split direct limit*.

If $\mathcal{I}$ is a nonempty class of complexes in $\mathcal{A}$ closed under isomorphism, then we say that $\mathcal{I}$ is *closed under split inverse limits* if the limit of every split inverse system whose objects belong to $\mathcal{I}$, also belongs to $\mathcal{I}$. Dually we define a class *closed under split direct limits*.

Remark 35. In this notation, Proposition 84 says that any split direct limit is also a homotopy colimit, and its dual says that any split inverse limit is a homotopy limit.

Proposition 86. *Let $\mathcal{A}$ be a cocomplete abelian category and $\mathcal{S}$ a localising subcategory of $K(\mathcal{A})$ or $\mathfrak{D}(\mathcal{A})$. Then $\mathcal{S}$ is closed under split direct limits.*

Proof. In both cases we only need $\mathcal{S}$ to be closed under countable coproducts. For $\mathfrak{D}(\mathcal{A})$ the result is Proposition 65, so let $\mathcal{L} \subseteq K(\mathcal{A})$ be a triangulated subcategory closed under countable coproducts. Suppose we have a split direct system of complexes (45) with every X_i in $\mathcal{S}$. The homotopy colimit fits into a triangle in $K(\mathcal{A})$

$$\bigoplus_{i\geq 1} X_i \longrightarrow \bigoplus_{i\geq 1} X_i \longrightarrow \underrightarrow{holim} X_i \longrightarrow \Sigma \bigoplus_{i\geq 1} X_i$$

so $\underrightarrow{holim} X_i$ certainly belongs to $\mathcal{S}$. But by Proposition 84 the usual direct limit $\varinjlim X_i$ in $\mathbf{C}(\mathcal{A})$ is isomorphic in $K(\mathcal{A})$ to $\underrightarrow{holim} X_i$, so it must also belong to $\mathcal{S}$. $\square$

Dually, we have

Proposition 87. *Let $\mathcal{A}$ be a complete abelian category and $\mathcal{S}$ a colocalising subcategory of $K(\mathcal{A})$. Then $\mathcal{S}$ is closed under split inverse limits.*

Example 6. Let $\mathcal{A}$ be an abelian category. Here are some easy consequences of the above results

(a) If $\mathcal{A}$ has exact coproducts the exact complexes form a localising subcategory $\mathcal{Z}$ of $K(\mathcal{A})$. Therefore any split direct limit of exact complexes is exact. If on the other hand $\mathcal{A}$ has exact products, $\mathcal{Z}$ is a colocalising subcategory and any split inverse limit of exact complexes is exact. In particular both statements apply when $\mathcal{A} = \mathbf{Ab}$ is the category of abelian groups.

(b) If $\mathcal{A}$ is complete then any split inverse limit of hoinjective complexes is hoinjective.

(c) If $\mathcal{A}$ is cocomplete then any split direct limit of hoprojective complexes is hoprojective.

Remark 36. Let $\mathcal{A}$ be an abelian category and W a complex in $\mathcal{A}$. Then for arbitrary $n \in \mathbb{Z}$ we can write W as the inverse limit of the following split inverse system of the brutal truncations of Definition 16

$$\cdots \longrightarrow {}_bW_{\leq(n+2)} \longrightarrow {}_bW_{\leq(n+1)} \longrightarrow {}_bW_{\leq n}$$

which is graphically

$$\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & W^{n-1} & \longrightarrow & W^{n} & \longrightarrow & W^{n+1} & \longrightarrow & W^{n+2} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & W^{n-1} & \longrightarrow & W^{n} & \longrightarrow & W^{n+1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & W^{n-1} & \longrightarrow & W^{n} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Thus any complex is the split inverse limit of bounded above complexes, whose objects all come from the original complex. Dually we can write W as the direct limit of the following split direct system of brutal truncations of Definition 17

$${}_bW_{\geq n} \longrightarrow {}_bW_{\geq(n-1)} \longrightarrow {}_bW_{\geq(n-2)} \longrightarrow \cdots$$

which is graphically

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & W^n & \longrightarrow & W^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & W^{n-1} & \longrightarrow & W^n & \longrightarrow & W^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & W^{n-2} & \longrightarrow & W^{n-1} & \longrightarrow & W^n & \longrightarrow & W^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Thus any complex is the split direct limit of bounded below complexes, whose objects all come from the original complex.

We already know that any bounded below complex of injectives is hoinjective, and any bounded above complex of projectives is hoprojective. These statements now become special cases of much more general results.

Proposition 88. *Let $\mathcal{A}$ be a cocomplete abelian category, $\mathcal{P} \subseteq \mathcal{A}$ a class of objects containing the zero objects, and $\mathcal{S}$ a localising subcategory of $K(\mathcal{A})$ or $\mathfrak{D}(\mathcal{A})$ containing the objects of $\mathcal{P}$. Then $\mathcal{S}$ contains any bounded above complex in $\mathcal{P}$.*

Proof. It is enough to assume $\mathcal{S}$ closed under countable coproducts in $K(\mathcal{A})$ or $\mathfrak{D}(\mathcal{A})$ respectively. From Lemma 79 we know that $\mathcal{S}$ contains any *bounded* complex in $\mathcal{P}$. Given a bounded above complex W in $\mathcal{P}$, we can by Remark 36 write W as the split direct limit of bounded complexes in $\mathcal{P}$. Each such complex belongs to $\mathcal{S}$, which is closed under split direct limits by Proposition 86. We conclude that $W \in \mathcal{S}$, as required. □

Proposition 89. *Let $\mathcal{A}$ be a complete abelian category, $\mathcal{I} \subseteq \mathcal{A}$ a class of objects containing the zero objects, and $\mathcal{S}$ a colocalising subcategory of $K(\mathcal{A})$ or $\mathfrak{D}(\mathcal{A})$ containing the objects of $\mathcal{I}$. Then $\mathcal{S}$ contains any bounded below complex in $\mathcal{I}$.*

Proof. As above, it is actually enough for $\mathcal{S}$ to be closed under countable products in $K(\mathcal{A})$. The result now follows by duality from Proposition 88. □

Combining these two results we have

Corollary 90. *Let $\mathcal{A}$ be a complete, cocomplete abelian category, $\mathcal{H} \subseteq \mathcal{A}$ a class of objects containing the zero objects, and $\mathcal{S}$ a localising, colocalising subcategory of $K(\mathcal{A})$ containing the objects of $\mathcal{H}$. Then $\mathcal{S}$ contains any complex in $\mathcal{H}$.*

As we observed in Remark 31, the problem with the construction of hoinjective resolutions in Proposition 75 is that inverse limits and products don't behave well with cohomology. The way to fix this is to use brutal truncations to replace inverse limits by direct limits, and Proposition 84 is the technical tool needed to make this transition work.

The last time we tried to construct resolutions on the right was using inverse limits, so when we constructed resolutions for bounded below complexes in Proposition 73 we were interested in functoriality going "backwards" (i.e. left on the page). For the proof of the next major result, we need functoriality going "forwards", which is what we acquire in the next lemma.

Lemma 91. *Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ a cosmothering class for $\mathcal{A}$. Suppose we are given a morphism $f : X \longrightarrow Y$ of bounded below complexes and a quasi-isomorphism $u : X \longrightarrow I$ with I a bounded below complex in $\mathcal{I}$. Then there exists a commutative diagram in $K(\mathcal{A})$*

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
I & \xrightarrow[m]{} & J
\end{array}$$

with $v : Y \longrightarrow J$ a quasi-isomorphism and J a bounded below complex in $\mathcal{I}$.

Proof. The idea is to take the homotopy pushout in $K(\mathcal{A})$ and then resolve the "nose" of the pushout. We have a morphism of complexes $\alpha = \left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right) : X \longrightarrow Y \oplus I$ of which we take the mapping cone C_α. We have a commutative diagram in $K(\mathcal{A})$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\big\downarrow & & \big\downarrow \\ I & \longrightarrow & C_\alpha \end{array} \tag{46}$$

which is by construction a homotopy pushout in the sense of (TRC,Definition 15). Since u is a quasi-isomorphism, we deduce from (TRC,Lemma 37) that $Y \longrightarrow C_\alpha$ is a quasi-isomorphism. Now C_α is a bounded below complex, so we can by Proposition 73 find a quasi-isomorphism $C_\alpha \longrightarrow J$ with J a bounded below complex in $\mathcal{I}$. Attaching this to the nose of (46) we have the required commutative diagram in $K(\mathcal{A})$. □

Remark 37. With the notation of Proposition 91 that there exists $N \in \mathbb{Z}$ with $X^i = 0, I^i = 0$ for $i < N$ and $Y^i = 0$ for $i < N - 1$. Going through the proof we observe that J can be found such that $J^i = 0$ for $i < N - 1$.

Proposition 92. *Let $\mathcal{A}$ be an abelian category with exact coproducts and $\mathcal{I}$ a cosmothering class that is closed under countable coproducts in $\mathcal{A}$. Then every complex X in $\mathcal{A}$ admits a quasi-isomorphism $X \longrightarrow I$ with I a complex in $\mathcal{I}$.*

Proof. Let M be any complex in $\mathcal{A}$ and for $n \leq 0$ let ${}_bM_{\geq n}$ be the brutal truncation of Definition 17. Since this complex is bounded below we can find by Proposition 73 a quasi-isomorphism $\rho_n : {}_bM_{\geq n} \longrightarrow I_n$ with I_n a bounded below complex in $\mathcal{I}$. In fact proceeding inductively and using Lemma 91 at each stage we can choose these resolutions in such a way that we have a commutative diagram in $K(\mathcal{A})$

$$\begin{array}{ccccccccc} {}_bM_{\geq 0} & \longrightarrow & {}_bM_{\geq -1} & \longrightarrow & {}_bM_{\geq -2} & \longrightarrow \cdots \longrightarrow & {}_bM_{\geq n} & \longrightarrow & \cdots \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ I_0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow \cdots \longrightarrow & I_n & \longrightarrow & \cdots \end{array} \tag{47}$$

Take the homotopy colimits of these sequences in $K(\mathcal{A})$. Then there is an induced morphism $\underrightarrow{holim}\, {}_bM_{\geq n} \longrightarrow \underrightarrow{holim}\, I_n$ fitting into a morphism of triangles in $K(\mathcal{A})$

$$\begin{array}{ccccccc} \bigoplus_{n \leq 0} {}_bM_{\geq n} & \xrightarrow{1-\nu} & \bigoplus_{n \leq 0} {}_bM_{\geq n} & \longrightarrow & \underrightarrow{holim}\, {}_bM_{\geq n} & \longrightarrow & \Sigma \bigoplus_{n \leq 0} {}_bM_{\geq n} \\ {\scriptstyle \oplus_n \rho_n}\big\downarrow & & {\scriptstyle \oplus_n \rho_n}\big\downarrow & & {\scriptstyle \underrightarrow{holim}\, \rho_n}\big\downarrow & & \big\downarrow \\ \bigoplus_{n \leq 0} I_n & \xrightarrow[1-\nu]{} & \bigoplus_{n \leq 0} I_n & \longrightarrow & \underrightarrow{holim}\, I_n & \longrightarrow & \Sigma \bigoplus_n I_n \end{array}$$

where we construct both homotopy colimits as mapping cones on the level of complexes. Since $\mathcal{A}$ has exact coproducts the morphism $\underrightarrow{holim}\, \rho_n$ is a quasi-isomorphism. The top row of (47) is a split direct system, with direct limit $M = \varinjlim_{n \leq 0} {}_bM_{\geq n}$. By Proposition 84 we have a quasi-isomorphism of complexes $g : M \longrightarrow \underrightarrow{holim}\, {}_bM_{\geq n}$. Composing with $\underrightarrow{holim}\, \rho_n$ we have finally a quasi-isomorphism $M \longrightarrow \underrightarrow{holim}\, I_n$. Since $\mathcal{I}$ is closed under countable coproducts it is easy to see that $\underrightarrow{holim}\, I_n$ is a complex in $\mathcal{I}$, which completes the proof. □

Corollary 93. *Let $\mathcal{A}$ be a locally noetherian grothendieck abelian category. Then every complex X in $\mathcal{A}$ admits a quasi-isomorphism $X \longrightarrow I$ with I a complex of injectives.*

Proof. Let $\mathcal{I}$ be the class of all injective objects in $\mathcal{A}$. This is certainly cosmothering, and it is closed under coproducts in $\mathcal{A}$ by a well-known result. So the conclusion follows from Proposition 92. For a completely different proof of this result using Brown representability, see Krause's paper [Kra05]. ☐

Corollary 94. *Let $\mathcal{A}$ be a locally noetherian grothendieck abelian category. Then every hoinjective complex X in $\mathcal{A}$ is a retraction in $K(\mathcal{A})$ of a complex of injectives.*

Proof. Let I be a hoinjective complex, and take a quasi-isomorphism $I \longrightarrow J$ with J a complex of injectives. By Proposition 51 this must be a coretraction in $K(\mathcal{A})$. ☐

Example 7. Let X be a quasi-noetherian topological space and $\mathcal{I}$ the class of all quasi-flasque sheaves of abelian groups on X. By (COS,Proposition 23) this is closed under arbitrary coproducts in $\mathfrak{Ab}(X)$. Since any injective sheaf of abelian groups is flasque, therefore quasi-flasque, $\mathcal{I}$ is a cosmothering class for $\mathfrak{Ab}(X)$. Therefore by Proposition 92 every complex $\mathscr{X}$ of sheaves of abelian groups on X admits a quasi-isomorphism $\mathscr{X} \longrightarrow \mathscr{F}$ with $\mathscr{F}$ a complex of quasi-flasque sheaves.

