Algebra in a Category

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In the topos **Sets** we build algebraic structures out of sets and operations (morphisms between sets) where the operations are required to satisfy various axioms. Once we have translated these axioms into a diagrammatic form, we can copy the definition into any category with finite products. The advantage of such constructions is that many of the complicated concepts turn out to be simple objects (such as rings or groups) built up inside a category other than **Sets**.

We begin this note by explaining how to build abelian groups, rings and modules in any category with finite products. We then show that the categories of these objects have various nice properties. Throughout we will be dealing with products A^n and since in a general category there may be no *canonical* product, there is some potential ambiguity. Our standing assumptions about the underlying set theory include a strong form of the axiom of choice, which allows us to select a product for every set of objects (including a terminal object). Throughout all categories are nonempty.

Contents

1 Definitions

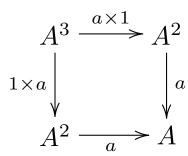
1.1 Abelian Groups

An abelian group is a set A with a binary operation (addition), a unary operation (additive inverse), and a nullary operation (zero). These are morphisms

$$\begin{aligned} a : A \times A \longrightarrow A, &\quad (x, y) \mapsto x + y \\ v : A \longrightarrow A, &\quad x \mapsto -x \\ u : 1 \longrightarrow A, &\quad * \mapsto 0 \end{aligned} \tag{1}$$

satisfying certain identities, which can be restated as the commutativity of the following diagrams:

Associativity of Addition



Additive Identity

$$\begin{array}{ccccc} A\times 1 & \Longleftarrow & A & \Longrightarrow & 1\times A \\ {\scriptstyle 1\times u}\downarrow & & \Downarrow & & \downarrow{\scriptstyle u\times 1} \\ A^2 & \xrightarrow[a]{} & A & \xleftarrow[a]{} & A^2 \end{array}$$

Additive Inverses

$$\begin{array}{ccccc} A & \xrightarrow{\left(\begin{smallmatrix}1\\ v\end{smallmatrix}\right)} & A^2 & \xleftarrow{\left(\begin{smallmatrix}v\\ 1\end{smallmatrix}\right)} & A \\ \downarrow & & {\scriptstyle a}\downarrow & & \downarrow \\ 1 & \xrightarrow[u]{} & A & \xleftarrow[u]{} & 1 \end{array}$$

Commutativity

$$\begin{array}{ccc} A^2 & \Longrightarrow & A^2 \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle a} \\ A^2 & \xrightarrow[a]{} & A \end{array}$$

where τ is defined by taking the projections $p_1, p_2 : A \times A \longrightarrow A$ and interchanging them - so $\tau = \left(\begin{smallmatrix}p_2\\ p_1\end{smallmatrix}\right)$.

Definition 1. Let $\mathcal{C}$ be a category with finite products. We define an *abelian group object* (or just *abelian group*) in $\mathcal{C}$ to be an object A together with three arrows $a : A \times A \longrightarrow A, v : A \longrightarrow A, u : 1 \longrightarrow A$ for which the four diagrams above commute. A *morphism* of abelian groups $(A, a, v, u) \longrightarrow (B, a', v', u')$ in $\mathcal{C}$ is a morphism $f : A \longrightarrow B$ which commutes with the structure maps. That is, the following three diagrams commute

$$\begin{array}{ccc} A\times A & \xrightarrow{a} & A \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ A'\times A' & \xrightarrow[a']{} & A' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{v} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[v']{} & A' \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{u} & A \\ \Downarrow & & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow[u']{} & A' \end{array}$$

This defines the category $\mathbf{Ab}(\mathcal{C})$ of *abelian groups in* $\mathcal{C}$. Note that $\mathbf{Ab}(\mathbf{Sets})$ is isomorphic to the normal category of abelian groups, and we will denote both by $\mathbf{Ab}$.

Remark 1. The definition of an abelian group in $\mathcal{C}$ includes a morphism $A \times A \longrightarrow A$. Here the domain is the object selected as part of the assignment of a specific product $C \times C$ to every object C of $\mathcal{C}$, referred to in the introduction. If we were to make a *different* selection of products, the resulting category of abelian groups in $\mathcal{C}$ would be isomorphic.

