CHAPTER 3

Statistical Mechanics

Most applications of conformal invariance pertain to statistical systems at criticality. A brief introduction to statistical mechanics is therefore required for those readers unfamiliar with the subject. The emphasis is put on the concepts underlying the hypothesis of conformal invariance in critical systems. Some parallels are to be drawn with the previous chapter, since quantum field theory and statistical mechanics walk hand in hand in the modern theory of critical phenomena. Section 3.1 reviews the notion of statistical ensemble of states and describes some basic models defined on the lattice or in the continuum. Section 3.2 explains the basic features of critical phenomena and how the scaling hypothesis provides a unified understanding of phenomena at or near the critical point. Section 3.3 justifies the scaling hypothesis with the idea of real-space renormalization. Section 3.4 applies the concepts of the renormalization group to continuum models and gives deeper meaning to the notion of scale invariance for Euclidian field theories. Finally, Sect. 3.5 briefly explains the transfer matrix method, a discrete analogue in statistical mechanics of the operator formalism of quantum theory.

§3.1. The Boltzmann Distribution

Statistical mechanics describes complex physical systems (i.e., systems made of a large number of atoms in interaction) whose exact states cannot be specified because of this complexity. Instead, macroscopic properties alone may be specified, and the role of the theory is to infer these properties from the microscopic Hamiltonian. Thus, statistical mechanics distinguishes *microscopic states* (or *microstates*) from *macroscopic states* (or *macrostates*). A microstate is specified by the quantum numbers of all the particles in the system or, classically, by the exact configuration (positions and momenta) of all the particles. It characterizes the system from a dynamical point of view in the sense that its future state is fixed by its present state through deterministic laws. A macrostate is specified by a finite number of macroscopic parameters, which characterize the system from the point

of view of observation, such as pressure, temperature, magnetization, and so on. To a given macrostate corresponds a large number of microstates, each leading to the same macroscopic properties. Having no more information about an isolated system than that given by the macroscopic parameters, we assume that all the microstates associated with the observed macrostate have equal probabilities to be the actual state of the system.

The basic idea behind the statistical study of a complex system is that any physical property—like the energy, the magnetization, and so on—may be regarded as a statistical average, calculated over a suitable *ensemble* of microstates. Of course, at any instant, the system is in a specific (but unknown) microstate. The replacement of this microstate by a statistical ensemble needs some justification. It has long been customary to justify this replacement by invoking the so-called *ergodic hypothesis*, which states that the time average of a quantity over the time evolution of a specific microstate is equal to the average of the same quantity, at fixed time, over some statistical ensemble of microstates. If one accepts this hypothesis, then the use of a statistical ensemble is justified provided the time necessary for an efficient sweep of the ensemble by any of its microstates is short enough compared with the time of measurement of the physical quantity of interest. This is far from obvious. A better justification for the use of statistical ensembles follows from dividing the system into a very large number of mesoscopic parts, each of them large enough to display the complex properties of the whole system. At any instant, each of these mesoscopic subsystems is characterized by its own microstate, but the properties of the whole system are obtained by averaging over all subsystems. Thus, the ensemble averaging amounts more to a spatial averaging than to a time averaging.

Which ensemble of states is most appropriate for averaging depends on how isolated the system is. If it is completely isolated, with no exchange of energy or particles with its surroundings, the relevant ensemble of microstates is made of all states on a given energy "shell", occurring with equal probabilities. It is called the *microcanonical* ensemble.

If, on the other hand, a system S is in thermal contact with its surroundings and hence is free to exchange energy with it, then all microstates of S do not have equal probabilities. However, all microstates of the "universe" (S plus its surroundings) have equal probabilities. This, in turn, provides us with a distribution of probabilities for the microstates of S: The probability that a specific microstate of S be the actual state of the system depends only on its energy and is given by the *Boltzmann distribution*:

$$P_i = \frac{1}{Z} \exp -\beta E_i \qquad \beta = \frac{1}{T} \tag{3.1}$$

where T is the absolute temperature[1] and Z is the normalization of the distribution, called the *partition function*:

[1] This definition of temperature includes the unit-dependent Boltzmann constant k_B. Thus T has the dimension of energy.

$$\boxed{Z = \sum_i \exp -\beta E_i} \tag{3.2}$$

The ensemble of microstates defined by the Boltzmann distribution is the *canonical ensemble.*

The partition function (3.2) is of central importance in statistical mechanics since macroscopic quantities are generically related to derivatives of Z. For instance, the average energy within the canonical ensemble is obtained by lowering a factor of E_i in the sum of Boltzmann weights through differentiation with respect to β:

$$\begin{aligned} U &= \frac{1}{Z} \sum_i E_i \exp -\beta E_i \\ &= -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \\ &= -T^2 \frac{\partial}{\partial T}(F/T) \end{aligned} \tag{3.3}$$

where we have introduced the *free energy*:

$$F = -T \ln Z \tag{3.4}$$

Similarly, the heat capacity C at constant volume is

$$C = \left(\frac{\partial U}{\partial T}\right)_V = -T \frac{\partial^2 F}{\partial T^2} \tag{3.5}$$

The *specific heat* is defined as the heat capacity per unit volume. Thus, the partition function is the generating function of all the thermodynamic functions of interest.

In practice, statistical mechanics studies systems composed of a large quantity of N identical components (atoms, molecules). The properties of each individual atom (e.g., energy, spin, etc.) fluctuate according to the Boltzmann distribution, but the physical quantities of interest are summed over all N components of the system. Because of the law of large numbers, their fluctuations vary as $1/\sqrt{N}$ and are completely negligible when N is large. The limit $N \to \infty$ is called the *thermodynamic limit* since then the variance of the macroscopic properties vanishes and their values cease to be random variables, becoming instead exact variables to be treated in the formalism of thermodynamics.

3.1.1. Classical Statistical Models

In practice the number of systems for which the partition function can be calculated, even in an approximate way, is very small. Confronted with the extreme complexity of most realistic systems one relies on simplified models to investigate finite-temperature properties. Some of these models are defined in terms of discrete, classical variables, which live on a lattice of sites. The best-known and simplest of these discrete models is the *Ising model.* It consists of a discrete lattice of spins σ_i,

each taking the value -1 or 1. Unless otherwise indicated, a square lattice is used and i stands for a lattice site. For a lattice with N sites the number of different spin configurations $[\sigma]$ is 2^N, and the energy of a given configuration is

$$E[\sigma] = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - h\sum_i \sigma_i \tag{3.6}$$

where the notation $\langle ij\rangle$ indicates that the summation is taken over pairs of nearest-neighbor lattice sites. The first term in the energy represents the interaction of neighboring spins through a ferromagnetic ($J > 0$) or antiferromagnetic ($J < 0$) coupling. The second term represents the interaction with an external magnetic field h. We shall not try to explain how such a simple model can arise from the microscopic quantum theory of magnetism but will be content in considering it for its own sake. We will assume that $J > 0$, although the case $J < 0$ is strictly equivalent at zero field ($h = 0$). In zero field, the lowest energy configuration is doubly degenerate: The spins can be either all up ($+1$) or all down (-1). If the field h is nonzero, the lowest energy configuration will have all spins aligned with h (i.e., of the same sign as h).

The first thermodynamic quantity of interest is the magnetization M, the mean value of a single spin. By translation invariance, this is the same for all spins, and we can write:

$$\begin{aligned} M &= \langle\sigma_j\rangle \qquad (\text{any } j) \\ &= \frac{1}{NZ}\sum_{[\sigma]}\left\{\sum_i \sigma_i\right\}\exp -\beta E[\sigma] \\ &= -\frac{1}{N}\frac{\partial F}{\partial h} \end{aligned} \tag{3.7}$$

where the notation $\langle\ldots\rangle$ denotes an ensemble average. Also of interest is the magnetic susceptibility, which indicates how the magnetization responds to a very small external field:

$$\begin{aligned} \chi &= \frac{\partial M}{\partial h}\Big|_{h=0} \\ &= \frac{1}{N}\frac{\partial}{\partial h}\left\{\frac{1}{Z}\sum_{[\sigma]}\left(\sum_i \sigma_i\right)\exp -\beta E[\sigma]\right\} \\ &= \frac{1}{NT}\left\{\langle\sigma_{\text{tot.}}^2\rangle - \langle\sigma_{\text{tot.}}\rangle^2\right\} \end{aligned} \tag{3.8}$$

where $\sigma_{\text{tot.}} = \sum_i \sigma_i$. The susceptibility is therefore proportional to the variance of the total spin, and measures its fluctuations.

The susceptibility is also related to the *pair correlation function* $\Gamma(i)$:

$$\Gamma(i-j) = \langle\sigma_i\sigma_j\rangle \tag{3.9}$$

Because of translation invariance, the correlator Γ can depend only on the difference of lattice sites. Moreover, for large distances $|i-j|$, the lattice structure is

less relevant, some rotation symmetry is restored and the correlators depend only on the distance $|i-j|$. The *connected* correlation function

$$\Gamma_c(i-j) = \langle \sigma_i \sigma_j \rangle_c = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \tag{3.10}$$

is a measure of the mutual statistical dependence of the spins σ_i and σ_j, in terms of which the susceptibility may be rewritten as

$$\chi = \beta \sum_{i=0}^{\infty} \Gamma_c(i) \tag{3.11}$$

We therefore expect the susceptibility to be a measure of the statistical coherence of the system, increasing with the statistical dependence of all the spins.

The Boltzmann distribution is ,of course, invariant under a constant shift of the energy. This allows us to write the Hamiltonian of the Ising model in a slightly different way. Indeed, since $\sigma_i \sigma_j = 2\delta_{\sigma_i,\sigma_j} - 1$, the configuration energy is, up to a constant,

$$E[\sigma] = -2J \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} - h \sum_i \sigma_i \tag{3.12}$$

This form lends itself to an immediate generalization of the Ising model, the so-called q-state Potts model, in which the spin σ_i takes q different integer values: $\sigma_i = 1, 2, \cdots, q$. To each possible value of σ we associate a unit vector $\boldsymbol{d}(\sigma)$ in $q-1$ dimensional space such that $\sum_\sigma^q \boldsymbol{d}(\sigma) = 0$. $\boldsymbol{d}(\sigma)$ plays the role of the magnetic dipole moment associated with the spin value σ. The configuration energy in an external field is

$$E[\sigma] = -\alpha \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} - \boldsymbol{h} \cdot \sum_i \boldsymbol{d}(\sigma_i) \tag{3.13}$$

Other generalizations of the Ising model are possible, wherein for instance the spins are regarded as "flavors" of atoms interacting with their nearest neighbors with coupling constants depending on which flavors are paired (Ashkin-Teller models) and so on.