Similarly if $(X, \mathcal{O}_X)$ is a quasi-noetherian ringed space the class $\mathcal{I}$ of all quasi-flasque sheaves of modules is cosmothering and closed under coproducts, so every complex of sheaves of modules has a resolution by a complex of quasi-flasque sheaves.

5.2 General Direct Limits

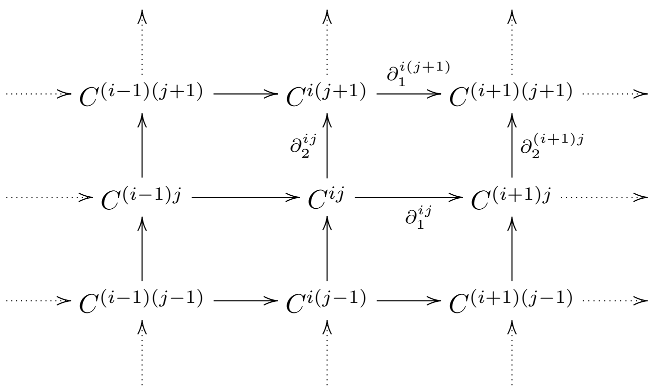
We saw in Proposition 65 that localising subcategories of $\mathfrak{D}(\mathcal{A})$ are closed under a special kind of direct limit in $\mathbf{C}(\mathcal{A})$. In this section we follow the proof of this statement for arbitrary direct limits given in [ATJLSS00]. Throughout this section $\mathcal{A}$ denotes an abelian category.

Definition 32. A *bicomplex* in $\mathcal{A}$ is a collection of objects $\{C^{ij}\}_{i,j\in\mathbb{Z}}$ of $\mathcal{A}$ together with morphisms $\partial_1^{ij} : C^{ij} \longrightarrow C^{(i+1)j}$ and $\partial_2^{ij} : C^{ij} \longrightarrow C^{i(j+1)}$ for every $i,j \in \mathbb{Z}$ such that for every $i,j \in \mathbb{Z}$

$$\partial_1^{(i+1)j}\partial_1^{ij} = 0, \quad \partial_2^{i(j+1)}\partial_2^{ij} = 0, \quad \partial_1^{i(j+1)}\partial_2^{ij} = \partial_2^{(i+1)j}\partial_1^{ij}$$

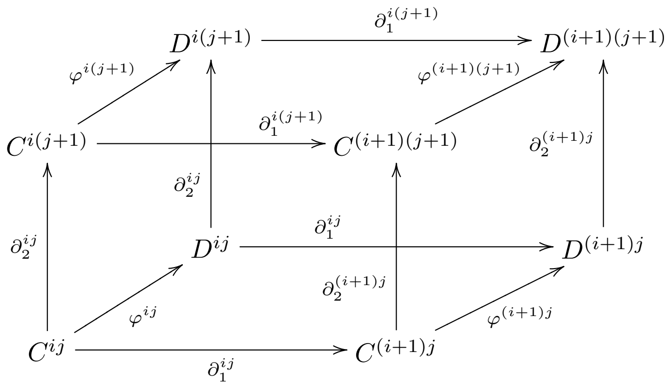
In other words we have a two-dimensional grid of objects of $\mathcal{A}$

(48)



which we require to be commutative with each row and column a complex in $\mathcal{A}$. Often we will just say that C is a bicomplex, and leave the differentials ∂_1, ∂_2 implicit. A *morphism* of bicomplexes $\varphi : C \longrightarrow D$ is a collection of morphisms $\{\varphi^{ij} : C^{ij} \longrightarrow D^{ij}\}_{i,j\in\mathbb{Z}}$ such that for every $i,j \in \mathbb{Z}$

every face of the following cube commutes



If we are given another morphism of bicomplexes $\psi : D \longrightarrow E$ then it is clear that the morphisms $\psi^{ij}\varphi^{ij}$ define a morphism of bicomplexes $\psi\varphi : C \longrightarrow E$. This defines the preadditive category $\mathbf{C}^2(\mathcal{A})$ of bicomplexes in $\mathcal{A}$.

Definition 33. Let $\mathcal{A}$ be a cocomplete abelian category and C a bicomplex in $\mathcal{A}$. The *totalisation complex* $Tot(C)$ of C is defined as follows. For $n \in \mathbb{Z}$ we have

$$Tot(C)^n = \bigoplus_{i+j=n} C^{ij}$$

Let $u_{ij} : C^{ij} \longrightarrow Tot(C)^{i+j}$ be the injection into the coproduct. Then for $n \in \mathbb{Z}$ we define a morphism $\partial^n : Tot(C)^n \longrightarrow Tot(C)^{n+1}$ on components by

$$\partial^n u_{ij} = u_{(i+1)j}\partial_1^{ij} + (-1)^i u_{i(j+1)}\partial_2^{ij}$$

for any $i, j \in \mathbb{Z}$ with $i+j = n$. One checks easily that $Tot(C)$ is indeed a complex in $\mathcal{A}$. Intuitively we are taking the direct sums along the diagonals in (48) and the differential does the obvious thing: at every object of the $(n+1)$th diagonal there are two incoming morphisms from objects of the nth diagonal, and at that point in our sequence we insert the sum of these two terms (modulo a sign).

Given a morphism of bicomplexes $\varphi : C \longrightarrow D$ we define a morphism of complexes $Tot(\varphi) : Tot(C) \longrightarrow Tot(D)$ by $Tot(\varphi)^n = \bigoplus_{i+j=n} \varphi^{ij}$. This makes the totalisation complex into an additive functor $Tot(-) : \mathbf{C}^2(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{A})$ from the category of bicomplexes in $\mathcal{A}$ to the category $\mathbf{C}(\mathcal{A})$.

Definition 34. Let $\mathcal{A}$ be an abelian category and D a complex in $\mathcal{A}$. We define a bicomplex $Grid(D)$ by $Grid(D)^{ij} = D^{i+j}$ and $\partial_1^{ij} = \partial_2^{ij} = \partial_D^{i+j}$. If $\varphi : D \longrightarrow E$ is a morphism of complexes then $Grid(\varphi) : Grid(D) \longrightarrow Grid(E)$ defined by $Grid(\varphi)^{ij} = \varphi^{i+j}$ is a morphism of bicomplexes. This defines a functor $Grid(-) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}^2(\mathcal{A})$.

Let $\mathcal{A}$ be a cocomplete abelian category, so that $\mathbf{C}(\mathcal{A})$ is also cocomplete (DF,Lemma 65). Let Γ be a directed set (AC,Definition 24) and let $\{G_s, \mu_{st} \,|\, s \in \Gamma\}$ be a direct system over Γ in $\mathbf{C}(\mathcal{A})$. Given $k \geq 0$ let W^k be the set of strictly ascending chains in Γ of length k. That is, all chains of the form $s_0 < s_1 < \cdots < s_k$. Our convention is that a chain of length zero is just an element of Γ, so that $W^0 = \Gamma$. Given a strictly ascending chain $w \in W^k$ we write w_i for the element of Γ in the ith position, so that w is $w_0 < \cdots < w_k$. With this data, we construct a bicomplex $B(G)$ as follows. For $j, k \in \mathbb{Z}$ define

$$B(G)^{kj} = \begin{cases} 0 & \text{if } k > 0 \\ \bigoplus_{w \in W^{-k}} G^j_{w_0} & \text{if } k \leq 0 \end{cases}$$

If we write $u^j_w : G^j_{w_0} \longrightarrow B(G)^{kj}$ for the injection, then for $k < 0$ and $w \in W^{-k}$ the horizontal differential $\partial_1^{kj} : B(G)^{kj} \longrightarrow B(G)^{(k+1)j}$ is defined by

$$\partial_1^{kj} u^j_w = u^j_{w_1 < \cdots < w_{-k}} \mu^j_{w_0 w_1} + \sum_{i=1}^{-k} (-1)^i u^j_{w_0 < \cdots < \hat{w}_i < \cdots < w_{-k}}$$

where $w_0 < \cdots < \hat{w}_i < \cdots < w_{-k}$ denotes w with the ith position deleted. The vertical differential $\partial_2^{kj} : B(G)^{kj} \longrightarrow B(G)^{k(j+1)}$ is defined for $k \le 0$ by $\partial_2^{jk} = \bigoplus_{w \in W^{-k}} \partial^j_{w_0}$ where ∂_s is the differential of the complex G_s for every $s \in \Gamma$. One checks that this defines a bicomplex in $\mathbf{C}(\mathcal{A})$.

Now let $\Lambda \subseteq \Gamma$ be a nonempty subset that is a directed set with the induced relation. Then $\{G_p, \mu_{pq}\}_{p \in \Lambda}$ is direct system over Λ in $\mathcal{A}$, which we denote by G_Λ. Given $k \ge 0$ let W^k_Λ denote the set of strictly ascending chains in Λ of length k. Then $W^k_\Lambda \subseteq W^k$ so we have a canonical morphism of bicomplexes $\rho : B(G_\Lambda) \longrightarrow B(G)$.

Remark 38. If $\mathcal{A}$ is grothendieck abelian and U a generator for $\mathcal{A}$, then a U-element x of G^j_w (that is, a morphism $x : U \longrightarrow G^j_w$) can be denoted by $(x; w)$ or $(x; w_0 < w_1 < \cdots < w_k)$. With this notation the horizontal differential takes the following form

$$\begin{aligned} \partial_1^{kj}(x; w_0 < w_1 < \cdots < w_{-k}) &= (\mu_{w_0 w_1}(x); w_1 < \cdots < w_{-k}) \\ &\quad + \sum_{i=1}^{-k} ((-1)^i x; w_0 < \cdots < \hat{w}_i < \cdots < w_{-k}) \end{aligned}$$

Definition 35. Let $\mathcal{A}$ be a cocomplete abelian category and $\{G_s, \mu_{st}\}_{s \in \Gamma}$ a direct system in $\mathbf{C}(\mathcal{A})$. The *homotopy direct limit* of this system is the totalisation of the bicomplex $B(G)$, which we denote by $\underrightarrow{hodlim}_{s \in \Gamma} G_s$. For each $s \in \Gamma$ there is a canonical morphism of complexes $G_s \longrightarrow \underrightarrow{hodlim}_{s \in \Gamma} G_s$ which for $n \in \mathbb{Z}$ is given by the following composite

$$G^n_s \longrightarrow \bigoplus_{s \in \Gamma} G^n_s = B(G)^{0n} \longrightarrow \bigoplus_{k+j=n} B(G)^{kj} = Tot(B(G))^n$$

Remark 39. We use the notation $\underrightarrow{hodlim}$ rather than $\underrightarrow{holim}$ in order to distinguish the object we have just constructed from the homotopy colimits defined in (TRC,Section 3). The only real difference is that our $\underrightarrow{hodlim}$ is constructed on the level of complexes, instead of in a triangulated category where colimits are less well behaved.

Lemma 95. *Let $\mathcal{A}$ be a cocomplete abelian category, F a finite directed set with maximum m and let $\{G_s, \mu_{st}\}_{s \in F}$ be a direct system in $\mathbf{C}(\mathcal{A})$. The canonical morphism of complexes $\psi : G_m \longrightarrow \underrightarrow{hodlim}_{s \in F} G_s$ is a homotopy equivalence.*

Proof. For each $n \in \mathbb{Z}$ the morphisms $\mu^n_{sm} : G^n_s \longrightarrow G^n_m$ induce a morphism $\bigoplus_{s \in \Gamma} G^n_s \longrightarrow G^n_m$. Compose with the projection $Tot(B(G))^n = \bigoplus_{k+j=n} B(G)^{kj} \longrightarrow B(G)^{0n}$ and we have defined a morphism of complexes $\phi : \underrightarrow{hodlim}_{s \in F} G_s \longrightarrow G_m$. It is clear that $\phi\psi = 1$.