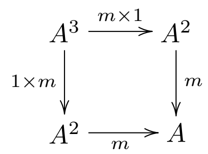
The reader who is uncomfortable with strong choice may prefer to leave the choice of products arbitrary, so that an abelian group in $\mathcal{C}$ is an object A together with a morphism $a : A \times A \longrightarrow A$ from *some* product of A with itself to A. This leaves open the possibility of the *same* abelian group occurring more than once in $\mathbf{Ab}(\mathcal{C})$, in the sense of the identity $1_A : A \longrightarrow A$ being an isomorphism of the two different objects, without the structure morphisms being the same! This happens because the morphism $a : A \times A \longrightarrow A$ is only "determined" up to a canonical isomorphism, in the sense that we could choose another product $A \times' A$ and a would uniquely determine $a' : A \times' A \longrightarrow A$.

In this approach the category $\mathbf{Ab}(\mathbf{Sets})$ would only be *equivalent* to $\mathbf{Ab}$, not isomorphic. One way around this is to define an equivalence relation on group objects that says $(A, a, v, u) \sim (A, a', v', u')$ iff. 1_A is an isomorphism of group objects. A far easier way is to accept a strong axiom of choice, which is in any case essential to many other aspects of category theory.

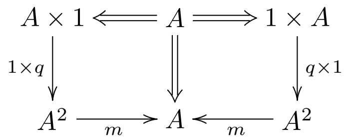
1.2 Rings

Throughout the rest of this section, $\mathcal{C}$ will be a category with finite products. A *ring object* (or just a *ring*) in $\mathcal{C}$ is an abelian group (A, a, v, u) together with another two morphisms $m : A \times A \longrightarrow A$ (multiplication) and $q : 1 \longrightarrow A$ (multiplicative identity) which make the following four diagrams commute:

Associativity of Multiplication



Multiplicative Identity

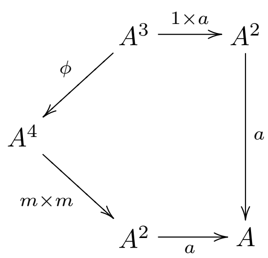


To describe diagrams for left and right distributivity, let $p_1, p_2, p_3 : A^3 \longrightarrow A$ be the projections from the product and consider the maps

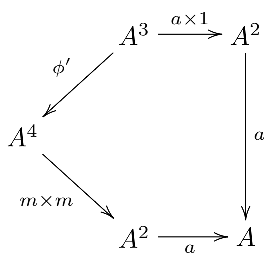
$$\phi = \begin{pmatrix} p_1 \\ p_2 \\ p_1 \\ p_3 \end{pmatrix} : A^3 \longrightarrow A^4, \qquad \phi' = \begin{pmatrix} p_2 \\ p_1 \\ p_3 \\ p_1 \end{pmatrix} : A^3 \longrightarrow A^4$$

Then we require that the following two diagrams commute:

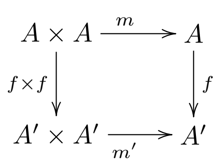
Left Distributivity

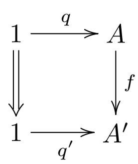


Right Distributivity



A morphism of rings $(A, a, v, u, m, q) \longrightarrow (B, a', v', u', m', q')$ is a morphism $f : A \longrightarrow B$ of abelian groups which in addition makes the following two diagrams commute:





This defines the category $\mathbf{nRng}(\mathcal{C})$ of (not necessarily commutative) rings in $\mathcal{C}$. Once again, the category $\mathbf{nRng}(\mathbf{Sets})$ is isomorphic to the usual category of not necessarily commutative rings, and we denote both by $\mathbf{nRng}$. We say a ring $R \in \mathbf{nRng}(\mathcal{C})$ is *commutative* if the following diagram commutes

Commutativity

$$\begin{array}{ccc} A^2 & \Longrightarrow & A^2 \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle m} \\ A^2 & \xrightarrow[m]{} & A \end{array}$$

Here τ is defined by taking the projections $p_1, p_2 : A \times A \longrightarrow A$ and interchanging them - so $\tau = \binom{p_2}{p_1}$.

Let $\mathbf{Rng}(\mathcal{C})$ denote the full subcategory of $\mathbf{nRng}(\mathcal{C})$ consisting of the commutative ring objects. The category $\mathbf{Rng}(\mathbf{Sets})$ is isomorphic to the usual category of commutative rings, and we denote both by $\mathbf{Rng}$.