In Ising-type models, the variables (spins) reside on the sites of the lattice whereas the interaction energy resides on the links between nearest-neighbor pairs. In systems such as the eight-vertex model the opposite is true: The variables are arrows living on the links, each taking one of two possible directions along the link. The interaction energy resides on the sites and its value depends on how the four arrows come together at that point, with the constraint that the number of arrows coming into (and out of) a site must be even.

Other statistical models involve continuous degrees of freedom rather than discrete ones. For instance, a more realistic treatment of classical ferromagnetism is obtained by assuming the local spin to be a unit vector $\boldsymbol{n}$, with the configuration energy

$$E[\boldsymbol{n}] = J \sum_{\langle ij \rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j - \sum_i \boldsymbol{h} \cdot \boldsymbol{n}_i \tag{3.14}$$

where $\boldsymbol{h}$ is some external magnetic field. This is the classical Heisenberg model, or the classical $O(n)$ model if the vector $\boldsymbol{n}$ is taken to have n components.

When discussing critical properties (in the next section) it is often more convenient to replace the lattice by a continuum, in which case the use of continuous degrees of freedom is mandatory. The above Hamiltonian is then equivalent to

$$E[\boldsymbol{n}] = \int d^d x \ \{J\partial_k \boldsymbol{n} \cdot \partial_k \boldsymbol{n} - \boldsymbol{h} \cdot \boldsymbol{n}\} \tag{3.15}$$

wherein $\boldsymbol{n}_i$ and $\boldsymbol{h}_i$ are replaced by $\boldsymbol{n}(x)$ and $\boldsymbol{h}(x)$. The gradient term is the equivalent of the nearest-neighbor interaction of the discrete case.

Because the constraint $\boldsymbol{n}^2(x) = 1$ at every position is difficult to implement in practical calculations, we may consider the simpler alternative in which it is replaced by the single constraint

$$\frac{1}{V}\int d^d x \, \boldsymbol{n}^2 = 1 \tag{3.16}$$

where V is the volume of the system. One then obtains the *spherical model*, which differs from the $O(n)$ model by the constraint imposed. Another way to approximate the constraint $\boldsymbol{n}^2(x) = 1$ is to make it energetically unfavorable for $\boldsymbol{n}^2(x)$ to be different from 1. This may be done with the help of a quartic potential $V(|\boldsymbol{n}|)$ having a minimum at $|\boldsymbol{n}| = 1$. After rescaling the field $\boldsymbol{n}$, the energy functional may be taken as

$$E[\boldsymbol{n}] = \int d^d x \ \{\tfrac{1}{2}\partial_k \boldsymbol{n} \cdot \partial_k \boldsymbol{n} - \tfrac{1}{2}\mu^2 \boldsymbol{n}^2 + \tfrac{1}{4}u(\boldsymbol{n}^2)^2\} \tag{3.17}$$

The position of the minimum of energy as a function of $|\boldsymbol{n}|$ depends on the relative values of μ and u. If $\boldsymbol{n}$ has a single component φ, this is termed the φ^4 model. The sign of the φ^2 term (positive or negative) determines whether the ground state value of φ vanishes or not. The case $u = 0$ is exactly solvable, and is called the Gaussian model since the partition function reduces to a product of Gaussian integrals. The associated configuration energy is

$$E[\varphi] = \int d^d x \ (\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}\mu^2\varphi^2) \tag{3.18}$$

All of these models were extensively studied and are discussed in great detail in most texts devoted to critical phenomena.

For models defined on the continuum, the analogy between statistical mechanics and quantum field theory is manifest. The partition function of the φ^4 model is a sum over the possible configurations of the field φ (i.e., a functional integral):

$$\begin{aligned} Z &= \int [d\varphi] \ \exp -\beta E[\varphi] \\ &= \int [d\varphi] \ \exp\left\{-\int d^d x \ \left[\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}r\varphi^2 + \tfrac{1}{4}u\varphi^4\right]\right\} \end{aligned} \tag{3.19}$$

Here we have rescaled the field φ by $\sqrt{\beta}$ and the φ^4 coupling u by $1/\beta$, so that the inverse temperature does not explicitly appear. The partition function of a

d-dimensional statistical model is thus entirely analogous to the generating functional of a quantum field in d space-time dimensions in the Euclidian formalism. Changing the temperature then amounts to scaling the field φ and modifying the φ^4 coupling.

3.1.2. Quantum Statistics

The statistical models described in the preceding subsection are all classical: All physical quantities have a definite value within each microstate of the statistical ensemble. In *quantum* statistical mechanics, we must deal with quantum indeterminacy as well as with thermal fluctuations. In that context, we define the *density operator*

$$\rho = \exp -\beta H \tag{3.20}$$

where H is the Hamiltonian of the system. The partition function may be expressed as a sum over the eigenstates of H:

$$Z = \sum_n e^{-\beta E_n} = \operatorname{Tr} \rho \tag{3.21}$$

The statistical average of an operator A is then

$$\langle A \rangle = \sum_n \langle n | e^{-\beta H} A | n \rangle = \operatorname{Tr} (\rho A) \tag{3.22}$$

The resemblance between the density operator $e^{-\beta H}$ and the evolution operator e^{-iHt} allows for the representation of the density operator as a functional integral. This introduces the Lagrangian formalism into statistical mechanics. Explicitly, consider the kernel of the density operator for a single degree of freedom:

$$\rho(x_f, x_i) = \langle x_f | e^{-\beta H} | x_i \rangle \tag{3.23}$$

The path integral is adapted to this kernel by substituting $t \to -i\tau$ (the Wick rotation), where τ is a real variable going from 0 to β. The action $S[x(t)]$ then becomes the Euclidian action $iS_E[x(\tau)]$. The kernel of the density operator ρ becomes then

$$\rho(x_f, x_i) = \int_{(x_i, 0)}^{(x_f, \beta)} [dx] \exp -S_E[x] \tag{3.24}$$

The partition function may be expressed as

$$Z = \int dx\, \rho(x, x) = \int [dx] \exp -S_E[x] \tag{3.25}$$

This time, the integration limits are no longer specified: all "trajectories" such that $x(0) = x(\beta)$ contribute. Here the "time" τ is merely an auxiliary variable introduced to take advantage of the analogy with path integrals. The expectation value of an operator A is

$$\langle A \rangle = \frac{1}{Z} \int dx\, \langle x | \rho A | x \rangle$$

$$= \frac{1}{Z} \int dxdy \, \langle x|\rho|y\rangle \langle y|A|x\rangle$$

$$= \frac{1}{Z} \int dxdy \int_{(x,0)}^{(y,\beta)} [dx] \, \langle y|A|x\rangle \exp -S_E[x]$$

$$= \frac{1}{Z} \int dxdy \int_{(x,0)}^{(y,\beta)} [dx] \, A(x)\delta(x-y) \exp -S_E[x]$$

$$= \frac{1}{Z} \int [dx] \, A(x(0)) \, \exp -S_E[x] \tag{3.26}$$

where we have supposed that A is a function of x only, so that

$$\langle y|A|x\rangle = A(x)\delta(x-y) \tag{3.27}$$

Hence, the expectation value of A is calculated as in the path-integral method. Note, however, that the operator A is evaluated at $\tau = 0$.

The generalization to a system with a continuum of degrees of freedom and to multipoint correlation functions is straightforward. The key point here is that the partition function of a quantum system in the path integral formalism is obtained from the ordinary path integral by a Wick rotation and by restricting the Euclidian time to a finite domain of extent β. At zero temperature this domain is infinite in extent and we recover the usual generating functional in Euclidian time. At finite temperatures, the quantum partition function of a d-dimensional system resembles that of a $(d+1)$-dimensional classical system defined on a strip of width β.

§3.2. Critical Phenomena

3.2.1. Generalities

Phase transitions are arguably the most interesting feature of statistical systems. They are characterized by a sudden and qualitative change in the macroscopic properties of the system as the temperature (or some other control parameter) is varied. We distinguish *first-order* transitions from *continuous* transitions. First-order transitions are characterized by a finite jump in the energy U (the *latent heat*) at the transition temperature. This means that the system must absorb or deliver a finite amount of energy before leaving the transition temperature. Liquid-gas transitions and other structural transitions are generally of this type. On the other hand, continuous phase transitions do not involve any latent heat, nor any abrupt change in the average value of microscopic variables, such as the magnetization. However, the derivatives of such quantities, such as the specific heat or the susceptibility, are discontinuous or display some singular behavior at continuous phase transitions.

Strictly speaking, phase transitions exist only in the thermodynamic limit. The reason is clear: In systems such as the Ising model in zero field, where the energy of any configuration is an integer multiple of a fundamental energy scale ε, the partition function for a finite number of lattice sites is a polynomial in $z = \exp -\beta\varepsilon$.

For instance, in the Ising model, one can choose $\varepsilon = -J$, and the configuration of highest energy has $E = 2N\varepsilon$. Each configuration contributes a power of z to the partition function, with unit coefficient. Therefore Z is a polynomial of degree $2N$ in z, whose roots lie away from the positive real axis, and occur as complex conjugated pairs. Singularities of the free energy or of its derivatives can occur only at those roots, which all lie outside of the physical domain of interest as long as N is finite. As $N \to \infty$, the number of these roots becomes infinite, and they tend to form various arcs, some of them touching the real positive axis. It is at these locations on the positive real axis that the behavior of thermodynamic quantities becomes singular in the thermodynamic limit.

Continuous phase transitions will be of central interest to us because of their relation to conformal invariance. The two-dimensional Ising model, of which the exact solution is known, exhibits such a transition. Let us describe this transition before commenting on the general case: The critical temperature T_c is related to the coupling J by

$$\sinh(2J/T_c) = 1 \tag{3.28}$$

Above T_c, the magnetization at zero field (or *spontaneous magnetization*) vanishes, whereas below T_c it takes a nonzero value, tending toward 1 at $T = 0$ and toward 0 as $T \to T_c$ according to the power law

$$M \sim (T_c - T)^{1/8} \tag{3.29}$$

The system is then in its *ferromagnetic phase*. The two directions of spontaneous magnetization (up and down) are energetically equivalent, and which one is actually realized depends on how the external field h was brought to zero. Although the magnetization is continuous at T_c, its derivative with respect to the magnetic field—the susceptibility χ—diverges as $T \to T_c$, according to

$$\chi = \frac{\partial M}{\partial h} \sim (T - T_c)^{-7/4} \tag{3.30}$$

Away from T_c, the correlations $\Gamma_c(i)$ decay exponentially with distance, with a temperature-dependent characteristic length ξ called the *correlation length*, expressed here in units of the lattice spacing:

$$\langle s_i s_j \rangle_c \sim \exp -|i-j|/\xi(T) \qquad |i-j| \gg 1 \tag{3.31}$$

As T approaches its critical value, the correlation length increases toward infinity, like the inverse power of $T - T_c$:

$$\xi(T) \sim \frac{1}{|T - T_c|} \tag{3.32}$$

As we shall see, this divergence of the correlation length is the most fundamental characteristic of continuous phase transitions. Such transitions are termed *critical phenomena* and occur at so-called *critical points* of the phase diagram.