Given integers $j + k = n$ with $k \le 0$ and $w \in W^{-k}$ we denote by $v_{k,j,w}$ the composite of the wth injection $G^j_{w_0} \longrightarrow \bigoplus_{w \in W^{-k}} G^j_{w_0}$ with the (k,j)th injection $B(G)^{kj} \longrightarrow Tot(B(G))^n$. The morphisms $\{v_{k,j,w}\}_{j+k=n, k \le 0, w \in W^{-k}}$ are obviously a coproduct in $\mathcal{A}$. We define a morphism $\Sigma^n : Tot(B(G))^n \longrightarrow Tot(B(G))^{n-1}$ on components by

$$\Sigma^n v_{k,j,w} = \begin{cases} 0 & \text{if } w_{-k} = m \\ (-1)^k v_{k-1,j,w_0 < \cdots < w_{-k} < m} & \text{if } w_{-k} \neq m \end{cases}$$

We claim that Σ is a homotopy $1 \longrightarrow \psi\phi$. The proof is straightforward but very tedious, with ample opportunity for error, so it seems prudent to include the details. For $k \le 0, k + j = n, w \in W^{-k}$ we set $\Lambda_{k,j,w} = (\partial^{n-1}\Sigma^n + \Sigma^{n+1}\partial^n) v_{k,j,w}$ to simply the notation. Then

$$\Lambda_{k,j,w} = \partial^{n-1}\Sigma^n v_{k,j,w} + \Sigma^{n+1} u_{(k+1)j} \partial_1^{kj} u^j_w + (-1)^k \Sigma^{n+1} u_{k(j+1)} \partial_2^{kj} u^j_w \tag{49}$$

We now divide into cases

- *Case* $k = 0$. We have $j = n$ and w is the sequence with one element w_0. The differential ∂_1^{kj} vanishes, so we have $\Lambda_{k,j,w} = \partial^{n-1}\Sigma^n v_{k,j,w} + \Sigma^{n+1}u_{k(j+1)}\partial_2^{kj}u_w^j$. Using the definition of Σ and ∂_2 we obtain

$$\Lambda_{k,j,w} = \begin{cases} 0 & \text{if } w_0 = m \\ \partial^{n-1}v_{-1,n,w_0<m} + v_{-1,j+1,w_0<m}\partial^n_{w_0} & \text{if } w_0 \neq m \end{cases}$$

  Using the definition of ∂^{n-1} we calculate

$$\partial^{n-1}v_{-1,n,w_0<m} = u_{0n}(u^n_m\mu^n_{w_0 m} - u^n_{w_0}) - v_{-1,j+1,w_0<m}\partial^n_{w_0}$$

  So finally

$$\Lambda_{k,j,w} = \begin{cases} 0 & \text{if } w_0 = m \\ u_{0n}(u^n_m\mu^n_{w_0 m} - u^n_{w_0}) & \text{if } w_0 \neq m \end{cases}$$

- *Case* $k < 0$ Expanding (49) using the definition of ∂_1 and ∂_2 we obtain

$$\begin{aligned}\Lambda_{k,j,w} &= \partial^{n-1}\Sigma^n v_{k,j,w_0<\cdots<w_{-k}} + \Sigma^{n+1}v_{k+1,j,w_1<\cdots<w_{-k}}\mu^j_{w_0w_1} \\ &+ \sum_{i=1}^{-k-1}(-1)^i\Sigma^{n+1}v_{k+1,j,w_0<\cdots<\hat{w_i}<\cdots<w_{-k}} \\ &+ (-1)^k\Sigma^{n+1}v_{k+1,j,w_0<\cdots<w_{-k-1}} + (-1)^k\Sigma^{n+1}v_{k,j+1,w_0<\cdots<w_{-k}}\partial^j_{w_0}\end{aligned}$$

  In the case where $w_{-k} = m$ all of these terms vanish except for the second last, which is equal to $-v_{k,j,w}$. Now assume that $w_{-k} \neq m$ and apply the definition of Σ to each term. Expanding using the definitions of ∂_1, ∂_2 and simplyfing, we find that $\Lambda_{k,j,w} = -v_{k,j,w}$ once again.

One checks more easily that $(\psi\phi - 1)^n v_{k,j,w}$ agrees with $\Lambda_{k,j,w}$ for all $k \leq 0, k + j = n, w \in W^{-k}$, by again splitting into the cases $k = 0$ and $k < 0$. We have now shown that for $n \in \mathbb{Z}$

$$\psi^n\phi^n - 1 = \partial^{n-1}\Sigma^n + \Sigma^{n+1}\partial^n$$

so Σ is a homotopy $1 \longrightarrow \psi\phi$ and the proof is complete. □

Theorem 96. *Let $\mathcal{A}$ be an abelian category with exact direct limits, and let $\{G_s, \mu_{st}\}_{s\in\Gamma}$ be a direct system in $\mathbf{C}(\mathcal{A})$. Then there is a canonical quasi-isomorphism $\underrightarrow{hodlim}_{s\in\Gamma}G_s \longrightarrow \varinjlim_{s\in\Gamma} G_s$.*

Proof. Denote by $M(\Gamma)$ the set of all finite subsets of Γ with maximum. This is a directed set ordered by inclusion. If $F \in M(\Gamma)$ then we denote by G_F the direct system $\{G_p, \mu_{pq}\}_{p\in F}$. If $F \subseteq H$ then there is a canonical morphism of bicomplexes $B(G_F) \longrightarrow B(G_H)$ and therefore also of the totalisations $Tot(B(G_F)) \longrightarrow Tot(B(G_H))$. This defines a direct system in $\mathbf{C}(\mathcal{A})$ over the directed set $M(\Gamma)$. The canonical morphisms $Tot(B(G_F)) \longrightarrow Tot(B(G))$ are a cocone on this system, and we claim they are actually a colimit in $\mathbf{C}(\mathcal{A})$.

Suppose that for every $F \in M(\Gamma)$ we are given a morphism of complexes $\phi_F : Tot(B(G_F)) \longrightarrow Q$ which is compatible with our direct system. Fix an integer $n \in \mathbb{Z}$. Then for $k \leq 0, j \in \mathbb{Z}$ with $k + j = n$ and $w \in W^{-k}$ we can choose $F \in M(\Gamma)$ containing all the elements of w, and consider the morphism

$$G^j_{w_0} \longrightarrow B(G_F)^{kj} \longrightarrow Tot(B(G_F))^n \xrightarrow{\phi^n_F} Q^n$$

This morphism does not depend on the choice of F, and taken together these morphisms induce a morphism $B(G)^{kj} \longrightarrow Q^n$. As k, j range over all integers with $k \leq 0$ and $k + j = n$ we induce another morphism $Tot(B(G))^n \longrightarrow Q^n$. This defines a morphism of complexes $\phi : Tot(B(G)) \longrightarrow Q$ which is the required unique factorisation. This shows that $Tot(B(G))$ is a direct limit of the $Tot(B(G_F))$ in $\mathbf{C}(\mathcal{A})$.

For each $F \in M(\Gamma)$ let m_F denote the maximum of F. Then by Lemma 95 we have a canonical homotopy equivalence

$$Tot(B(G_F)) = \underrightarrow{hodlim}_{s\in F} G_s \longrightarrow G_{m_F}$$

It is easily checked that these morphisms are compatible with the morphisms in the direct systems $\{Tot(B(G_F))\}_{F\in M(\Gamma)}$ and $\{G_{m_F}\}_{F\in M(\Gamma)}$. Taking direct limits and using the fact that cohomology commutes with direct limits (DF,Lemma 68), we obtain the desired quasi-isomorphism $Tot(B(G)) \longrightarrow \varinjlim_{s\in\Gamma} G_s$. □

Definition 36. Let $\mathcal{A}$ be a cocomplete abelian category and X an object of $\mathcal{A}$. A *filtration* of X is a sequence of subobjects $\{u_i : X_i \longrightarrow X\}_{i\geq 0}$ in $\mathcal{A}$ with $u_i \leq u_{i+1}$ for every $i \geq 0$. This is clearly a direct family of subobjects, and we say the filtration is *exhaustive* if $X = \varinjlim_{i\geq 0} X_i$.

Remark 40. Let $\mathcal{A}$ be an abelian category, X, Y two complexes in $\mathcal{A}$. When we refer to a morphism $u : X \longrightarrow Y$ *of graded objects* we just mean a collection of morphisms $u^n : X^n \longrightarrow Y^n$ (which do not necessarily commute with the differentials).

Lemma 97. *Let $\mathcal{A}$ be a cocomplete abelian category and B a bicomplex in $\mathcal{A}$ which is bounded on the right. That is, there exists $i_0 \geq 0$ with $B^{ij} = 0$ for all $i > i_0$. Then the complex $Tot(B)$ has an exhaustive filtration.*

Proof. By the complex $B^{i\bullet}$ we mean the ith column of B. First we construct complexes F_n for $n \geq 0$, and then we show that they are a filtration of $X = Tot(B)$ with the required property. We let F_0 be the complex $B^{i_0\bullet}[-i_0]$ (the shift is so that F_0 will map into $Tot(B)$). There is a morphism of complexes $B^{(i_0-1)\bullet}[-i_0] \longrightarrow F_0$ and we let F_1 be the mapping cone of this morphism. Suppose that for some $n > 0$ we have constructed the complex F_n together with a triangle

$$B^{(i_0-n)\bullet}[n - i_0 - 1] \longrightarrow F_{n-1} \longrightarrow F_n \longrightarrow B^{(i_0-n)\bullet}[n - i_0]$$

with the property that the composite $B^{(i_0-n-1)\bullet}[n - i_0 - 1] \longrightarrow B^{(i_0-n)\bullet}[n - i_0 - 1] \longrightarrow F_{n-1}$ vanishes. That is, the second morphism is the mapping cone of the first, and the third morphism is the canonical projection from the mapping cone. The previous construction of F_1 means that we have already done this for $n = 1$.

There is a canonical morphism of graded objects $B^{(i_0-n)\bullet}[n - i_0] \longrightarrow F_n$ which we compose with the morphism of complexes $B^{(i_0-n-1)\bullet}[n - i_0] \longrightarrow B^{(i_0-n)\bullet}[n - i_0]$ to obtain a morphism of graded objects $B^{(i_0-n-1)\bullet}[n - i_0] \longrightarrow F_n$. One checks this is actually a morphism of complexes. We define F_{n+1} to be the mapping cone of this morphism, so that we have a triangle

$$B^{(i_0-n-1)\bullet}[n - i_0] \longrightarrow F_n \longrightarrow F_{n+1} \longrightarrow B^{(i_0-n-1)\bullet}[n - i_0 + 1]$$

and it is clear that the first morphism vanishes on $B^{(i_0-n-2)\bullet}[n - i_0] \longrightarrow B^{(i_0-n-1)\bullet}[n - i_0]$. This completes the inductive step, and shows how to construct the complex F_n for $n \geq 0$.

The construction also provides a monomorphism $F_n \longrightarrow F_{n+1}$ for every $n \geq 0$. In fact, for $k \in \mathbb{Z}$ the object F^k_{n+1} is the coproduct $B^{(i_0-n-1)(k+n-i_0+1)} \oplus F^k_n$. Using induction one shows that for $n \geq 0$ and $k \in \mathbb{Z}$

$$F^k_n = B^{(i_0-n)(k-i_0+n)} \oplus \cdots \oplus B^{(i_0-1)(k-i_0+1)} \oplus B^{i_0(k-i_0)} \tag{50}$$

which is a subobject of $Tot(B)^k$ in a canonical way. This defines a monomorphism of complexes $F_n \longrightarrow Tot(B)$ and the morphisms $F_n \longrightarrow F_{n+1}$ show that $F_n \leq F_{n+1}$ as subobjects of $Tot(B)$ for $n \geq 1$. This defines the filtration $\{F_n\}_{n\geq 0}$, and it only remains to show that it is exhaustive.

By (DF,Lemma 3) it suffices to show that the morphisms $F^k_n \longrightarrow Tot(B)^k$ are a colimit in $\mathcal{A}$ for every $k \in \mathbb{Z}$. But the definition of F^k_n in (50) makes it obvious that $\varinjlim_{n\geq 0} F^k_n = \bigoplus_{i+j=k} B^{ij}$, so the proof is complete. □

Theorem 98. *Let $\mathcal{A}$ be a grothendieck abelian category, $\mathcal{L}$ a localising subcategory of $\mathfrak{D}(\mathcal{A})$ and $\{G_s, \mu_{st}\}_{s\in\Gamma}$ a direct system in $\mathbf{C}(\mathcal{A})$ such that $G_s \in \mathcal{L}$ for every $s \in \Gamma$. Then the homotopy direct limit $\underrightarrow{hodlim}_{s\in\Gamma} G_s$ also belongs to $\mathcal{L}$.*

Proof. By definition $\underrightarrow{hodlim}_{s\in\Gamma} G_s$ is the totalisation $Tot(B(G))$. Here $B(G)$ is a bicomplex whose columns are either zero, or for $k \leq 0$ are given by the following coproduct in $\mathbf{C}(\mathcal{A})$

$$B(G)^{k\bullet} = \bigoplus_{w\in W^{-k}} G_{w_0}$$

These complexes all belong to $\mathcal{L}$, so once again applying Proposition 44 we see that the columns of $B(G)$ belong to $\mathcal{L}$. By Lemma 97 the complex $Tot(B(G))$ is a direct limit $Tot(B(G)) = \varinjlim_{n\geq 0} F_n$ of certain complexes F_n, which we constructed earlier (in the present case $i_0 = 0$). By Proposition 65 to complete the proof it suffices to show that each complex F_n belongs to $\mathcal{L}$. The complex F_0 is a column of $Tot(B(G))$, so this is trivial for $n = 0$. For $n > 0$ we have a triangle in $\mathfrak{D}(\mathcal{A})$

$$B^{(-n)\bullet}[n-1] \longrightarrow F_{n-1} \longrightarrow F_n \longrightarrow B^{(-n)\bullet}[n]$$

where by induction and our earlier comments the first two objects belong to $\mathcal{L}$. Therefore so does F_n, and the proof is complete. □

As an immediate consequence we have the desired generalisation of Proposition 65.