1.3 Right Modules

A *right module object* (or just *right module*) over a ring (R, a, v, u, m, q) in $\mathcal{C}$ is a group (A, a', v', u') together with a morphism $c : A \times R \longrightarrow A$ making the following diagrams commute. Diagrams are labelled with the usual law they generalise:

$(a + b) \cdot r = a \cdot r + b \cdot r$

$$\begin{array}{ccc} A^2 \times R & \longrightarrow & A \times R \times A \times R \\ & & \downarrow{\scriptstyle c\times c} \\ {\scriptstyle a'\times 1}\downarrow & & A^2 \\ & & \downarrow{\scriptstyle a'} \\ A \times R & \xrightarrow[c]{} & A \end{array}$$

$a \cdot 1 = a$

$$\begin{array}{ccc} A \times 1 & & \\ {\scriptstyle 1\times q}\downarrow & \searrow & \\ A \times R & \xrightarrow[c]{} & A \end{array}$$

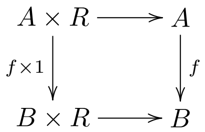
$a \cdot (r + s) = a \cdot r + a \cdot s$

$$\begin{array}{ccc} A \times R^2 & \longrightarrow & A \times R \times A \times R \\ & & \downarrow{\scriptstyle c\times c} \\ {\scriptstyle 1\times a}\downarrow & & A^2 \\ & & \downarrow{\scriptstyle a'} \\ A \times R & \xrightarrow[c]{} & A \end{array}$$

$(a \cdot r) \cdot s = a \cdot (rs)$

$$\begin{array}{ccc} A \times R^2 & \xrightarrow{1\times m} & A \times R \\ {\scriptstyle c\times 1}\downarrow & & \downarrow{\scriptstyle c} \\ A \times R & \xrightarrow[c]{} & A \end{array}$$

A morphism of modules $(A, a', v', u', c) \longrightarrow (B, a'', v'', u'', c')$ is a morphism of abelian groups which in addition makes the following diagram commute:

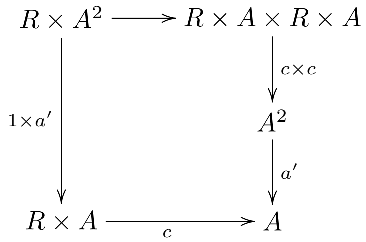


This defines the category of right modules over the ring object $R \in \mathbf{nRng}(\mathcal{C})$, which is denoted $\mathbf{Mod}(\mathcal{C}; R)$ or more often just $\mathbf{Mod}R$. To avoid an explosion in notation we will refer to group, ring and modules by the object, and only refer to the structure morphisms when necessary.

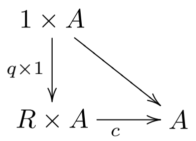
1.4 Left Modules

A *left module object* (or just *left module*) over a ring (R, a, v, u, m, q) in $\mathcal{C}$ is a group (A, a', v', u') together with a morphism $c : R \times A \longrightarrow A$ making the following diagrams commute. Diagrams are labelled with the usual law they generalise:

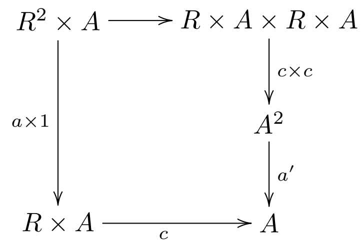
$r \cdot (a + b) = r \cdot a + r \cdot b$



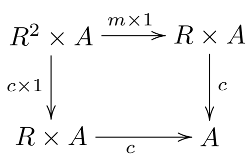
$1 \cdot a = a$



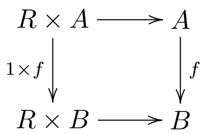
$(r + s) \cdot a = r \cdot a + s \cdot a$



$s \cdot (r \cdot a) = (sr) \cdot a$



A morphism of modules $(A, a', v', u', c) \longrightarrow (B, a'', v'', u'', c')$ is a morphism of abelian groups which in addition makes the following diagram commute:



This defines the category of left modules over the ring object $R \in \mathbf{nRng}(\mathcal{C})$, which is denoted $(\mathcal{C}; R)\mathbf{Mod}$ or more often just $R\mathbf{Mod}$.

2 Sheaves of Groups, Rings and Modules

In this section the reader will need to know something about topoi, in particular about grothendieck topologies. However, the reader only interested in sheaves on a topological space can keep in mind the central example of a small site which we will describe in a moment.

