Basic Set Theory

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Throughout we work with the foundation provided by standard ZFC (FCT,Section 3). In particular we do not assume we are working inside any fixed grothendieck universe. It seems to me that some basic proofs about ordinals in standard references are flawed, so since the end result is the same we adopt a slightly different definition to make life easier.

Contents

1 Ordinal Numbers

Definition 1. Given a set x, we let x^+ denote the set $x \cup \{x\}$.

Definition 2. A set x is *transitive* if whenever $y \in x$ we have $y \subseteq x$. A set x is *urtransitive* if it is transitive, and if in addition whenever a proper subset $y \subset x$ is transitive we have $y \in x$. We say x is an *ordinal* if it is urtransitive, and if every element of x is urtransitive. We tend to use lower case greek letters $\alpha, \beta, \gamma, \ldots$ to represent ordinals. Given ordinals α, β we write $\alpha \prec \beta$ for $\alpha \in \beta$. One checks that this is a transitive irreflexive relation on ordinals. We write $\alpha \preceq \beta$ if $\alpha \prec \beta$ or $\alpha = \beta$, and this defines a partial order on ordinals.

Remark 1. The following observations are immediate

- The empty set is an ordinal. The empty set is a member of any nonempty ordinal.
- Any element of an ordinal is an ordinal.
- If α, β are ordinals then $\alpha \prec \beta$ if and only if $\alpha \subset \beta$.

Lemma 1. *If α is an ordinal, then so is α^+.*

Proof. The set α^+ is clearly transitive. To see that it is urtransitive, let $y \subset \alpha \cup \{\alpha\}$ be a transitive proper subset. We want to show that $y \in \alpha \cup \{\alpha\}$. Suppose that $y \neq \alpha$. First we claim that $\alpha \notin y$. For if this were the case, transitivity of y implies $\alpha \subset y$. Suppose $t \in y \setminus \alpha$. Then $t \in y \subset \alpha \cup \{\alpha\}$ so we must have $t = \alpha$. But then $y \supseteq \alpha \cup \{\alpha\}$, which is a contradiction. This shows that $\alpha \notin y$. But then y must be a proper subset of α, and since α is an ordinal this means $y \in \alpha$, as required. This shows that α^+ is urtransitive, and it is clear that every element of α^+ is urtransitive, so α^+ is an ordinal. □

Lemma 2. *If a set contains an ordinal, then it contains a minimal ordinal.*

Proof. Let X be a set and suppose $\alpha \in X$ for some ordinal α. Then the set Z of elements of X which are ordinals is nonempty, and applying the Axiom of Foundation to this set we obtain an ordinal $\beta \in X$ with the property that no ordinal in β is an element of X. $\square$

Proposition 3 (Minimal element)**.** *Let $\mathscr{B}(x)$ be a wf with x free, and suppose there exists an ordinal α with $\mathscr{B}(\alpha)$. Then there a* minimal *such ordinal. That is, there exists an ordinal α with $\mathscr{B}(\alpha)$ but not $\mathscr{B}(\beta)$ for any ordinal $\beta \prec \alpha$.*

Proof. Given an ordinal α with $\mathscr{B}(\alpha)$ we can apply Lemma 2 to the set of all ordinals $\beta \prec \alpha$ with $\mathscr{B}(\beta)$ (of course if this set is empty, then α already has the right property). $\square$

Theorem 4 (Trichotomy)**.** *Given two ordinals α, β we have $\alpha \prec \beta, \beta \prec \alpha$ or $\alpha = \beta$.*

Proof. Suppose not, and let α, β be two ordinals satisfying none of these conditions. We fix α, and let $\beta' \prec \beta$ be an ordinal minimal with the property that $\alpha \not\prec \beta', \beta' \not\prec \alpha, \alpha \neq \beta'$. By minimality for any $\gamma \prec \beta'$ we must $\alpha \prec \gamma, \alpha = \gamma$ or $\gamma \prec \alpha$. The first two alternatives would imply $\alpha \prec \beta$, which contradicts our hypothesis. We deduce that $\gamma \in \alpha$, and since γ was arbitrary we have $\beta' \subset \alpha$. By definition of an ordinal this implies $\beta' \in \alpha$, a contradiction. This shows that for any two ordinals α, β trichotomy holds. $\square$

Remark 2. Most references would define an ordinal as a transitive set whose elements are all transitive. But it seems to me there is a subtle difficulty in proving trichotomy for this definition. In any case, if you believe trichotomy for these ordinals, then you can easily show that a set is an ordinal (in this sense) if and only if it is an ordinal in our sense.

Corollary 5. *Let α, β be ordinals. Then $\alpha \prec \beta$ if and only if $\alpha^+ \prec \beta^+$. If $\alpha \prec \beta$ then $\alpha^+ \preceq \beta$.*

Proposition 6. *If X is a set of ordinals, the union set $\bigcup X$ is also an ordinal.*

Proof. Let us treat some easy cases first

- If $\bigcup X$ is empty then it is trivially an ordinal, so we can assume that X is nonempty and also that X contains a nonempty set.

- If X contains an ordinal α which is *maximal*, that is, there is no ordinal $\beta \in X$ with $\beta \succ \alpha$, then by trichotomy every ordinal in X belongs to α (or *is* α) and so $\bigcup X = \alpha$ is certainly an ordinal. So we can assume that for every ordinal $\alpha \in X$ there is another ordinal $\beta \in X$ with $\beta \succ \alpha$. One easy consequence of this assumption is that $\bigcup \bigcup X = \bigcup X$; that is, $\bigcup X$ is the union of all its elements.

The set $\bigcup X$ is clearly transitive, and every one of its elements is urtransitive, so to complete the proof it suffices to show that any proper transitive subset $y \subset \bigcup X$ is actually an element of $\bigcup X$. We assume otherwise, and produce a contradiction. Of course trivially $\phi \in \bigcup X$, so we may assume y is nonempty.