Corollary 99. *Let $\mathcal{A}$ be a grothendieck abelian category, $\mathcal{L}$ a localising subcategory of $\mathfrak{D}(\mathcal{A})$ and $\{G_s, \mu_{st}\}_{s\in\Gamma}$ a direct system in $\mathbf{C}(\mathcal{A})$ such that $G_s \in \mathcal{L}$ for every $s \in \Gamma$. Then the direct limit $\varinjlim_{s\in\Gamma} G_s$ also belongs to $\mathcal{L}$.*

Proof. By Theorem 96 we have an isomomrphism $\underrightarrow{hodlim}_{s\in\Gamma} G_s \cong \varinjlim_{s\in\Gamma} G_s$ in $\mathfrak{D}(\mathcal{A})$, so the claim follows from Theorem 98. □

6 Bousfield Subcategories

Among the thick localising subcategories of triangulated categories, the *bousfield subcategories* (TRC,Definition 40) play a special role. In this section we prove some important results about bousfield subcategories of derived categories, following [ATJLSS00]. The first is Proposition 102, which says that if $\mathcal{A}$ is a grothendieck abelian category with enough projectives, then any localising subcategory of $\mathfrak{D}(\mathcal{A})$ generated by a set of objects is bousfield. The required background for this section includes (TRC,Section 4).

In this section we will need to use some transfinite recursion arguments, so the careful reader may want to see the introduction of our Triangulated Categories Part II notes, where we discuss the interaction between ordinals, cardinals and universes. In particular we say what we mean by a *small ordinal*.

One important insight in the theory of triangulated categories is the following: given a triangulated category $\mathcal{T}$ and a triangulated subcategory $\mathcal{S}$ we have a sequence of triangulated functors

$$\mathcal{S} \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}/\mathcal{S}$$

Often $\mathcal{S}$ is a category of "exact" complexes in some sense, so that morphisms in $\mathcal{T}$ whose mapping cones belong to $\mathcal{S}$ are "quasi-isomorphisms". A left adjoint to G takes objects of $\mathcal{T}$ to a "projective" resolution and a right adjoint takes objects of $\mathcal{T}$ to an "injective" resolution. The existence of adjoints to G is therefore equivalent with the existence of these types of resolutions, and we know that (modulo some technical details) the functor G has a right (left) adjoint if and only if $\mathcal{S}$ is bousfield (cobousfield). These conditions are therefore of great interest if we are to construct derived functors, which make essential use of the existence of such resolutions.

Corollary 100. *Let $\mathcal{A}$ be an abelian category with exact products and coproducts, a projective generator and enough injectives. Then $\mathcal{Z}$ is a bousfield subcategory of $K(\mathcal{A})$*

Proof. As usual $\mathcal{Z}$ denotes the exact complexes in $\mathcal{A}$, which form a thick localising subcategory of $K(\mathcal{A})$ by Proposition 44. By definition $K(I) = \mathcal{Z}^\perp$ and Proposition 75 shows that the composite $\mathcal{Z}^\perp \longrightarrow K(\mathcal{A}) \longrightarrow K(\mathcal{A})/\mathcal{Z}$ (which is always fully faithful) is an equivalence. It now follows from (TRC,Proposition 99) that the subcategory $\mathcal{Z}$ is bousfield. □

Remark 41. In particular this means that if $\mathcal{A}$ is an abelian category satisfying the conditions of Corollary 100 then the canonical triangulated functor $K(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})$ has a right triadjoint.

Corollary 101. *Let $\mathcal{A}$ be a grothendieck abelian category with enough projectives. Then $K(P)$ is a bousfield subcategory of $K(\mathcal{A})$.*

Proof. We showed in Proposition 71 that any complex X in $\mathcal{A}$ fits into a triangle in $K(\mathcal{A})$ of the form $P \longrightarrow X \longrightarrow Z \longrightarrow \Sigma P$ with $P \in K(P)$ and $Z \in \mathcal{Z} \subseteq K(P)^{\perp}$. It follows from (TRC,Proposition 99) that $K(P)$ is a bousfield subcategory of $K(\mathcal{A})$. □

Given the sea of technical detail the reader will soon encounter, it might be wise to briefly read the proof of Proposition 102 and then refer to Remark 44 which explains the simple idea underlying the proof.

Remark 42. Let $\mathcal{C}$ be a category, I, J two directed sets and $\{D_i, \mu_{ij}\}_{i\in I}, \{E_i, \lambda_{ij}\}_{i\in J}$ two direct systems in $\mathcal{C}$ over I, J respectively. We say these direct systems are *equivalent* if there is a bijection $f : I \longrightarrow J$ compatible with the ordering (that is, $f(i) \leq f(j)$ iff. $i \leq j$) such that $D_i = E_{f(i)}$ and $\mu_{ij} = \lambda_{f(i)f(j)}$ whenever $i \leq j$. In other words, the direct systems are made up of the same objects and morphisms in $\mathcal{C}$, indexed in the "same" way, up to some trivial colouring of the indices.

Remark 43. For the duration of this remark we drop the conglomerate convention (FCT,Definition 5). In the proof of Proposition 102 we are going to use a construction by transfinite recursion. To do this carefully we have to define two "constructions" $\tau(x), \mu(x)$ on all sets x (BST,Remark 3). We define the construction τ as follows

(i) If $x = (\mathfrak{U}, \mathcal{A}, S, \mathcal{Q}, \alpha, \{B_\gamma, \mu_{\gamma\tau}\}_{\gamma\preceq\alpha})$ is a tuple consisting of a universe $\mathfrak{U}$, a grothendieck abelian category $\mathcal{A}$ with enough projectives, a nonempty set $S \subseteq \mathbf{C}(\mathcal{A})$, an assignment $\mathcal{Q}$ of canonical set-indexed coproducts and direct limits to $\mathbf{C}(\mathcal{A})$ in which equivalent direct systems receive the same colimit, a small ordinal α and a direct system $\{B_\gamma, \mu_{\gamma\tau}\}_{\gamma\preceq\alpha}$ in $\mathbf{C}(\mathcal{A})$ over $\alpha+1$ with the obvious ordering (throughout the meaning of "category", "small" and "set" are relative to $\mathfrak{U}$) then we define $\tau(x)$ as follows: define the following set

$$\Omega = \bigcup_{k\in\mathbb{Z}, Y\in S} Hom_{\mathbf{C}(\mathcal{A})}(\Sigma^k Y, B_\alpha)$$

Given a morphism $s \in \Omega$ let $d(s)$ denote the domain of s. Let $\varphi : \bigoplus_{s\in\Omega} d(s) \longrightarrow B_\alpha$ be the induced morphism out of the canonical coproduct in $\mathbf{C}(\mathcal{A})$. That is, $\varphi u_s = s$. Let $B_{\alpha+1}$ be the canonical mapping cone of φ, and $\mu_{\alpha(\alpha+1)} : B_\alpha \longrightarrow B_{\alpha+1}$ be the canonical morphism into the mapping cone. For $\gamma \prec \alpha$ we define $\mu_{\gamma(\alpha+1)} = \mu_{\alpha(\alpha+1)}\mu_{\gamma\alpha}$. Then $\{B_\gamma, \mu_{\gamma\tau}\}_{\gamma\preceq\alpha+1}$ is a direct system in $\mathbf{C}(\mathcal{A})$ over $\alpha+2$ and we define

$$\tau(x) = (\mathfrak{U}, \mathcal{A}, S, \mathcal{Q}, \alpha+1, \{B_\gamma, \mu_{\gamma\tau}\}_{\gamma\preceq\alpha+1})$$

(ii) If x is not of this form, then $\tau(x)$ is the empty set.

We define the construction μ as follows

(a) If x is nonempty and consists of tuples of the form given in (i) above, all of which are equal to each other in the first four places $\{(\mathfrak{U}, \mathcal{A}, S, \mathcal{Q}, \alpha, \{B^\alpha_\gamma, \mu^\alpha_{\gamma\tau}\}_{\gamma\preceq\alpha})\}_\alpha$, and if the following conditions are satisfied

 – The direct families are all "compatible" in the sense that for occurring ordinals $\alpha \preceq \beta$ the restriction of the direct family $\{B^\beta_\gamma, \mu^\beta_{\gamma\tau}\}_{\gamma\preceq\beta}$ to the vertices of $\alpha+1$ agrees with the direct family $\{B^\alpha_\gamma, \mu^\alpha_{\gamma\tau}\}_{\gamma\preceq\alpha}$.
 – The union λ of all the ordinals α occurring in tuples in x is small (with respect to the common universe $\mathfrak{U}$ of all the tuples).

Then $\mu(x)$ is constructed as follows: for any $\gamma \prec \lambda$ choose α occurring in a tuple of x such that $\gamma \prec \alpha$, and define $C_\gamma = B^\alpha_\gamma$. This does not depend on the choice of α. Given another $\gamma' \preceq \gamma$ we define $\mu_{\gamma'\gamma} = \mu^\alpha_{\gamma'\gamma}$, which again does not depend on α. This defines a direct system $\{C_\gamma, \mu_{\gamma'\gamma}\}_{\gamma\prec\lambda}$ in $\mathbf{C}(\mathcal{A})$ over the directed set λ. Define

$$C_\lambda = \varinjlim_{\gamma\prec\lambda} C_\lambda$$

to be the canonical direct limit in $\mathbf{C}(\mathcal{A})$ of this direct system (if λ is empty, take C_λ to be a canonical zero object). Define $\mu_{\gamma\lambda} : C_\gamma \longrightarrow C_\lambda$ to be the injection into the colimit for any $\gamma \prec \lambda$. Then $\{C_\gamma, \mu_{\gamma\tau}\}_{\gamma\preceq\lambda}$ is a direct system in $\mathbf{C}(\mathcal{A})$ and we define

$$\mu(x) = (\mathfrak{U}, \mathcal{A}, S, \mathcal{Q}, \lambda, \{C_\gamma, \mu_{\gamma\tau}\}_{\gamma\preceq\lambda})$$

(b) If x is not of this form, then $\mu(x)$ is the empty set.

Proposition 102. *Let $\mathcal{A}$ be a grothendieck abelian category with enough projectives, S a nonempty set of objects of $\mathbf{C}(\mathcal{A})$, and $\mathcal{L} = \langle S\rangle$ the smallest localising subcategory of $\mathfrak{D}(\mathcal{A})$ containing these complexes. Then $\mathcal{L}$ is a bousfield subcategory of $\mathfrak{D}(\mathcal{A})$.*

Proof. Following [ATJLSS00] Proposition 4.5. By Proposition 71 every object of S is isomorphic in $\mathfrak{D}(\mathcal{A})$ to a hoprojective complex, so we may as well assume that S consists of hoprojective complexes. Therefore for every $k \in \mathbb{Z}$ and $Y \in S$ the complex $\Sigma^k Y$ is hoprojective, and $Hom_{\mathfrak{D}(\mathcal{A})}(\Sigma^k Y, X) \cong Hom_{K(\mathcal{A})}(\Sigma^k Y, X)$ by Corollary 50. We infer from (TRC,Proposition 99) that to complete the proof it is enough to show that for any $M \in \mathfrak{D}(\mathcal{A})$ there is a triangle in $\mathfrak{D}(\mathcal{A})$

$$N_M \longrightarrow M \longrightarrow B_M \longrightarrow \Sigma N_M$$

with $N_M \in \mathcal{L}$ and $B_M \in \mathcal{L}^\perp$. We construct the object B_M by transfinite recursion. Choose an assignment $\mathcal{Q}$ of set-indexed coproducts and direct limits to $\mathbf{C}(\mathcal{A})$, which we can arrange so that equivalent direct systems (in the sense of Remark 42) receive the same colimit. We define z to be the following conglomerate

$$z = (\mathfrak{U}, \mathcal{A}, S, \mathcal{Q}, 0, \{M\})$$

where $\mathfrak{U}$ is our fixed grothendieck universe. Using (BST,Theorem 17) with initial conglomerate z we produce a function f_α for every ordinal α. For every *small* ordinal α one checks by transfinite induction that

$$f_\alpha(\alpha) = (\mathfrak{U}, \mathcal{A}, S, \mathcal{Q}, \alpha, \{B^\alpha_\gamma, \mu^\alpha_{\gamma\tau}\}_{\gamma\preceq\alpha})$$

for some direct system in the last position, and we define $B_\alpha = B^\alpha_\alpha$. Let us make the following observations:

- For small ordinals $\alpha \preceq \beta$ there is a morphism of complexes $\mu_{\alpha\beta} = \mu^\beta_{\alpha\beta} : B_\alpha \longrightarrow B_\beta$. We have $\mu_{\alpha\alpha} = 1$ and $\mu_{\alpha\beta}\mu_{\gamma\alpha} = \mu_{\gamma\beta}$. In particular $B_0 = M$ so we have a morphism of complexes $\mu_{0\alpha} : M \longrightarrow B_\alpha$.
- For any small limit ordinal λ we have $B_\lambda = \varinjlim_{\alpha\prec\lambda} B_\alpha$ by construction, and for $\alpha \prec \lambda$ the morphism $\mu_{\alpha\lambda} : B_\alpha \longrightarrow B_\lambda$ is the canonical injection into the colimit.
- It follows by transfinite induction and (AC,Corollary 54) that for any small ordinals $\alpha \preceq \beta$ the morphism $\mu_{\alpha\beta} : B_\alpha \longrightarrow B_\beta$ is a monomorphism.
- For small ordinals $\alpha \prec \beta$, any $k \in \mathbb{Z}, Y \in S$ and morphism $f : \Sigma^k Y \longrightarrow B_\alpha$ the morphism $\mu_{\alpha\beta} f$ is null-homotopic. This was the whole point of defining τ above in the way we did.

