An Adjunction for Modules over Projective Schemes

Daniel Murfet

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For modules over a ring there is an adjunction between the associated sheaf functor $A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(SpecA)$ and the global sections functor $\mathfrak{Mod}(SpecA) \longrightarrow A\mathbf{Mod}$. In this note we develop the graded version of this result. All of this material is taken from EGA.

Contents

1 Introduction

Let S be a graded ring generated by S_1 as an S_0-algebra and set $X = ProjS$. If $\mathscr{F}$ is a sheaf of $\mathcal{O}_X$-modules then for $n, d \in \mathbb{Z}$ there is an isomorphism of modules :

$$\kappa^{n,d} : \mathscr{F}(n)(d) \longrightarrow \mathscr{F}(n+d)$$
$$(\mathscr{F} \otimes \mathcal{O}_X(n)) \otimes \mathcal{O}_X(d) \cong \mathscr{F} \otimes (\mathcal{O}_X(n) \otimes \mathcal{O}_X(d)) \cong \mathscr{F} \otimes \mathcal{O}_X(n+d)$$

This isomorphism is natural in $\mathscr{F}$, in the sense that if $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of modules then for $n, d \in \mathbb{Z}$ the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(n)(d) & \Longrightarrow & \mathscr{F}(n+d) \\ {\scriptstyle \phi(n)(d)}\downarrow & & \downarrow{\scriptstyle \phi(n+d)} \\ \mathscr{G}(n)(d) & \Longrightarrow & \mathscr{G}(n+d) \end{array}$$

For $n \in \mathbb{Z}$ we let κ^n denote the isomorphism $\mathscr{F}(n)(-n) \cong \mathscr{F}(0) \cong \mathscr{F}$. This is also natural in $\mathscr{F}$. For any graded S-module M and $n \in \mathbb{Z}$ there is a natural isomorphism of modules

$$\rho^n : M(n)^\sim \cong (M \otimes S(n))^\sim \cong \widetilde{M} \otimes S(n)^\sim = \widetilde{M}(n)$$

Let $m \in M$ be homogenous of degree n and and consider the global section $\mathfrak{p} \mapsto m/1$ of $M(n)^\sim$, which we denote by $\dot{m}$. Then the corresponding global section of $M^\sim(n)$ is defined as follows: let $\mathfrak{p}$ be given and find $f \in S_1$ with $f \notin \mathfrak{p}$. Then

$$\rho^n_X(\dot{m})(\mathfrak{p}) = (D_+(f), \dot{m/f^n} \otimes \dot{f^n})$$

Where we handle $n < 0$ by replacing m/f^n by $f^{-n}m$ and f^n by $1/f^{-n}$, as usual. That is, m/f^n denotes the action of f^n of $m/1$ in $T^{-1}M$, which is the localisation of M at the homogenous elements not in $\mathfrak{p}$.

2 The Adjunction

Throughout this section S denotes a graded ring generated by S_1 as an S_0-algebra. Associated to any $\mathcal{O}_X$-module $\mathscr{F}$ is the graded S-module

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n))$$
$$(s\cdot m)_i = \sum_{\substack{d\geq 0, j\in\mathbb{Z}\\ d+j=i}} \kappa^{j,d}(m_j \dot{\otimes} \dot{s_d})$$

for $s \in S$ and $m \in \Gamma_*(\mathscr{F})$. The homogenous part of $\Gamma_*(\mathscr{F})$ of degree n is the subgroup $\Gamma(X, \mathscr{F}(n))$. Let $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of $\mathcal{O}_X$-modules. We define a morphism of graded S-modules

$$\Gamma_*(\phi) : \Gamma_*(\mathscr{F}) \longrightarrow \Gamma_*(\mathscr{G})$$
$$\Gamma_*(\phi)(m)_i = \phi(i)_X(m_i)$$

This defines an additive functor $\Gamma_* : \mathfrak{Mod}(X) \longrightarrow S\mathbf{GrMod}$. The aim of this note is to show that this functor is right adjoint to the additive functor $\widetilde{-} : S\mathbf{GrMod} \longrightarrow \mathfrak{Mod}(X)$. First we have some technical observations to make.

The definition of the module structure on $\Gamma_*(-)$ uses the isomorphisms $\kappa^{i,j}$ on global sections. We extend this definition to all open sets. Let $U \subseteq X$ be open, $\mathscr{F}$ a $\mathcal{O}_X$-module, $i, j \in \mathbb{Z}$ and $r \in \mathcal{O}_X(j)(U), m \in \mathscr{F}(i)(U)$. Then we define

$$r \cdot m = \kappa_U^{i,j}(m\dot{\otimes}r) \in \mathscr{F}(i+j)(U)$$

We now collect some properties of this action. First of all if $r \in \mathcal{O}_X(0)(U) = \mathcal{O}_X(U)$ then $r \cdot m \in \mathscr{F}(j)(U)$ agrees with the normal action of $\mathcal{O}_X$ on $\mathscr{F}(j)$. If $V \subseteq U$ is open it is easy to see that $(r\cdot m)|_V = r|_V \cdot m|_V$. We have additivity in each variable:

$$(r + r')\cdot m = r\cdot m + r'\cdot m$$
$$r\cdot(m + m') = r\cdot m + r\cdot m'$$

Suppose $x, y \in \mathbb{Z}$ and $s \in \mathcal{O}_X(x)(U), r \in \mathcal{O}_X(y)(U)$ are given. Let sr denote the image of $s\dot{\otimes}r$ in $\mathcal{O}_X(x+y)(U)$ under the isomorphism $\tau : \mathcal{O}_X(x) \otimes \mathcal{O}_X(y) \cong \mathcal{O}_X(x+y)$. Then we claim that

$$(sr)\cdot m = s\cdot(r\cdot m)$$

Both sides are elements of $\mathscr{F}(x+y+i)(U)$. The fact that they are equal follows from diagram (3) of our note on compatibility of the associator and $\widetilde{-}$. Referring to the explicit definition of κ it is not difficult to check that if $1 \in \mathcal{O}_X(0)(U) = \mathcal{O}_X(U)$ then $1\cdot m = m$ for any $m \in \mathscr{F}(U)$.