Suppose $\alpha \in \bigcup X$ is an ordinal with $\alpha \notin y$. We must also have $y \notin \alpha$, since otherwise $y \in \bigcup X$ which contradicts our hypothesis. Since α is urtransitive, we also deduce that $y \not\subseteq \alpha$. It follows that there exists an ordinal $\beta \in y \setminus \alpha$. By trichotomy we must have $\alpha \prec \beta$ and therefore $\alpha \subseteq y$ by transitivity of y. This shows that any element of $\bigcup X$ *not* in y is a subset of y. But of course by transitivity any element of $\bigcup X$ which *is* in y is also a subset of y, so as $\bigcup X$ is the union of its elements we deduce that $\bigcup X \subseteq y$, a contradiction. This shows that $y \in \bigcup X$, so $\bigcup X$ is urtransitive and therefore an ordinal. $\square$

Corollary 7. *Any transitive set X of ordinals is an ordinal.*

Proof. If X is empty this is trivial, so assume otherwise. Since X is transitive we have $\bigcup X \subseteq X$. If this is an equality then by Proposition 6 we are done, so suppose to the contrary that there exists $\alpha \in X \setminus \bigcup X$. That is, $\alpha \not\prec \beta$ for every $\beta \in X$. By trichotomy this means $\beta \prec \alpha$ for every $\beta \neq \alpha$ in X. In particular $X = \alpha \cup \{\alpha\} = \alpha^+$ is an ordinal. $\square$

Definition 3. An ordinal α is a *successor* ordinal if there is an ordinal β with $\beta^+ = \alpha$. An nonempty ordinal which is not a successor ordinal is called a *limit* ordinal. An ordinal α is a *finite ordinal* if $\alpha = \emptyset$ or α is a successor ordinal and every element of α is either empty or a successor ordinal. If α is a finite ordinal and $\beta \prec \alpha$ then β is a finite ordinal. Obviously the successor of any finite ordinal is a finite ordinal.

Lemma 8. *Let α be a finite ordinal and W a set with the property that $\phi \in W$ and $x^+ \in W$ whenever $x \in W$. Then $\alpha \in W$.*

Proof. Suppose $\alpha \notin W$ and let $\beta \prec \alpha$ be minimal with $\beta \notin W$. Since β is a finite ordinal it must be a successor ordinal (since $\phi \in W$). But if $\beta = \gamma^+$ then γ is a finite ordinal not belonging to W, contradicting minimality of β. □

Definition 4. By the Axiom of Infinity at least one set of the form described in Lemma 8 exists, so we can define a set ω by

$$\omega = \{\alpha \,|\, \alpha \text{ is a finite ordinal}\}$$

If we write $0 = \emptyset, 1 = 0^+, 2 = 1^+$ and so on, then intuitively ω is the set $\{0, 1, 2, 3, \dots\}$.

Proposition 9. *A nonempty ordinal α is a limit ordinal if and only if $\alpha = \bigcup \alpha$.*

Proof. Let α be a nonempty ordinal and suppose $\alpha = \bigcup \alpha$. Let β be an ordinal. Then by trichotomy we have $\alpha \prec \beta, \beta \prec \alpha$ or $\alpha = \beta$. If $\alpha = \beta$ then certainly $\alpha \neq \beta^+$. If $\alpha \prec \beta$ then $\alpha \prec \beta^+$ so the two ordinals cannot be equal. Finally if $\beta \prec \alpha = \bigcup \alpha$ then there is $\gamma \prec \alpha$ with $\beta \prec \gamma$. Then $\beta^+ \preceq \gamma \prec \alpha$ so $\alpha \neq \beta^+$. This shows that α is a limit ordinal.

Conversely if α is a limit ordinal, then of course $\bigcup \alpha \subseteq \alpha$. If $\beta \prec \alpha$ then by trichotomy $\beta^+ \prec \alpha$ and therefore $\beta \in \bigcup \alpha$. This shows that $\alpha \subseteq \bigcup \alpha$ and completes the proof. □

Definition 5. A relation $<$ on a set Y is a *well-ordering* if it is irreflexive and if for every nonempty subset $Z \subseteq Y$ there exists $y \in Z$ such that whenever $v \in Z$ with $v \neq y$ we have $y < v$ and not $v < y$. It is clear that if $<$ is a well-ordering, it is also a total order. A *morphism* of well-ordered sets $(Y, <) \longrightarrow (Y', <')$ is a function $f : Y \longrightarrow Y'$ with the property that whenever $x < y$ in Y we have $f(x) < f(y)$ in Y'. We say f is an *isomorphism* of well-ordered sets if there is a morphism of well-ordered sets $g : (Y', <') \longrightarrow (Y, <)$ such that $fg = 1, gf = 1$. Since any well-ordering is total, it is easy to check that f is an isomorphism of well-ordered sets if and only if it is a bijection.

Lemma 10. *Any ordinal α is well-ordered by the relation $\prec$ on its elements.*

Proof. If α is empty this is trivial, so suppose otherwise. Then every element of α is an ordinal, so $\prec$ is certainly an irreflexive relation on the set α. The fact that $\prec$ is a well-ordering follows from Lemma 2 and trichotomy. □

Lemma 11. *There is no largest ordinal, and there is no set of all ordinals.*

Proof. For any ordinal α we have $\alpha \prec \alpha^+$, so there can be no "largest" ordinal. If there existed a set X of all ordinals, the set $\bigcup X$ would be a largest ordinal. □

Lemma 12. *If α, β are ordinals and $f : \alpha \longrightarrow \beta$ a function preserving the relation $\prec$, then for every $\gamma \prec \alpha$ we have $\gamma \preceq f(\gamma)$.*

Proof. When we say that f preserves the relation $\prec$ we mean that for ordinals $\delta \prec \gamma \prec \alpha$ we have $f(\delta) \prec f(\gamma)$. Suppose to the contrary that there exists some ordinal $\gamma \prec \alpha$ with $f(\gamma) \prec \gamma$. We may assume γ is minimal with this property. Then for every $\delta \prec \gamma$ we must have $\delta \preceq f(\delta) \prec f(\gamma)$ and consequently $\delta \in f(\gamma)$. This shows that $\gamma \subseteq f(\gamma)$. But this is impossible, since $f(\gamma) \prec \gamma$, and from this contradiction we deduce the result. □

Lemma 13. *If two ordinals α, β are isomorphic as sets ordered by $\prec$, then $\alpha = \beta$.*

Proof. That is, if there exist functions $f : \alpha \longrightarrow \beta, g : \beta \longrightarrow \alpha$ preserving $\prec$ with $fg = 1$ and $gf = 1$ then we claim $\alpha = \beta$. This follows easily from the previous result. □

2 Transfinite Pain

Proposition 14 (Transfinite Induction, First Form)**.** *Let $\mathscr{B}(x)$ be a wf with x free. Suppose that for every ordinal α, whenever we have $\mathscr{B}(\beta)$ for every ordinal $\beta \prec \alpha$ we have $\mathscr{B}(\alpha)$. Then $\mathscr{B}(\alpha)$ for every ordinal α.*

Proof. Suppose that the inductive condition is satisfied. In particular we have $\mathscr{B}(\emptyset)$ vacuously. Suppose for a contradiction that there exists an ordinal α for which $\mathscr{B}(\alpha)$ does not hold. Then the set of ordinals $\beta \prec \alpha$ with $\neg\mathscr{B}(\beta)$ is nonempty (since otherwise the inductive condition would force $\mathscr{B}(\alpha)$) and we can let β' be minimal among these ordinals. This minimality together with the induction hypothesis clearly implies $\mathscr{B}(\beta')$, which is the required contradiction. □

Proposition 15 (Transfinite Induction, Second Form)**.** *Let $\mathscr{B}(x)$ be a wf with x free. Suppose*

(i) $\mathscr{B}(\emptyset)$.