For a nonempty small category $\mathcal{C}$ we denote by $P(\mathcal{C})$ the topos of contravariant functors $\mathcal{C} \longrightarrow \mathbf{Sets}$. We call the objects of $\mathbf{Ab}(P(\mathcal{C}))$ *presheaves of abelian groups*, the objects of $\mathbf{nRng}(P(\mathcal{C}))$ *presheaves of rings*, and the objects of $\mathbf{Rng}(P(\mathcal{C}))$ *presheaves of commutative rings.*

Lemma 1. *For any small category $\mathcal{C}$ there is a canonical isomorphism of categories*

$$E : \mathbf{Ab}(P(\mathcal{C})) \longrightarrow \mathbf{Ab}^{\mathcal{C}^{op}}$$
$$E(F, a, v, u)(C) = (F(C), a_C, v_C, u_C)$$

Proof. Let (F, a, v, u) be an abelian group object in $P(\mathcal{C})$. Then for $C \in \mathcal{C}$, $(F(C), a_C, v_C, u_C)$ is an abelian group in $\mathbf{Sets}$, since diagrams in functor categories commute iff. they commute pointwise. Similarly, if $\phi : C \longrightarrow C'$ then $F(\phi)$ is a morphism of the appropriate groups, since a, v, u are natural. Also, if $f : F \longrightarrow G$ is a morphism of group objects, then it is also a natural transformation of the group valued functors.

In the opposite direction, if $F' : \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Ab}$ then the structure morphisms on each $F'(C)$ (which are functions between sets) piece together to form natural transformations a, v, u (of set-valued functors), and the tuple (F', a, v, u) is in $\mathbf{Ab}(P(\mathcal{C}))$. A morphism $F' \longrightarrow G'$ of functors is naturally a morphism of the group objects produced from F', G'. Once we check that these two assignments are in fact inverse, we see that we have the desired isomorphism. □

Lemma 2. *For any small category $\mathcal{C}$ there are canonical isomorphisms of categories*

$$E : \mathbf{nRng}(P(\mathcal{C})) \longrightarrow \mathbf{nRng}^{\mathcal{C}^{op}}$$
$$E : \mathbf{Rng}(P(\mathcal{C})) \longrightarrow \mathbf{Rng}^{\mathcal{C}^{op}}$$
$$E(F, a, v, u, m, q)(C) = (F(C), a_C, v_C, u_C, m_C, q_C)$$

Using the previous two Lemmas, we henceforth identify the categories $\mathbf{Ab}(P(\mathcal{C}))$ and $\mathbf{Ab}^{\mathcal{C}^{\text{op}}}$, with a similar convention holding for rings and commutative rings.

Corollary 3. *Let $\mathcal{C}$ be a small category. Then $\mathbf{Ab}(P(\mathcal{C}))$ is a complete grothendieck abelian category, $\mathbf{Rng}(P(\mathcal{C}))$ is complete and cocomplete and $\mathbf{nRng}(P(\mathcal{C}))$ is complete.*

Proof. These statements follow immediately from the fact that $\mathbf{Ab}$ is complete grothendieck abelian, $\mathbf{Rng}$ is complete and cocomplete and $\mathbf{nRng}$ is complete. □

Definition 2. A *small site* is a pair $(\mathcal{C}, J)$ consisting of a small category $\mathcal{C}$ and a grothendieck topology J on $\mathcal{C}$. Let X be a topological space and $\mathcal{C}$ the category of open sets of X. Then the open cover topology J is a grothendieck topology, and the pair $(\mathcal{C}, J)$ is an example of a small site. A presheaf of sets on $\mathcal{C}$ is a sheaf in the usual sense if and only if it is a J-sheaf. The reader not familiar with topoi can substitute this example every time we mention "small sites".

Definition 3. Let $(\mathcal{C}, J)$ be a small site. A *sheaf of abelian groups* on $\mathcal{C}$ is an object $A \in \mathbf{Ab}(P(\mathcal{C}))$ which is a J-sheaf when considered as a presheaf of sets. Similarly we define *sheaves of rings* and *sheaves of commutative rings* on $\mathcal{C}$. Since limits in $Sh_J(\mathcal{C})$ are computed pointwise, the objects of $\mathbf{Ab}(Sh_J(\mathcal{C}))$ are precisely the sheaves of abelian groups, and similarly for rings and commutative rings. Therefore $\mathbf{Ab}(Sh_J(C))$ is a full subcategory of $\mathbf{Ab}(P(\mathcal{C}))$, and similarly for $\mathbf{nRng}$ and $\mathbf{Rng}$.