The importance of the correlation length in the behavior of thermodynamic quantities near the critical point is intuitively clear. Near a critical point, a spin

system such as the Ising model is an aggregate of domains (or *droplets*) of different magnetizations. At first thought, the typical size of such droplets should be ξ, roughly the maximum scale over which the spins should be correlated. But in fact, droplets of all sizes up to the correlation length must be present, and droplets within droplets, etc. Otherwise the connected correlation functions $\Gamma_c(n)$ would have a peak near $n \sim \xi$ but would be small below that scale, which is not true: This can be seen from the observed divergence of the susceptibility χ as $T \to T_c$ and the expression (3.11) for χ. In other words, the spins fluctuate over all length scales between the lattice spacing and ξ. The free energy F will receive contributions from the domain walls separating spin droplets, integrated from the lattice spacing up to ξ, and it is plausible that its singular behavior (or, rather, that of its derivatives) be governed by the "upper integration bound", which is ξ.

At T_c or sufficiently close to it, the correlation length exceeds the physical dimension L of the system (we suppose, for the sake of argument, that the system lives in a square box of side L). At this point the free energy no longer depends on the correlation length but is limited by the box volume.[2] The pair correlation function does not have enough room to decay exponentially within the box, and its spatial dependence is algebraic (d is the dimension of space):

$$\Gamma(n) \sim \frac{1}{|n|^{d-2+\eta}} \tag{3.33}$$

The behavior of thermodynamic functions near or at the critical point is characterized by *critical exponents* defining power laws as $T \to T_c$. The most common exponents are defined in Table 3.1.

Table 3.1. Definitions of the most common critical exponents and their exact value within the two-dimensional Ising model. Here d is the dimension of space.

Exponent		Definition	Ising Value
α	C	$\propto (T - T_c)^{-\alpha}$	0
β	M	$\propto (T_c - T)^{\beta}$	1/8
γ	χ	$\propto (T - T_c)^{-\gamma}$	7/4
δ	M	$\propto h^{1/\delta}$	15
ν	ξ	$\propto (T - T_c)^{-\nu}$	1
η	$\Gamma(n)$	$\propto \lvert n\rvert^{2-d-\eta}$	1/4

[2] In real systems, the correlation length is limited not by the physical size of the sample, but by the presence of sample inhomogeneities. It rarely goes beyond a thousand lattice sites, even in very pure samples.

We conclude this section by a remark on the relevance of classical statistical mechanics in a quantum world. Classical statistical mechanics is an approximation to quantum statistical mechanics, valid in the context of critical phenomena when the statistical coherence length ξ exceeds the characteristic de Broglie wavelength of the system. For a system with a characteristic velocity v (e.g., the speed of light, the Fermi velocity or the speed of some other excitation), the de Broglie wavelength at temperature T is $\lambda_T = v\hbar/k_B T \propto \beta$. Classical statistics takes over at large enough temperatures, or close to a finite-temperature critical point, where the classical correlation length ξ exceeds λ_T. This justifies the extensive use of classical models in a realistic study of critical phenomena. The exception to this rule occurs when $T_c = 0$, which happens in a large class of low-dimensional systems.

3.2.2. Scaling

The critical exponents of Table 3.1 can be related to each other by use of the *scaling hypothesis*, which stipulates that the free energy density (or the free energy per site, in the discrete case) near the critical point is a homogeneous function of its parameters, the external field h, and the *reduced temperature* $t = T/T_c - 1$. In other words, there should be exponents a and b such that

$$f(\lambda^a t, \lambda^b h) = \lambda f(t,h) \tag{3.34}$$

This hypothesis will be justified below, but for now let us derive its consequences on critical exponents.

First, the homogeneity relation (3.34) implies that the function $t^{-1/a} f$ is invariant under the scalings $t \to \lambda^a t$ and $h \to \lambda^b h$. Therefore it must depend only on the scale-invariant variable $y = h/t^{b/a}$, and the free energy density may be expressed as

$$f(t,h) = t^{1/a} g(y) \qquad y = h/t^{b/a} \tag{3.35}$$

where g is some function. The spontaneous magnetization near criticality is then

$$M = -\frac{\partial f}{\partial h}\Big|_{h=0} = t^{(1-b)/a} g'(0) \tag{3.36}$$

One more derivative yields the magnetic susceptibility:

$$\chi = \frac{\partial^2 f}{\partial h^2}\Big|_{h=0} = t^{(1-2b)/a} g''(0) \tag{3.37}$$

Similarly, the specific heat (heat capacity per unit volume) is

$$c = -T\frac{\partial^2 f}{\partial T^2}\Big|_{h=0} = -\frac{1}{T_c} t^{1/a-2} g''(0) \tag{3.38}$$

Finally, in the limit $t \to 0$, the behavior of M as a function of h is $M \sim h^{1/\delta}$, which implies the asymptotic behavior $g(y) \sim y^{1/\delta}$ as $y \to \infty$, and imposes the constraint $1 - b - b/\delta = 0$, if the limit $t \to 0$ is to be finite and nonzero. We

have therefore obtained a set of four constraints on some of the critical exponents introduced in Table 3.1:

$$\begin{aligned} \alpha &= 2 - 1/a \\ \beta &= (1-b)/a \\ \gamma &= -(1-2b)/a \\ \delta &= b/(1-b) \end{aligned} \tag{3.39}$$

We now justify the scaling hypothesis, and at the same time express a and b in terms of the remaining exponents ν and η, both pertaining to the pair correlation function. Following Kadanoff, we focus our attention on the Ising model on a hypercubic lattice, with the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{3.40}$$

We now reduce the number of degrees of freedom of the system by grouping spins into *blocks* of side r (in units of lattice spacings), as indicated in Fig. 3.1. If d is the dimension of space there are r^d elementary spins within a block and the sum of spins therein can take values ranging from $-r^d$ to r^d. Accordingly, we define a *block spin* variable Σ_I as

$$\Sigma_I = \frac{1}{R} \sum_{i \in I} \sigma_i \tag{3.41}$$

where the sum is taken over the sites i within the block I and where R is some normalization factor introduced so that Σ_I can effectively take the values ± 1. For instance, R would be equal to r^d if the spins within the block were always perfectly aligned (since this is not true, R will be lower than that).

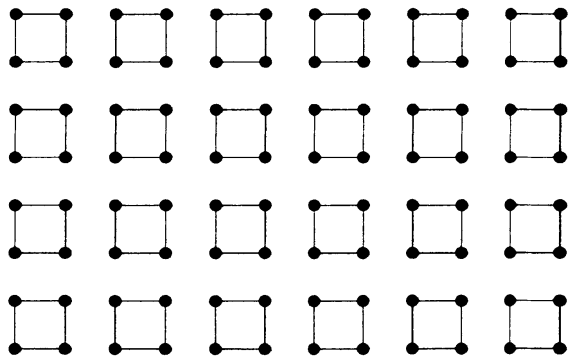

Figure 3.1. Block spins: an illustration of how four spins may be combined into a single site variable.

We will assume that the cooperative phenomena observed near the critical point can be accounted for equally well by a description in terms of block spins with a

nearest-block Hamiltonian of the same form as the original Ising Hamiltonian,

$$H' = -J' \sum_{\langle IJ \rangle} \Sigma_I \Sigma_J - h' \sum_i \Sigma_I \tag{3.42}$$

but with different parameters J' and h'. This is plausible since near criticality the correlation length ξ is much larger than the block side r. The correlation length of the blocks (the number of blocks over which the block spins are correlated) is, of course, ξ/r, which means that the effective reduced temperature t' is different from the original reduced temperature by a factor $r^{1/\nu}$:

$$t' = r^{1/\nu} t \tag{3.43}$$

The two Hamiltonians H and H' should involve the same interaction energy with an external field, and therefore

$$\begin{aligned} h \sum_i \sigma_i &= h' \sum_I \Sigma_I \\ &= h' R^{-1} \sum_i \sigma_i \end{aligned} \tag{3.44}$$

which implies $h' = Rh$. Since our grouping procedure should in no way affect the total free energy of the system, the free energy per block should be r^d times the original free energy per site, and should moreover have the same functional dependence because H and H' have the same form:

$$\begin{aligned} &f(t', h') = r^d f(t, h) \quad \text{or} \\ &f(t, h) = r^{-d} f(r^{1/\nu} t, Rh) \end{aligned} \tag{3.45}$$

It remains to find R as a function of r in order to recover the scaling hypothesis (3.34). This is done by looking at the pair correlation function at criticality: The block-spin correlation function is then

$$\begin{aligned} \Gamma'(n) &= \langle \Sigma_I \Sigma_J \rangle - \langle \Sigma_I \rangle \langle \Sigma_J \rangle \\ &= R^{-2} \sum_{i \in I} \sum_{j \in J} \Big\{ \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \Big\} \\ &= R^{-2} r^{2d} \Gamma(rn) \\ &= \frac{R^{-2} r^{2d}}{|rn|^{d-2+\eta}} \\ &= \frac{R^{-2} r^{d+2-\eta}}{|n|^{d-2+\eta}} \end{aligned} \tag{3.46}$$

which implies

$$R = r^{(d+2-\eta)/2} \quad \text{so that} \quad h' = r^{(d+2-\eta)/2} h \tag{3.47}$$

Looking back at the scaling hypothesis (3.34) and letting $r = \lambda^{1/d}$, we conclude that

$$a = 1/(\nu d) \quad \text{and} \quad b = (d + 2 - \eta)/(2d) \tag{3.48}$$

The critical exponents α through δ can thus be expressed in terms of η and ν:

$$\begin{aligned} \alpha &= 2 - \nu d \\ \beta &= \frac{1}{2}\nu(d - 2 + \eta) \\ \gamma &= \nu(2 - \eta) \\ \delta &= (d + 2 - \eta)/(d - 2 + \eta) \end{aligned} \tag{3.49}$$

We have succeeded in expressing all six critical exponents in terms of two of them (η and ν) pertaining more directly to the correlation functions. Of course, these relations can be written with a different set of "independent exponents." Table 3.2 gives the four scaling relations in their original form, with their accepted names.

Table 3.2. Summary of the scaling laws.