(ii) $\mathscr{B}(\alpha)$ implies $\mathscr{B}(\alpha^+)$ for any ordinal α.

(iii) If α is a limit ordinal and $\mathscr{B}(\beta)$ for every ordinal $\beta \prec \alpha$, then $\mathscr{B}(\alpha)$.

Then $\mathscr{B}(\alpha)$ for every ordinal α.

Proof. Follows trivially from Proposition 14. □

Remark 3. Let $\mathscr{B}(x, y)$ be a wf with x, y free. Suppose that the wf $\forall x \exists_1 y \mathscr{B}(x, y)$ is a theorem in ZFC (here $\exists_1$ means "exists a unique"). Then as usual we can introduce a new function symbol $\varphi(x)$ and axiom $\forall x \mathscr{B}(x, \varphi(x))$. Formally this is how one introduces the notation $A \cap B$, $A \cup B$, $A \times B$ and so on for sets. Intuitively $\mathscr{B}(x, y)$ expresses that y is some "construction" starting from x, and we have introduced the notation $\varphi(x)$ to denote the end result of this construction. Throughout the remainder of this section, if we say "φ is a construction on sets" we mean that φ has the meaning just elaborated.

Theorem 16 (Construction by Transfinite Recursion)**.** *Let φ be a construction on sets. For any ordinal α there is a unique function f defined on α^+ with the property that*

$$f(\beta) = \varphi\big(\{f(\gamma) \,|\, \gamma \prec \beta\}\big)$$

for every ordinal $\beta \preceq \alpha$.

Proof. By transfinite induction it suffices to show that if the theorem is true for every ordinal $\beta \prec \alpha$, it is true for α. So let an ordinal α be given, and assume that the theorem is true for each ordinal $\beta \prec \alpha$. Let f_β be the function defined on β^+. We define a function f on α^+ by first defining it on ordinals $\beta \prec \alpha$ by

$$f(\beta) = \varphi\big(\{f_\beta(\gamma) \,|\, \gamma \prec \beta\}\big)$$

Then defining

$$f(\alpha) = \varphi\big(\{f(\beta) \,|\, \beta \prec \alpha\}\big)$$

This defines a function f on α^+, but we have yet to show it has the required property or that it is unique. We prove each statement separately:

Proof of existence. The first step is to show that for for ordinals $\gamma \prec \beta \prec \alpha$ and $\delta \prec \gamma$ we have $f_\beta(\delta) = f_\gamma(\delta)$. If an ordinal δ existed without this property, there would be a minimal such δ. Then for every $\kappa \prec \delta$ we would have $f_\beta(\kappa) = f_\gamma(\kappa)$ and consequently

$$\begin{aligned} f_\beta(\delta) &= \varphi\big(\{f_\beta(\kappa) \,|\, \kappa \prec \delta\}\big) \\ &= \varphi\big(\{f_\gamma(\kappa) \,|\, \kappa \prec \delta\}\big) \\ &= f_\gamma(\delta) \end{aligned}$$

which is a contradiction. Hence $f_\beta(\delta) = f_\gamma(\delta)$ for every $\delta \prec \gamma$. Consequently

$$\begin{aligned} f(\gamma) &= \varphi\big(\{f_\gamma(\delta) \,|\, \delta \prec \gamma\}\big) \\ &= \varphi\big(\{f_\beta(\delta) \,|\, \delta \prec \gamma\}\big) \\ &= f_\beta(\gamma) \end{aligned}$$

and $f(\beta) = \varphi(\{f_\beta(\gamma) \,|\, \gamma \prec \beta\}) = \varphi(\{f(\gamma) \,|\, \gamma \prec \beta\})$, so f is a function on α^+ with the correct property. It remains to prove uniqueness.

Proof of uniqueness. Supose f^* were another function defined on α^+ with the stated property. If there existed an ordinal $\beta \prec \alpha$ with $f^*(\beta) \neq f(\beta)$ then there would exist a minimal such β. But then

$$f^*(\beta) = \varphi\big(\{f^*(\gamma) \,|\, \gamma \prec \beta\}\big) = \varphi\big(\{f(\gamma) \,|\, \gamma \prec \beta\}\big) = f(\beta)$$

a contradiction. Hence $f^*(\beta) = f(\beta)$ for every $\beta \prec \alpha$, and it follows immediately that $f^*(\alpha) = f(\alpha)$ as well, completing the proof. □

Remark 4. Let φ be a construction on sets, $\alpha \prec \beta$ ordinals and f_α, f_β the induced functions on α^+, β^+ respectively. Then $\alpha^+ \prec \beta^+$ so f_β restricts to a function on α^+. One checks easily that in fact $f_\beta|_{\alpha^+} = f_\alpha$.

Theorem 17. *Let τ, μ be constructions on sets. Then for any ordinal α and set z there is a unique function f defined on α^+ with the property that*

(i) $f(\emptyset) = z$.

(ii) $f(\beta^+) = \tau(f(\beta))$ *for any ordinal* $\beta \prec \alpha$.

(iii) $f(\beta) = \mu\big(\{f(\gamma) \,|\, \gamma \prec \beta\}\big)$ *for any limit ordinal* $\beta \preceq \alpha$.

Proof. Just to make the exposition clearer, we say a set q is *groovy* if it is a pair (x, y) with x an ordinal. A set x is *awesome* if it is nonempty, all its elements are groovy, and the same ordinal never occurs twice as the first element of a pair in x. Given an awesome set x, let x_1 denote all the ordinals occurring in the first position of some pair in x, and similarly let x_2 be all the second coordinates in x. We define a new construction on sets φ as follows (the careful reader may write out the corresponding explicit wf $\mathscr{B}(x, y)$ of ZFC if they desire)

- If x is the empty set then $\varphi(x)$ is the set $(0, z)$.
- If x is nonempty but not awesome, then $\varphi(x)$ is the set $(0, 0)$.
- Assume that x is awesome. There are two cases
  (a) The set of ordinals x_1 contains a maximal element. That is, there is some ordinal $\alpha \in x_1$ with $\beta \preceq \alpha$ for every $\beta \in x_1$. If the pair (α, m) is the (unique) pair in x with first coordinate α then we define $\varphi(x)$ to be $(\alpha^+, \tau(m))$.
  (b) The set of ordinals x_1 contains no maximal element. Then $\bigcup x_1$ is an ordinal, and we define $\varphi(x)$ to be $(\bigcup x_1, \mu(x_2))$.

