Concentrated Schemes

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The original reference for quasi-compact and quasi-separated morphisms is EGA IV.1.

Definition 1. A morphism of schemes $f : X \longrightarrow Y$ is *quasi-compact* if there is a nonempty open cover $\{V_i\}_{i\in I}$ of Y by open affines such that $f^{-1}V_i$ is quasi-compact for each $i \in I$. This property is stable under isomorphism on either end. Any morphism out of a noetherian scheme is quasi-compact. Any affine morphism is quasi-compact (RAS,Definition 1).

Proposition 1. *A morphism of schemes $f : X \longrightarrow Y$ is quasi-compact if and only if for every open affine subset $V \subseteq Y$ the space $f^{-1}V$ is quasi-compact.*

Proof. See (H, Ex. 3.2). □

Corollary 2. *A morphism of schemes $f : X \longrightarrow Y$ is quasi-compact if and only if for every quasi-compact open subset $V \subseteq Y$ the space $f^{-1}V$ is quasi-compact.*

Proof. The condition is clearly sufficient. It is also necessary, since we can cover any quasi-compact open subset $V \subseteq Y$ with a finite collection of affine open sets, apply Proposition 1 and use the fact that a finite union of quasi-compact open subsets is quasi-compact. □

Lemma 3. *Let $f : X \longrightarrow Y$ be a morphism of schemes. If f is quasi-compact and $V \subseteq Y$ open then the induced morphism $f^{-1}V \longrightarrow V$ is quasi-compact. If $\{V_i\}_{i\in I}$ is a nonempty open cover of Y then f is quasi-compact if and only if the induced morphism $f^{-1}V_i \longrightarrow V_i$ is quasi-compact for every $i \in I$.*

Proof. It is clear that if $V \subseteq Y$ is open and f quasi-compact then $f^{-1}V \longrightarrow V$ is quasi-compact. For the second statement the condition is therefore necessary. To show it is sufficient, suppose that every morphism $f^{-1}V_i \longrightarrow V_i$ is quasi-compact. Given a point $y \in Y$ we can find an affine open subset $y \in U \subseteq V_i$ for some $i \in I$. Then by assumption $f^{-1}U = f_i^{-1}U$ is quasi-compact, and the proof is complete. □

Proposition 4. *We have the following properties of quasi-compact morphisms*

(i) *A closed immersion is quasi-compact.*

(ii) *The composition of two quasi-compact morphisms is quasi-compact.*

(iii) *Quasi-compactness is stable under base extension.*

(iv) *The product of two quasi-compact morphisms is quasi-compact.*

(v) *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, and if $g \circ f$ is quasi-compact and g separated, then f is quasi-compact.*

Proof. (i) A closed immersion is affine and therefore quasi-compact. (ii) Immediate from Corollary 2 (iii) Suppose we are given a pullback diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y' & \xrightarrow[g]{} & Y \end{array} \tag{1}$$

with g quasi-compact. We claim that f is also quasi-compact. It suffices to find, for every $x \in X$, an affine open neighborhood V of x such that $f^{-1}V$ is quasi-compact. Let S be an affine open neighborhood of $q(x)$ and $V \subseteq p^{-1}S$ an affine open neighborhood of x. Then we have another pullback diagram

$$\begin{array}{ccc} f^{-1}V & \longrightarrow & V \\ \downarrow & & \downarrow \\ g^{-1}S & \longrightarrow & S \end{array}$$

So we have reduced to the case in (1) where X, Y are affine and Y' quasi-compact. If we then cover Y' with a finite collection $V_1, \dots, V_n$ of affine open subsets, then from the pullback

$$\begin{array}{ccc} p^{-1}V_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ V_i & \longrightarrow & Y \end{array}$$

we deduce that $p^{-1}V_i$ is affine and therefore quasi-compact. As the finite union of the quasi-compact open subsets $p^{-1}V_i$, the scheme X' must also be quasi-compact, which is what we needed to show. The remaining statements (iv) and (v) now follow formally from (SEM,Proposition 11). □

Definition 2. A morphism of schemes $f : X \longrightarrow Y$ is *quasi-separated* if the diagonal morphism $\Delta : X \longrightarrow X \times_Y X$ is quasi-compact. This property is stable under isomorphism on either end. We say that a scheme X is *quasi-separated* if it is quasi-separated over $Spec(\mathbb{Z})$. A separated morphism is quasi-separated, so in particular a separated scheme is quasi-separated.

Proposition 5. *We have the following properties of quasi-separated morphisms*

(i) A monomorphism of schemes is quasi-separated.

(ii) The composition of two quasi-separated morphisms is quasi-separated.