Rushbrooke's law	$\alpha + 2\beta + \gamma = 2$
Widom's law	$\gamma = \beta(\delta - 1)$
Fisher's law	$\gamma = \nu(2 - \eta)$
Josephson's law	$\nu d = 2 - \alpha$

3.2.3. Broken Symmetry

Phase transitions are generally associated with broken symmetries. By *broken symmetry*, we mean a symmetry of the configuration energy (or the action, in the quantum case) that is no longer reflected in the macrostate of the statistical system (or the ground state of the quantum system). For instance, the configuration energy of the two-dimensional Ising model at zero field is invariant with respect to the reversal of spins $\sigma_i \to -\sigma_i$. We say that this symmetry is broken if quantities that are not invariant under this symmetry operation have a nonvanishing expectation value. The magnetization $\langle \sigma_i \rangle$ is nonzero in the low temperature phase of the Ising model in the limit of zero external field, and the spin reversal symmetry is then broken. The simplest quantity that is not invariant under the symmetry considered and has a nonzero expectation value, such as the magnetization here, is called an *order parameter*. The phase with broken symmetry is often called the *ordered phase*. On the other hand, the high-temperature phase, in which the symmetry in unbroken, is often called the *symmetric phase*. We notice that in field theories, the analogue of temperature, after a rescaling of the fields, is some nonlinear coupling constant. Phase transitions in this case occur as a function of coupling; the interpretation is different, but the underlying physics is identical.

The spin-reversal symmetry of the Ising model has a discrete character. On the other hand, the $O(n)$ model (3.15) is endowed with a continuous symmetry: Its configuration energy is invariant under a rotation of its order parameter

$\boldsymbol{n}$ by a uniform $O(n)$ matrix. The average $\langle \boldsymbol{n} \rangle$ would be nonzero in the ordered phase, except that a slow, continuous change of $\langle \boldsymbol{n} \rangle$ throughout the system would cost very little energy. The consequence of this is the impossibility to break a continuous symmetry in a classical statistical system in one or two dimensions: this is the Mermin-Wagner-Coleman theorem. Simply put, long-wavelength thermal fluctuations of the order parameter take too much place in the phase space of low-dimensional systems (infrared divergence), and these fluctuations always succeed in destroying the order. The implications of this theorem to quantum statistical systems follow from the analogy between a quantum system in d spatial dimensions and a classical system in $d + 1$ dimensions, where the extra (imaginary time) dimension is limited in extent by the inverse temperature β. At any nonzero temperature, a certain class of fluctuations of the continuous order parameter occurs on a length scale greater than $v\beta$ (v is the characteristic velocity), and these long-wavelength fluctuations are thus governed by classical statistical mechanics. The Mermin-Wagner-Coleman theorem then implies that no continuous symmetry can be broken in two dimensions except at zero temperature. In a one-dimensional quantum system, such breaking is impossible even at zero temperature.

We point out that the Mermin-Wagner-Coleman theorem does not forbid all transitions implying a continuous order parameter. Such transitions are possible, provided they do not imply an expectation value of the order parameter. The best-known example is the Kosterlitz-Thouless transition in the $O(2)$ model defined on a plane (the two-dimensional XY model). In this model, the local order parameter is a planar, fixed-length vector $\boldsymbol{n}$, and topological defects (vortices) play an important role. These vortices are bound in pairs below some critical temperature and are *deconfined* above that temperature. In both phases the average $\langle \boldsymbol{n} \rangle$ vanishes.

§3.3. The Renormalization Group: Lattice Models

The scaling hypothesis of Sect. 3.2.2 has been motivated by the introduction of block spins with an effective Hamiltonian having the same form as the original Hamiltonian, albeit with different values of the couplings (this last step has not been demonstrated, but seems plausible; in fact it is only approximately valid). This procedure is called *block-spin renormalization* or *real-space renormalization* and defines a map between an original Hamiltonian H and a new *scaled* Hamiltonian H'. This map and its iterations form what we call the *renormalization group*, the most powerful tool at our disposal in the analysis of critical phenomena. In this section we present a survey of the basic concepts, along with a more detailed calculation within the Ising model on a triangular lattice. An exhaustive presentation of the renormalization group lies outside the scope of this review chapter and may be found in many good texts.

3.3.1. Generalities

We consider a general d-dimensional lattice model with N spins σ_i and Hamiltonian

$$H(\mathbf{J},[s],N) = J_0 + J_1 \sum_i \sigma_i + J_2 \sum_{\langle ij\rangle}^{(1)} \sigma_i\sigma_j + J_3 \sum_{\langle ij\rangle}^{(2)} \sigma_i\sigma_j + \cdots \tag{3.50}$$

$\mathbf{J}$ represents the collection of couplings $J_0, J_1, \cdots$ and the symbol $\sum_{\langle ij\rangle}^{(1)}$ means a summation over nearest neighbors, while $\sum_{\langle ij\rangle}^{(2)}$ means a summation over next-to-nearest neighbors, etc. Other couplings can possibly be included, with three-spin couplings and so on. We then define block spins Σ_I, along with a set of independent variables collectively denoted by ξ_I and describing the remaining degrees of freedom within each block. The Hamiltonian can in principle be rewritten in terms of these variables, and the partition function is

$$Z(\mathbf{J},N) = \sum_{[\Sigma][\xi]} \exp -H(\mathbf{J},[\Sigma],[\xi],N) \tag{3.51}$$

The inverse temperature β has been absorbed in the couplings J_i. Each block is of size r in units of the lattice spacing, and the number of blocks is therefore Nr^{-d}. The block Hamiltonian $H'(\mathbf{J}',[\Sigma],Nr^{-d})$ is obtained by tracing over the internal variables ξ:

$$\exp -H'(\mathbf{J}',[\Sigma],Nr^{-d}) = \sum_{[\xi]} \exp -H(\mathbf{J},[\Sigma],[\xi],N) \tag{3.52}$$

We have assumed that H' has the same functional form as H, and this fixes the value of the effective coupling $\mathbf{J}'$. This assumption is only approximately valid, but the closer we are to the critical point, the better this approximation is. Its validity can also be improved with the inclusion of a more complete set of couplings in the theory. The partition function is then

$$\begin{aligned} Z(\mathbf{J},N) &= \sum_{[\Sigma]} \exp -H'(\mathbf{J}',[\Sigma],Nr^{-d}) \\ &= Z(\mathbf{J}',Nr^{-d}) \end{aligned} \tag{3.53}$$

The free energy per site is therefore mapped as

$$f(\mathbf{J}) = r^{-d} f(\mathbf{J}') \tag{3.54}$$

The map $\mathbf{J} \to \mathbf{J}'$ from the original set of couplings to the set of effective block couplings generates the renormalization group.[3] We write

$$\mathbf{J}' = \mathbf{T}(\mathbf{J}) \tag{3.55}$$

Iterations of this map generate a sequence of points in the space of couplings, which we call a renormalization group (RG) trajectory. Since the correlation length is

[3] In fact, some information is lost during the process of tracing over the internal variables σ. Thus the map $\mathbf{J} \to \mathbf{J}'$ is not reversible and the renormalization group is only a *semi-group*.

reduced by a factor r at each step, a typical renormalization-group trajectory tends to take the system away from criticality. Because the correlation length is infinite at the critical point, it takes an infinite number of iterations to leave that point. In general, a system is critical not only at a given point in coupling space but on a whole "hypersurface", which we call the *critical surface*, or sometimes the *critical line*. Under renormalization-group flow, a point on the critical surface stays on the critical surface. A point $\mathbf{J}_c$ on the critical surface that is stationary under renormalization-group flow is called a *fixed point* of the renormalization group:

$$\mathbf{J}_c = \mathbf{T}(\mathbf{J}_c) \tag{3.56}$$

In general, the map (3.55) is nonlinear and its exact analysis is difficult. What is most important, however, is its behavior near a fixed point, which can be obtained by *linearizing* the renormalization-group map around $\mathbf{J}_c$. This is done by defining the difference $\delta\mathbf{J} = \mathbf{J} - \mathbf{J}_c$ and expanding $\mathbf{T}$ to first order in a multivariable Taylor series. The resulting truncation is a linear map of the differences $\delta\mathbf{J}$:

$$\delta\mathbf{J}' = A\,\delta\mathbf{J} \qquad A_{ij} = \frac{\partial T_i}{\partial J_j} \tag{3.57}$$

The matrix A may be diagonalized, with eigenvalues λ_i and eigenvectors $\mathbf{u}_i$. These eigenvectors form a basis of coupling space, that is,

$$\mathbf{J} = \mathbf{J}_c + \sum_i t_i \mathbf{u}_i \tag{3.58}$$

with the t_i's playing the role of "proper couplings." In terms of these, the renormalization-group linearized action is diagonal:

$$\begin{aligned} t_i' &= \lambda_i t_i \\ &= r^{y_i} t_i \end{aligned} \tag{3.59}$$

The exponents y_i are precisely the scaling exponents[4] a and b (times d) of Eq. (3.34), since the singular part of the free energy density transforms like

$$f(t_1, t_2, \cdots) = r^{-d} f(r^{y_1} t_1, r^{y_2} t_2, \cdots) \tag{3.60}$$

Therefore all critical exponents can be obtained from the eigenvalues of the linearized renormalization-group transformation at the fixed point. To find these eigenvalues is the prime objective of renormalization-group calculations.

The character of a fixed point is determined by whether the eigenvalues λ_i are greater or smaller than 1, or equivalently whether the exponents y_i are positive or negative. A fixed point with positive and negative exponents is called *hyperbolic* because of the shape of renormalization-group trajectories near $\mathbf{J}_c$. A two-parameter example is illustrated in Fig. 3.2. The critical surface (which is a line on the figure)

[4] The reduced temperature may undergo a sign change, since $J_i \propto 1/T$, but this does not affect the critical exponents.

is the set of points in coupling space whose renormalization-group trajectories end up at the fixed point:

$$\lim_{n\to\infty} \mathbf{T}^n(\mathbf{J}) = \mathbf{J}_c \tag{3.61}$$

The critical surface near $\mathbf{J}_c$ is a vector space spanned by the eigenvectors $\mathbf{u}_i$ such that $\lambda_i < 1$. Off the critical surface, the system is taken away from it by the renormalization-group flow.

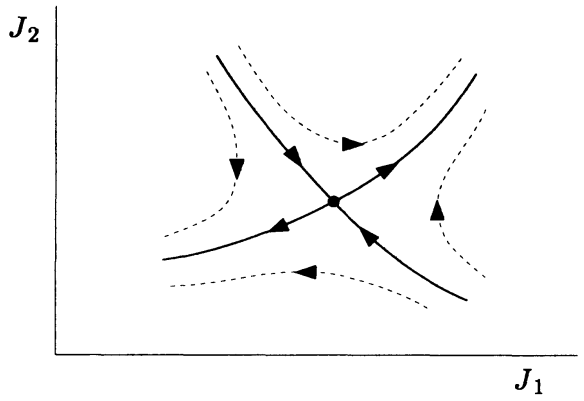

Figure 3.2. Schematic renormalization-group flow around a generic hyperbolic fixed point.