By Theorem 16 for any given ordinal α there exists a unique function g on α^+ with

$$g(\beta) = \varphi\big(\{g(\gamma) \,|\, \gamma \prec \beta\}\big)$$

for every ordinal $\beta \preceq \alpha$. By definition of φ its "output" is always a pair, so we can define a new function f on α^+ by defining $f(\beta)$ to be the second component of the pair $g(\beta)$. We claim that this is the desired function.

Clearly $f(\emptyset) = z$. Given an ordinal $\beta \prec \alpha$ we have

$$\begin{aligned} g(\beta^+) &= \varphi\big(\{g(\gamma) \,|\, \gamma \prec \beta^+\}\big) \\ &= \varphi\big(\{(\gamma, f(\gamma)) \,|\, \gamma \prec \beta^+\}\big) \end{aligned}$$

But β occurs as a maximal ordinal in these pairs, so by definition of φ we have $g(\beta^+) = (\beta^+, \tau(f(\beta)))$ and so $f(\beta^+) = \tau(f(\beta))$ as required. If $\beta \preceq \alpha$ is a limit ordinal, then the set of ordinals occurring in the first position of pairs in $\{(\gamma, f(\gamma)) \,|\, \gamma \prec \beta\}$ has no maximal element, so we have

$$g(\beta) = \varphi\big(\{(\gamma, f(\gamma)) \,|\, \gamma \prec \beta\}\big) = \Big(\bigcup \beta, \mu\big(\{f(\gamma) \,|\, \gamma \prec \beta\}\big)\Big)$$

and therefore $f(\beta) = \mu\big(\{f(\gamma) \,|\, \gamma \prec \beta\}\big)$ as required. This proves that a function f with the desired properties exists, and uniqueness follows from a simple transfinite induction. □

Theorem 18. *Let A be a set well-ordered by some relation $<$. Then there exists a unique ordinal τ such that $(A, <)$ and $(\tau, \prec)$ are isomorphic as well-ordered sets.*

Proof. Uniqueness is immediate by Lemma 13, so it suffices to show existence. If A is empty we may take the empty ordinal, so assume $A \neq \emptyset$. Let q be an arbitrary set *not* in A. Let z be a "least member" of A. That is, since A is well-ordered by $<$, we can in particular find $z \in A$ with the property that for every $v \in A$ with $v \neq z$ we have $z < v$ and not $v < z$. Define a construction φ on sets as follows:

- If $A \setminus x$ is empty then $\varphi(x)$ is the set q.
- If $A \setminus x$ is nonempty then $\varphi(x)$ is the least member of this set.

By transfinite recursion (i.e. Theorem 16) for any ordinal β there is a function f_β on β^+ with the following property for $\gamma \preceq \beta$

$$f_\beta(\gamma) = \begin{cases} \text{least member of } A \setminus \{f(\delta) \,|\, \delta \prec \gamma\} & A \setminus \{f_\beta(\delta) \,|\, \delta \prec \gamma\} \neq \emptyset \\ q & \text{otherwise} \end{cases}$$

In particular $f_\beta(0) = z$. Clearly either $f_\beta(\gamma) = q$ or $f_\beta(\gamma) \in A$, with these two possibilities being mutually exclusive. So for every ordinal β we have the set $f_\beta(\beta)$. Suppose that α, β are distinct ordinals with $f_\alpha(\alpha), f_\beta(\beta) \in A$. We can assume wlog that $\alpha \prec \beta$. Then $f_\beta(\beta)$ is an element of $A \setminus \{f_\beta(\gamma) \,|\, \gamma \prec \beta\}$ and in particular $f_\beta(\beta) \neq f_\beta(\alpha)$. But by Remark 4 we have $f_\beta(\alpha) = f_\alpha(\alpha)$, so we have shown that $f_\alpha(\alpha) \neq f_\beta(\beta)$. Since an element of A is therefore of the form $f_\alpha(\alpha)$ for *at most* one ordinal α, we can form the following set

$$\tau = \{\beta \,|\, f_\beta(\beta) \in A\}$$

To show that τ is an ordinal, it suffices by Corollary 7 to show that it is transitive. So take ordinals $\gamma \prec \beta$ with $\beta \in \tau$. Since $f_\beta(\beta) \in A$ the set $A \setminus \{f_\beta(\delta) \,|\, \delta \prec \beta\}$ must have been nonempty, so then clearly $A \setminus \{f_\gamma(\varepsilon) \,|\, \varepsilon \prec \gamma\}$ must also be nonempty. Therefore $f_\gamma(\gamma) \in A$ and $\gamma \in \tau$, and we have shown that τ is an ordinal. We claim that the set

$$\mathscr{F} = \{(\beta, f_\beta(\beta)) \,|\, \beta \in \tau\}$$

is an order isomorphism $\tau \longrightarrow A$. It is certainly a function, which is injective by our earlier comments. If this map were not surjective then the set $A \setminus \{f_\tau(\beta) \,|\, \beta \prec \tau\}$ would be nonempty, from which we deduce the contradiction $\tau \in \tau$. So our map is a bijection. To see that this is an isomorphism of well-ordered sets, it suffices to show that it preserves the ordering on τ. So let $\gamma \prec \beta \prec \tau$ be ordinals. We already know $A = \{f_\beta(\beta) \,|\, \beta \prec \tau\}$ so

$$A \setminus \{f_\gamma(\delta) \,|\, \delta \prec \gamma\} = \{f_\tau(\delta) \,|\, \gamma \preceq \delta \prec \tau\}$$

and by definition $f_\gamma(\gamma)$ is the least member of this set. From this we deduce $f_\gamma(\gamma) < f_\beta(\beta)$ which is what we wanted to show. We have produced an ordinal τ and an isomorphism of well-ordered sets $\tau \longrightarrow A$, so the proof is complete. □

3 Cardinal Numbers

Definition 6. Two sets x, y have the same *cardinality* if there is a bijection $f : x \longrightarrow y$, and we write $x \cong y$. This is an equivalence relation on sets. We write $x \leq y$ if there is an injective map $f : x \longrightarrow y$. We write $x < y$ if $x \leq y$ and not $x \cong y$.