Let $(\mathcal{C}, J)$ be a small site and P a presheaf of sets on $\mathcal{C}$. Let P^+ be the presheaf of sets obtained by the plus-construction and $\eta : P \longrightarrow P^+$ the canonical morphism of presheaves of sets. If P is a presheaf of abelian groups, then P^+ is a presheaf of abelian groups with addition

$$\{x_f \,|\, f \in S\} + \{y_g \,|\, g \in T\} = \{x_h + y_h \,|\, h \in S \cap T\}$$

and η is a morphism of presheaves of abelian groups. Similarly if P is a presheaf of rings, so is P^+, with the multiplication

$$\{x_f \,|\, f \in S\} \cdot \{y_g \,|\, g \in T\} = \{x_h y_h \,|\, h \in S \cap T\}$$

and η is a morphism of presheaves of rings. If P is a presheaf of commutative rings, so is P^+. If $\phi : P \longrightarrow Q$ is a morphism of presheaves of abelian groups, rings or commutative rings, so is $\phi^+ : P^+ \longrightarrow Q^+$. Therefore applying this construction twice gives functors

$$\begin{aligned} \mathbf{a} : \mathbf{Ab}(P(\mathcal{C})) &\longrightarrow \mathbf{Ab}(Sh_J(\mathcal{C})) \\ \mathbf{a} : \mathbf{nRng}(P(\mathcal{C})) &\longrightarrow \mathbf{nRng}(Sh_J(\mathcal{C})) \\ \mathbf{a} : \mathbf{Rng}(P(\mathcal{C})) &\longrightarrow \mathbf{Rng}(Sh_J(\mathcal{C})) \end{aligned}$$

Fix one of the three types of structure and let $\mathbf{i}$ denote the inclusion. Then $P \longrightarrow P^+ \longrightarrow (P^+)^+$ gives a natural transformation $\eta : 1 \longrightarrow \mathbf{ia}$ which is easily seen to be the unit of an adjunction $\mathbf{a} \longrightarrow\!\!\!\!\!\dashv \mathbf{i}$. Since products and pullbacks in $Sh_J(\mathcal{C})$ are computed pointwise, and pointwise acquire group and ring structures, the functors $\mathbf{a}$ all preserve finite limits. Therefore $\mathbf{Ab}(Sh_J(\mathcal{C}))$ is a Giraud subcategory of $\mathbf{Ab}(P(\mathcal{C}))$ and similarly for rings and commutative rings.

Remark 2. The sheafification functor just defined is the one typically introduced in the theory of topoi. In the special case where our small site is the open cover topology on a topological space, the sheafification functor is usually described differently. Both constructions must yield naturally equivalent functors, so this difference is nothing to worry about.

Corollary 4. *Let $(\mathcal{C}, J)$ be a small site. Then $\mathbf{Ab}(Sh_J(\mathcal{C}))$ is a complete grothendieck abelian category, $\mathbf{Rng}(Sh_J(\mathcal{C}))$ is complete and cocomplete and $\mathbf{nRng}(Sh_J(\mathcal{C}))$ is complete. In particular this is true for sheaves of abelian groups, rings and commutative rings on a topological space X.*

Definition 4. If $(\mathcal{C}, J)$ is a small site and P a sheaf of abelian groups on $\mathcal{C}$, a *subsheaf of abelian groups* is a monomorphism $\eta : Q \longrightarrow P$ of sheaves of abelian groups with the property that $\eta_C : Q(C) \longrightarrow P(C)$ is the inclusion of a *subset* for every $C \in \mathcal{C}$. Every subobject of P is equivalent to a subsheaf. If Q, Q' are subsheaves of abelian groups of P then $Q \leq Q'$ if and only if $Q(p) \subseteq Q'(p)$ for every $p \in \mathcal{C}$.

Corollary 5. *Let $(\mathcal{C}, J)$ be a small site. The structures on the abelian category $\mathbf{Ab}(Sh_J(\mathcal{C}))$ are described as follows*

Zero *The zero object is the presheaf $Z(p) = 0$.*

Kernel *If $\phi : M \longrightarrow N$ is a morphism of sheaves of abelian groups, then $K(p) = Ker(\phi_p)$ defines a sheaf of abelian groups, and the inclusion $K \longrightarrow M$ is the kernel of ϕ.*

Cokernel *If $\phi : M \longrightarrow N$ is a morphism of sheaves of abelian groups, then $C(p) = N(p)/Im(\phi_p)$ defines a presheaf of abelian groups, and the canonocal morphism $N \longrightarrow C \longrightarrow \mathbf{a}C$ is the cokernel of ϕ.*

Limits *If D is a diagram of presheaves of abelian groups, then define $L(p)$ to be the limit of the diagram $D(p)$ of abelian groups. This becomes a sheaf of abelian groups and the projections $L \longrightarrow D_i$ are a limit for the diagram.*