Let U be a projective generator of $\mathcal{A}$ and κ be the cardinal successor of the cardinality of the set $\bigsqcup_{p\in\mathbb{Z}, Y\in S} Hom_{\mathcal{A}}(U, Y^p)$ (this is the disjoint union of sets). Then κ is a small infinite regular cardinal (TRC2,Lemma 1) (BST,Proposition 38) and we claim that the complex $B_M = B_\kappa$ has the required properties. First we check that B_M is $\mathcal{L}$-local. By Lemma 55 it is enough to check that $Hom_{\mathfrak{D}(\mathcal{A})}(\Sigma^k Y, B_M) \cong Hom_{K(\mathcal{A})}(\Sigma^k Y, B_M) = 0$ for every $k \in \mathbb{Z}$ and $Y \in S$.

Given $k \in \mathbb{Z}, Y \in S$ and a morphism of complexes $g : \Sigma^k Y \longrightarrow B_M$ we have to show that g is null-homotopic. By construction of the morphisms $\mu_{\alpha\beta}$ it would suffice to show that g factors through some $\mu_{i\kappa} : B_i \longrightarrow B_M$ with $i \prec \kappa$. That is, we want to show that $Im(g) \subseteq B_i$ for some $i \prec \kappa$. Construct the following pullback diagram for every ordinal $i \preceq \kappa$

$$\begin{array}{ccc} B'_i & \xrightarrow{\mu'_{i\kappa}} & Im(g) \\ \downarrow & & \downarrow \\ B_i & \xrightarrow[\mu_{i\kappa}]{} & B_M \end{array}$$

and $\mu'_{ij} : B'_i \longrightarrow B'_j$ for ordinals $i \preceq j \preceq \kappa$ by the universal property of the pullback. Clearly $\mu'_{ii} = 1$ and $\mu'_{js}\mu'_{ij} = \mu'_{is}$. In fact, since $\mathbf{C}(\mathcal{A})$ is grothendieck abelian, for any limit ordinal $t \preceq \kappa$ we have

$$\varinjlim_{i \prec t} B'_i = \sum_{i \prec t} (B_i \cap Im(g)) = \left(\sum_{i \prec t} B_i \right) \cap Im(g) = B_t \cap Im(g) = B'_t$$

That is, the morphisms $\mu'_{it} : B'_i \longrightarrow B'_t$ are a direct limit of the direct system $\{B'_i, \mu'_{ij}\}_{i \prec t}$. In particular the morphisms $\mu'_{i\kappa}$ are a direct limit. We want to show that $\mu'_{s\kappa}$ is an isomorphism for some $s \prec \kappa$. For this it would be enough to show that μ'_{st} (which is already a monomorphism) is an epimorphism for every ordinal t with $s \preceq t \prec \kappa$. Suppose for a contradiction that for every $s \prec \kappa$ there exists some $s \prec t \prec \kappa$ with μ'_{st} *not* an epimorphism. We claim that for every $s \prec \kappa$ there is $s \preceq t \prec \kappa$ wth $\mu'_{t(t+1)}$ not an epimorphism.

> *Proof of claim.* Fix an ordinal $s \prec \kappa$ and assume to the contrary that for every $s \preceq t \prec \kappa$ the morphism $\mu'_{t(t+1)}$ is an isomorphism. By our standing hypothesis there is some $s \prec t \prec \kappa$ for which μ'_{st} is not an epimorphism. We may as well assume t is minimal with this property. That is, whenever $s \prec q \prec t$ the morphism μ'_{sq} is an isomorphism. If $t = q + 1$ were a successor ordinal then μ'_{st} would be the composite $\mu'_{q(q+1)}\mu'_{sq}$ which is an isomorphism, a contradiction. Therefore t is a limit ordinal. The ordinals q with $s \prec q \prec t$ form a cofinal subset of all ordinals in t, and we have
>
> $$B'_t = \varinjlim_{q \prec t} B'_q = \varinjlim_{s \prec q \prec t} B'_q$$
>
> Given ordinals $s \prec q \prec q' \prec t$ the morphisms $\mu'_{sq}, \mu'_{sq'}$ are isomorphisms, and therefore so is $\mu'_{qq'} = \mu'_{sq'}(\mu'_{sq})^{-1}$. Since B'_t is the colimit of a direct system of isomorphisms, for $s \prec q \prec t$ the morphism $\mu'_{qt} : B'_q \longrightarrow B'_t$ is an isomorphism. But μ'_{sq} is an isomorphism, and hence so is μ'_{st}, which is the desired contradiction.

Having proved the claim, we know that for every $s \prec \kappa$ the set

$$J_s = \{t \prec \kappa \,|\, t \succeq s \text{ and } \mu'_{t(t+1)} \text{ is not an epimorphism}\}$$

is nonempty. In particular $J = J_0$ is a nonempty cofinal subset of κ. Since κ is regular we have $cf(\kappa) = \kappa$, and therefore $|J| = \kappa$ by definition of cofinality. For each $t \in J$ there exists $p \in \mathbb{Z}$ such that $(\mu'_{t(t+1)})^p$ is not an epimorphism, so we can choose a morphism $h_t : U \longrightarrow (B'_{t+1})^p$ which does not factor through $(B'_t)^p$. Define

$$\phi : J \longrightarrow \coprod_{p \in \mathbb{Z}} Hom_{\mathcal{A}}(U, Im(g)^p)$$

by mapping $t \in J$ to the composite $(\mu'_{(t+1)\kappa})^p h_t$. This morphism is injective, since if $\phi(t) = \phi(q)$ with $t \neq q$ then wlog $t \prec q$ and so also $t + 1 \preceq q$. One deduces easily that $U \longrightarrow (B'_{q+1})^p$ factors through $(B'_{t+1})^p$ and therefore also $(B'_q)^p$, a contradiction. So we have shown

$$\kappa = |J| \leq card\Big(\coprod_{p \in \mathbb{Z}} Hom_{\mathcal{A}}(U, Im(g)^p)\Big) < \kappa$$

where the last inequality follows from the fact that $Im(g)^p$ is a quotient of Y^{k+p} (here we use the fact that U is projective). This contradiction shows that $\mu'_{s\kappa} : B'_s \longrightarrow Im(g)$ is an isomorphism for some $s \prec \kappa$. Hence $g : \Sigma^k Y \longrightarrow B_M$ factors through B_s, which implies that g is null-homotopic. This completes the proof that B_M is $\mathcal{L}$-local in $\mathfrak{D}(\mathcal{A})$.

For every small ordinal i we have a triangle in $\mathfrak{D}(\mathcal{A})$ of the following form

$$N_i \longrightarrow M \xrightarrow{\mu_{0i}} B_i \longrightarrow \Sigma N_i$$

where by Proposition 20 we can take $B_i \longrightarrow \Sigma N_i$ to be the cokernel in $\mathbf{C}(\mathcal{A})$ of μ_{0i}. On the level of chain complexes for $i \preceq j$ the μ_{ij} induce morphisms $N_i \longrightarrow N_j$ making the obvious diagrams commute. Since cokernels commute with direct limits we deduce that for any small limit ordinal t these morphisms are a direct limit $N_t = \varinjlim_{i \prec t} N_i$. We claim that $N_i \in \mathcal{L}$ for any small ordinal i.

Since $N_0 = 0$ we have $N_0 \in \mathcal{L}$ trivially. Suppose that i is a small ordinal with $N_i \in \mathcal{L}$. By definition of N_i, N_{i+1} and B_{i+1} we have the following triangles in $\mathfrak{D}(\mathcal{A})$

$$M \xrightarrow{\mu_{0i}} B_i \longrightarrow \Sigma N_i \longrightarrow \Sigma M$$

$$M \xrightarrow{\mu_{0(i+1)}} B_{i+1} \longrightarrow \Sigma N_{i+1} \longrightarrow \Sigma M$$

$$B_i \xrightarrow{\mu_{i(i+1)}} B_{i+1} \longrightarrow \Sigma \bigoplus\nolimits_{s \in \Omega_i} d(s) \longrightarrow \Sigma B_i$$

By the octahedral axiom (TRC,Proposition 30) we deduce a triangle in $\mathfrak{D}(\mathcal{A})$

$$\Sigma N_i \longrightarrow \Sigma N_{i+1} \longrightarrow \Sigma \bigoplus_{s \in \Omega_i} d(s) \longrightarrow \Sigma^2 N_i$$

the first and third objects in this triangle belong to $\mathcal{L}$, and therefore so does N_{i+1}, which is what we wanted to show. Now suppose that t is a small limit ordinal and that $N_i \in \mathcal{L}$ for every $i \prec t$. Since $N_t = \varinjlim_{i \prec t} N_i$ it follows from Corollary 99 that $N_t \in \mathcal{L}$. By transfinite induction this shows that $N_i \in \mathcal{L}$ for every small ordinal i. In particular if we set $N_M = N_\kappa$ then $N_M \in \mathcal{L}$ and we have a triangle in $\mathfrak{D}(\mathcal{A})$

$$N_M \longrightarrow M \longrightarrow B_M \longrightarrow \Sigma N_M$$

with $N_M \in \mathcal{L}, B_M \in \mathcal{L}^\perp$, so the proof is complete. □

Remark 44. We make a few comments to clarify the technique used in the proof of Proposition 102. The idea is actually very simple. We are given a complex M and we want to construct a morphism $M \longrightarrow B_M$ in $\mathfrak{D}(\mathcal{A})$ with B_M an $\langle S \rangle$-local complex. The idea is that we begin with M and then keep "modding out" all the morphisms from objects of S. That is, we form the set Ω of all morphisms $\Sigma^k Y \longrightarrow M$ for various $Y \in S$, and let $\varphi : \bigoplus_{s \in \Omega} d(s) \longrightarrow M$ be the induced morphism, where $d(s)$ denotes the domain of s. Then we form the triangle

$$\bigoplus_{s \in \Omega} d(s) \longrightarrow M \longrightarrow B_1 \longrightarrow \Sigma \bigoplus_{s \in \Omega} d(s)$$

We apply the same procedure to B_1 to produce B_2, and in this way produce the sequence $M = B_0, B_1, B_2, \ldots$. We define B_ω by taking the direct limit of this sequence, then produce $B_{\omega+1}, B_{\omega+2}$ and so on. This defines a complex B_κ for any ordinal κ, and by taking a sufficiently enormous cardinal κ one forces B_κ to be $\langle S \rangle$-local, because any morphism from a $Y \in S$ to X_κ has to factor through some smaller B_λ.

Remark 45. In fact Proposition 102 is true even without the hypothesis that $\mathcal{A}$ have enough projectives, as we will see in a moment.

Recall that a *ringoid* $\mathcal{R}$ is a small preadditive category, and a right $\mathcal{R}$-module is an additive functor $\mathcal{R}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$. We denote by $\mathbf{Mod}\mathcal{R}$ the category of right $\mathcal{R}$-modules and refer the reader to our Rings with Several Objects (RSO) notes for further details. In particular $\mathbf{Mod}\mathcal{R}$

is a grothendieck abelian category with enough projectives. Throughout we use the notation of our RSO notes. An *additive topology* on a ringoid $\mathcal{R}$ is a right additive topology J as defined in (LOR,Definition 4). Associated to any additive topology is the corresponding *localisation* $\mathbf{Mod}(\mathcal{R}, J)$ which is a giraud subcategory of $\mathbf{Mod}\mathcal{R}$ (LOR,Corollary 17). This defines a bijection between additive topologies on $\mathcal{R}$ and giraud subcategories of $\mathbf{Mod}\mathcal{R}$ (LOR,Theorem 21).

Remark 46. We work in the generality of ringoids since there is no reason not to, but the reader who wants to work only with rings is perfectly welcome to do so. All the results will go through, since the Gabriel-Popecu theorem needs only a module category over a ring to embed any grothendieck abelian category.