Lemma 19. *Let S be a set and $f : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ a function with the property that $X \subseteq Y$ implies $f(X) \subseteq f(Y)$. Then f has a fixed point; that is, $f(T) = T$ for some $T \in \mathcal{P}(S)$.*

Proof. The set $A = \{X \in \mathcal{P}(S) \,|\, X \subseteq f(X)\}$ is nonempty, and if $X \in A$ then also $f(X) \in A$. We set $T = \bigcup A$ and claim that $f(T) = T$. To begin with, if $X \in A$ then $X \subseteq T$ so $f(X) \subseteq f(T)$, and hence $X \subseteq f(X) \subseteq f(T)$. It follows that $T \subseteq f(T)$, whence $T \in A$ and $f(T) \in A$, showing that $f(T) \subseteq T$, which completes the proof. □

Theorem 20 (Schröder-Bernstein)**.** *If a, b are two sets with $a \leq b$ and $b \leq a$, then $a \cong b$.*

Proof. Suppose we are given injective maps $f : a \longrightarrow b$ and $g : b \longrightarrow a$. We want to find a subset $t \subseteq a$ so that f on t pieced together with g^{-1} on $a \setminus t$ gives a bijection. To this end, for each $x \subseteq a$ we define a subset $x^* \subseteq a$ by

$$x^* = a \setminus g(b \setminus f(x))$$

Then x^* measures "how far" we are from t. Now

$$\begin{aligned} x_0 \subseteq x_1 \subseteq \Rightarrow\ & f(x_0) \subseteq f(x_1) \\ \Rightarrow\ & b \setminus f(x_0) \supseteq b \setminus f(x_1) \\ \Rightarrow\ & g(b \setminus f(x_0)) \supseteq g(b \setminus f(x_1)) \\ \Rightarrow\ & a \setminus g(b \setminus f(x_0)) \subseteq a \setminus g(b - f(x_1)) \end{aligned}$$

In other words, $x_0 \subseteq x_1$ implies $x_0^* \subseteq x_1^*$. Hence by Lemma 19 there is t such that $t = t^*$. That is,

$$a \setminus t = g(b \setminus f(t))$$

We define a function $h : a \longrightarrow b$ piecewise by

$$h(x) = \begin{cases} f(x) & x \in t \\ g^{-1}(x) & x \in a \setminus t \end{cases}$$

Now h is onto since if $y \in b \setminus f(t)$, let $x = g(y) \in a \setminus t$, then $h(x) = y$. And h is injective since if $h(x) = h(x')$ and x, x' both belong to either t or $a \setminus t$ then trivially $x = x'$. If one belongs to t and one to its complement, then we would have an element in the intersection of $f(t)$ and $b \setminus f(t)$, a contradiction. Therefore h is a bijection, as required. □

Theorem 21 (Hartog)**.** *For any set a there is an ordinal α which does* not *satisfy $\alpha \leq a$.*

Proof. That is, there is an ordinal α which does not admit an injection into a. Suppose to the contrary that every ordinal injects into a. In particular for every ordinal α there is a subset $b \subseteq a$ with $\alpha \cong b$. In particular this induces a well-ordering $<$ on b with $(b, <) \cong (\alpha, \prec)$ as well-ordered sets. For each ordinal α we define a set

$$F_\alpha = \{(b, <) \,|\, b \subseteq a \text{ and } < \text{ well-orders } b \text{ and } (b, <) \cong (\alpha, \prec) \text{ as well-ordered sets}\}$$

So our hypothesis implies that F_α is nonempty. Using Lemma 13 we deduce that if $F_\alpha = F_\beta$ then $\alpha = \beta$. Since each F_α is a subset of $\mathcal{P}(a) \times \mathcal{P}(a \times a)$ and any subset of this set is of the form F_α for *at most* one α, we can form the following set

$$C = \{F_\alpha \,|\, \alpha \text{ is an ordinal}\}$$

But using the Axiom of Replacement we can replace each F_α by α and still have a set. This shows that there is only a set of ordinals, which we already know to be false. This contradiction shows that there must exist an ordinal α not injecting into a. □

Theorem 22. *The following statements are true in ZFC*

- *Zorn's Lemma (ZL).*
- *Given any set Y there exists a well-ordering on Y (WO).*
- *Every set is in bijection with some ordinal (OT).*

Proof. (ZL) Let $(A, \leq)$ be a nonempty partially ordered set in which every chain has an upper bound and let f be a choice function for $\mathcal{P}(A)$. Let q be a set not in A, and define by transfinite recursion the following function G_α on α^+ for any ordinal α (where we agree that $f(0) = q$)

$$G_\alpha(\beta) = \begin{cases} f\big(\{y \in A \,|\, \forall \gamma \prec \beta (G_\alpha(\gamma) < y)\}\big) & G_\alpha(\gamma) \in A \text{ for every } \gamma \prec \alpha \\ q & \text{otherwise} \end{cases}$$

The idea is that at each stage we produce an element of A strictly larger than all those previously selected. By Remark 4 there is no harm in dropping the subscript from G and simply talking about G as a "function" on ordinals. We say an ordinal α is *tame* if $G(\alpha) \in A$. If α is tame, then β is tame for every $\beta \prec \alpha$. For an ordinal α all of whose elements are tame, we define

$$U_\alpha = \{y \in A \,|\, \forall \beta \prec \alpha (G(\beta) < y)\}$$

Then $G(\alpha) \in U_\alpha$, provided $U_\alpha \neq \emptyset$. If U_α is empty then $G(\alpha) = q$ so α is not tame. So we encounter untame ordinals when we "run out" of things in A. Of course $G(0) = f(A)$ so the zero ordinal is certainly tame.

We claim that not every ordinal is tame. Suppose otherwise. Then given ordinals $\beta \prec \alpha$ the ordinal α has tame elements and by assumption $U_\alpha \neq \emptyset$ so one deduces $G(\beta) < G(\alpha)$. In particular every ordinal corresponds to a distinct element of A, and by a standard argument this would imply there is only a set of ordinals, which is false. Therefore some ordinal σ is not tame, and we may assume σ is minimal. Then every element of σ is tame, so for σ to be not tame, we must have $U_\sigma = \emptyset$.

Clearly $\{G(\delta) \,|\, \delta \prec \sigma\}$ is a chain, and hence has an upper bound w in A. We claim that w is maximal in A. For if not, we have $v \in A$ with $w < v$. Then for all $\delta \prec \sigma$ we have $G(\delta) \leq w < v$ and so $v \in U_\sigma$ which is a contradiction. This completes the proof of Zorn's Lemma.