(iii) Quasi-separatedness is stable under base extension.

(iv) The product of two quasi-separated morphisms is quasi-separated.

(v) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms with $g \circ f$ quasi-separated then f is quasi-separated.

Proof. (i) is trivial, since a monomorphism of schemes is separated. (ii) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be quasi-separated morphisms. Then from (SPM,Lemma 2) and Proposition 4(iii) we deduce that i, Δ_f are quasi-compact, and hence so is their composite Δ_{gf}. Therefore $g \circ f$ is quasi-separated, as required. (iii) follows from (SPM,Lemma 1) since pullbacks of quasi-compact morphisms are quasi-compact. (iv) now follow formally from $(i), (ii), (iii)$. The same is true for (v), but we need to observe that since any monomorphism is quasi-separated, the graph morphism Γ_f is already quasi-separated (without any assumption on g), so the proof of (SEM,Proposition 11)(e) still goes through. □

Corollary 6. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Then*

(i) If X is quasi-separated then f is quasi-separated.

(ii) If Y is quasi-separated then f is quasi-separated if and only if X is quasi-separated.

Proof. Immediate from Proposition 5 (ii) and (v). □

Corollary 7. *A scheme X is quasi-separated if and only if it is quasi-separated over some affine scheme.*

Proposition 8. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two morphisms with $g \circ f$ quasi-compact and g quasi-separated. Then f is also quasi-compact.*

Proof. The proof is the same as (SEM,Proposition 11)(e). $\square$

Proposition 9. *Let $f : X \longrightarrow Y$ be a morphism of schemes, and $\{V_i\}_{i \in I}$ a nonempty open cover of Y. Then f is quasi-separated if and only if the induced morphism $f^{-1}V_i \longrightarrow V_i$ is quasi-separated for every $i \in I$.*

Proof. The condition is clearly necessary by Proposition 5(iii). Suppose that every morphism $f^{-1}V_i \longrightarrow V_i$ is quasi-separated. Using Lemma 3 and the argument of (SPM,Proposition 20) the proof is straightforward. $\square$

Corollary 10. *Let $f : X \longrightarrow Y$ be a morphism of schemes, and $\{V_i\}_{i \in I}$ a nonempty open cover of Y by quasi-separated open subsets. Then f is quasi-separated if and only if each open subset $f^{-1}V_i$ is quasi-separated.*

Proof. Follows immediately from Proposition 9 and Corollary 6. $\square$

Lemma 11. *Suppose we have a pullback diagram of schemes*

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

where S is affine and X, Y quasi-compact. Then $X \times_S Y$ is also quasi-compact.

Proof. We can cover both X, Y by a finite number of affine open subsets. Pullbacks of affine schemes are affine and therefore quasi-compact, so $X \times_S Y$ has a finite cover by quasi-compact open subsets. It is therefore itself quasi-compact, as required. $\square$

Proposition 12. *Let X be a scheme. Then the following conditions are equivalent:*

(a) X is quasi-separated.

(b) For every quasi-compact open subset $U \subseteq X$ the inclusion $U \longrightarrow X$ is quasi-compact.

(c) The intersection of two quasi-compact open subsets of X is quasi-compact.

Proof. It is clear that $(b) \Leftrightarrow (c)$, and $(a) \Rightarrow (c)$ for the same reason that in a separated scheme the intersection of affine open subsets is affine (using Lemma 11). It only remains to show that $(c) \Rightarrow (a)$. Any point of $X \times_{\mathbb{Z}} X$ has an affine open neighborhood of the form $U \times_{\mathbb{Z}} V$ for affine open subsets $U, V \subseteq X$. By assumption the intersection $U \cap V$ is quasi-compact, and since this is the inverse image of $U \times_{\mathbb{Z}} V$ under the diagonal $\Delta : X \longrightarrow X \times_{\mathbb{Z}} X$ we deduce that the diagonal is quasi-compact, as required. $\square$

Corollary 13. *If a scheme X is quasi-separated, then so is any open subset $U \subseteq X$.*

Remark 1. Let X be a scheme whose underlying topological space is noetherian. Every open subset of X is quasi-compact, so in particular X is quasi-separated. In particular any noetherian scheme is quasi-compact and quasi-separated. This latter notion is preferable, because it is *relative* (i.e. a property of morphisms) which the notion of a noetherian scheme is not. In fact, most interesting theorems about noetherian schemes are also true for quasi-compact quasi-separated schemes (see EGA IV 1.7), and this is the generality in which many theorems in the literature are proven. Following Lipman, we give these schemes a special name to simplify the notation.