Suppose we are given a ringoid $\mathcal{R}$ and an additive topology J and let $\mathcal{D} = \mathbf{Mod}(\mathcal{R}, J)$ be the associated giraud subcategory. By definition the inclusion $i : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{R}$ has an exact left adjoint a and this adjunction extends by Lemma 25 to a triadjunction of the homotopy categories

$$K(\mathcal{D}) \underset{K(a)}{\overset{K(i)}{\rightleftarrows}} K(\mathbf{Mod}\mathcal{R}) \qquad K(a) \dashv K(i) \tag{51}$$

By Lemma 45 the functor $K(a)$ preserves coproducts, so its kernel K_J is a thick localising subcategory of $K(\mathbf{Mod}\mathcal{R})$. Since $a : \mathbf{Mod}\mathcal{R} \longrightarrow \mathcal{D}$ is exact it lifts to a triangulated functor $\mathfrak{D}(a) : \mathfrak{D}(\mathbf{Mod}\mathcal{R}) \longrightarrow \mathfrak{D}(\mathcal{D})$ which by Lemma 46 must preserve coproducts. Its kernel $\mathcal{L}_J$ is therefore a thick localising subcategory of $\mathfrak{D}(\mathbf{Mod}\mathcal{R})$. By Remark 13 the subcategories $K_J, \mathcal{L}_J$ depend only on the topology J and not on the choice of exact left adjoint a. Also observe that since $ai \cong 1$ there is a trinatural equivalence $K(a)K(i) \cong 1$.

Lemma 103. *Let $\mathcal{R}$ be a ringoid and J an additive topology. Then the thick localising subcategory $K_J \subseteq K(\mathbf{Mod}\mathcal{R})$ is bousfield.*

Proof. Set $\mathcal{A} = \mathbf{Mod}\mathcal{R}$ and observe that by Lemma 38 we can consider $K(\mathcal{D})$ as a fragile triangulated subcategory of $K(\mathcal{A})$, whose inclusion has a left adjoint. Let $\mathcal{C}$ denote its replete closure (that is, all objects of $K(\mathcal{A})$ isomorphic to an object of $K(\mathcal{D})$). Firstly we show that $\mathcal{C} = K_J^{\perp}$. If $X \in K_J$ and $Y \in K(\mathcal{D})$ then we have

$$Hom(X, Y) = Hom(X, iY) \cong Hom(aX, Y) = 0$$

so it is clear that $\mathcal{C} \subseteq K_J^{\perp}$. Let $\eta : 1 \longrightarrow ia$ denote the unit of the adjunction $a \dashv i$. In Section 3.1 we saw how this unit gives rise in the obvious way to the unit of the adjunction $K(a) \dashv K(i)$. Let X be a complex in $\mathcal{A}$ and $\eta_X : X \longrightarrow aX$ the unit morphism, which we can extend to a triangle in $K(\mathcal{A})$

$$X \longrightarrow aX \longrightarrow Y \longrightarrow \Sigma X \tag{52}$$

Since $a(\eta_X)$ is an isomorphism (LOR,Lemma 15), it is clear that $Y \in K_J$. If X happens to belong to $K_J^{\perp}$ then we can we can apply $Hom(Y, -)$ to this triangle and deduce from the resulting exact sequence that $Hom_{K(\mathcal{A})}(Y, Y) = 0$. Therefore $Y = 0$ and $X \cong aX$ in $K(\mathcal{A})$, which proves that $\mathcal{C} = K_J^{\perp}$. Shifting (52) we see that every object $X \in K(\mathcal{A})$ fits into a triangle

$$N \longrightarrow X \longrightarrow B \longrightarrow \Sigma N$$

with $N \in K_J, B \in K_J^{\perp}$, so by (TRC,Proposition 99) the subcategory K_J is bousfield. ☐

Lemma 104. *A complex X belongs to $\mathcal{L}_J$ if and only if all its cohomology objects are J-torsion. That is, $X \in \mathcal{L}_J$ if and only if $aH^j(X) = 0$ for every $j \in \mathbb{Z}$.*

Proof. The complex X belongs to $\mathcal{L}_J$ if and only if aX is exact, which is if and only if $H^j(aX) = 0$ for $j \in \mathbb{Z}$. Exactness of a implies $aH^j(X) \cong H^j(aX)$ so the proof is complete. ☐

In a moment we will prove that $\mathcal{L}_J$ can be generated by a set of objects. But first we record some categorical trivialities that we will make use of in the proof.

Lemma 105. *Let F be a nonzero J-torsion right $\mathcal{R}$-module. Then there is $A \in \mathcal{R}$ and $\mathfrak{a} \in J(A)$ together with a nonzero monomorphism $H_A/\mathfrak{a} \longrightarrow F$.*

Proof. In some sense the quotients $H_A/\mathfrak{a}$ with $\mathfrak{a}$ in the topology generate the subcategory of torsion objects. Recall that F is J-torsion if and only if $aF = 0$ (LOR,Lemma 14). Since F is nonzero, there is $A \in \mathcal{R}$ and a nonzero morphism $x : H_A \longrightarrow F$, whose kernel $Ann(x)$ belongs to $J(A)$ by assumption. The factorisation $H_A/Ann(x) \longrightarrow F$ is the desired monomorphism. $\square$

Lemma 106. *Let $\mathcal{A}$ be an abelian category and suppose we are given a pullback diagram in which the horizontal morphisms are monomorphisms*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

The induced morphism $B'/A' \longrightarrow B/A$ is a monomorphism.

Proof. By the embedding theorem (DCAC,Theorem 1) (DCAC,Lemma 2) we can reduce to the case where $\mathcal{A} = \mathbf{Ab}$, which is trivial. $\square$

Remark 47. We can avoid the use of embedding theorems in the proof of Lemma 106 if we assume $\mathcal{A}$ has a generating family of projectives, in which case the proof proceeds in the obvious way.

Definition 37. Let $\mathcal{R}$ be a ringoid. The *size* of $\mathcal{R}$ is the cardinality of its set of morphisms, and we denote this by $size(\mathcal{R})$. If F is a right $\mathcal{R}$-module then the *size* of F, also denoted by $size(F)$, is defined to be the cardinality of the following disjoint union

$$size(F) = card\Big(\bigsqcup_{A \in \mathcal{R}} F(A) \Big)$$

The cardinal $size(\mathcal{R})$ is small, as is $size(F)$ for any right $\mathcal{R}$-module F.

Theorem 107. *Let $\mathcal{R}$ be a ringoid and J an additive topology. Then the thick localising subcategory $\mathcal{L}_J \subseteq \mathfrak{D}(\mathbf{Mod}\mathcal{R})$ is bousfield.*

Proof. Following [ATJLSS00] Proposition 5.1. We prove the result by showing that there is a set of objects $S \subseteq \mathfrak{D}(\mathbf{Mod}\mathcal{R})$ with $\mathcal{L}_J = \langle S \rangle$, and then applying Proposition 102.

Set $\mathcal{A} = \mathbf{Mod}\mathcal{R}$ and let β be a small infinite cardinal with $\beta \geq size(\mathcal{R})$ and select a set Q with $|Q| = \beta$. Since coproducts in $\mathcal{A}$ can be computed pointwise, for any given set-indexed family of objects $\{F_i\}_{i \in I}$ of $\mathcal{A}$ we have a canonical coproduct $\bigoplus_{i \in I} F_i$ in $\mathcal{A}$. We say that a complex $E \in \mathbf{C}(\mathcal{A})$ is *cloned* if it has the following properties

(i) $E^j = 0$ for $j > 0$. Here we mean that E^j is actually *equal* to the *canonical* zero object.

(ii) $E^0 = H_A$ for some $A \in \mathcal{R}$.

(iii) For $j < 0$ the object E^j is the canonical coproduct $\bigoplus_{i \in I} H_{A_i}$ for some nonempty family of objects $\{A_i\}_{i \in I}$ of $\mathcal{R}$, with $I \subseteq Q$.

(iv) $aH^j(E) = 0$ for all $j \in \mathbb{Z}$. That is, the cohomology objects are torsion.

It is clear that the class $S \subseteq \mathbf{C}(\mathcal{A})$ of all cloned complexes is small (here we use (AC,Proposition 51) to see that the differentials in E can be written in terms of components). We must show that $\mathcal{L}_J = \langle S \rangle$. The functor a is exact, so for any $E \in S$ we have $H^j(aE) \cong aH^j(E) = 0$, so aE is an exact complex and is therefore zero in $\mathfrak{D}(\mathcal{D})$. It is therefore clear that $\langle S \rangle \subseteq \mathcal{L}_J$, and it remains to prove the reverse inclusion.

Given $M \in \mathcal{L}_J$ there is by Proposition 102 a triangle in $\mathfrak{D}(\mathcal{A})$ of the form

$$N \longrightarrow M \longrightarrow B \longrightarrow \Sigma N$$

with $N \in \langle S \rangle$ and $B \in \langle S \rangle^{\perp}$. Since $M, N \in \mathcal{L}_J$ we have $B \in \mathcal{L}_J$. To show that $M \in \langle S \rangle$ it therefore suffices to show that B is an exact complex. If B is not exact, then there exists some $k \in \mathbb{Z}$ with $H^k(B) \neq 0$. Since $aH^k(B) = 0$ we are in the situation of Lemma 105: the right $\mathcal{R}$-module $H^k(B)$ is nonzero but J-torsion, so there exists $A \in \mathcal{R}, \mathfrak{a} \in J(A)$ and a nonzero monomorphism $g : H_A/\mathfrak{a} \longrightarrow H^k(B)$. Using g we will construct a nonzero morphism in $\mathfrak{D}(\mathcal{A})$ from a complex in $\langle S \rangle$ to B, which is impossible. This contradiction shows that B is exact and completes the proof.

Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & H_A & \longrightarrow & H_A/\mathfrak{a} & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \downarrow g & & \\ 0 & \longrightarrow & Im(\partial_B^{k-1}) & \longrightarrow & Ker(\partial_B^k) & \longrightarrow & H^k(B) & \longrightarrow & 0 \end{array}$$

Where we use projectivity of H_A to find a morphism $H_A \longrightarrow Ker(\partial_B^k)$ making the right hand square commute, and then induce a morphism $\mathfrak{a} \longrightarrow Im(\partial_B^{k-1})$ on the kernels. It is clear that $size(\mathfrak{a}) \leq size(\mathcal{R}) \leq \beta$, so we can find a nonempty family $\{A_i\}_{i \in I_{k-1}}$ of objects of $\mathcal{R}$ indexed by a set $I_{k-1} \subseteq Q$ and an epimorphism $P^{k-1} = \bigoplus_{i \in I_{k-1}} H_{A_i} \longrightarrow \mathfrak{a}$, where P^{k-1} is clearly projective. Define $P^k = H_A$, let $f^k : P^k \longrightarrow B^k$ be the composite $H_A \longrightarrow Ker(\partial_B^k) \longrightarrow B^k$ and let $f^{k-1} : P^{k-1} \longrightarrow B^{k-1}$ be some lifting of the composite $P^{k-1} \longrightarrow \mathfrak{a} \longrightarrow Im(\partial_B^{k-1})$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} & & P^{k-1} & \xrightarrow{\partial_P^{k-1}} & P^k & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow f^{k-1} & & \downarrow f^k & & \downarrow & & \\ \cdots & \longrightarrow B^{k-2} \longrightarrow & B^{k-1} & \longrightarrow & B^k & \longrightarrow & B^{k+1} & \longrightarrow & \cdots \end{array}$$

Observe that $a(P^k/Im(\partial_P^{k-1})) \cong a(H_A/\mathfrak{a}) = 0$ since $\mathfrak{a} \in J(A)$ and vanishing of such quotients characterises the ideals in the topology (LOR, Theorem 21). The idea is to iterate this construction.

Suppose that for some $j < k$ we have constructed a cloned complex P down to degree j (that is, modules $P^j, \ldots, P^k$ and morphisms $\partial_P^j, \ldots, \partial_P^{k-1}$ such that the resulting sequence satisfies conditions $(i), (ii), (iii)$ and (iv) of a cloned complex, with the understanding that (iv) is applied at positions $> j$) together with a morphism $P \longrightarrow B$. Form the following diagram

$$\begin{array}{ccccc} C^j & \xrightarrow{q} & Ker(\partial_P^j) & \longrightarrow & P^j \\ \downarrow & & \downarrow & & \downarrow f^j \\ Im(\partial_B^{j-1}) & \longrightarrow & Ker(\partial_B^j) & \longrightarrow & B^j \end{array}$$

where the right hand square is induced on kernels on the usual way, and the left hand square is formed by pullback. Therefore q is a monomorphism and we have

$$\begin{aligned} size(C^j) \leq size(P^j) &= size(\bigoplus_{i \in I_j} H_{A_i}) \\ &= card\Big(\coprod_{A \in \mathcal{R}} \bigoplus_{i \in I_j} Hom_{\mathcal{R}}(A, A_i)\Big) \\ &\leq card\Big(\coprod_{A \in \mathcal{R}} \prod_{i \in I_j} Mor\mathcal{R}\Big) \\ &\leq card\Big(\mathcal{R} \times I_j \times Mor\mathcal{R}\Big) \\ &\leq \beta \cdot \beta \cdot \beta = \beta \end{aligned}$$

where we use in a crucial way the fact that β is an infinite cardinal. We can therefore find a nonempty family $\{A_i\}_{i\in I_{j-1}}$ of objects of $\mathcal{R}$ indexed by a set $I_{j-1} \subseteq Q$ and an epimorphism $P^{j-1} = \bigoplus_{i\in I_{j-1}} H_{A_i} \longrightarrow C^j$. Define the morphisms ∂_P^{j-1} and f^{j-1} in the obvious way. By Lemma 106 the cohomology $H^j(P)$ is a subobject of $H^j(B)$ and therefore $aH^j(P) \subseteq aH^j(B) = 0$ must vanish, so we have extended our complex P to degree $j-1$. Since this recursive step involves noncanonical choices, we invoke Zorn's Lemma to guarantee the construction of a complex P and a morphism of complexes $f : P \longrightarrow B$. As P is the shift of a cloned complex, we have $P \in \langle S\rangle$.