(WO) We prove that ZL $\Rightarrow$ WO. If our set Y is empty this is trivial, so assume Y is nonempty and let $\mathcal{A}$ be the set of all pairs $(A, <)$ where $A \subseteq Y$ and $<$ is a well-ordering on A (this nonempty since it contains $\emptyset$ with the trivial ordering). We define a relation $(A, <) \leq (B, <')$ to mean that $A \subseteq B$, the inclusion is a morphism of well-ordered sets, and $a <' b$ for every $a \in A, b \in B \setminus A$. This is a partial order on $\mathcal{A}$.

Given a chain $\mathcal{C} \subseteq \mathcal{A}$, let R be the union of all the sets A occurring in pairs $(A, <) \in \mathcal{C}$ and define a relation $\sqsubset$ on R as follows: given $a, b \in R$ find a pair $(A, <) \in \mathcal{C}$ with $a, b \in A$ and define $a \sqsubset b$ in R iff. $a < b$. This doesn't depend on the chosen $(A, <)$ and defines an irreflexive relation on R. To see that this is a well-ordering on R, let $Z \subseteq R$ be a nonempty subset and let $(A, <) \in \mathcal{C}$ be arbitrary with $Z \cap A \neq \emptyset$. A least element of $Z \cap A$ will also be a least element of Z, so we can assume $Z \subseteq A$. But then A is well-ordered by $<$, so Z certainly has a least element in A and therefore also in $(R, \sqsubset)$. This shows that every chain in $\mathcal{C}$ has an upper bound, so by ZL it must have a maximal element $(M, <_M)$.

We claim that $M = Y$, in which case the proof is complete. If not, choose $a \in Y \setminus M$ and extend $<_M$ to $M \cup \{a\}$ by declaring that a is larger than everything else in M. One checks that this new set is well-ordered, and is strictly larger than $(M, <_M)$ in $\mathcal{A}$, which is impossible. Therefore $M = Y$ and we are done.

(OT) Since we have shown WO, this now follows immediately from Theorem 18. □

Definition 7. An ordinal α is a *cardinal* if there is no ordinal $\beta \prec \alpha$ with $\beta \cong \alpha$. Intuitively, α is the "smallest" ordinal in its equivalence class under bijection of sets. Bijective cardinals are equal. For any ordinal α there is a *unique* cardinal $\kappa \preceq \alpha$ with $\kappa \cong \alpha$.

Lemma 23. *Let x, y be sets. Then $x^+ \cong y^+$ if and only if $x \cong y$.*

Proof. Let $f : x^+ \longrightarrow y^+$ be a bijection. If $f(x) = y$ then the restriction of f to x gives a bijection $x \longrightarrow y$. If $f(x) = g \in y$ then there is some $d \in x$ with $f(d) = y$ and we define a bijection $f' : x \longrightarrow y$ to be f on every ordinal in x except for d, which we map to g. This is clearly a bijection. □

Lemma 24. *Every finite ordinal is a cardinal.*

Proof. If there exist finite ordinals which are not cardinals, then there exists a minimal such ordinal α. Since the empty set is clearly a cardinal, α is nonempty. Therefore it is of the form β^+ for some ordinal $\beta \prec \alpha$, which by minimality must be a cardinal. Since α is not a cardinal, there exists $\gamma \prec \alpha$ with $\gamma \cong \alpha$. Since α is nonempty, so is γ, so $\gamma = \tau^+$ is a successor ordinal. From this we deduce $\tau^+ \cong \beta^+$ and therefore $\tau \cong \beta$. But $\tau \prec \gamma \preceq \beta$ which contradicts the fact that β is a cardinal. Therefore every finite ordinal is a cardinal. □

Lemma 25. *There is a bijection $\omega \cong \omega^+$. In particular the ordinal ω^+ is not a cardinal.*

Proof. Define a function $f : \omega^+ \longrightarrow \omega$ by $f(\omega) = 0$ and $f(n) = n + 1$ (here of course we write $n + 1$ for the ordinal n^+). This is easily checked to be a bijection. □

Lemma 26. *The ordinal ω is a cardinal.*

Proof. If not, there is some nonempty finite ordinal α with $\alpha \cong \omega$. We deduce $\alpha^+ \cong \omega^+ \cong \omega \cong \alpha$ which contradicts the fact that the finite ordinal n^+ is a cardinal. □

Lemma 27. *Given two cardinals κ, λ we have $\kappa \preceq \lambda$ if and only if $\kappa \leq \lambda$. Similarly $\kappa \prec \lambda$ if and only if $\kappa < \lambda$.*

Proof. Certainly $\kappa \preceq \lambda$ implies $\kappa \leq \lambda$. The converse follows from the definition of a cardinal and Theorem 20. □

Lemma 28. *If X is a set of cardinals, the union set $\bigcup X$ is also a cardinal.*

Proof. We know from Proposition 6 that $\bigcup X$ is an ordinal, which we may as well assume is nonempty. Suppose for a contradiction that there is an ordinal $\beta \prec \bigcup X$ with $\beta \cong \bigcup X$. By definition we have $\beta \prec \alpha$ for some cardinal $\alpha \in X$. This implies $\beta \leq \alpha$, and from $\alpha \subseteq \bigcup X$ we deduce $\alpha \leq \bigcup X$ and therefore $\alpha \leq \beta$. By Theorem 20 we have $\alpha \cong \beta$ which is impossible because α is an ordinal. This contradiction shows that $\bigcup X$ must be a cardinal. □

Definition 8. Let x be a set. By Theorem 22 the set x is in bijection with some ordinal, and therefore also with a cardinal κ. The unique cardinal κ with $x \cong \kappa$ is called the *cardinality* of x, and is denoted $|x|$ or $card(x)$. Obviously any cardinal is its own cardinality.

Definition 9. A set x is *finite* if $|x| \in \omega$, otherwise it is *infinite*. We say x is *countable* if $|x| \preceq \omega$. An ordinal α is a finite ordinal if and only if it is a finite set. An infinite ordinal contains the successor of each of its finite elements.

