Triangulated Categories Part I

Daniel Murfet

April 11, 2007

Triangulated categories are important structures lying at the confluence of several exciting areas of mathematics (and even physics). Our notes on the subject are divided into three parts which, if named by the major construction occurring within them, would be titled "Verdier quotients", "Thomason localisaton" and "Brown representability". There are many places to learn about triangulated categories, but these notes are mostly influenced by Neeman's excellent book [Nee01], with elements from [BN93] and [ATJLSS00].

Contents

1 Triangulated Categories

For our conventions regarding categories the reader should consult (AC,Section 1). In particular we work with the set-theoretic foundations outlined there. Throughout if we say a conglomerate X is *small* we mean that there is a set x and a bijection $X \cong x$. All categories are nonempty unless specified otherwise. As usual we write $X = 0$ to indicate an object X is a zero object, not that it is necessarily equal to any particular canonical zero. In the first two sections of these notes there is nothing new, and the reader should consult [Nee01] for the canonical treatment. We include the material here for completeness.

Definition 1. Let $\mathcal{C}$ be a category. Isomorphism defines an equivalence relation on the class of objects of $\mathcal{C}$, and we denote the conglomerate of equivalence classes of this relation by $I(\mathcal{C})$. We say that $\mathcal{C}$ is *essentially small* if the conglomerate $I(\mathcal{C})$ is small.

1.1 Pretriangulated Categories

Definition 2. Let $\mathcal{C}$ be an additive category and $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ an additive automorphism. A *candidate triangle* in $\mathcal{C}$ (with respect to Σ) is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that the composites $v \circ u, w \circ v$ and $\Sigma u \circ w$ are zero. A *morphism* of candidate triangles is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

This defines the category of candidate triangles in $\mathcal{C}$ (with respect to Σ).

Definition 3. A *pretriangulated category* is an additive category $\mathcal{T}$ together with an additive automorphism Σ, and a class of candidate triangles (with respect to Σ) called *distinguished triangles*. The following conditions must hold:

TR0: Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished triangle. For any object X the candidate triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

is distinguished.

TR1: For any morphism $f : X \longrightarrow Y$ in $\mathcal{T}$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

TR2: Suppose we have a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

Then the following two candidate triangles are also distinguished

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

$$\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} w} X \xrightarrow{-u} Y \xrightarrow{-v} Z$$

TR3: For any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & \Sigma X \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

where the rows are distinguished triangles, there is a morphism $h : Z \longrightarrow Z'$, not necessarily unique, which makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd} \tag{1}$$

Throughout when we say "triangle" we mean "distinguished triangle" unless we say explicitly that we are talking about a candidate triangle. Since the functor Σ is an isomorphism, it preserves all limits and colimits and sends epimorphisms (resp. monomorphisms) to epimorphisms (resp. monomorphisms).

Remark 1. Let $\mathcal{T}$ be a pretriangulated category. Then using TR3 and TR2 it is not difficult to check we have the following results

TR3': For any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

where the rows are distinguished triangles, there is a morphism $g : Y \longrightarrow Y'$, not necessarily unique, which makes (1) commute.

TR3": For any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

where the rows are distinguished triangles, there is a morphism $f : X \longrightarrow X'$, not necessarily unique, which makes (1) commute.

Remark 2. Let $\mathcal{T}$ be a pretriangulated category and suppose we have a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

Using TR0 one checks that if you change the sign on any two of u, v, w then the resulting candidate triangle is still distinguished. For any $n \in \mathbb{Z}$ it follows from TR2 that the following triangle is distinguished

$$\Sigma^n X \xrightarrow{(-1)^n \Sigma^n u} \Sigma^n Y \xrightarrow{(-1)^n \Sigma^n v} \Sigma^n Z \xrightarrow{(-1)^n \Sigma^n w} \Sigma^{n+1} X$$

By the previous comment we can replace the first two morphisms with $\Sigma^n u, \Sigma^n v$ and still have a triangle.