Hoprojectivity of P means that to check f is nonzero in $\mathfrak{D}(\mathcal{A})$ it suffices by Corollary 50 to check it is nonzero in $K(\mathcal{A})$, that is, it is not null-homotopic. If it were, then in particular the morphism $f^0 : H_A \longrightarrow Ker(\partial_B^k) \longrightarrow B^k$ would factor through B^{k-1}, and it follows that $H_A \longrightarrow Ker(\partial_B^k)$ factors through $Im(\partial_B^{k-1})$. Then g composed with $H_A \longrightarrow H_A/\mathfrak{a}$ is zero, which contradicts the fact that g is nonzero.

Therefore f is a nonzero morphism in $\mathfrak{D}(\mathcal{A})$ from an object of $\langle S\rangle$ to $B \in \langle S\rangle^\perp$, which is a contradiction. This shows that B is exact, and completes the proof. □

7 Existence of Resolutions

Let $\mathcal{A}$ be a grothendieck abelian category, so that we have a sequence of triangulated functors

$$\mathcal{Z} \longrightarrow K(\mathcal{A}) \xrightarrow{Q} \mathfrak{D}(\mathcal{A})$$

The subcategory $\mathcal{Z}$ of exact complexes is bousfield if and only if Q has a right adjoint, which by the formalism of bousfield subcategories means that for every complex X in $\mathcal{A}$ there is a triangle in $K(\mathcal{A})$ of the form

$$Z \longrightarrow X \longrightarrow I \longrightarrow \Sigma Z$$

with $I \in \mathcal{Z}^\perp$ and Z an exact complex. This a resolution of X by a hoinjective complex, so that if the subcategory $\mathcal{Z}$ is bousfield then $\mathcal{A}$ has enough hoinjectives. In this section we show, following [ATJLSS00], that any grothendieck abelian category $\mathcal{A}$ satisfies this condition.

Throughout this section $\mathcal{R}$ is a ringoid, J an additive topology on $\mathcal{R}$, $\mathcal{D} = \mathbf{Mod}(\mathcal{R}, J)$ the associated giraud subcategory with inclusion $i : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{R}$ and exact left adjoint a. As before we write K_J for the kernel of the triangulated functor $K(a) : K(\mathbf{Mod}\mathcal{R}) \longrightarrow K(\mathcal{D})$, which is a thick localising subcategory of $K(\mathbf{Mod}\mathcal{R})$. We write $\mathcal{L}_J$ for the kernel of $\mathfrak{D}(a)$, which is a thick localising subcategory of $\mathfrak{D}(\mathbf{Mod}\mathcal{R})$. The reader should be familiar with the notion of a *weak verdier quotient* (TRC,Section 2.3).

Proposition 108. *The triangulated functor $K(a) : K(\mathbf{Mod}\mathcal{R}) \longrightarrow K(\mathcal{D})$ is a weak verdier quotient of $K(\mathbf{Mod}\mathcal{R})$ by K_J.*

Proof. Set $\mathcal{A} = \mathbf{Mod}\mathcal{R}$ and recall that by Lemma 103 the subcategory $K_J \subseteq K(\mathbf{Mod}\mathcal{R})$ is bousfield. As in the proof of Lemma 103 we write $\mathcal{C}$ for the replete closure in $K(\mathcal{A})$ of $K(\mathcal{D})$, which we know is also equal to $K_J^\perp$. If we write $j : \mathcal{C} \longrightarrow K(\mathcal{A})$ for the inclusion and $Q : K(\mathcal{A}) \longrightarrow K(\mathcal{A})/K_J$ for the verdier quotient, then Qj is a triequivalence (TRC,Corollary 100). Let $r : K(\mathcal{A})/K_J \longrightarrow \mathcal{C}$ be a triangulated functor together with trinatural equivalences $rQj \cong 1, Qjr \cong 1$. Then in the proof of (TRC,Corollary 100) we showed that rQ is left triadjoint to j.

Denote by q the inclusion $K(\mathcal{D}) \longrightarrow \mathcal{C}$. This is a triequivalence: that is, there is a triangulated functor $q' : \mathcal{C} \longrightarrow K(\mathcal{D})$ together with trinatural equivalences $qq' \cong 1, q'q \cong 1$.

Since $K(a)$ is left triadjoint to the inclusion $K(\mathcal{D}) \longrightarrow K(\mathcal{A})$ and q is left triadjoint to q' (TRC,Remark 36), the functor $qK(a)$ is left triadjoint to j (TRC,Proposition 46) (TRC,Lemma 45). We deduce a trinatural equivalence $qK(a) \cong rQ$ (TRC,Lemma 43) and so finally we have a trinatural equivalence $K(a) \cong q'rQ$. The functor $q'rQ$ is a weak verdier quotient of $K(\mathcal{A})$ by K_J, since it is the composite of a weak verdier quotient Q (TRC,Proposition 74) with a triequivalence $q'r$ (TRC,Proposition 76). Therefore $K(a)$ is also a weak verdier quotient (TRC,Lemma 77) and the proof is complete. □

Corollary 109. *There is a canonical triequivalence* $K(\mathbf{Mod}\mathcal{R})/K_J \longrightarrow K(\mathcal{D})$*.*

Proof. Both functors $K(\mathbf{Mod}\mathcal{R}) \longrightarrow K(\mathbf{Mod}\mathcal{R})/K_J$ and $K(a)$ are weak verdier quotients, so the unique triangulated functor T making the following diagram commute (using the universal property of the verdier quotient)

$$\begin{array}{ccc} & K(\mathbf{Mod}\mathcal{R}) & \\ \swarrow & & \searrow \\ K(\mathbf{Mod}\mathcal{R})/K_J & \xrightarrow[T]{} & K(\mathcal{D}) \end{array}$$

is a triequivalence, by the "weak" uniqueness property of the weak verdier quotient. □

Proposition 110. *The triangulated functor* $\mathfrak{D}(a) : \mathfrak{D}(\mathbf{Mod}\mathcal{R}) \longrightarrow \mathfrak{D}(\mathcal{D})$ *is a weak verdier quotient of* $\mathfrak{D}(\mathbf{Mod}\mathcal{R})$ *by* $\mathcal{L}_J$*.*

Proof. Set $\mathcal{A} = \mathbf{Mod}\mathcal{R}$ and observe that we have the following commutative diagram of triangulated functors

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{K(a)} & K(\mathcal{D}) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle P} \\ \mathfrak{D}(\mathcal{A}) & \xrightarrow[\mathfrak{D}(a)]{} & \mathfrak{D}(\mathcal{D}) \end{array} \tag{53}$$

Suppose we are given a triangulated functor $G : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathcal{S}$ into a portly triangulated category with $\mathcal{L}_J \subseteq Ker(G)$. Then GQ sends objects of K_J to zero so by Proposition 108 there is a triangulated functor $H : K(\mathcal{D}) \longrightarrow \mathcal{S}$ and a trinatural equivalence $HK(a) \cong GQ$. Suppose we are given an exact complex X in $\mathcal{D}$. Since $K(a)K(i) \cong 1$ we have $X \cong aX$ and therefore we deduce from commutativity of (53) that $X \in \mathcal{L}_J$. Hence

$$H(X) \cong HK(a)(X) \cong GQ(X) = 0$$

since by assumption G vanishes on $\mathcal{L}_J$. The functor H therefore factors uniquely (as a triangulated functor) through P. Let $H' : \mathfrak{D}(\mathcal{D}) \longrightarrow \mathcal{S}$ be this factorisation. Then we have a trinatural equivalence

$$H'\mathfrak{D}(a)Q = H'PK(a) = HK(a) \cong GQ$$

and since Q is itself a weak verdier quotient we infer from (TRC,Remark 52) that there is a trinatural equivalence $H'\mathfrak{D}(a) \cong G$. This proves the existence of weak factorisations. To show weak uniqueness, suppose that $H'' : \mathfrak{D}(\mathcal{D}) \longrightarrow \mathcal{S}$ is another triangulated functor together with a trinatural equivalence $H''\mathfrak{D}(a) \cong G$. Then we have a trinatural equivalence

$$H''PK(a) = H''\mathfrak{D}(a)Q \cong GQ \cong HK(a) = H'PK(a)$$

By Proposition 108 the functor $K(a)$ is a weak verdier quotient, so we deduce a trinatural equivalence $H''P \cong H'P$. But P is also a weak verdier quotient, so there is a trinatural equivalence $H'' \cong H'$ as required. □

Corollary 111. *There is a canonical triequivalence* $\mathfrak{D}(\mathbf{Mod}\mathcal{R})/\mathcal{L}_J \longrightarrow \mathfrak{D}(\mathcal{D})$*.*

Proof. Same proof as Corollary 109. □

Corollary 112. *The triangulated functor* $\mathfrak{D}(a) : \mathfrak{D}(\mathbf{Mod}\mathcal{R}) \longrightarrow \mathfrak{D}(\mathcal{D})$ *has a right triadjoint.*

Proof. Set $\mathcal{A} = \mathbf{Mod}\mathcal{R}$. By Theorem 107 the subcategory $\mathcal{L}_J \subseteq \mathfrak{D}(\mathbf{Mod}\mathcal{R})$ is bousfield, so the verdier quotient $Q : \mathfrak{D}(\mathcal{A}) \longrightarrow \mathfrak{D}(\mathcal{A})/\mathcal{L}_J$ has a right triadjoint M (TRC,Corollary 100). But by Corollary 111 there is a commutative diagram of triangulated functors

$$\begin{array}{ccc} & \mathfrak{D}(\mathcal{A}) & \\ \swarrow & & \searrow \\ \mathfrak{D}(\mathcal{A})/\mathcal{L}_J & \xrightarrow[T]{} & \mathfrak{D}(\mathcal{D}) \end{array}$$

with T a triequivalence. Let $T' : \mathfrak{D}(\mathcal{D}) \longrightarrow \mathfrak{D}(\mathcal{A})/\mathcal{L}_J$ be a triangulated functor together with natural triequivalences $TT' \cong 1, T'T \cong 1$. Then T' is right triadjoint to T (TRC,Remark 36) and therefore MT' is right triadjoint to $TQ = \mathfrak{D}(a)$ (TRC,Proposition 46) as required. $\square$

Remark 48. If we knew *a priori* that $\mathcal{D}$ had enough hoinjectives, then it would follow immediately from (TRC,Theorem 122) that $\mathfrak{D}(a)$ had a right adjoint, namely the right derived functor $\mathbb{R}(i)$ where $i : \mathcal{D} \longrightarrow \mathbf{Mod}\mathcal{R}$ is the inclusion. But in our approach it is Corollary 112 that allows us to prove the existence of hoinjective resolutions for $\mathcal{D}$. We will return to this point after we have developed the theory of derived functors in (DTC2,Section 2).

One should be very careful to note that the right adjoint of $\mathfrak{D}(a)$ is *not* a lift of the inclusion i to the derived category. In general no such lift can exist, because i is not exact (for example, the inclusion of sheaves in presheaves). Nonetheless the right derived functor of $\mathfrak{D}(a)$ does have some good properties, as we will show in Lemma 116.

Earlier we proved the existence of hoinjective resolutions only under very restrictive hypothesis (see Proposition 75). We are now prepared to prove the major theorem, which establishes the existence of hoinjective resolutions for arbitrary grothendieck abelian categories.