Lemma 1. *Let S be a graded ring generated by S_1 as an S_0-algebra and set $X = ProjS$. Let $\mathscr{F}$ be a sheaf of modules on X. Then for every $\mathfrak{p} \in X$ there is a canonical morphism of $S_{(\mathfrak{p})}$-modules*

$$\kappa_{\mathfrak{p}} : \Gamma_*(\mathscr{F})_{(\mathfrak{p})} \longrightarrow \mathscr{F}_{\mathfrak{p}}$$
$$\kappa_{\mathfrak{p}}(m/s) = (D_+(s), \nu(\dot{1/s}\cdot m|_{D_+(s)}))$$

for $s \in S_n$ and $m \in \Gamma(X, \mathscr{F}(n))$, where $\nu : \mathscr{F}(0) \longrightarrow \mathscr{F}$ is the canonical isomorphism.

Proof. For $s \in S_n$, $\mathfrak{p} \mapsto s/1$ defines a global section of $\mathcal{O}_X(n)$ which we denote by $\dot{s}$, and $\mathfrak{p} \mapsto 1/s$ defines a section of $\mathcal{O}_X(-n)$ over $D_+(s)$ which we denote by $\dot{1/s}$. To show that $\kappa_{\mathfrak{p}}$ is well-defined, we must show that if $m/s = m'/s'$ with m, s homogenous of degree i and m', s' of degree j then

$$(\dot{1/s}\cdot m|_{D_+(s)})|_Q = (\dot{1/s'}\cdot m'|_{D_+(s')})|_Q \tag{1}$$

in $\mathscr{F}(Q)$ for some open $Q \subseteq D_+(s) \cap D_+(s')$ with $\mathfrak{p} \in Q$. By definition $m/s = m'/s'$ implies that there is homogenous $t \in S_k$ with $t \notin \mathfrak{p}$ and $t\dot{s'} \cdot m = t\dot{s} \cdot m'$. Let $Q = D_+(t) \cap D_+(s) \cap D_+(s')$. Then

$$\dot{t}|_Q \cdot (\dot{s'}|_Q \cdot m|_Q) = \dot{t}|_Q \cdot (\dot{s}|_Q \cdot m'|_Q)$$

Acting on both sides with $1\dot{/}t|_Q \in \mathcal{O}_X(-k)(Q)$ and using the properties of the action gives $\dot{s'}|_Q \cdot m|_Q = \dot{s}|_Q \cdot m'|_Q$. Using $1\dot{/}s|_Q$ and $1\dot{/}s'|_Q$ we end up with $1\dot{/}s|_Q \cdot m|_Q = 1\dot{/}s'|_Q \cdot m'|_Q$ which implies (1), so $\kappa_{\mathfrak{p}}$ is well-defined. The group $\Gamma_*(\mathscr{F})_{(\mathfrak{p})}$ has a canonical $S_{(\mathfrak{p})}$-module structure, as does $\mathscr{F}_{\mathfrak{p}}$ via the isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong S_{(\mathfrak{p})}$, and it is readily seen that $\kappa_{\mathfrak{p}}$ is a morphism of $S_{(\mathfrak{p})}$-modules. □

Proposition 2. *Let S be a graded ring generated by S_1 as an S_0-algebra and set $X = ProjS$. Then we have a diagram of adjoints*

$$S\mathbf{GrMod} \underset{\Gamma_*(-)}{\overset{\sim}{\rightleftarrows}} \mathfrak{Mod}(X) \qquad \sim \dashv \Gamma_*$$

For a graded S-module M and a sheaf of modules $\mathscr{F}$ on X the unit and counit are defined by

$$\eta : M \longrightarrow \Gamma_*(\widetilde{M}) \qquad\qquad \varepsilon : \Gamma_*(\mathscr{F})^{\sim} \longrightarrow \mathscr{F}$$

$$\eta(m)_i = \rho^i_X(\dot{m}_i) \qquad\qquad germ_{\mathfrak{p}}\varepsilon_U(s) = \kappa_{\mathfrak{p}}(s(\mathfrak{p}))$$

In particular if $m \in \Gamma(X, \mathscr{F}(i))$ and $s \in S_i$ then $\varepsilon_{D_+(s)}(m\dot{/}s) = \nu(1\dot{/}s \cdot m|_{D_+(s)})$.

Proof. First we check that the given unit $\eta : M \longrightarrow \Gamma_*(\widetilde{M})$ is a morphism of graded S-modules. It is clearly a graded morphism of abelian groups, and to prove it is a morphism of S-modules it suffices to prove that $\eta(s \cdot m) = s \cdot \eta(m)$ for homogenous $s \in S_d$ and $m \in M_e$. Both sides of this equation are sequences whose ith elements are global sections of $M^{\sim}(i)$. Let $\mathfrak{p}$ be given and find $f \in S_1$ with $f \notin \mathfrak{p}$. Then using the definition of κ we have

$$\eta(s \cdot m)_{d+e}(\mathfrak{p}) = \rho^{d+e}_X((s \dot{\cdot} m))(\mathfrak{p}) = (D_+(f), (s \cdot m)\dot{/}f^{d+e} \otimes \dot{f}^{d+e}) = (D_+(f), m\dot{/}f^e \otimes s\dot{f}^e)$$

$$(s \cdot \eta(m))_{d+e}(\mathfrak{p}) = \kappa^{e,d}(\rho^e_X(\dot{m})\dot{\otimes}\dot{s})(\mathfrak{p}) = (D_+(f), m\dot{/}f^e \otimes s\dot{f}^e)$$

Hence η is a morphism of graded S-modules. Naturality of η in M follows from naturality of the isomorphism ρ.