A parameter t_i associated with a positive scaling exponent ($\lambda_i > 1$) is called *relevant*, since it grows under renormalization-group flow (i.e., when the system is scaled away from criticality). If, on the contrary, $y_i < 0$ ($\lambda_i < 1$), t_i is said to be *irrelevant*, whereas if $y_i = 0$ ($\lambda_i = 1$) it is *marginal*. Marginal operators do not scale with a power law behavior near a critical point, but rather logarithmically; the linear approximation around the fixed point $\mathbf{J}_c$ is then invalid.

The existence of critical surfaces and fixed points is thought to explain the *universality* of critical exponents (i.e., that many different systems are characterized by the same critical exponents). In other words, statistical systems seem to fit into *universality classes* whose members share the same critical behavior. This can be understood if different systems live on submanifolds of one large coupling space, and if these submanifolds intersect the same critical surface. At criticality, all of these systems will be (presumably) driven toward the same fixed point, with the same scaling exponents.

3.3.2. The Ising Model on a Triangular Lattice

In order to illustrate some of the previous statements we will perform an explicit real-space renormalization-group calculation for the Ising model living on a triangular lattice.

The block structure is indicated on Fig. 3.3. The Ising Hamiltonian is written as

$$H(k,h) = -k \sum_{\langle ij \rangle}^{(1)} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{3.62}$$

Each lattice site has 6 nearest neighbors. A block I is made of three spins, which we call σ_1^I, σ_2^I and σ_3^I . We define the block spin Σ_I as

$$\Sigma_I = \operatorname{sgn}(\sigma_1^I + \sigma_2^I + \sigma_3^I) \tag{3.63}$$

In other words, Σ_I adopts the sign of the majority. The three spins within a block lead to $2^3 = 8$ different states, which makes four different states for the internal variable ξ_I and two for the block spin Σ_I. The four states are chosen to be

$$\xi_I : \quad (+,+,-)\,,\ (+,-,+)\,,\ (-,+,+)\,,\ (+,+,+) \tag{3.64}$$

and the actual state of the spins σ_i is obtained by multiplying by $\Sigma_I = \pm 1$.

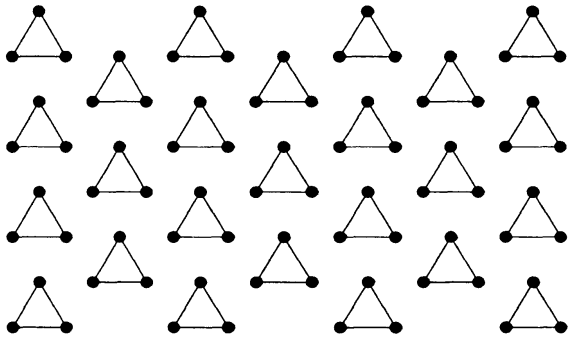

Figure 3.3. Block spins on the triangular lattice.

We decompose the Hamiltonian into the sum of a "free" part H_0 containing only the interaction within blocks, plus an "interaction" part V containing the interaction between blocks and with the external field:

$$\begin{aligned} H_0 &= -k \sum_I \sum_{\substack{\langle ij \rangle \\ (i,j \in I)}} \sigma_i \sigma_j \\ V &= -k \sum_{\langle IJ \rangle} \sum_{\substack{\langle ij \rangle \\ i \in I, j \in J}} \sigma_i \sigma_j - h \sum_I \sum_{i \in I} \sigma_i \end{aligned} \tag{3.65}$$

We also define the following expectation values in which only the variables internal to a block are summed:

$$\langle F[S] \rangle = Z_f^{-1} \sum_{[\xi]} F[\Sigma, \xi] \exp -H_0([\Sigma], [\xi]) \tag{3.66}$$

$$Z_f = \sum_{[\xi]} \exp -H_0([\Sigma], [\xi]) \tag{3.67}$$

According to (3.53), the block Hamiltonian $H(k', h')$ is defined by

$$\exp -H(k', h') = Z_f \langle e^V \rangle \tag{3.68}$$

The "free" partition function Z_f is easily calculated, since different blocks do not interact within H_0:

$$Z_f = Z_0^{N/3}$$

where Z_0 is the sum over states within a given block:

$$\begin{aligned} Z_0 &= \sum_{\sigma_I} \exp\left\{k(\Sigma_1^I \Sigma_2^I + \Sigma_2^I \Sigma_3^I + \Sigma_3^I \Sigma_1^I)\right\} \\ &= 3e^{-k} + e^{3k} \end{aligned} \tag{3.69}$$

This last step follows from Eq. (3.64), wherein three states have energy k and one state has energy $-3k$.

The expectation value $\langle e^V \rangle$ can be expressed as a *cumulant expansion*:

$$\langle e^V \rangle = \exp\left\{\langle V \rangle + \frac{1}{2}(\langle V^2 \rangle - \langle V \rangle^2) + \cdots\right\} \tag{3.70}$$

At this point we will make the approximation of keeping only the first term of this expansion. This amounts to neglecting the fluctuations of the interaction term within each block. The expectation value $\langle V \rangle$ is relatively easy to calculate. We start with the block-block interaction V_{IJ}. There are two elementary links between a pair of nearest-neighbor blocks and, as shown in Fig. 3.4, the interaction V_{IJ} is

$$V_{IJ} = -k\Sigma_3^J(\Sigma_1^I + \Sigma_2^I) \tag{3.71}$$

Since the expectation value within different blocks factorizes, we have

$$\langle V_{IJ} \rangle = -2k\langle \Sigma_3^J \rangle \langle \Sigma_3^I \rangle \tag{3.72}$$

where $\langle \Sigma_i^I \rangle$ is the same for all $i = 1, 2, 3$. The expectation value $\langle \Sigma_3^I \rangle$ is readily calculated:

$$\begin{aligned} \langle \Sigma_3^I \rangle &= Z_0^{-1} \sum_{\xi_I} \Sigma_3^I \exp -k(\Sigma_1^I \Sigma_2^I + \Sigma_2^I \Sigma_3^I + \Sigma_3^I \Sigma_1^I) \\ &= Z_0^{-1}(e^{3k} + e^{-k})\Sigma_I \end{aligned} \tag{3.73}$$

where we have used the definition (3.63) for the block spin Σ_I. Consequently, the mean interaction term between blocks is

$$\langle V_{IJ} \rangle = -2k\left(\frac{e^{3k} + e^{-k}}{e^{3k} + 3e^{-k}}\right)^2 \Sigma_I \Sigma_J \tag{3.74}$$

Since the average interaction with the external field involves only the expectation value $\langle \Sigma_3^I \rangle$, we find

$$\langle V \rangle = -2k\left(\frac{e^{3k} + e^{-k}}{e^{3k} + 3e^{-k}}\right)^2 \sum_{\langle IJ \rangle} \Sigma_I \Sigma_J - 3\left(\frac{e^{3k} + e^{-k}}{e^{3k} + 3e^{-k}}\right) h \sum_I \Sigma_I \tag{3.75}$$

To first order in the cumulant expansion, the block-spin Hamiltonian is therefore

$$H(k',h') = 3\ \ln Z_0 + \langle V \rangle \tag{3.76}$$

The first term is independent of Σ_I and may be ignored (except if one is interested in the value of the free energy F). We therefore end up with the following map between the block-spin couplings and the original ones:

$$\begin{aligned} k' &= 2k\left(\frac{e^{3k}+e^{-k}}{e^{3k}+3e^{-k}}\right)^2 \\ h' &= 3h\left(\frac{e^{3k}+e^{-k}}{e^{3k}+3e^{-k}}\right) \end{aligned} \tag{3.77}$$

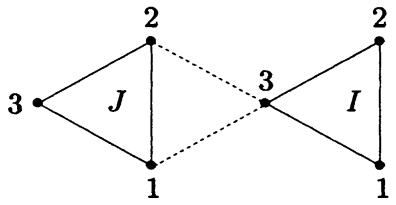

Figure 3.4. Interaction between block spins in the nearest-neighbor Ising model on the triangular lattice.

The renormalization-group (RG) flow associated with the above map is illustrated schematically in Fig. 3.5. There are 9 fixed points on this diagram, corresponding to the possible combinations of $h = 0, -\infty, \infty$ and $k = 0, k_c, \infty$, where k_c is determined by the equation

$$\frac{1}{2} = \left(\frac{e^{3k_c}+e^{-k_c}}{e^{3k_c}+3e^{-k_c}}\right) \quad \Rightarrow \quad k_c = \frac{1}{4}\ln(1+2\sqrt{2}) \approx 0.336 \tag{3.78}$$

The fixed point $(k,h) = (k_c, 0)$ is unstable in both directions and corresponds to a continuous phase transition. Near this point, the RG flow admits the following linearization:

$$\begin{pmatrix} \delta k' \\ \delta h' \end{pmatrix} = \begin{pmatrix} 1.62 & 0 \\ 0 & 2.12 \end{pmatrix} \begin{pmatrix} \delta k \\ \delta h \end{pmatrix} \tag{3.79}$$

with the eigenvalues $\lambda_k = 1.62$ and $\lambda_h = 2.12$. Since the scale factor for the triangular matrix is $r = \sqrt{3}$, the free energy density scales as

$$f(k,h) = r^{-d} f(r^{0.88}k, r^{1.37}h) \tag{3.80}$$

The critical exponents can be calculated from (3.39) and from the scaling laws of Table 3.2. We list them here, together with the exponents obtained in the exact

solution of the same model:

	α	β	γ	δ	ν	η
RG:	−0.27	0.72	0.84	2.17	1.13	1.26
exact:	0	$\frac{1}{8}$	$\frac{7}{4}$	15	1	$\frac{1}{4}$

Notice that the simplest RG calculation described here is not very successful at predicting the exponent η. The difference between its predictions and the exact exponents is attributed to the approximation made in neglecting higher-order terms in the cumulant expansion. If these terms were considered, more couplings would have to be included in order for the effective block Hamiltonian to have the same form as the original Hamiltonian, but a better agreement with the exact result would be found.

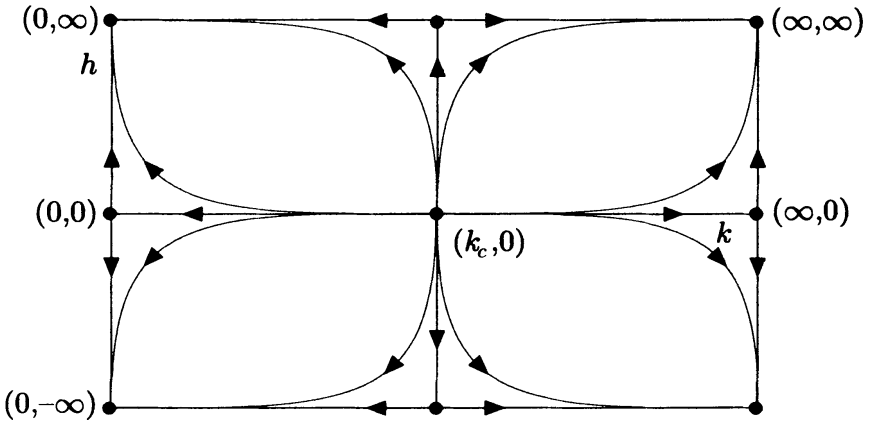

Figure 3.5. Schematic renormalization-group flow for the Ising model on a triangular lattice. The k and h axes have been contracted to display the points at infinity. The completely unstable fixed point $(k_c, 0)$ corresponds to the continuous phase transition, whereas the other fixed points are associated with phases (with or without an external field).