Colimits *If D is a diagram of presheaves of abelian groups, then define $C(p)$ to be the colimit of the diagram $D(p)$ of abelian groups. This becomes a presheaf of abelian groups and the morphisms $D_i \longrightarrow C \longrightarrow \mathbf{a}C$ are a colimit for the diagram.*

Image *Let $\phi : M \longrightarrow N$ be a morphism of sheaves of abelian groups and I the subsheaf of abelian groups of N defined by $n \in I(p)$ if and only if there exists $T \in J(p)$ such that $n|_h \in Im(\phi_q)$ for every $h : q \longrightarrow p$ in T. Then $I \longrightarrow N$ is the image of ϕ.*

Inverse Image *Let $\phi : M \longrightarrow N$ be a morphism of sheaves of abelian groups and $T \longrightarrow N$ a subsheaf of abelian groups of N. Then the inverse image $\phi^{-1}T$ is the subsheaf of abelian groups of M defined by $(\phi^{-1}T)(p) = \phi_p^{-1}(T(p))$.*

Proof. Since $\mathbf{Ab}(Sh_J(\mathcal{C}))$ is a Giraud subcategory of $\mathbf{Ab}(P(\mathcal{C}))$ the claims are all easily checked (AC,Section 3). □

Definition 5. Let $\mathcal{C}$ be a small category and R a presheaf of rings on $\mathcal{C}$. A *presheaf of right modules* over R is an object of $\mathbf{Mod}(P(\mathcal{C}); R)$. This consists of the following data: a presheaf of abelian groups $M : \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Ab}$ together with a right $R(C)$-module structure on $M(C)$ for every $C \in \mathcal{C}$, with the property that $(m \cdot r)|_f = m|_f \cdot r|_f$ for $f : D \longrightarrow C$, $r \in R(C)$ and $m \in M(C)$. We use the $-|_f$ notation for $M(f)(-)$ and $R(f)(-)$ to make the notation neater.

Similarly a *presheaf of left modules* over R is an object of $(P(\mathcal{C}); R)\mathbf{Mod}$, which consists of a presheaf of abelian groups M together with a left $R(C)$-module structure on $M(C)$ for every $C \in \mathcal{C}$, with the property that $(r \cdot m)|_f = r|_f \cdot m|_f$.

Definition 6. Let $(\mathcal{C}, J)$ be a small site and R a sheaf of rings on $\mathcal{C}$. A *sheaf of right modules* over R is an object of $\mathbf{Mod}(Sh_J(\mathcal{C}); R)$, which is just a presheaf of right modules over R which happens to be a J-sheaf. Therefore $\mathbf{Mod}(Sh_J(\mathcal{C}); R)$ is a full subcategory of $\mathbf{Mod}(P(\mathcal{C}); R)$. Similarly a *sheaf of left modules* over R is an object of $(Sh_J(\mathcal{C}); R)\mathbf{Mod}$, which is a presheaf of left R-modules that is a J-sheaf, so $(Sh_J(\mathcal{C}); R)\mathbf{Mod}$ is a full subcategory of $(P(\mathcal{C}); R)\mathbf{Mod}$.

Let $(\mathcal{C}, J)$ be a small site and R a sheaf of rings on $\mathcal{C}$. If P is a presheaf of right R-modules, then the presheaf of abelian groups P^+ obtained from the plus-construction is a presheaf of right R-modules with action

$$\{x_f \,|\, f \in S\} \cdot r = \{x_f \cdot r|_f \,|\, f \in S\}$$

Similarly if P is a presheaf of left R-modules, then P^+ becomes a presheaf of left R-modules. The morphism $P \longrightarrow P^+$ is then a morphism of presheaves of modules. If $\phi : P \longrightarrow Q$ is a morphism of presheaves of modules, then so is ϕ^+. Therefore applying this construction twice gives functors

$$\mathbf{a} : \mathbf{Mod}(P(\mathcal{C}); R) \longrightarrow \mathbf{Mod}(Sh_J(\mathcal{C}); R)$$
$$\mathbf{a} : (P(\mathcal{C}); R)\mathbf{Mod} \longrightarrow (Sh_J(\mathcal{C}); R)\mathbf{Mod}$$

In both cases $P \longrightarrow P^+ \longrightarrow P^{++}$ gives a natural transformation $\eta : 1 \longrightarrow \mathbf{ia}$ which is easily seen to be the unint of an adjunction $\mathbf{a} \longrightarrow\!\!| \; \mathbf{i}$. As for sheaves of groups, one checks that $\mathbf{a}$ preserves all finite limits, and therefore $\mathbf{Mod}(Sh_J(\mathcal{C}); R)$ is a Giraud subcategory of $\mathbf{Mod}(P(\mathcal{C}); R)$ and similarly for left modules.