To show that $\eta : M \longrightarrow \Gamma_*(\widetilde{M})$ is universal, let $\phi : M \longrightarrow \Gamma_*(\mathscr{F})$ be a morphism of graded S-modules. For $\mathfrak{p} \in X$ we have a morphism of $S_{(\mathfrak{p})}$-modules $\phi_{(\mathfrak{p})} : M_{(\mathfrak{p})} \longrightarrow \Gamma_*(\mathscr{F})_{(\mathfrak{p})}$. When composed with $\kappa_{\mathfrak{p}}$ this gives us a morphism of $S_{(\mathfrak{p})}$-modules

$$\psi_{\mathfrak{p}} : M_{(\mathfrak{p})} \longrightarrow \mathscr{F}_{\mathfrak{p}}$$

$$\psi_{\mathfrak{p}}(m/s) = (D_+(s), \nu(1\dot{/}s \cdot \phi(m)|_{D_+(s)}))$$

Here m, s are homogenous of the same degree i, so $\phi(m)$ is a sequence with a single nonzero entry in $\Gamma(X, \mathscr{F}(i))$, and by abuse of notation we denote this element by $\phi(m)$. We define

$$\psi : \widetilde{M} \longrightarrow \mathscr{F}$$

$$germ_{\mathfrak{p}}\psi_U(s) = \psi_{\mathfrak{p}}(s(\mathfrak{p}))$$

This is well-defined since $\mathscr{F}$ is a sheaf and the germs are locally those of a section of $\mathscr{F}$. It is clear that ψ is a morphism of $\mathcal{O}_X$-modules. Next we show that the following diagram commutes:

$$\begin{array}{ccc} & M & \\ {}^{\eta}\swarrow & & \searrow^{\phi} \\ \Gamma_*(\widetilde{M}) & \xrightarrow[\Gamma_*\psi]{} & \Gamma_*(\mathscr{F}) \end{array} \qquad (2)$$

Let $m \in M$ be given. Then for $i \in \mathbb{Z}$ and $\mathfrak{p} \in X$ we have

$$(\Gamma_*(\psi)\eta(m))_i(\mathfrak{p}) = \psi(i)_X(\eta(m)_i)(\mathfrak{p}) = \psi(i)_X(\rho_X^i(\dot{m}_i))(\mathfrak{p})$$

Find $f \in S_1$ such that $f \notin \mathfrak{p}$. Then $\rho_X^i(\dot{m}_i)(\mathfrak{p}) = (D_+(f), \dot{m_i/f^i} \otimes \dot{f^i})$, and so

$$(\Gamma_*(\psi)\eta(m))_i(\mathfrak{p}) = (D_+(f), \psi_{D_+(f)}(\dot{m_i/f^i}) \otimes \dot{f^i}) = (D_+(f), \nu(\dot{1/f^i} \cdot \phi(m_i)|_{D_+(f)}) \otimes \dot{f^i})$$

Consider the following pentagon, which is commutative by earlier notes:

$$\begin{array}{ccccc}
(\mathscr{F}(i) \otimes \mathcal{O}_X(-i)) \otimes \mathcal{O}_X(i) & & \longrightarrow & & \mathscr{F}(i) \otimes (\mathcal{O}_X(-i) \otimes \mathcal{O}_X(i)) \\
\downarrow & & & & \downarrow \\
\mathscr{F}(0) \otimes \mathcal{O}_X(i) & & & & \mathscr{F}(i) \otimes \mathcal{O}_X(0) \\
& \searrow & & \swarrow & \\
& & \mathscr{F}(i) & &
\end{array}$$

Applied to $(\phi(m_i)|_{D_+(f)} \dot{\otimes} \dot{1/f^i}) \dot{\otimes} \dot{f^i}$ this shows that $\nu(\dot{1/f^i} \cdot \phi(m_i)|_{D_+(f)}) \dot{\otimes} \dot{f^i} = \phi(m_i)|_{D_+(f)}$ and therefore $(\Gamma_*(\psi)\eta(m))_i(\mathfrak{p}) = \phi(m)_i(\mathfrak{p})$, as required, so (2) commutes. It only remains to show that ψ is unique with this property. Suppose $\psi' : M^\sim \longrightarrow \mathscr{F}$ is another morphism making (2) commute. Using naturality of various isomorphisms we get a commutative diagram:

$$\begin{array}{ccc}
\widetilde{M}(i) \otimes \mathcal{O}_X(-i) & \Longrightarrow & \widetilde{M} \\
{\scriptstyle \psi'(i)\otimes 1}\downarrow & & \downarrow{\scriptstyle \psi'} \\
\mathscr{F}(i) \otimes \mathcal{O}_X(-i) & \Longrightarrow & \mathscr{F}
\end{array}$$

Let $s \in S$ and $m \in M$ be homogenous of degree i. Then the section $\dot{m/s} \in \widetilde{M}(D_+(s))$ is the image of $\rho_X^i(\dot{m})|_{D_+(s)} \dot{\otimes} \dot{1/s}$, and denoting by α the isomorphism $\mathscr{F}(i) \otimes \mathcal{O}_X(-i) \longrightarrow \mathscr{F}$ we have

$$\psi'_{D_+(s)}(\dot{m/s}) = \alpha_{D_+(s)}\left(\psi'(i)_X(\rho_X^i(\dot{m}))|_{D_+(s)} \dot{\otimes} \dot{1/s}\right)$$

But $\psi'(i)_X(\rho_X^i(\dot{m})) = (\Gamma_*(\psi')\eta(m))_i$ and by assumption $\Gamma_*(\psi)\eta = \Gamma_*(\psi')\eta$ so we have

$$\psi'_{D_+(s)}(\dot{m/s}) = \psi_{D_+(s)}(\dot{m/s})$$

Using the isomorphism $M_{(\mathfrak{p})} \cong \widetilde{M}_{\mathfrak{p}}$ we see that the morphisms ψ, ψ' determine the same map $M_{\mathfrak{p}}{}^\sim \longrightarrow \mathscr{F}_{\mathfrak{p}}$ for every $\mathfrak{p} \in X$ and are thus equal, proving uniqueness. This completes the proof that $\tilde{-}$ is left adjoint to Γ_*. The natural isomorphism

$$\beta : Hom_S(M, \Gamma_*(\mathscr{F})) \longrightarrow Hom_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F})$$

Is given by the association $\phi \mapsto \psi$ given in the proof. Note that $\psi_U(\dot{m/s}) = \nu(\dot{1/s} \cdot \phi(m)|_U)$. In particular the counit has the form given in the statement of the Proposition. $\square$

By definition of the counit, for any module $\mathscr{F}$ if we let $M = \Gamma_*(\mathscr{F})$ then the unit $M \longrightarrow \Gamma_*(\widetilde{M})$ followed by $\Gamma_*(\varepsilon)$ is the identity. If $n \in \mathbb{Z}$ and $m \in \Gamma(X, \mathscr{F}(n))$, let $\dot{m}$ denote the canonical global section of $M(n)^\sim$. Then the composite

$$\widetilde{M(n)} \stackrel{\rho^n}{\Longrightarrow} \widetilde{M}(n) \stackrel{\varepsilon(n)}{\longrightarrow} \mathscr{F}(n)$$

Maps $\dot{m}$ to a global section of $\mathscr{F}(n)$. This global section is m, since

$$\varepsilon(n)_X(\rho_X(\dot{m})) = \varepsilon(n)_X(\eta(m)_n) = \Gamma_*(\varepsilon)(\eta(m))_n = m$$

Where, as usual, we also denote by m the sequence in M whose only nonzero entry is the element $m \in \Gamma(X, \mathscr{F}(n))$.