Theorem 113. *If $\mathcal{C}$ is a grothendieck abelian category then the exact complexes $\mathcal{Z}$ form a bousfield subcategory of $K(\mathcal{C})$.*

Proof. Following [ATJLSS00] Theorem 5.4. We know from Proposition 44 that $\mathcal{Z}$ is a thick localising subcategory of $K(\mathcal{C})$, and we have to show that the canonical functor $K(\mathcal{C}) \longrightarrow \mathfrak{D}(\mathcal{C})$ has a right adjoint. Choose a family of generators for $\mathcal{C}$, and let $\mathcal{R}$ be the small full subcategory whose objects are the generators. By the Gabriel-Popescu theorem (LOR,Theorem 25) there is an additive topology J on $\mathcal{R}$ and an equivalence

$$\ell : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{R}, J)$$

If we set $\mathcal{D} = \mathbf{Mod}(\mathcal{R}, J)$ then by Remark 13 there is an induced commutative diagram of triangulated functors in which the rows are triequivalences

$$\begin{array}{ccc} K(\mathcal{C}) & \xrightarrow{K(\ell)} & K(\mathcal{D}) \\ \downarrow & & \downarrow Q' \\ \mathfrak{D}(\mathcal{C}) & \xrightarrow[\mathfrak{D}(\ell)]{} & \mathfrak{D}(\mathcal{D}) \end{array}$$

Equivalences are both left and right adjoint to their inverses and we can compose adjoints, so to prove that $K(\mathcal{C}) \longrightarrow \mathfrak{D}(\mathcal{C})$ has a right adjoint, it suffices to show that $Q' : K(\mathcal{D}) \longrightarrow \mathfrak{D}(\mathcal{D})$ has a right adjoint. We have a commutative diagram of triangulated functors

$$\begin{array}{ccc} K(\mathbf{Mod}\mathcal{R}) & \xrightarrow{K(a)} & K(\mathcal{D}) \\ Q \downarrow & & \downarrow Q' \\ \mathfrak{D}(\mathbf{Mod}\mathcal{R}) & \xrightarrow[\mathfrak{D}(a)]{} & \mathfrak{D}(\mathcal{D}) \end{array}$$

The abelian category $\mathbf{Mod}\mathcal{R}$ satisfies the conditions of Corollary 100 and therefore the functor $K(\mathbf{Mod}\mathcal{R}) \longrightarrow \mathfrak{D}(\mathbf{Mod}\mathcal{R})$ has a right triadjoint. By Corollary 112 the functor $\mathfrak{D}(a)$ has a right triadjoint and by Proposition 108 the functor $K(a)$ is a weak verdier quotient. We deduce that the composite $Q'K(a) = \mathfrak{D}(a)Q$ has a right triadjoint (TRC,Proposition 46). We are now in the situation of (TRC,Proposition 79) which implies that Q' has a right triadjoint and completes the proof. $\square$

Remark 49. For any grothendieck abelian category $\mathcal{C}$ we have the following consequences of Theorem 113

- The canonical triangulated functor $K(\mathcal{C}) \longrightarrow \mathfrak{D}(\mathcal{C})$ has a right triadjoint.
- The composite $K(I) \longrightarrow K(\mathcal{C}) \longrightarrow \mathfrak{D}(\mathcal{C})$ is a triequivalence, and in particular by Remark 22 every complex in $\mathcal{C}$ has a hoinjective resolution. In other words, *any grothendieck abelian category has enough hoinjectives.*
- The inclusion $K(I) \longrightarrow K(\mathcal{C})$ has a right triadjoint and ${}^{\perp}K(I) = \mathcal{Z}$. That is, a complex Z is exact *if and only if* every morphism of complexes $Z \longrightarrow I$ with I hoinjective is null-homotopic.
- The inclusion $K(P) \longrightarrow K(\mathcal{C})$ has a right triadjoint.

Corollary 114. *If $\mathcal{C}$ is a grothendieck abelian category then the portly triangulated category $\mathfrak{D}(\mathcal{C})$ has small morphism conglomerates.*

Proof. Since $\mathcal{C}$ has enough hoinjectives, this is a consequence of Lemma 61. □

Lemma 115. *Let $\mathcal{T}$ be a triangulated category and (ℓ, η) a localisation in $\mathcal{T}$. Given a triangulated functor $\ell' : \mathcal{T} \longrightarrow \mathcal{T}$ and a trinatural equivalence $m : \ell \longrightarrow \ell'$, the pair $(\ell', m \circ \eta)$ is also a localisation.*

Proof. See (TRC,Definition 39) for the definition of a *localisation* in a triangulated category. Verification of the claim is straightforward. □

Lemma 116. *Suppose that $\mathbf{i}$ is right triadjoint to $\mathfrak{D}(a)$ with unit η and counit ε*

$$\mathfrak{D}(\mathbf{Mod}\mathcal{R}) \underset{\mathbf{i}}{\overset{\mathfrak{D}(a)}{\rightleftarrows}} \mathfrak{D}(\mathcal{D}) \qquad (\eta, \varepsilon) : \mathfrak{D}(a) \dashv \mathbf{i}$$

Then we have the following

(i) $(\mathbf{i}\mathfrak{D}(a), \eta)$ *is a localisation of* $\mathfrak{D}(\mathbf{Mod}\mathcal{R})$ *with kernel* $\mathcal{L}_J$.

(ii) The image of the functor $\mathbf{i}$ *is contained in* $\mathcal{L}_J^{\perp}$ *and the induced functor* $\mathfrak{D}(\mathcal{D}) \longrightarrow \mathcal{L}_J^{\perp}$ *is a triequivalence.*

(iii) $\varepsilon : \mathfrak{D}(a)\mathbf{i} \longrightarrow 1$ *is a trinatural equivalence and* $\eta_X : X \longrightarrow \mathbf{i}\mathfrak{D}(a)X$ *is an isomorphism if and only if* $X \in \mathcal{L}_J^{\perp}$.

Proof. Using Lemma 115 we may as well assume the right triadjoint $\mathbf{i}$ is constructed as in Corollary 112. That is, we have a commutative diagram of triangulated functors

$$\begin{array}{ccc} & \mathfrak{D}(\mathbf{Mod}\mathcal{R}) & \\ \swarrow & & \searrow \\ \mathfrak{D}(\mathcal{A})/\mathcal{L}_J & \xrightarrow[T]{} & \mathfrak{D}(\mathcal{D}) \end{array}$$

together with a triangulated functor $T' : \mathfrak{D}(\mathcal{D}) \longrightarrow \mathfrak{D}(\mathbf{Mod}\mathcal{R})/\mathcal{L}_J$ and trinatural equivalences $\nu : 1 \longrightarrow T'T, \rho : TT' \longrightarrow 1$, which we can assume are the unit and counit of a triadjunction $T \dashv T'$. We choose a right triadjoint M of Q with triadjunction (η', ε') and set $\mathbf{i} = MT'$. Then $\mathbf{i}$ is right triadjoint to $\mathfrak{D}(a)$ with unit $\eta : 1 \longrightarrow \mathbf{i}\mathfrak{D}(a)$ given by $\eta = (M\nu Q) \circ \eta'$ and counit $\varepsilon : \mathfrak{D}(a)\mathbf{i} \longrightarrow 1$ given by $\varepsilon = \rho \circ (T\varepsilon' T')$.

It follows from (TRC,Remark 61) that the pair (MQ, η') is a localisation of $\mathfrak{D}(\mathbf{Mod}\mathcal{R})$ with kernel $\mathcal{L}_J$. But $M\nu Q$ is a trinatural equivalence $MQ \longrightarrow MT'TQ = \mathbf{i}\mathfrak{D}(a)$, so we deduce from Lemma 115 that the pair $(\mathbf{i}\mathfrak{D}(a), \eta)$ is a localisation of $\mathfrak{D}(\mathbf{Mod}\mathcal{R})$ with kernel $\mathcal{L}_J$. It is now straightforward to check the remaining claims. □

Remark 50. Let $F : \mathcal{T} \longrightarrow \mathcal{T}'$ be a triequivalence of triangulated categories. If $\mathcal{S} \subseteq \mathcal{T}$ is a triangulated subcategory, we write $F[\mathcal{S}]$ for the replete closure in $\mathcal{T}'$ of the class $\{F(S) \,|\, S \in \mathcal{S}\}$. One checks that this is a triangulated subcategory of $\mathcal{T}'$. In fact this sets up a bijection between triangulated subcategories of $\mathcal{T}$ and $\mathcal{T}'$, which identifies thick subcategories with thick subcategories. For any triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ we have $F[\mathcal{S}^{\perp}] = F[\mathcal{S}]^{\perp}$.

If $\mathcal{T}, \mathcal{T}'$ have coproducts and $\mathcal{S}$ is localising then $F[\mathcal{S}]$ is localising, so we have a bijection between localising subcategories as well. This bijection identifies bousfield subcategories with bousfield subcategories. Given a nonempty class $Q \subseteq \mathcal{T}$ it is clear that $F[\langle Q \rangle] = \langle F(Q) \rangle$ is the smallest localising subcategory of $\mathcal{T}'$ containing the class $\{F(k) \,|\, k \in Q\}$.

Remark 51. Let $F : \mathcal{T} \longrightarrow \mathcal{T}'$ be a triangulated functor. If $\mathcal{S}$ is a triangulated subcategory of $\mathcal{T}'$ then so is the full subcategory of $\mathcal{T}$ whose objects are those $X \in \mathcal{T}$ with $F(X) \in \mathcal{S}$. We denote this triangulated subcategory by $F^{-1}\mathcal{S}$. If $\mathcal{S}$ is thick then so is $F^{-1}\mathcal{S}$, and if F preserves coproducts then $\mathcal{S}$ localising implies $F^{-1}\mathcal{S}$ localising.

We can now prove the most general version of Proposition 102.

Theorem 117. *Let $\mathcal{C}$ be a grothendieck abelian category, S a nonempty set of objects of $\mathbf{C}(\mathcal{C})$, and $\mathcal{L} = \langle S \rangle$ the smallest localising subcategory of $\mathfrak{D}(\mathcal{C})$ containing these complexes. Then $\mathcal{L}$ is a bousfield subcategory of $\mathfrak{D}(\mathcal{C})$.*

Proof. Following [ATJLSS00] Theorem 5.7. By the Gabriel-Popescu theorem and Remark 50 it suffices to prove this in the case where $\mathcal{C} = \mathcal{D}$ is the localisation of a category of modules over a ringoid $\mathcal{R}$. Set $\mathcal{A} = \mathbf{Mod}\mathcal{R}$ and adopt the standard notation of this section. By Corollary 112 the triangulated functor $\mathfrak{D}(a)$ has a right triadjoint, which we denote by $\mathbf{i}$. So we have a diagram of triangulated functors

$$\mathfrak{D}(\mathcal{A}) \underset{\mathbf{i}}{\overset{\mathfrak{D}(a)}{\rightleftarrows}} \mathfrak{D}(\mathcal{D})$$

Let η and ε be the unit and counit of this triadjunction. Then by Lemma 116 the triangulated functor $\ell = \mathbf{i}\mathfrak{D}(a)$ together with η is a localisation of $\mathfrak{D}(\mathcal{A})$ with kernel $\mathcal{L}_J$, and ε is a trinatural equivalence of $\mathfrak{D}(a)\mathbf{i}$ with the identity.

We showed in the proof of Theorem 107 that $\mathcal{L}_J = \langle E \rangle$ for some object $E \in \mathbf{C}(\mathcal{A})$ (by taking coproducts over the set of cloned complexes). It also suffices to consider the case where $S = \{Y\}$ consists of a single object. Let $\mathcal{M}$ be the smallest localising subcategory of $\mathfrak{D}(\mathcal{A})$ containing the complexes $E, \mathbf{i}Y$. By Proposition 102 this is bousfield, so for any $X \in \mathfrak{D}(\mathcal{D})$ there is a triangle in $\mathfrak{D}(\mathcal{A})$

$$N \longrightarrow X \longrightarrow B \longrightarrow \Sigma N$$

with $N \in \mathcal{M}$ and $B \in \mathcal{M}^{\perp}$. Applying the triangulated functor $\mathfrak{D}(a)$ we have a triangle in $\mathfrak{D}(\mathcal{D})$

$$aN \longrightarrow X \longrightarrow aB \longrightarrow \Sigma aN$$

Applying the observation of Remark 51 it is clear that $\mathfrak{D}(a)$ sends objects of $\mathcal{M}$ into $\mathcal{L}$. In particular $aN \in \mathcal{L}$, and it only remains to show that $aB \in \mathcal{L}^{\perp}$. But for $k \in \mathbb{Z}$ we have

$$Hom_{\mathfrak{D}(\mathcal{D})}(\Sigma^k Y, \mathfrak{D}(a)B) \cong Hom_{\mathfrak{D}(\mathcal{A})}(\Sigma^k \mathbf{i}Y, \mathbf{i}\mathfrak{D}(a)B) \cong Hom_{\mathfrak{D}(\mathcal{A})}(\Sigma^k \mathbf{i}Y, B) = 0$$

since $\mathbf{i}$ is fully faithful and η is an isomorphism on objects of $\mathcal{L}_J^{\perp}$ (and clearly $\mathcal{M}^{\perp} \subseteq \mathcal{L}_J^{\perp}$). It follows from Lemma 55 that $aB \in \mathcal{L}^{\perp}$, as required. □

If we are willing to restrict ourselves to *compact* objects, we can prove the same result but for arbitrary triangulated categories. The reader not familiar with Brown representability can provide an elementary proof of the next result along the lines of (HRT,Theorem 33).

Theorem 118. *Let $\mathcal{T}$ be a triangulated category with coproducts, $S \subseteq \mathcal{T}$ a nonempty set of compact objects, and $\mathcal{L} = \langle S \rangle$ the smallest localising subcategory of $\mathcal{T}$ containing S. Then $\mathcal{L}$ is a bousfield subcategory of $\mathcal{T}$.*

Proof. The triangulated category $\mathcal{L}$ is compactly generated (TRC3,Definition 9) by the set of suspensions $\{\Sigma^k Y \,|\, Y \in S, k \in \mathbb{Z}\}$ and therefore satisfies the representability theorem. It is now clear that $\mathcal{L}$ is a bousfield subcategory (TRC3,Lemma 29). □

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