Remark 3. There are probably a number of ways to understand these axioms intuitively. Here is an algebraic way. The idea is that given a morphism $u : X \longrightarrow Y$, a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

provides us with morphisms $v : Y \longrightarrow Z$ and $\Sigma^{-1}w : \Sigma^{-1}Z \longrightarrow X$, which we think of as being the *homotopy cokernel* and *homotopy kernel* respectively. In this language, the axioms loosely correspond to the following statements

- TR0 : The identity has homotopy kernel and cokernel equal to zero.
- TR1 : Every morphism has a homotopy kernel and cokernel.
- TR2: Any morphism is the homotopy kernel of its homotopy cokernel (up to sign), and any morphism is the homotopy cokernel of its homotopy kernel (up to sign).
- TR3: Homotopy kernels and cokernels are weakly functorial.

Remark 4. Let $\mathcal{T}$ be a pretriangulated category. Triangles in $\mathcal{T}$ are stable under isomorphism at any of their vertices, in the sense that if you replace one of X, Y, Z with an isomorphic object (and modify the morphisms appropriately) the result is still a triangle.

Remark 5. If $\mathcal{T}$ is a pretriangulated category then so is $\mathcal{T}^{\mathrm{op}}$, where we replace Σ by Σ^{-1}. We define the distinguished triangles of $\mathcal{T}^{\mathrm{op}}$ as follows: given a distinguished triangle of $\mathcal{T}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

we define the following candidate triangle of $\mathcal{T}^{\mathrm{op}}$ (with respect to Σ^{-1}) to be distinguished

$$\Sigma^{-1}Z \xleftarrow{\Sigma^{-1}w} X \xleftarrow{u} Y \xleftarrow{v} Z$$

With these structures, it is easy to check that $\mathcal{T}^{\mathrm{op}}$ is a pretriangulated category. Moreover the double dual $(\mathcal{T}^{\mathrm{op}})^{\mathrm{op}}$ is equal *as a pretriangulated category* to the original $\mathcal{T}$.

Remark 6. Let $\mathcal{T}$ be a pretriangulated category and suppose we have two candidate triangles

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$
$$X' \longrightarrow Y' \longrightarrow Z' \longrightarrow \Sigma X'$$

Since $\Sigma(X \oplus X')$ is a direct sum of $\Sigma X, \Sigma X'$ we have a candidate triangle

$$X \oplus X' \longrightarrow Y \oplus Y' \longrightarrow Z \oplus Z' \longrightarrow \Sigma(X \oplus X')$$

which we call the *direct sum* of the two candidate triangles.

Definition 4. Let $\mathcal{T}$ be a pretriangulated category and $H : \mathcal{T} \longrightarrow \mathcal{A}$ an additive functor into an abelian category $\mathcal{A}$. Then H is called *homological* if for every triangle of $\mathcal{T}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the following sequence is exact

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

Remark 7. Let $\mathcal{T}$ be a pretriangulated category and $H : \mathcal{T} \longrightarrow \mathcal{A}$ a homological functor as above. By TR2 we can produce the following sequence in $\mathcal{T}$, where every consecutive triple of morphisms is a triangle (if we modify the signs appropriately)

$$\cdots \longrightarrow \Sigma^{-1}X \longrightarrow \Sigma^{-1}Y \longrightarrow \Sigma^{-1}Z \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma Z \longrightarrow \cdots$$

Then by definition of H (here we use additivity) the following infinite sequence is exact

$$\cdots \longrightarrow H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}w)} H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(\Sigma X) \longrightarrow \cdots$$

In particular, the additive functor $H \circ \Sigma^n$ is homological for any $n \in \mathbb{Z}$.

Definition 5. Let $\mathcal{T}$ be a pretriangulated category and $H : \mathcal{T} \longrightarrow \mathcal{A}$ a contravariant additive functor into an abelian category $\mathcal{A}$. Then H is *cohomological* if it is homological as a covariant functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}$. Equivalently, for every triangle of $\mathcal{T}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the following sequence is exact

$$H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X)$$

Lemma 1. *Let $\mathcal{T}$ be a pretriangulated category and U an object of $\mathcal{T}$. Then the representable functor $Hom(U, -) : \mathcal{T} \longrightarrow \mathbf{Ab}$ is homological.*

Proof. Suppose we are given a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