Corollary 3. *Let S be a graded ring generated by S_1 as an S_0-algebra and set $X = ProjS$. Then the functor $\Gamma_* : \mathfrak{Mod}(X) \longrightarrow S\mathbf{GrMod}$ is left exact.*

Lemma 4. *Let $\mathscr{F}$ be a sheaf of modules on X. Then for $n \in \mathbb{Z}$ there is a canonical isomorphism $\Gamma_*(\mathscr{F}(n)) \cong \Gamma_*(\mathscr{F})(n)$ of graded S-modules natural in $\mathscr{F}$.*

Proof. We have a degree preserving isomorphism of abelian groups

$$\begin{aligned}\Gamma_*(\mathscr{F}(n)) &= \bigoplus_{m\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n)(m)) \\ &\cong \bigoplus_{m\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n+m)) \\ &\cong \Gamma_*(\mathscr{F})(n)\end{aligned}$$

It follows from commutativity of the diagram (3) on $p.11$ of our Section 2.5($Proj$) notes that this is an isomorphism of graded S-modules. Naturality in $\mathscr{F}$ is easily checked. ☐

3 Functorial Properties

Lemma 5. *Let $\phi : S \longrightarrow T$ be a morphism of graded rings. Set $X = ProjS, Y = ProjT$ and let $\Phi : U \longrightarrow X$ be the induced morphism of schemes. For $n \in \mathbb{Z}$ there is a canonical morphism of sheaves of modules*

$$\chi^n : \mathcal{O}_X(n) \longrightarrow \Phi_*(\mathcal{O}_Y(n)|_U)$$
$$a\dot{/}s \mapsto \phi(a)\dot{/}\phi(s)$$

If ϕ is an isomorphism then so is χ^n. If S is generated by S_1 as an S_0-algebra and T is generated by T_1 as a T_0-algebra then there is a canonical isomorphism of sheaves of modules natural in M

$$\omega^n : ({}_SM)^\sim(n) \longrightarrow \Phi_*(\widetilde{M}(n)|_U)$$
$$m\dot{/}s \,\dot{\otimes}\, a\dot{/}t \mapsto m\dot{/}\phi(s) \,\dot{\otimes}\, \phi(a)\dot{/}\phi(t)$$

Proof. Let χ^n be the following composite

$$\mathcal{O}_X(n) = S(n)^\sim \longrightarrow T(n)^\sim \cong \Phi_*(T(n)^\sim|_U) = \Phi_*(\mathcal{O}_Y(n)|_U)$$

This has the required property and is clearly an isomorphism if ϕ is. Let ω^n be the following isomorphism

$$({}_SM)^\sim(n) \cong ({}_SM(n))^\sim \cong \Phi_*(M(n)^\sim|_U) \cong \Phi_*(\widetilde{M}(n)|_U)$$

Using (H, 5.12b) and (MPS,Proposition 7) it is clear that ω^n has the required property and is natural in M. ☐

Lemma 6. *Let $\phi : S \longrightarrow T$ be a morphism of graded rings with S generated by S_1 as an S_0-algebra and T generated by T_1 as a T_0-algebra. If M is a graded T-module then for $n, d \in \mathbb{Z}$ the following diagram commutes*

$$\begin{array}{ccc}
({}_SM)^\sim(n) \otimes \mathcal{O}_X(d) & \overset{\kappa^{n,d}}{\Longrightarrow} & ({}_SM)^\sim(n+d) \\
\downarrow{\scriptstyle \omega^n\otimes\chi^d} & & \Big\Vert \\
\Phi_*(\widetilde{M}(n)|_U) \otimes \Phi_*(\mathcal{O}_Y(d)|_U) & & \Big\Vert \\
\downarrow & & \Big\Vert{\scriptstyle \omega^{n+d}} \\
\Phi_*(\widetilde{M}(n)|_U \otimes \mathcal{O}_Y(d)|_U) & & \Big\Vert \\
\Big\Downarrow & & \Big\Downarrow \\
\Phi_*(\widetilde{M}(n) \otimes \mathcal{O}_Y(d))|_U & \Longrightarrow & \Phi_*(\widetilde{M}(n+d)|_U)
\end{array}$$

Proof. It suffices to check commutativity on sections of the form $(m\dot{/}s \,\dot{\otimes}\, a\dot{/}t) \,\dot{\otimes}\, c\dot{/}q$ which is straightforward. $\square$

Lemma 7. *Let $\phi : S \longrightarrow T$ be a surjective morphism of graded rings where S is generated by S_1 as an S_0-algebra. Let $\Phi : ProjT \longrightarrow ProjS$ the induced closed immersion. If M is a graded T-module then there is a canonical isomorphism of graded S-modules natural in M*

$$\mu : \Gamma_*((_SM)^\sim) \longrightarrow {}_S\Gamma_*(\widetilde{M})$$

Proof. Set $X = ProjS$ and $Y = ProjT$. Then using Lemma 5 we have the following isomorphism of abelian groups

$$\begin{aligned}\mu : \Gamma_*((_SM)^\sim) &= \bigoplus_{n\in\mathbb{Z}} \Gamma(X, (_SM)^\sim(n)) \\ &\cong \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \Phi_*(\widetilde{M}(n))) \\ &= \bigoplus_{n\in\mathbb{Z}} \Gamma(Y, \widetilde{M}(n)) \\ &= \Gamma_*(\widetilde{M})\end{aligned}$$

It follows from Lemma 6 that this is a morphism of graded S-modules. Naturality in M is easily checked. $\square$

Throughout the last part of this section $\phi : S \longrightarrow T$ is an isomorphism of graded rings with S generated by S_1 as an S_0-algebra and T generated by T_1 as a T_0-algebra. Set $X = ProjS, Y = ProjT$ and let $\Phi : Y \longrightarrow X$ be the induced isomorphism of schemes.