The other fixed points in Fig. 3.5 do not have the interpretation of phase transitions governed by temperature. Recall that the physical inverse temperature $\beta = 1/T$ is included in the definitions of the coupling k and of the field h. The "physical" field is rather $\tilde{h} = Th$. Thus, the fixed point $(k, h) = (0, 0)$ corresponds to infinite temperature and small field $\tilde{h}$ and describes a disordered phase. This point is unstable when an "infinite" field $\tilde{h}$ is turned on and a nonzero magnetization then appears, in one direction or the other. These ordered states are described by the points $(0, \pm\infty)$. At the other extreme, the fixed point $(k, h) = (\infty, 0)$ corresponds to zero temperature and describes an ordered phase in the absence of a field. It is unstable against an infinitesimal field $\tilde{h}$, which drives the system into a state of nonzero magnetization, described by the points $(\infty, \pm\infty)$. In general, stable fixed points describe stable phases of the system. This interpretation is natural since the correlation length decreases along the RG trajectory and the statistical mechanics of the system becomes simpler, since more and more degrees of freedom have been eliminated. The unstable fixed points located between the

basins of attraction of stable fixed points are, on the contrary, associated with phase transitions governed by temperature (e.g., $(k,h) = (k_c, 0)$) or by other parameters (e.g., $(k,h) = (0,0)$).

§3.4. The Renormalization Group: Continuum Models

Block-spin—or *real-space*—renormalization is an intuitive procedure designed for lattice models. If we want to apply renormalization ideas to continuum models, be it in the context of statistical mechanics or that of quantum field theory, a different procedure is needed, namely *momentum-space renormalization.* In what follows, the term *action* functional is used instead of *energy* functional, as it should be in statistical mechanics, since we have quantum field theory in mind and will refer to scale transformations as defined in Chap. 2.

3.4.1. Introduction

For the sake of introduction, we consider a statistical model defined in terms of a single scalar field $\varphi(\boldsymbol{x})$ in d-dimensional space (boldface letters denote vectors). The field $\varphi(\boldsymbol{x})$ may be Fourier decomposed as follows:

$$\varphi(\boldsymbol{x}) = \int (d\boldsymbol{k})\, \tilde{\varphi}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \qquad (d\boldsymbol{k}) \equiv \frac{d^d k}{(2\pi)^d} \tag{3.81}$$

The action functional $S[\varphi]$ may be expressed in terms of the Fourier components $\tilde{\varphi}(\boldsymbol{k})$. For instance, the action for the φ^4 theory in Eq. (3.19) becomes

$$\begin{aligned} S[\varphi; r, u] = &\int (d\boldsymbol{k})\, \frac{1}{2}\tilde{\varphi}(-\boldsymbol{k})\tilde{\varphi}(\boldsymbol{k})(\boldsymbol{k}^2 + r) \\ &+ \frac{1}{4}u \int (d\boldsymbol{k}_1)(d\boldsymbol{k}_2)(d\boldsymbol{k}_3)\, \tilde{\varphi}(-\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\tilde{\varphi}(\boldsymbol{k}_1)\tilde{\varphi}(\boldsymbol{k}_2)\tilde{\varphi}(\boldsymbol{k}_3) \end{aligned} \tag{3.82}$$

In general, we write the action as $S[\varphi; u_i]$, where u_i stands for the collection of parameters multiplying the various terms of the Lagrangian density.

Naturally, the continuum theory is defined only through some regularization procedure, which we take here as a cutoff Λ, meaning that the integration is restricted to the region of momentum space such that all arguments $\boldsymbol{k}$ of $\tilde{\varphi}(\boldsymbol{k})$ lie within the cutoff: $|\boldsymbol{k}| < \Lambda$. The Fourier decomposition (3.81) amounts to a unitary transformation of the degrees of freedom, as could easily be seen in a discrete version of the Fourier transform for a finite lattice of points. Therefore, the functional integration measure may be formally written as

$$[d\varphi]_\Lambda = \prod_{\boldsymbol{x}} d\varphi(\boldsymbol{x}) = \prod_{|\boldsymbol{k}|<\Lambda} d\tilde{\varphi}(\boldsymbol{k}) \tag{3.83}$$

since no Jacobian arises from the change of integration variables $\varphi(\boldsymbol{x}) \to \tilde{\varphi}(\boldsymbol{k})$.

The first step of the renormalization procedure[5] consists in integrating out the Fourier components $\tilde{\varphi}(\boldsymbol{k})$ such that $\Lambda/s < |\boldsymbol{k}| < \Lambda$ (the so-called *fast modes*), where s is some dilation factor ($s > 1$). The number of degrees of freedom is then effectively reduced, with a new cutoff equal to Λ/s. The remaining degrees of freedom (the *slow modes*) are governed by a modified action $S'[\varphi; u_i]$:

$$\exp -S'[\varphi; u_i] = \int \prod_{\Lambda/s<|\boldsymbol{k}|<\Lambda} d\tilde{\varphi}(\boldsymbol{k}) \exp -S[\varphi; u_i] \tag{3.84}$$

As long as we are interested in correlation functions of slow modes only, the effective action S' is entirely equivalent to the original action S which includes fast modes.

The second step of the renormalization procedure is a scale transformation on the slow-mode action, as defined in Eq. (2.121):

$$\boldsymbol{k} \to \boldsymbol{k}' = s\boldsymbol{k} \quad \text{or} \quad \boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x}/s \tag{3.85}$$

Here the scaling factor λ is $1/s$. In general such a transformation also affects the field:

$$\varphi(\boldsymbol{x}) \to \varphi'(\boldsymbol{x}/s) = s^{\Delta}\varphi(\boldsymbol{x}) \quad \text{or} \quad \tilde{\varphi}'(s\boldsymbol{k}) = s^{\Delta-d}\tilde{\varphi}(\boldsymbol{k}) \tag{3.86}$$

The exponent Δ is the *scaling dimension* of the field φ and is related to the exponent η: $\Delta = \eta/2$. Such a transformation of the field affects the functional integration measure only through a multiplication factor. After this rescaling, the modified action S' can be rightfully compared with the initial action S, because they now have the same cutoff Λ, that is, the same set of degrees of freedom (this was not true before rescaling). As said above, the two actions S and S' are equivalent as far as the slow modes are concerned: they describe the same long-distance properties. However, the parameters u_i defining these two action functionals are different in general: $S'[\varphi] = S[\varphi; u_i']$.[6] We thus generate a curve $u_i(s)$ in parameter space (s is the rescaling factor), and each point on this curve defines an action functional with the same long-distance properties. The outcome of the renormalization procedure can be expressed in a set of coupled flow equations in parameter space:

$$\frac{du_i}{d\ln s} = \beta_i(u_j) \tag{3.87}$$

where β_i is commonly referred to as the *beta function* associated with the parameter u_i. Like before, a *fixed point* u_j^* of the renormalization group is a point in parameter space that is unaffected by the renormalization procedure. In other words, it is characterized by a vanishing beta function:

$$\beta_i(u_j^*) = 0 \tag{3.88}$$

[5] This procedure is known as the *Wilson-Kadanoff renormalization scheme.*

[6] This equation supposes that the scaling dimension Δ has been chosen appropriately; otherwise, the two actions are not equal, but differ by a multiplicative constant. Also, the number of parameters needed for the new action to be of the same form as the old action is in principle infinite. In practice, however, one keeps only a finite subset of parameters: relevant and marginal ones. Irrelevant parameters (in the RG sense) rapidly decrease under RG flow.

To summarize, a renormalization-group transformation amounts to a scale transformation applied both to the action and to the integration measure (i.e., the Fourier modes that would be scaled beyond the cutoff Λ are integrated out). A fixed point of the renormalization-group transformation thus defines a theory that has scale invariance at the quantum level.

THE GAUSSIAN MODEL

The simplest example of a continuum model for which the renormalization procedure can be carried out exactly is the free boson, or *Gaussian model*, obtained from Eq. (3.82) by setting $u = 0$:

$$S[\varphi; r] = \int_\Lambda (d\boldsymbol{k})\, \frac{1}{2}\tilde{\varphi}^*(\boldsymbol{k})\tilde{\varphi}(\boldsymbol{k})(\boldsymbol{k}^2 + r) \tag{3.89}$$

In this model the fast and slow modes are decoupled, since different values of the wavevector do not mix in the action. Therefore, integrating the fast modes produces only an irrelevant multiplicative constant in front of the partition function. The effective slow-mode action is then

$$\begin{aligned} S'[\varphi] &= \int_{\Lambda/s} (d\boldsymbol{k})\, \frac{1}{2}\tilde{\varphi}(-\boldsymbol{k})\tilde{\varphi}(\boldsymbol{k})(\boldsymbol{k}^2 + r) \\ &= s^{-d} \int_\Lambda (d\boldsymbol{k}')\, \frac{1}{2}\tilde{\varphi}(-\boldsymbol{k}'/s)\tilde{\varphi}(\boldsymbol{k}'/s)(\boldsymbol{k}'^2/s^2 + r) \\ &= s^{d-2\Delta-2} \int_\Lambda (d\boldsymbol{k}')\, \frac{1}{2}\tilde{\varphi}'(-\boldsymbol{k}')\tilde{\varphi}'(\boldsymbol{k}')(\boldsymbol{k}'^2 + s^2 r) \end{aligned} \tag{3.90}$$

We immediately see that S', in terms of φ', has the same form as $S[\varphi]$, provided $r = 0$ and $\Delta = \frac{1}{2}d - 1$. This we knew already from Eq. (2.124). In this particular case, the scale transformation on the path-integral measure brings nothing new and the scaling properties all follow from the action alone. Thus, the massless ($r = 0$) Gaussian model is a fixed point of the renormalization group—in fact, the simplest of all fixed points from the present point of view.