Lemma 29. *Let x, y be sets. Then $x \leq y$ if and only if $|x| \leq |y|$ and $x < y$ if and only if $|x| < |y|$. For any two sets x, y we have $x \cong y$, $x < y$ or $y < x$.*

Lemma 30. *For any infinite ordinal λ there is a bijection $\lambda \cong \lambda^+$.*

Proof. We define a map $f : \lambda^+ \longrightarrow \lambda$ by $f(\lambda) = 0$, $f(n) = n + 1$ for finite ordinals $n \prec \lambda$ and $f(\alpha) = \alpha$ for any infinite ordinal $\alpha \prec \lambda$. It is straightforward to check this is a bijection. □

Lemma 31. *Any infinite cardinal is a limit ordinal.*

Proof. Let $\kappa \geq \omega$ be an infinite cardinal. Then κ is nonempty, so it suffices to show it is not a successor. Suppose to the contrary that $\kappa = \lambda^+$ for some ordinal $\lambda \prec \kappa$. If $\kappa = \omega$ then this would imply κ is a finite ordinal, which it is not. So $\kappa > \omega$ and therefore $\omega \preceq \lambda \prec \kappa$ implies λ is an infinite ordinal, so $\lambda \cong \lambda^+ = \kappa$, but this is impossible because κ is an ordinal. □

Let β be an ordinal. By Theorem 21 there exists at least one ordinal α which does not satisfy $\alpha \leq \beta$ (that is, it does not admit an injection into β). Let $H(\beta)$ denote the unique ordinal which is minimal with this property (that is, it does not admit an injection into β, but all its members do). Clearly $\beta \prec H(\beta)$ by trichotomy. We make this into a construction on sets by declaring that $H(x)$ is the emptyset if x is not an ordinal. Then by Theorem 17 we can define sets $\aleph_\alpha$ for every ordinal α with the following properties

$$\begin{aligned}\aleph_0 &= \omega\\ \aleph_{\alpha^+} &= H(\aleph_\alpha)\\ \aleph_\lambda &= \bigcup\{\aleph_\delta \,|\, \delta \prec \lambda\}\end{aligned}$$

where λ is a limit ordinal. To be precise, for every ordinal α we obtain a uniquely defined function f_α on α^+ (using $z = \omega, \tau = H, \mu = \bigcup$) and we define $\aleph_\alpha = f_\alpha(\alpha)$.

Proposition 32. *For any ordinal α the set $\aleph_\alpha$ is a cardinal.*

Proof. Firstly, we observe that for any set x the set $H(x)$ is a cardinal. Secondly, the union set of any set of cardinals is a cardinal by Theorem 28, so in our definition by transfinite recursion of the alephs, everything in sight is a cardinal. Therefore the end product $\aleph_\alpha$ must also be a cardinal (more precisely, if $\aleph_\alpha$ were not a cardinal we could assume α was minimal with this property, and derive a contradiction). □

Corollary 33. *For any ordinals $\beta \prec \alpha$ we have $\aleph_\beta < \aleph_\alpha$. In particular every aleph is an infinite cardinal.*

Proof. Note that since alephs are cardinals, the statements $\aleph_\beta \prec \aleph_\alpha$ and $\aleph_\beta < \aleph_\alpha$ are equivalent. We prove the result by the second form of transfinite induction on the ordinal α (with β ranging over all ordinals at each stage). The result holds vacuously for $\alpha = 0$. Now we suppose the result is true for α, and prove it for α^+. So let an ordinal $\gamma \prec \alpha^+$ be given. If $\gamma \prec \alpha$ then by hypothesis $\aleph_\gamma < \aleph_\alpha < \aleph_{\alpha^+}$, and if $\gamma = \alpha$ then $\aleph_\gamma = \aleph_\alpha < \aleph_{\alpha^+}$. So the result is true for α^+.

Now suppose that α is a limit ordinal. In this case we don't even need to use our inductive hypothesis. Since α is a limit ordinal, for any $\gamma \prec \alpha$ we have $\gamma^+ \prec \alpha$ and so both $\aleph_\gamma$ and $\aleph_{\gamma^+}$ occur in the set $Q = \{\aleph_\gamma \,|\, \gamma \prec \alpha\}$. In particular $\aleph_\gamma$ is an element of the following union

$$\aleph_\alpha = \bigcup\{\aleph_\gamma \,|\, \gamma \prec \alpha\}$$

so $\aleph_\gamma \prec \aleph_\alpha$ and therefore $\aleph_\gamma < \aleph_\alpha$, as required. □

4 Cardinal Operations

Definition 10. Let $\{x_i\}_{i\in I}$ be a nonempty family of sets. We define the *disjoint union* of this indexed family to be the set $\coprod_{i\in I} x_i = \{(i,y) \,|\, i \in I \text{ and } y \in x_i\}$. In particular we can define the disjoint union $x \amalg y$ of any two sets x, y.

Definition 11. Let κ, λ be cardinals. We define their cardinal sum and product as follows

$$\begin{aligned}\kappa + \lambda &= |\kappa \amalg \lambda|\\ \kappa\lambda &= |\kappa \times \lambda|\\ 2^\kappa &= |\mathcal{P}(\kappa)|\end{aligned}$$

In fact one can check that if x, y are sets with $|x| = \kappa, |y| = \lambda$ then $\kappa\lambda = |x \times y|$ and if x, y are in addition disjoint then $\kappa + \lambda = |x \cup y|$. More generally given a nonempty family $\{\kappa_i\}_{i\in I}$ of cardinals we define

$$\begin{aligned}\sum_{i\in I} \kappa_i &= card\Big(\coprod_{i\in I} \kappa_i\Big)\\ \prod_{i\in I} \kappa_i &= card\big(\times_{i\in I}\, \kappa_i\big)\end{aligned}$$

Theorem 34. *For any ordinal α we have $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$. For any ordinals $\alpha \prec \beta$ we have $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$.*

Proposition 35. *For any cardinal κ we have $\kappa < 2^\kappa$.*

5 Regular Cardinals

Definition 12. Let α be an ordinal. A subset $A \subseteq \alpha$ is *cofinal* in α if for every ordinal $\beta \prec \alpha$ there exists $\gamma \in A$ with $\beta \preceq \gamma$.

Definition 13. For any ordinal α there is a *unique* ordinal β which is minimal with the property that it admits a function $\beta \longrightarrow \alpha$ whose image is a cofinal subset of α. This ordinal is denoted $cf(\alpha)$ and is called the *cofinality* of α. By definition we have $cf(\alpha) \preceq \alpha$.

Definition 14. We say that a cardinal κ is *regular* if $cf(\kappa) = \kappa$, and *singular* otherwise. The confinality of a cardinal measures how "accessible" it is from below.

Example 1. If α is a successor ordinal then $cf(\alpha) = 1$ so the ordinal 1 is the only regular successor ordinal. In particular the only finite ordinals which are regular are $0, 1$. The ordinal ω is regular, but $\aleph_\omega$ is not since $cf(\aleph_\omega) = \omega$ (and since $0 \prec \omega$ we have $\omega = \aleph_0 \prec \aleph_\omega$).