Lemma 8. *If $\mathscr{F}$ is a sheaf of modules on Y then the following diagram of sheaves of modules on X commutes for $n, d \in \mathbb{Z}$*

$$\begin{array}{ccc}
(\Phi_*\mathscr{F})(n) \otimes \mathcal{O}_X(d) & \xRightarrow{\kappa^{n,d}} & (\Phi_*\mathscr{F})(n+d) \\
\Downarrow & & \Downarrow \\
(\Phi_*\mathscr{F} \otimes \Phi_*(\mathcal{O}_Y(n))) \otimes \mathcal{O}_X(d) & & \\
\Downarrow & & \\
\Phi_*(\mathscr{F}(n)) \otimes \mathcal{O}_X(d) & & \Phi_*\mathscr{F} \otimes \Phi_*(\mathcal{O}_Y(n+d)) \\
\Downarrow & & \Downarrow \\
\Phi_*(\mathscr{F}(n)) \otimes \Phi_*(\mathcal{O}_Y(d)) & & \\
\Downarrow & & \\
\Phi_*(\mathscr{F}(n)(d)) & \xRightarrow[\Phi_*\kappa^{n,d}]{} & \Phi_*(\mathscr{F}(n+d))
\end{array}$$

Proof. It suffices to check commutativity on sections of the form $(m \,\dot{\otimes}\, a\dot{/}b) \,\dot{\otimes}\, c\dot{/}d$ which is straightforward. $\square$

Lemma 9. *Let M be a graded T-module. Then the following diagram of sheaves of modules on*

X *commutes for* $n \in \mathbb{Z}$

$$\begin{array}{ccc}
\widetilde{\{_S M\}}(n) & \Longrightarrow & \widetilde{_S M}(n) \\
\Downarrow & & \Downarrow \\
_S\widetilde{\{M(n)\}} & & (\Phi_* \widetilde{M})(n) \\
\Downarrow & & \Downarrow \\
\Phi_*(\widetilde{M(n)}) & \Longrightarrow & \Phi_*(\widetilde{M}(n))
\end{array}$$

Proof. Inverting the top morphism, we can reduce to checking the modified diagram commutes on sections $\dot{m}/s \otimes \dot{a}/b$ of $(_S M)^{\sim}(n)$, which is straightforward. $\square$

Lemma 10. *The the following diagram of functors commutes up to a canonical natural equivalence*

$$\begin{array}{ccc}
\mathfrak{Mod}(Y) & \xrightarrow{\Phi_*} & \mathfrak{Mod}(X) \\
\Gamma_* \downarrow & & \downarrow \Gamma_* \\
T\mathbf{GrMod} & \Longrightarrow & S\mathbf{GrMod}
\end{array}$$

Moreover for a graded T*-module* M *the isomorphism* $\Gamma_*(\widetilde{_S M}) \cong \Gamma_*(\Phi_* \widetilde{M}) \cong {_S\Gamma_*}(\widetilde{M})$ *fits into the following commutative diagram of graded* S*-modules*

$$\begin{array}{ccc}
_S M & \longrightarrow & \Gamma_*(\widetilde{_S M}) \\
& \searrow & \Downarrow \\
& & _S\Gamma_*(\widetilde{M})
\end{array} \tag{3}$$

Proof. Let $\mathscr{F}$ be a sheaf of modules on Y. Then we have an isomorphism of abelian groups

$$\begin{aligned}
\nu : \Gamma_*(\Phi_* \mathscr{F}) &= \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \Phi_* \mathscr{F} \otimes \mathcal{O}_X(n)) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \Phi_* \mathscr{F} \otimes \Phi_* \mathcal{O}_Y(n)) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \Phi_*(\mathscr{F} \otimes \mathcal{O}_Y(n))) \\
&= \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \mathscr{F} \otimes \mathcal{O}_Y(n)) \\
&= \Gamma_*(\mathscr{F})
\end{aligned}$$

To check this is a morphism of graded S-modules it suffices to show that $\nu(s \cdot m) = s \cdot \nu(m)$ for $s \in S_d$ and $m \in \Gamma(X, \Phi_* \mathscr{F} \otimes \mathcal{O}_X(n))$, which follows from Lemma 8. Naturality of ν in $\mathscr{F}$ is easily checked. The diagram (3) says that the isomorphism ν identifies the unit morphisms $\eta : M \longrightarrow \Gamma_* \circ \widetilde{(-)}$ of T and S. Using Lemma 9 it is not difficult to check that (3) commutes. $\square$

4 The Quasicoherent Case

Throughout this section S denotes a graded ring generated by S_1 as an S_0-algebra and we set $X = ProjS$.

Lemma 11. *Let* $f \in S$ *be homogenous of degree* $e > 0$ *and* $\dot{f} \in \Gamma(X, \mathcal{O}_X(e))$ *as above. Then* $X_f = D_+(f)$, *where* X_f *is the canonical open set associated to the global section* $\dot{f}$ *of the invertible sheaf* $\mathcal{O}_X(e)$.

Proof. By assumption X is covered by open sets $D_+(g)$ for $g \in S_1$, so it suffices to show that $X_f \cap D_+(g) = D_+(f) \cap D_+(g)$ for $g \in S_1$. We showed in (5.12) that $\mathcal{O}_X(e)|_{D_+(g)} \cong \mathcal{O}_X|_{D_+(g)}$ and under this isomorphism $\mathfrak{p} \mapsto f/1$ corresponds to the section $\mathfrak{p} \mapsto f/g^e$ of $\mathcal{O}_X$ over $D_+(g)$. Under $\varphi : \mathcal{O}_X|_{D_+(g)} \cong SpecS_{(g)}$ this corresponds to the element f/g^e of $S_{(f)}$. We showed in our notes on locally free sheaves that $X_f \cap D_+(g) = \varphi^{-1}D(f/g^e)$. But $\varphi(\mathfrak{p}) = (\mathfrak{p}S_f) \cap S_{(f)}$ so it is clear that $\varphi^{-1}D(f/g^e) = D_+(f) \cap D_+(g)$, as required. □

Lemma 12. *Let $e \in \mathbb{Z}$ and $n > 0$ be given. Then there is an isomorphism of $\mathcal{O}_X$-modules*

$$\zeta : \mathcal{O}_X(e)^{\otimes n} \longrightarrow \mathcal{O}_X(ne)$$
$$f_1\dot{/}s_1 \dot{\otimes} \cdots \dot{\otimes} f_n\dot{/}s_n \mapsto f_1 \cdots f_n\dot{/}s_1 \cdots s_n$$

where $f_1, \ldots, f_n, s_1, \cdots, s_n \in S$ are homogenous with s_i of degree d_i and f_i of degree $d_i + e$.