3.4.2. Dimensional Analysis

We consider a field ϕ (not necessarily a scalar field) governed by an action functional $S[\phi]$ and let us assume that there exists a fixed-point action $S_0[\phi]$ (not necessarily Gaussian) at some point in parameter space, which we take, for convenience, as the origin. In the vicinity of this fixed point, the generic action $S[\phi]$ may be expressed as

$$S[\phi] = S_0[\phi] + \sum_i u_i \int dx\, O_i(x) \tag{3.91}$$

where the $O_i(x)$'s are some local operators, expressible in terms of the field ϕ. The couplings u_i must be small if we are close to the fixed point. Under a renormalization-group (scale) transformation, the field ϕ transforms like

$\phi'(x) = s^{\Delta}\phi(sx)$ and only $S_0[\phi]$ is invariant. The other terms are modified through their couplings:

$$S'[\phi] = S_0[\phi] + \sum_i u_i'(s) \int dx\, O_i(x) \tag{3.92}$$

In principle, the series on the r.h.s. may be infinite, and the transformed couplings u_i' may depend on s in a complicated way, because of the functional integration of the fast modes. We assume, however, that the couplings u_i are so small that they have a negligible effect on the fast mode integration. In this approximation, the new couplings u_i' may be obtained simply from the behavior of the operators O_i under a scale transformation, which follows from the expression of O_i in terms of ϕ:

$$\begin{aligned} O_i'(x) &= s^{\Delta_i} O_i(sx) \\ u_i \int dx\, O_i'(x) &= u_i s^{\Delta_i - d} \int dx\, O_i(x) \end{aligned} \tag{3.93}$$

Therefore

$$u_i' = u_i s^{d-\Delta_i} \tag{3.94}$$

In other words, in this zeroth-order approximation, the dimensions of couplings are obtained from the scaling dimension Δ of ϕ by applying *dimensional analysis*.

Adopting the terminology of the previous section, a coupling is said to be *relevant* if $\Delta_i < d$: It will grow as the fast modes are integrated. An *irrelevant coupling* is such that $\Delta_i > d$, and will shrink as the fast modes are integrated. At last, a *marginal* coupling will stay the same, or rather vary logarithmically near the fixed point.[7]

For instance, we now look at some operators within the Gaussian model. The first operator that comes to mind is the mass term $O_2 = \frac{1}{2}\varphi^2$, with coupling r. With Gaussian scaling (i.e., $\Delta = \frac{1}{2}d - 1$) we find that $\Delta_2 - d = -2$, and hence $r' = s^2 r$. This, of course, was already known from Eq. (3.90). Thus, the mass term is relevant at the Gaussian fixed point, in all dimensions. This is a trivial statement since we know from Chap. 2 that the mass is the inverse correlation length ($\mu \sim \xi^{-1}$) and that ξ decreases under scaling ($\xi' = \xi/s$). The quartic coupling of the φ^4 theory is associated with the operator $O_4 = \varphi^4$, with $\Delta_4 - d = d - 4$. Thus the quartic coupling u is relevant in dimensions smaller than four, irrelevant in $d > 4$, and marginal in $d = 4$ (still at zeroth order). At this order, it looks as if any (positive) value of u yields a fixed point in $d = 4$.

[7] It is important to keep in mind that the scaling dimensions of operators, or the relevance or irrelevance of couplings, depends not only on the form of these operators in terms of ϕ, but also on the fixed point considered.

3.4.3. Beyond Dimensional Analysis: The φ^4 Theory

To go beyond dimensional analysis, we generally use perturbation theory: We expands the exponential $\exp -S$ in powers of the perturbing coupling. The problem is then reduced to the calculation of Gaussian correlators, which can be done using Wick's theorem. Since we will make little use of perturbation theory in this work, this method is not reviewed in these introductory chapters; again we refer the reader to the standard texts on quantum field theory. Here we simply cite known results.

To first order in u and r, perturbation theory leads to the following renormalization-group transformation of the couplings:

$$\begin{aligned} r' &= s^2(r + ub(1 - s^{2-d})) \\ u' &= s^{4-d}u \end{aligned} \tag{3.95}$$

with

$$b = K_d \Lambda^{d-2}/(2d-4) \quad , \quad K_d^{-1} = (4\pi)^{d/2}\Gamma(d/2)/2 \tag{3.96}$$

In matrix form this becomes

$$\begin{pmatrix} r \\ u \end{pmatrix}' = \begin{pmatrix} s^2 & b(s^2 - s^{4-d}) \\ 0 & s^{4-d} \end{pmatrix} \begin{pmatrix} r \\ u \end{pmatrix} \tag{3.97}$$

We recall that the proper couplings t_i of Eq. (3.58) are obtained by diagonalizing this matrix. The eigenvalues and eigenvectors are

$$\begin{aligned} \lambda_1 &= s^2 & \mathbf{u}_1 &= (1,0) \\ \lambda_2 &= s^{4-d} & \mathbf{u}_2 &= (-b,1) \end{aligned} \tag{3.98}$$

Since by definition $(r,u) = t_1\mathbf{u}_1 + t_2\mathbf{u}_2$, we have the proper couplings

$$t_1 = r + bu \qquad t_2 = u \tag{3.99}$$

At this order, there is a critical line in $d > 4$ specified by the equation $t_1 = 0$, or $r = -bu$. In $d = 4$, it still looks as if any value of $t_2 = u$ constitutes a fixed point.

However, this picture breaks down once we take into account higher orders of u in the perturbation expansion. At second order, we find that

$$\begin{aligned} r' &= s^2\left[r + \frac{u}{16\pi^2}\left(\frac{1}{2}\Lambda^2(1 - s^{-2}) - r\ln s\right)\right] \\ u' &= s^{4-d}\left[u - \frac{3u^2}{16\pi^2}\ln s\right] \end{aligned} \tag{3.100}$$

The quartic coupling then receives logarithmic corrections in $d = 4$. This RG mapping is better expressed by the corresponding beta functions:

$$\begin{aligned} \frac{dr}{d\ln s} &= 2r - \frac{ur}{16\pi^2} + \frac{u\Lambda^2}{16\pi^2} \\ \frac{du}{d\ln s} &= (4-d)u - \frac{3}{16\pi^2}u^2 \end{aligned} \tag{3.101}$$

This shows the emergence of a new (non-Gaussian) fixed point at $r, u \neq 0$, whose location is readily found from the above beta functions:

$$u^* = \frac{16\pi^2}{3}(4-d) \qquad r^* = \frac{d-4}{6}\Lambda^2 \tag{3.102}$$

It is a straightforward exercise to linearize the flow (3.101) around this new fixed point and to find the critical exponents. For reasons that will not be explained here, the critical exponents of the φ^4 theory (and of other Gaussian-like models) are calculated in the form of a series in powers of $\varepsilon \equiv 4 - d$ (the so-called ε-expansion). Each additional order in perturbation theory leads to the correct evaluation of a new term of this expansion. To order ε^2, the exponents of the φ^4 theory are calculated to be

$$\nu = \frac{1}{2} + \frac{1}{12}\varepsilon + O(\varepsilon^2) \qquad \eta = 0 + O(\varepsilon^2) \tag{3.103}$$

The φ^4 model in $d = 4$ clearly illustrates that scale invariance of the action (here on the line $r = 0$) does not guarantee scale invariance at the quantum level (i.e., a renormalization-group fixed point). This breakdown of dimensional analysis is due to interactions.

§3.5. The Transfer Matrix

A powerful way to solve the Ising model and other related statistical models is the *transfer matrix* method, which is the analogue in statistical mechanics of the operator formalism in quantum field theory. In this section we will describe this formalism and indicate how it can lead to an analogy between quantum field theories and statistical systems near criticality.

Again, we turn to the Ising model on a square lattice with m rows and n columns. A spin is here indexed by two integers[8] for the row number and column number, respectively, and we will impose periodic boundary conditions

$$\sigma_{i,j+n} = \sigma_{ij} \qquad \sigma_{i+m,j} = \sigma_{ij} \tag{3.104}$$

thereby defining the lattice on a torus. Let us denote by μ_i the configuration of spins on the i-th row:

$$\mu_i = \{\sigma_{i1}, \sigma_{i2}, \cdots, \sigma_{in}\} \tag{3.105}$$

There are 2^n such configurations. The row configuration μ_i has an energy of its own:

$$E[\mu_i] = \sum_{k=1}^{n} \sigma_{ik}\sigma_{i,k+1} \tag{3.106}$$

[8] The two indices will be separated by a comma only when necessary to avoid confusion.

as well as an interaction energy with the neighboring rows:

$$E[\mu_i, \mu_j] = \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} \tag{3.107}$$

We next define a formal vector space V of row configurations spanned by the $|\mu_i\rangle$, for which we introduce a "bra-ket" notation in analogy with quantum mechanics. On this space, we define the action of the *transfer matrix* T by its matrix elements:

$$\langle\mu|T|\mu'\rangle = \exp -\beta\big(E[\mu, \mu'] + \frac{1}{2}E[\mu] + \frac{1}{2}E[\mu']\big) \tag{3.108}$$

In terms of the operator T, the partition function has the following simple form:

$$\begin{aligned} Z &= \sum_{\mu_1,\cdots,\mu_m} \langle\mu_1|T|\mu_2\rangle\langle\mu_2|T|\mu_3\rangle\cdots\langle\mu_m|T|\mu_1\rangle \\ &= \text{Tr } T^m \end{aligned} \tag{3.109}$$

The transfer matrix defined in (3.108) is manifestly symmetric, and therefore diagonalizable. The partition function may be expressed in terms of the 2^n eigenvalues Λ_k of T:

$$Z = \sum_{k=0}^{2^n-1} \Lambda_k^m \tag{3.110}$$

The thermodynamic limit is obtained when $m, n \to \infty$. In this limit, the free energy can be extracted by keeping only the largest eigenvalue of T, assuming, for the sake of argument, that it is nondegenerate. Indeed, the free energy per site f is given by

$$\begin{aligned} -f/T &= \lim_{m,n\to\infty} \frac{1}{mn} \ln(\Lambda_0^m + \Lambda_1^m + \cdots) \\ &= \lim_{m,n\to\infty} \frac{1}{mn}\{m \ln \Lambda_0 + \ln(1 + (\Lambda_1/\Lambda_0)^m + \cdots)\} \\ &= \lim_{n\to\infty} \frac{\ln \Lambda_0}{n} \end{aligned} \tag{3.111}$$

since $\Lambda_1/\Lambda_0 < 1$. The calculation of more complicated thermodynamic quantities requires the knowledge of more eigenvalues.