Now let $(\mathcal{C}, J)$ be a small site and R a *presheaf* of rings on $\mathcal{C}$. If P is a presheaf of right R-modules, then the presheaf of abelian groups P^+ becomes a presheaf of right R^+-modules, where R^+ is the presheaf of rings obtained from R, via the action

$$\{x_f \,|\, f \in S\} \cdot \{r_g \,|\, g \in T\} = \{x_h \cdot r_h \,|\, h \in S \cap T\}$$

Similarly if P is a presheaf of left R-modules, then P^+ becomes a presheaf of left R-modules. The morphism $P \longrightarrow P^+$ is then a morphism of presheaves of abelian groups compatible with the morphism of presheaves of rings $R \longrightarrow R^+$. If $\phi : P \longrightarrow Q$ is a morphism of presheaves of modules, then ϕ^+ is a morphism of presheaves of R^+-modules. Therefore applying this construction twice, and denoting the sheaf of rings R^{++} by $\mathbf{a}R$, we have functors

$$\mathbf{a} : \mathbf{Mod}(P(\mathcal{C}); R) \longrightarrow \mathbf{Mod}(Sh_J(\mathcal{C}); \mathbf{a}R)$$
$$\mathbf{a} : (P(\mathcal{C}); R)\mathbf{Mod} \longrightarrow (Sh_J(\mathcal{C}); \mathbf{a}R)\mathbf{Mod}$$

Definition 7. Let $\mathcal{C}$ be a small category and R a presheaf of rings on $\mathcal{C}$. Define another presheaf of rings $R^{\text{op}} : \mathcal{C}^{\text{op}} \longrightarrow \mathbf{nRng}$ by $R^{\text{op}}(C) = R(C)^{\text{op}}$, the opposite ring of $R(C)$. We let R^{op} agree with R on morphisms. Clearly $R = (R^{\text{op}})^{\text{op}}$ and R is commutative if and only if $R = R^{\text{op}}$. If $(\mathcal{C}, J)$ is a small site then R is a sheaf of rings if and only if R^{op} is.

Lemma 6. *Let $\mathcal{C}$ be a small category and R a presheaf of rings on $\mathcal{C}$. Then there is a canonical isomorphism of categories*

$$T : (P(\mathcal{C}); R)\mathbf{Mod} \longrightarrow \mathbf{Mod}(P(\mathcal{C}); R^{op})$$
$$T(A, a, v, u, c) = (A, a, v, u, c\tau)$$

where $\tau : A \times R \longrightarrow R \times A$ is the canonical twist. If $(\mathcal{C}, J)$ is a small site and R a sheaf of rings on $\mathcal{C}$, then there is a canonical isomorphism of categories

$$T : (Sh_J(\mathcal{C}); R)\mathbf{Mod} \longrightarrow \mathbf{Mod}(Sh_J(\mathcal{C}); R^{op})$$
$$T(A, a, v, u, c) = (A, a, v, u, c\tau)$$

Proof. It suffices to check the first statement. Using the pointwise criterion for the data to determine a presheaf of right modules, it is easy to see that T is well-defined functor, which is an isomorphism since the twist is invertible. $\square$

Definition 8. Let $(\mathcal{C}, J)$ be a small site. A *sheaf of $\mathbb{Z}$-graded rings* on $\mathcal{C}$ is a sheaf of rings R together with a set of subsheaves of abelian groups $R_d, d \in \mathbb{Z}$ such that the morphisms $R_d \longrightarrow R$ induce an isomorphism of sheaves of abelian groups $\bigoplus_{d \in \mathbb{Z}} R_d \cong R$ and for $d, e \in \mathbb{Z}$, $C \in \mathcal{C}$ and $s \in R_d(C), t \in R_e(C)$ we must have $st \in R_{d+e}(C)$. We also require that $1 \in R_0(C)$. We say R is *positive* or is a *sheaf of graded rings* if $R_d = 0$ for all $d < 0$.

Definition 9. Let $(\mathcal{C}, J)$ be a small site and R a sheaf of $\mathbb{Z}$-graded rings. A *sheaf of graded left R-modules* is a sheaf of left R-modules M together with a set of subsheaves of abelian groups $\{M_n\}_{n \in \mathbb{Z}}$ such that the morphisms $M_n \longrightarrow M$ induce an isomorphism of sheaves of abelian groups $\bigoplus_{n \in \mathbb{Z}} M_n \cong M$ and such that for $C \in \mathcal{C}, d, n \in \mathbb{Z}$ and $s \in R_d(C), m \in M_n(C)$ we have $s \cdot m \in M_{n+d}(C)$. A *morphism* of sheaves of graded left R-modules is a morphism of sheaves of left R-modules $M \longrightarrow N$ which carries M_n into N_n for all $n \in \mathbb{Z}$. This makes the sheaves of graded left R-modules into a preadditive category, denoted $(Sh_J(\mathcal{C}); R)\mathbf{GrMod}$.