Proof. The proof is by induction on n. If $n = 1$ the result is trivial, so assume $n > 1$ and the result is true for $n - 1$. Let ζ be the isomorphism

$$\mathcal{O}_X(e)^{\otimes n} = \mathcal{O}_X(e) \otimes \mathcal{O}_X(e)^{\otimes(n-1)} \cong \mathcal{O}_X(e) \otimes \mathcal{O}_X(ne-e) \cong \mathcal{O}_X(ne)$$

Using the inductive hypothesis and the explicit form of the isomorphism $\tau : \mathcal{O}_X(e) \otimes \mathcal{O}_X(ne-e) \cong \mathcal{O}_X(ne)$ one checks that we have the desired isomorphism. □

Proposition 13. *Let S be a graded ring finitely generated by S_1 as an S_0-algebra and set $X = ProjS$. If $\mathscr{F}$ is a sheaf of modules on X then the counit $\varepsilon : \Gamma_*(\mathscr{F})^{\sim} \longrightarrow \mathscr{F}$ is an isomorphism if and only if $\mathscr{F}$ is a quasi-coherent.*

Proof. If the generators of S over S_0 are $f_1, \ldots, f_n \in S_1$ then the open sets $D_+(f_i)$ cover $X = ProjS$, and consequently X is quasi-compact. If S is generated by the empty set over S_0, it is generated by the single element $0 \in S_1$, so we can always assume $n \geq 1$.

If the counit is an isomorphism then $\mathscr{F}$ is trivially quasi-coherent, so it only remains to prove the converse. So assume that $\mathscr{F}$ is a quasi-coherent sheaf of modules on X and let M be the graded S-module $\Gamma_*(\mathscr{F})$. For homogenous $f \in S$ of degree $e > 0$ define

$$\varphi : M_{(f)} \longrightarrow \mathscr{F}(D_+(f))$$
$$\varphi(m/f^n) = \nu(1\dot{/}f^n \cdot m|_{D_+(f)})$$

Where $1\dot{/}f^n \in \mathcal{O}_X(-ne)(D_+(f))$ and $m \in \Gamma(X, \mathscr{F}(ne))$. One checks this is well-defined morphism of groups in the same way as for $\kappa_{\mathfrak{p}}$ earlier. We now show that φ is bijective.

If $\varphi(m/f^n) = 0$ then $m|_{D_+(f)} = 0$, and Lemma 11 implies that $D_+(f) = X_{\dot{f}}$, where $\dot{f}$ is a global section of the invertible sheaf $\mathcal{O}_X(e)$. Moreover X is quasi-compact and since tensor products of quasi-coherent modules are quasi-coherent, the sheaf $\mathscr{F}(ne)$ is quasi-coherent. So we can apply (5.14a) to the global section $m \in \Gamma(X, \mathscr{F}(ne))$ to see that for some $M > 0$ we have $m\dot{\otimes}\dot{f}^{\otimes M} = 0$ as a global section of $\mathscr{F}(ne) \otimes \mathcal{O}_X(e)^{\otimes M}$. By Lemma 12 there is an isomorphism $\mathscr{F}(ne) \otimes \mathcal{O}_X(e)^{\otimes M} \cong \mathscr{F}(ne) \otimes \mathcal{O}_X(Me)$ which identifies $m\dot{\otimes}\dot{f}^{\otimes M}$ with $m\dot{\otimes}\dot{f^M}$. Thus $f^M \cdot m = \kappa^{ne,Me}(m\dot{\otimes}\dot{f^M}) = 0$ and so $m/f^n = 0$ in $M_{(f)}$, showing that φ is injective.

To see that φ is surjective let $t \in \mathscr{F}(0)(D_+(f))$ be given. By assumption X is covered by the affine open sets $D_+(f_1), \ldots, D_+(f_n)$ and $\mathcal{O}_X(e)$ is an invertible sheaf which restricts to a free sheaf on each of these sets. Each $D_+(f_i) \cap D_+(f_j)$ is affine and thus quasi-compact. So by (5.14b) for some $M > 0$ the section $t\dot{\otimes}\dot{f}|^{\otimes M}_{D_+(f)}$ of $\mathscr{F}(0) \otimes \mathcal{O}_X(e)^{\otimes M}$ extends to a global section α. There is an isomorphism

$$\mathscr{F}(0) \otimes \mathcal{O}_X(e)^{\otimes M} \cong \mathscr{F}(0) \otimes \mathcal{O}_X(Me) \cong \mathscr{F}(Me)$$

Let $m \in \Gamma(X, \mathscr{F}(Me))$ be the global section corresponding to α. Then $m|_{D_+(f)} = \dot{f^M} \cdot t$, where the action of $\mathcal{O}_X(Me)(D_+(f))$ on $\mathscr{F}(0)(D_+(f))$ is the canonical one. It follows that $1\dot{/}f^M \cdot m|_{D_+(f)} = t$, so φ is surjective.

To show that ε is an isomorphism it suffices to show that $\varepsilon_{D_+(f)}$ is an isomorphism for every homogenous $f \in S_+$. But this follows from bijecitivity of φ and commutativity of the following diagram

$$\begin{array}{ccc} \widetilde{M}(D_+(f)) & \xrightarrow{\varepsilon_{D_+(f)}} & \mathscr{F}(D_+(f)) \\ \Uparrow & \nearrow_{\varphi} & \\ M_{(f)} & & \end{array}$$

Hence ε is an isomorphism, as required. $\square$

5 Sheaves of Algebras

Throughout this section S denotes a graded ring generated by S_1 as an S_0-algebra and $X = ProjS$. In previous sections we defined the functor $\Gamma_* : \mathfrak{Mod}(X) \longrightarrow S\mathbf{GrMod}$, which is the projective analogue of the global sections functor $\Gamma : \mathfrak{Mod}(SpecA) \longrightarrow A\mathbf{Mod}$. In the affine case there are also functors (SOA,Definition 6)

$$\Gamma : \mathfrak{nAlg}(SpecA) \longrightarrow A\mathbf{nAlg}$$
$$\Gamma : \mathfrak{Alg}(SpecA) \longrightarrow A\mathbf{Alg}$$

In this section we take the first steps towards defining a projective version of these functors by defining the structure of a commutative $\mathbb{Z}$-graded ring on the graded S-module $\Gamma_*(\mathcal{O}_X)$.