In order to express correlation functions in terms of the transfer matrix, we introduce a spin operator $\hat{\sigma}_i$ acting on V and giving the value of the spin on the i-th column when acting on basis vector $|\mu\rangle$:

$$\hat{\sigma}_i|\mu\rangle = \sigma_i|\mu\rangle \tag{3.112}$$

Then

$$\begin{aligned}\langle\sigma_{ij}\sigma_{i+r,k}\rangle &= \frac{1}{Z}\sum_{\mu_1,\cdots,\mu_m}\langle\mu_1|T|\mu_2\rangle\cdots\langle\mu_i|\hat{\sigma}_j T|\mu_{i+1}\rangle\cdots \\ &\qquad \cdots\langle\mu_{i+r}|\hat{\sigma}_k T|\mu_{i+r+1}\rangle\cdots\langle\mu_m|T|\mu_1\rangle \qquad (3.113)\\ &= \frac{\mathrm{Tr}\left(T^{m-r}\,\hat{\sigma}_j\,T^r\,\hat{\sigma}_k\right)}{\mathrm{Tr}\,T^m}\end{aligned}$$

This should be reminiscent of the passage from the operator formalism to the path integral formalism in Euclidian quantum field theory. The transfer matrix here plays the role of the evolution operator $U(a)$ over a "distance of time" equal to the lattice spacing a. In other words, one can define a Hamiltonian operator $\hat{H}$ as

$$T = \exp -a\hat{H} \qquad (3.114)$$

The eigenstates of T are the analogue of the energy eigenstates of quantum mechanics, the eigenvalues E_r of $\hat{H}$ (the energy levels) being expressed as

$$E_r = -\frac{1}{a}\ln\Lambda_r \qquad (3.115)$$

in terms of the eigenvalues of T. Therefore, the free energy density f/a^2 is proportional to the vacuum energy per site, or the vacuum energy density in field theoretic language:

$$f/a^2 = \lim_{n\to\infty}\frac{E_0}{na} \qquad (3.116)$$

The magnetization $\langle\sigma_{ij}\rangle$ in the thermodynamic limit is

$$\begin{aligned}\langle\sigma_{11}\rangle &= \lim_{m\to\infty}(\mathrm{Tr}\;T^m)^{-1}\,\mathrm{Tr}\;(\hat{\sigma}\,T^m)\\ &= \lim_{m\to\infty}e^{-ma(E_l-E_0)}\sum_l\langle 0|\hat{\sigma}_1|l\rangle \qquad (3.117)\\ &= \langle 0|\hat{\sigma}_1|0\rangle\end{aligned}$$

where we have inserted a complete set of T eigenstates, which reduces to $|0\rangle\langle 0|$ in the limit $m\to\infty$ because of the exponential factor. The statistical average of the spin is therefore given by the "vacuum expectation value" of the corresponding operator S. This applies to any local quantity and its operator.

Likewise, the pair correlation function can be expressed in the thermodynamic limit:

$$\begin{aligned}\langle s_{11}s_{1+r,1}\rangle &= \lim_{m\to\infty}(\mathrm{Tr}\;T^m)^{-1}\,\mathrm{Tr}\;(T^{m-r}\,S_1\,T^r\,S_1)\\ &= \lim_{m\to\infty}e^{maE_0}\sum_l\langle 0|e^{(m-r)aE_0}S_1|l\rangle\langle l|e^{-raE_l}S_1|0\rangle \qquad (3.118)\\ &= \langle s_{11}\rangle^2 + |\langle 0|S_1|1\rangle|^2\exp -ra(E_1-E_0)+\cdots\end{aligned}$$

The connected correlation function in the long distance limit ($r \gg 1$) is therefore

$$\langle s_{11} s_{1+r,1} \rangle \sim |\langle 0|S_1|1\rangle|^2 \exp -ra(E_1 - E_0) \tag{3.119}$$

The energy gap $E_1 - E_0$ is the mass m of the field quantum: It is the energy of a particle at rest. The relation between the correlation length and the mass of the associated Euclidian quantum field theory is therefore

$$\xi = \frac{1}{ma} \tag{3.120}$$

Near a critical point the correlation length grows without bounds and correspondingly the mass goes to zero (for fixed a). In other words, the largest eigenvalues of the transfer matrix coalesce at the critical point.

To summarize, we have shown how a lattice model can be described in an operator formalism, which makes clear the very close analogy with Euclidian quantum field theories. The free energy density is then the vacuum energy density, the pair correlation function is the field's propagator, and the correlation length is proportional to the inverse mass. A system at the critical point is therefore equivalent to a massless field theory, provided the lattice spacing a is not exactly zero.

Exercises

3.1 *The binomial distribution*
Consider a set of N particles moving almost freely in a box of volume V, with occasional collisions among themselves. The probability that a given particle be within the left half of the box at any moment is $\frac{1}{2}$. If we neglect the volume of the particles, i.e., if the density of the gas is not too large, then the fact that a particle is in the left half of the box is independent of the situation of other particles, and the number n of particles in the left half obeys a binomial probability distribution:

$$P(n) = \frac{N!}{n!(N-n)!} 2^{-N}$$

a) Compute the expectation value of the binomial distribution, namely the quantity $\langle n \rangle = \sum_{n=0}^{N} nP(n)$, which represents the average number of particles in the left half of the box.

b) Compute the standard deviation $\Delta n = \left\langle (n - \langle n \rangle)^2 \right\rangle^{1/2}$.

c) By expanding the probability $P(n)$ around the mean value $\langle n \rangle$, find the thermodynamic limit of the distribution $P(n)$.
Result: Writing $n = \frac{N}{2} + \varepsilon$, and using Stirling's formula

$$\ln x! = (x + \frac{1}{2}) \ln x - x + \frac{1}{2} \ln 2\pi + O(1/x)$$

for large x, we find that

$$P(\frac{N}{2} + \varepsilon) \sim \sqrt{\frac{2}{\pi N}} e^{-2(2N-1)\varepsilon^2/N^2}$$

Hence, in terms of the scaling variable $x = 2\varepsilon/\sqrt{N}$, the thermodynamic distribution becomes the Gaussian distribution

$$P(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

3.2 *The one-dimensional Ising model*
We consider the one-dimensional Ising model, with energy (3.6). We introduce the scaled variables $K = -J/k_BT$ and $H = h/k_BT$.

a) Show that the partition function on a chain of N sites $i = 1, .., N$, with periodic boundary conditions $N + 1 \equiv 1$, can be expressed as the trace

$$\begin{aligned} Z_N(K,H) &= \sum_{\substack{s_i=\pm 1 \\ s_{N+1}=s_1}} \exp\left\{ K \sum_{(ij)} s_i s_j + H \sum_i s_i \right\} \\ &= \mathrm{Tr}\left(T(K,H)^N \right) \end{aligned}$$

where $T(K,H)$ is the 2×2 transfer matrix of the model. Show that $T(K,H)$ is

$$T(\beta,H) = \begin{pmatrix} e^{K+H} & e^{-K} \\ e^{-K} & e^{K-H} \end{pmatrix}$$

in the basis $(+1,-1)$ for s.

b) Compute the thermodynamic free energy

$$f(K,H) = \lim_{N\to\infty} -(1/N) \ln Z_N(K,H)$$

Hint: $(Z_N)^{1/N}$ is dominated by the largest eigenvalue of the transfer matrix T, namely

$$\lambda_{\max} = e^K \cosh(H) + \sqrt{e^{-2K} + e^{2K} \sinh(H)}$$

c) Compute the magnetization $M = -\partial f/\partial K$. Show in particular that the magnetization is linear for h small ($M \sim he^{2K}$). Deduce that the magnetic susceptibility diverges at zero temperature. Show that there is no phase transition at finite temperature for the one-dimensional Ising model.

d) Compute the spin-spin correlation in the thermodynamic limit.

3.3 *Free energy of the one-dimensional Potts model*
In the q-state Potts model, the spin variable s_i takes q possible values, in the set $\{0, 1, ..., q-1\}$. The energy of a configuration reads

$$E(s_1, ..., s_N) = -J \sum_{\langle ij \rangle} \delta_{s_i, s_j}$$

and we use the scaled variable $K = J/k_BT$.

a) Write the transfer matrix T of the one-dimensional model with periodic boundary conditions in terms of the $q \times q$ matrix J, with all entries equal to 1.
Result: $T = (e^K - 1)\mathbb{I} + J$.

b) Compute the thermodynamic free energy of the one-dimensional q-state Potts model.
Hint: Note that $J^2 = qJ$, and use this fact to compute $\mathrm{Tr}(T^N)$.

3.4 *Transfer matrix for the two-dimensional Ising model*
The two-dimensional Ising model with spins s_{ij} sitting at the vertices (i,j) of a square lattice of size $N \times L$ in zero magnetic field has the energy

$$E[s] = -J \sum_{((i,j)(k,l))} s_{ij}s_{kl}$$

where the sum extends over all the bonds of the lattice. We use the scaled variable $K = J/k_BT$.
Write the row-to-row transfer matrix for this model, namely the $2^L \times 2^L$ matrix $T_L(K)$, such that the partition function $Z_{N,L}$ with periodic boundary conditions reads

$$Z_{N,L}(K) = \mathrm{Tr}\left(T_L(K)^N\right)$$

3.5 *Numerical diagonalization of transfer matrices*
a) Given a symmetric indecomposable $r \times r$ matrix T, show that it has a unique maximal eigenvalue $\lambda_{\max}$. Let $v_{\max}$ denote the corresponding (normalized) eigenvector.
b) We define the sequence of vectors $v_0, v_1, v_2, \cdots$ where v_0 is arbitrary and the other members of the sequence are defined by recursion: $v_{n+1} = Tv_n/|Tv_n|$ ($|x|$ denotes the Euclidian norm of x). Show that if the scalar product $v_0 \cdot v_{\max}$ does not vanish, then the sequence v_n converges exponentially fast to $v_{\max}$.
Hint: Decompose v_0 in the orthonormal diagonalization basis of T.
c) Using the above, write a computer program to extract the largest eigenvalue of a symmetric matrix T.
d) Application: Evaluate numerically the thermodynamic free energy of the two-dimensional Ising model on an infinite strip of width L, at the known critical value of the coupling $K = K_c = -(1/2)\ln(\sqrt{2}-1)$. (Use Ex. 3.4 above for the definition of the relevant transfer matrix.) Plot the results for various widths L. Fit the results with the ansatz

$$f_L = Lf_0 - \frac{\pi}{6L}c + O(\frac{1}{L^2})$$

and evaluate the constants f_0 and c. The quantity c is the central charge of the corresponding conformal field theory. Its exact value for the two-dimensional Ising model is $c = 1/2$.

Notes

There are many excellent texts on statistical mechanics; we cannot list them all here. The very thorough and pedagogical text by Diu and collaborators [106] deserves special mention. Texts by Ma [261], Huang [194] and Pathria [292] are widely used. Among texts emphasizing critical phenomena are those of Amit [13], Binney et al. [47], Le Bellac [253], Ma [260] and Parisi [287].

Some discrete statistical models are described and solved using transfer matrix techniques in Baxter's text [31]. The scaling hypothesis for the free energy was introduced by Widom [355]. The idea of introducing block spins to calculate critical exponents is due to Kadanoff [222]. Applications of the renormalization group to critical phenomena were initiated by Wilson and are described in Ref. [357]. The real-space renormalization group treatment of the Ising model on a triangular lattice was done by Niemeijer and van Leeuwen [282]. The emergence of conformal invariance at critical points was shown by Polyakov [295].