Similarly a sheaf of graded right R-modules is a sheaf of right R-modules M together with a set of subsheaves of abelian groups $\{M_n\}_{n \in \mathbb{Z}}$ such that the morphisms $M_n \longrightarrow M$ induce an isomorphism of sheaves of abelian groups $\bigoplus_{n \in \mathbb{Z}} M_n \cong M$ and such that for $C \in \mathcal{C}, d, n \in \mathbb{Z}$ and $s \in R_d(C), m \in M_n(C)$ we have $m \cdot s \in M_{n+d}(C)$. A *morphism* of sheaves of graded right R-modules is a morphism of sheaves of right R-modules preserving grade. This makes the sheaves of graded right R-modules into a preadditive category, denoted $\mathbf{GrMod}(Sh_J(\mathcal{C}); R)$.

If $\phi : M \longrightarrow N$ is a morphism of sheaves of graded R-modules (right or left) then for $n \in \mathbb{Z}$ the maps $\phi_p : M(p) \longrightarrow N(p)$ restrict to give morphisms of abelian groups $M_n(p) \longrightarrow N_n(p)$ which define a morphism of sheaves of abelian groups $\phi_n : M_n \longrightarrow N_n$. Then $\phi = \bigoplus_n \phi_n$ in the sense that ϕ is the unique morphism of sheaves of abelian groups making the following diagram commute for every $n \in \mathbb{Z}$

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \uparrow & & \uparrow \\ M_n & \xrightarrow[\phi_n]{} & N_n \end{array}$$

Let $(\mathcal{C}, J)$ be a small site, S a sheaf of $\mathbb{Z}$-graded rings and consider the coproduct of presheaves of abelian groups $A = \bigoplus_{n \in \mathbb{Z}} S_n$. For $p \in \mathcal{C}$ we define a ring structure on the abelian group $A(p) = \bigoplus_n S_n(p)$ by

$$\{(s_n)(t_n)\}_i = \sum_{x+y=i} s_x t_y$$

using the product in the ring $S(p)$. This makes A into a presheaf of rings on $\mathcal{C}$ (clearly $A(p)$ is commutative if $S(p)$ is). Let Q be the sheaf of rings on $\mathcal{C}$ obtained by sheafifying A, which is a $\mathbb{Z}$-graded ring with the subsheaves given by the images of $S_d \longrightarrow A \longrightarrow Q$. The canonical morphism of presheaves of rings $A \longrightarrow S$ given pointwise by the morphism $\bigoplus_n S_n(p) \longrightarrow S(p)$ induced by the inclusions $S_n(p) \longrightarrow S(p)$ induces an isomorphism of sheaves of $\mathbb{Z}$-graded rings $\varphi : Q \longrightarrow S$. This shows that a sheaf of $\mathbb{Z}$-graded rings is determined by the groups $S_n(p)$ for $n \in \mathbb{Z}, p \in \mathcal{C}$ and the multiplication rules for homogenous elements. For more along this line see (LC,Section 3).

Definition 10. Let $(\mathcal{C}, J)$ be a small site and R a sheaf of $\mathbb{Z}$-graded rings. Then the sheaf of rings R^{op} is a sheaf of $\mathbb{Z}$-graded rings with the same grading, which we call the *opposite* $\mathbb{Z}$*-graded ring.*

Lemma 7. *Let* $(\mathcal{C}, J)$ *be a small site and* R *a sheaf of* $\mathbb{Z}$*-graded rings. Then there is a canonical isomorphism of categories*

$$T : (Sh_J(\mathcal{C}); R)\mathbf{GrMod} \longrightarrow \mathbf{GrMod}(Sh_J(\mathcal{C}); R^{op})$$

Proof. Given a sheaf of graded left R-modules M, let $T(M)$ be the canonical sheaf of right R^{op}-modules as defined above. The subsheaves M_n make $T(M)$ into a sheaf of graded right R-modules. The functor T acts as the identity on morphisms, and it is obvious that T is an isomorphism. □

References

[1] B. Mitchell, "Theory of Categories", *Academic Press* (1965).