A subtle point is that we must distinguish between $\mathcal{O}_X(n) = S(n)^{\sim}$and $\widehat{\mathcal{O}}_X(n) = \mathcal{O}_X \otimes \mathcal{O}_X(n)$. Of course using the natural isomorphism $\mathcal{O}_X \otimes \mathscr{F} \cong \mathscr{F}$ there is an isomorphism $\widehat{\mathcal{O}}_X(n) \cong \mathcal{O}_X(n)$. With this distinction, we have

$$\Gamma_*(\mathcal{O}_X) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \widehat{\mathcal{O}}_X(n))$$

For $m, n \in \mathbb{Z}$ let $\Lambda^{m,n}$ denote the following isomorphism of modules:

$$\Lambda^{m,n} : \widehat{\mathcal{O}}_X(m) \otimes \widehat{\mathcal{O}}_X(n) \cong \mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n) \cong \widehat{\mathcal{O}}_X(m+n)$$

Where we use the isomorphism $\tau^{m,n} : \mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \longrightarrow \mathcal{O}_X(m+n)$ defined earlier in notes. If $r, s \in \Gamma(X, \mathcal{O}_X)$ and $u \in \Gamma(X, \mathcal{O}_X(m)), v \in \Gamma(X, \mathcal{O}_X(n))$ then

$$\Lambda^{m,n}_X((r\dot{\otimes}u)\dot{\otimes}(s\dot{\otimes}v)) = 1\dot{\otimes}\tau^{m,n}_X((r \cdot u)\dot{\otimes}(s \cdot v))$$

For $a, b \in \Gamma_*(\mathcal{O}_X)$ we define

$$(a \cdot b)_i = \sum_{m+n=i} \Lambda^{m,n}_X(a_m\dot{\otimes}b_n)$$

It is easy enough to check that this operation is commutative and left and right distributive over addition. The identity element is the sequence I with $I_i = 0$ for $i \neq 0$ and $I_0 = 1\dot{\otimes}1$. By reduction to homogenous elements, associativity follows from commutativity of the following pentagon for integers $x, y, z \in \mathbb{Z}$

$$\begin{array}{ccc} (\widehat{\mathcal{O}}_X(x) \otimes \widehat{\mathcal{O}}_X(y)) \otimes \widehat{\mathcal{O}}_X(z) & \Longrightarrow & \widehat{\mathcal{O}}_X(x) \otimes (\widehat{\mathcal{O}}_X(y)) \otimes \widehat{\mathcal{O}}_X(z) \\ \Downarrow & & \Downarrow \\ \widehat{\mathcal{O}}_X(x+y) \otimes \widehat{\mathcal{O}}_X(z) & & \widehat{\mathcal{O}}_X(x) \otimes \widehat{\mathcal{O}}_X(y+z) \\ & \searrow \quad \swarrow & \\ & \widehat{\mathcal{O}}_X(x+y+z) & \end{array}$$

Commutativity of this diagram follows from commutativity of the pentagon for $\mathcal{O}_X(-)$. This structure makes $\Gamma_*(\mathcal{O}_X)$ is a $\mathbb{Z}$-graded commutative ring. Consider the unit morphism of graded S-modules

$$\eta : S \longrightarrow \Gamma_*(\mathcal{O}_X)$$
$$\eta(s)_n = 1 \dot{\otimes} \dot{s_n}$$

It is not hard to see that η is a morphism of $\mathbb{Z}$-graded rings.

Definition 1. Let S be a graded ring generated by S_1 as an S_0-algebra. We denote by $\Gamma(\mathcal{O}_X)'$ the subring $\bigoplus_{n\geq 0} \Gamma(X, \widehat{\mathcal{O}}_X(n))$ of $\Gamma_*(\mathcal{O}_X)$. This is a graded ring and there is a canonical morphism of graded rings $\eta : S \longrightarrow \Gamma_*(\mathcal{O}_X)'$. If A is a ring and S a graded A-algebra then $\Gamma_*(\mathcal{O}_X)'$ becomes a graded A-algebra via the ring morphism $A \longrightarrow S \longrightarrow \Gamma_*(\mathcal{O}_X)'$.

Lemma 14. *For $n \in \mathbb{Z}$ there is a canonical isomorphism $\Gamma_*(\mathcal{O}_X(n)) \cong \Gamma_*(\mathcal{O}_X)(n)$ of graded S-modules.*

Proof. This follows immediately from Lemma 4. □

Proposition 15. *Let A be a ring, $S = A[x_0, \ldots, x_r]$ for $r \geq 1$ so $X = \mathbb{P}^n_A$. Then the morphism of $\mathbb{Z}$-graded rings $\eta : S \longrightarrow \Gamma_*(\mathcal{O}_X)$ is an isomorphism. Therefore for all $n \in \mathbb{Z}$ there is an isomorphism of A-modules*

$$S_n \longrightarrow \Gamma(X, \mathcal{O}_X(n))$$
$$s \mapsto \dot{s}$$

In particular $\Gamma(X, \mathcal{O}_X(n)) = 0$ for any $n < 0$ and $\Gamma(X, \mathcal{O}_X) \cong A$ as rings.

Proof. The morphism η defined above agrees with the one obtained in the proof of Proposition 5.13 of Hartshorne, and we proved there that this was an isomorphism. □

Corollary 16. *Let A be a nonzero ring and set $X = \mathbb{P}^r_A$ for $r \geq 1$. Then for $n \geq 0$ the A-module $\Gamma(X, \mathcal{O}_X(n))$ is free of rank $\binom{n+r}{r}$.*

Proof. This follows from the fact the A-module S_n is free on the set of monomials of degree n. □

Proposition 17. *Let S be a graded domain generated by S_1 as an S_0-algebra with $S_+ \neq 0$. Then*

(i) The morphism $S \longrightarrow \Gamma_*(\mathcal{O}_X)$ is injective.

(ii) If S is finitely generated by S_1 as an S_0-algebra then $\Gamma_*(\mathcal{O}_X)$ is a domain. If further S is noetherian, $\Gamma_*(\mathcal{O}_X)'$ is integral over the subring S.