Theorem 36 (König)**.** *Suppose we are given a nonempty set I and for each $i \in I$ cardinals $1 \leq \mu_i < \nu_i$. Then*

$$\sum_{i \in I} \mu_i < \prod_{i \in I} \nu_i$$

Proposition 37. *Any cardinal κ can be written as a sum of $cf(\kappa)$ cardinals all $< \kappa$. That is, we can find cardinals $\lambda_\gamma < \kappa$ for $\gamma \prec cf(\kappa)$ such that*

$$\kappa = \sum_{\gamma \prec \kappa} \lambda_\gamma$$

Proposition 38. *For any ordinal α the cardinal $\aleph_{\alpha^+}$ is regular.*

Lemma 39. *Any infinite cardinal is of the form $\aleph_\alpha$ for a unique ordinal α.*

Definition 15. Let α be a cardinal. The *cardinal successor* of α is the unique cardinal $\beta > \alpha$ with the property that there is no cardinal γ with $\alpha < \gamma < \beta$. If α is a finite cardinal then the cardinal successor is the usual ordinal successor α^+. Otherwise $\alpha = \aleph_\delta$ for some ordinal δ, and the cardinal successor to α is $\aleph_{\delta^+}$.

Definition 16. A cardinal α is called a *successor cardinal* if it is the cardinal successor of some cardinal β. A nonzero cardinal which is not a successor cardinal is called a *limit cardinal.* Every nonzero finite cardinal is a successor cardinal. Given an infinite cardinal α, we can write $\alpha = \aleph_\delta$ for a unique ordinal δ. Clearly α is a successor cardinal if and only if δ is a successor ordinal (resp. α is a limit cardinal iff. δ is a limit ordinal or zero).

Lemma 40. *Let κ be a nonzero cardinal. Then κ is a limit cardinal if and only if for every cardinal $\alpha < \kappa$ there exists a cardinal β with $\alpha < \beta < \kappa$.*

Lemma 41. *Any regular ordinal is a cardinal.*

Proof. See p.134 of Azriel Levy's "Basic Set Theory". □

Proposition 42. *Let κ be an infinite cardinal. Then the following conditions are equivalent*

(a) κ is regular.

(b) κ cannot be written as the union of $< \kappa$ cardinals all of which are $< \kappa$.

(c) κ cannot be written as the sum of $< \kappa$ cardinals all of which are $< \kappa$.

(d) Any sum of $< \kappa$ cardinals all of which are $< \kappa$ is also $< \kappa$.

Proof. See p.135 of Azriel Levy's "Basic Set Theory". $\square$

We already know that the cardinal $\aleph_\alpha$ is regular whenever α is a successor ordinal. Are there infinite regular cardinals which are *not* successor cardinals? We give such cardinals a special name.

Definition 17. Let $\kappa > \omega$ be an uncountable cardinal. We say that κ is *weakly inaccessible* if it is a regular limit cardinal. An uncountable cardinal κ is called *strongly inaccessible* if it is regular and satisfies the following condition

- For every cardinal $\gamma < \kappa$, we have $2^\gamma < \kappa$.

Clearly a strongly inaccessible cardinal is also weakly inaccessible.

Remark 5. Assuming that ZFC is consistent, the existence of (strongly or weakly) inaccessible cardinals cannot be proved in ZFC. In fact the existence of strongly inaccessible cardinals is often adopted as an additional axiom in foundations for category theory (this being equivalent to the existence of universes). See SGA4 for more details in this direction.

In the remainder of this section we discuss the interaction of grothendieck universes and large cardinals. The reader should consult (FCT,Section 4) for the definition of a grothendieck universe. In particular our convention is that all universes are infinite (that is, contain $\mathbb{N}$).

Definition 18. Let $\mathfrak{U}$ be a universe and Q be the following set of cardinals

$$Q = \{|x| \,|\, x \in \mathfrak{U}\}$$

By (BST,Lemma 28) the union set $c(\mathfrak{U}) = \bigcup Q$ is also a cardinal. Since any element of $\mathfrak{U}$ is also a subset, we have $c(\mathfrak{U}) \leq |\mathfrak{U}|$ (in general this is not an equality).

Lemma 43. *Let $\mathfrak{U}$ be a universe and κ a cardinal. Then $\kappa < c(\mathfrak{U})$ if and only if there is $x \in \mathfrak{U}$ with $\kappa = |x|$. In particular if $x \in \mathfrak{U}$ then $|x| < c(\mathfrak{U})$.*

Proof. Suppose that $\kappa < c(\mathfrak{U})$. Then by definition there exists $y \in \mathfrak{U}$ with $\kappa < |y| \leq c(\mathfrak{U})$. Therefore κ is in bijection with some subset $x \subseteq y$, which must also belong to $\mathfrak{U}$. For the converse, we need only observe that if $x \in \mathfrak{U}$ then $\mathcal{P}(x) \in \mathfrak{U}$, so $|x| < 2^{|x|} = |\mathcal{P}(x)| \leq c(\mathfrak{U})$. $\square$

Proposition 44. *Let $\mathfrak{U}$ be a universe. Then $c(\mathfrak{U})$ is a strongly inaccessible cardinal.*

Proof. By convention $\omega = \mathbb{N} \in \mathfrak{U}$, so $\omega < c(\mathfrak{U})$ by Lemma 43. First we have to show that $c(\mathfrak{U})$ is regular. By Proposition 42 it suffices to show that if $I \in \mathfrak{U}$ is nonempty and $\{\alpha_i\}_{i \in I}$ a family of cardinals all $< c(\mathfrak{U})$ then $\sum_i \alpha_i < c(\mathfrak{U})$. But each α_i is in bijection with some $x_i \in \mathfrak{U}$, and therefore $\sum_i \alpha_i$ is in bijection with the disjoint union $\coprod_{i \in I} x_i$ which belongs to $\mathfrak{U}$. So by Lemma 43 we have $\sum_i \alpha_i < c(\mathfrak{U})$, as required. Now suppose we are given a cardinal $\gamma < c(\mathfrak{U})$. Find a set $x \in \mathfrak{U}$ with $\gamma = |x|$. Then $\mathcal{P}(x) \in \mathfrak{U}$ so by Lemma 43 we have $|\mathcal{P}(x)| < c(\mathfrak{U})$, and therefore $2^\gamma = 2^{|x|} = |\mathcal{P}(x)| < c(\mathfrak{U})$. This completes the proof that $c(\mathfrak{U})$ is a strongly inaccessible cardinal. $\square$