Definition 3. A morphism $f : X \longrightarrow Y$ of schemes is said to be *concentrated* if it is quasi-compact and quasi-separated. This property is stable under isomorphism on either end. A scheme X is *concentrated* if it is concentrated over $Spec(\mathbb{Z})$ (or equivalently, over any affine scheme). Equivalently, X is concentrated if it is quasi-compact and quasi-separated. This property is stable under isomorphism.

A noetherian scheme is concentrated, and any morphism of schemes $f : X \longrightarrow Y$ with X noetherian is concentrated. An affine scheme is concentrated. Any quasi-compact open subset of a concentrated scheme is concentrated.

We have the following immediate consequences of the earlier results.

Proposition 14. *Let $f : X \longrightarrow Y$ be a morphism of schemes. If f is concentrated and $V \subseteq Y$ open then the induced morphism $f^{-1}V \longrightarrow V$ is concentrated. If $\{V_i\}_{i \in I}$ is a nonempty open cover of Y then f is concentrated if and only if the induced morphism $f^{-1}V_i \longrightarrow V_i$ is concentrated for every $i \in I$.*

Proposition 15. *We have the following properties of concentrated morphisms*

(i) A closed immersion is concentrated.

(ii) The composition of two concentrated morphisms is concentrated.

(iii) Concentrated morphisms are stable under base extension.

(iv) The product of two concentrated morphisms is concetrated.

(v) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms with $g \circ f$ concentrated and g quasi-separated, then f is concentrated.

Lemma 16. *If $f : X \longrightarrow Y$ is a morphism of schemes with Y concentrated, then X is concentrated if and only if f is concentrated.*

Lemma 17. *A morphism of schemes $f : X \longrightarrow Y$ is concentrated if and only if for every affine open $V \subseteq Y$ the scheme $f^{-1}V$ is concentrated.*

Proof. If $f : X \longrightarrow Y$ is concentrated then so is the induced morphism $f^{-1}V \longrightarrow V$, so we deduce from Lemma 16 that $f^{-1}V$ is concentrated. For the converse, let $\{V_i\}_{i \in I}$ be an affine open cover of Y. By hypothesis and Lemma 16 each morphism $f^{-1}V_i \longrightarrow V_i$ is concentrated, so it follows from Proposition 14 that f is concentrated. □

Proposition 18. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes. If $\mathscr{F}$ is a quasi-coherent sheaf of modules on X then $f_*\mathscr{F}$ is quasi-coherent on Y.*

Proof. The current hypothesis are sufficient for the proof of (H, II.5.8). □

Lemma 19. *If $f : X \longrightarrow Y$ is an affine morphism of schemes then $f_* : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$ is exact.*

Proof. An affine morphism is concentrated, so $f_* : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$ is a well-defined additive functor (CON,Proposition 18). We can reduce immediately to the case where $X = SpecA$ and $Y = SpecB$, so that f is induced by a ring morphism $B \longrightarrow A$. Then f_* corresponds to the functor $A\mathbf{Mod} \longrightarrow B\mathbf{Mod}$ that acts by restriction of scalars. Since this latter functor is trivially exact, we have the desired conclusion. □

1 Semi-separated schemes

Recall that a *full basis* (COS,Definition 3) of a topological space X is a basis of X which is closed under finite intersections.

Definition 4. A scheme X is *semi-separated* if it possesses a full basis $\mathfrak{B}$ consisting of affine open subsets. The basis $\mathfrak{B}$ is called a *semi-separating affine basis*. Any open subset of a semi-separated scheme is semi-separated, and this property is stable under isomorphism. A separated scheme is semi-separated.

A nonempty open cover $\mathfrak{U} = \{V_\alpha\}_{\alpha \in \Lambda}$ of a scheme X is a *semi-separating cover* if all the V_α, and also all the pairwise intersections $V_\alpha \cap V_\beta$, are affine schemes. Then the inclusions $V_\alpha \longrightarrow X$ are affine, so any finite intersection of elements of $\mathfrak{U}$ is affine. A scheme X is semi-separated if and only if it has a semi-separating cover: take the collection $\mathfrak{B}$ of all affine open subsets of X contained in some V_α.

Lemma 20. *If a scheme X is semi-separated then it is quasi-separated.*

Proof. By Proposition 12 it suffices to show that the intersection of two quasi-compact open subsets U, V of X is quasi-compact. But we write $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cup \cdots \cup V_m$ for affine open sets U_i, V_i in a semi-separating affine basis for X. Since the intersections $U_i \cap V_j$ are all affine, it is clear that $U \cap V$ is quasi-compact. $\square$