Proof. (i) To show that the morphism of rings $S \longrightarrow \Gamma_*(\mathcal{O}_X)$ is injective it suffices to show that the map $S_n \longrightarrow \Gamma(X, \mathcal{O}_X(n))$ defined by $s \mapsto \dot{s}$ is injective. Suppose that $\dot{s} = 0$ in $\Gamma(X, \mathcal{O}_X(n))$. Since $S_+ \neq 0$ we can find a homogenous prime ideal $\mathfrak{p} \in ProjS$, and by assumption $s/1 = 0$ in $S(n)_{(\mathfrak{p})}$. Therefore $qs = 0$ for some $q \notin \mathfrak{p}$ which implies $s = 0$, as required. We therefore identify S with a subring of $\Gamma_*(\mathcal{O}_X)'$.

(ii) Assume that S is generated by over S_0 by nonzero elements $x_0, \ldots, x_r \in S_1$. Let $\alpha_i^n : (S_{x_i})_n \longrightarrow \Gamma(D_+(x_i), \widehat{\mathcal{O}}_X(n))$ be the isomorphism of groups given by the composite $(S_{x_i})_n = S(n)_{(x_i)} \cong \Gamma(D_+(x_i), S(n)^{\sim}) \cong \Gamma(D_+(x_i), \widehat{\mathcal{O}}_X(n))$. So $\alpha_i^n(a/x_i^m) = 1\dot{\otimes}(a/\dot{x_i^m})$ and for any i, j the following diagram commutes

$$\begin{array}{ccc} (S_{x_i})_n & \overset{\alpha_i^n}{\Longrightarrow} & \Gamma(D_+(x_i), \widehat{\mathcal{O}}_X(n)) \\ \downarrow & & \downarrow \\ (S_{x_i x_j})_n & \underset{\alpha_j^n}{\Longrightarrow} & \Gamma(D_+(x_i x_j), \widehat{\mathcal{O}}_X(n)) \end{array} \tag{4}$$

Let Ω be the set of tuples $(t_0, \dots, t_r)$ where $t_i \in S_{x_i}$ and t_i, t_j have the same image in $S_{x_i x_j}$ for all i, j. Under the pointwise operations this becomes a $\mathbb{Z}$-graded ring, with graded piece Ω_n given by those sequences with $t_i \in (S_{x_i})_n$ for all i. Commutativity of (4) implies that there is a bijection between global sections of $\widehat{\mathcal{O}}_X(n)$ and the set Ω_n. This gives rise to an isomorphism of $\mathbb{Z}$-graded rings

$$\beta : \Omega \longrightarrow \Gamma_*(\mathcal{O}_X)$$
$$\beta(t_0, \dots, t_r)_n|_{D_+(x_i)} = \alpha_i^n((t_i)_n)$$

Since the x_i are all nonzero and S is a domain, the canonical ring morphisms $S_{x_i} \longrightarrow S_{x_i x_j}$ and $\gamma_i : S_{x_i} \longrightarrow S_{x_0 \cdots x_r}$ are all injective. Moreover γ_i factors through $S_{x_i x_j} \longrightarrow S_{x_0 \cdots x_r}$ for any j. The map $\Omega \longrightarrow S_{x_0 \cdots x_r}$ given by $(t_0, \dots, t_r) \mapsto \gamma_i(t_i)$ is independent of i and is an injection of rings (that is, if two tuples in Ω agree in any position they agree in all positions) since $\gamma_i(t_i) = \gamma_j(t_j)$ for any i, j by the assumption on the tuples in Ω. Consequently for any $n \in \mathbb{Z}$ if two global sections of $\widehat{\mathcal{O}}_X(n)$ (or $\mathcal{O}_X(n)$) agree when restricted to $D_+(x_i)$ for any i, then they are equal globally.

We can thus identify $\Gamma_*(\mathcal{O}_X)$ with a subring of the domain $S_{x_0 \cdots x_r}$ containing S and contained in the intersection of all the localisations S_{x_i} (the composite $S \longrightarrow \Gamma_*(\mathcal{O}_X) \cong \Omega \longrightarrow S_{x_0 \cdots x_r}$ agrees with the canonical injection $S \longrightarrow S_{x_0 \cdots x_r}$). In particular the rings $\Gamma_*(\mathcal{O}_X), \Gamma_*(\mathcal{O}_X)'$ are integral domains.

Now assume that S is noetherian and let a homogenous element $b \in \Gamma_*(\mathcal{O}_X)'_n = \Gamma(X, \widehat{\mathcal{O}}_X(n))$ be given. Then for $0 \leq i \leq r$ we have $b|_{D_+(x_i)} = 1 \dot{\otimes} (a/x_i^m)$ for some $a/x_i^m \in (S_{x_i})_n$. Since $\eta(x_i)^m b$ and $\eta(a)$ agree on $D_+(x_i)$ they are equal. Now identify S with a graded subring of $\Gamma_*(\mathcal{O}_X)'$. We have shown that for every homogenous element b of the larger ring and $1 \leq i \leq r$ there is $m > 0$ with $x_i^m b \in S$. Choose one m that works for all i. Since the x_i generate S over S_0 any element $y \in S_m$ is a linear sum of monomials in $x_0, \dots, x_r$ of degree m with coefficients from S_0. So by making m sufficiently large we can assume that $yb \in S$ for all $y \in S_d$ with $d \geq m$. In fact since $n \geq 0$ we can say that for any $y \in S_{\geq m} = \bigoplus_{e \geq n} S_e$, $yb \in S_{\geq m}$. Now it follows inductively that $y(b^q) \in S$ for any $q \geq 1$ and $y \in S_{\geq m}$. In particular for every $q \geq 1$ we have $x_0^m b^q \in S$. Let Q be the quotient field of $\Gamma_*(\mathcal{O}_X)'$. Then $(1/x_0^m)S$ is a finitely generated sub-S-module of Q, which contains the ring $S[b]$. Since S is noetherian the S-module $S[b]$ is finitely generated, so by A & M Proposition 5.1, b is integral over S. Since the integral closure of S in $\Gamma_*(\mathcal{O}_X)'$ is a ring, it follows that $\Gamma_*(\mathcal{O}_X)'$ is integral over S, as required. $\square$