We need to show that the following sequence of abelian groups is exact

$$Hom(U, X) \longrightarrow Hom(U, Y) \longrightarrow Hom(U, Z)$$

Suppose that $f : U \longrightarrow Y$ is a morphism with $v \circ f = 0$. Then we have a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & \Sigma U \arrow[r, "-1"] & \Sigma U \arrow[d, "\Sigma f"] \\
Y \arrow[r, "-v"'] & Z \arrow[r, "-w"'] & \Sigma X \arrow[r, "-\Sigma u"'] & \Sigma Y
\end{tikzcd}$$

The bottom row is a triangle by TR2, the top row by TR0 and TR2. By TR3 there is a morphism $h : U \longrightarrow X$ such that Σh makes the diagram commute.In particular $f = u \circ h$, which is what we wanted to show. $\square$

Corollary 2. *Let $\mathcal{T}$ be a pretriangulated category and U an object of $\mathcal{T}$. Then the contravariant representable functor $Hom(-, U) : \mathcal{T} \longrightarrow \mathbf{Ab}$ is cohomological.*

Definition 6. Let $\mathcal{T}$ be a pretriangulated category and $\mathcal{A}$ a complete abelian category with exact products. Then a homological functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ is called *decent* if it preserves products.

Example 1. If $\mathcal{T}$ is a pretriangulated category and U an object of $\mathcal{T}$, then the homological functor $H : \mathcal{T} \longrightarrow \mathbf{Ab}$ is decent, since $\mathbf{Ab}$ is complete with exact products, and $Hom(U, -)$ certainly preserves products.

Definition 7. Let $\mathcal{T}$ be a pretriangulated category. A candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called a *pretriangle* if for every decent homological functor $H : \mathcal{T} \longrightarrow \mathcal{A}$, the following long sequence is exact

$$\cdots \longrightarrow H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}w)} H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(\Sigma X) \longrightarrow \cdots$$

Clearly any triangle is a pretriangle. Any direct summand of a pretriangle is a pretriangle.

Lemma 3. *Let $\mathcal{T}$ be a pretriangulated category, Λ a nonempty set and suppose that for every $\lambda \in \Lambda$ we have a pretriangle*

$$X_\lambda \longrightarrow Y_\lambda \longrightarrow Z_\lambda \longrightarrow \Sigma X_\lambda$$

Suppose further that products exist for the families $\{X_\lambda\}_{\lambda \in \Lambda}, \{Y_\lambda\}_{\lambda \in \Lambda}, \{Z_\lambda\}_{\lambda \in \Lambda}$. Then the following candidate triangle is a pretriangle

$$\prod X_\lambda \longrightarrow \prod Y_\lambda \longrightarrow \prod Z_\lambda \longrightarrow \Sigma\Big\{ \prod X_\lambda \Big\} \tag{2}$$

Proof. To be clear, given any products $\prod X_\lambda, \prod Y_\lambda, \prod Z_\lambda$ the morphisms $\Sigma \prod X_\lambda \longrightarrow \Sigma X_\lambda$ are also a product in $\mathcal{T}$, so the last morphism in the candidate triangle is the usual one induced between products. Evaluating any decent homological functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ on this triangle and using the exact products in $\mathcal{A}$ it is easily checked that (2) is a pretriangle. $\square$

Lemma 4. *Let H be a decent homological functor $\mathcal{T} \longrightarrow \mathcal{A}$ and suppose we have a morphism of pretriangles*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

Suppose that for every $n \in \mathbb{Z}$, $H(\Sigma^n f)$ and $H(\Sigma^n g)$ are isomorphisms. Then the $H(\Sigma^n h)$ are all isomorphisms.

Proof. Given $n \in \mathbb{Z}$ we have a commutative diagram in $\mathcal{A}$ with exact rows

$$\begin{tikzcd} H(\Sigma^n X) \arrow[r] \arrow[d, "H(\Sigma^n f)"'] & H(\Sigma^n Y) \arrow[r] \arrow[d, "H(\Sigma^n g)"'] & H(\Sigma^n Z) \arrow[r] \arrow[d, "H(\Sigma^n h)"'] & H(\Sigma^{n+1} X) \arrow[r] \arrow[d, "H(\Sigma^{n+1} f)"'] & H(\Sigma^{n+1} Y) \arrow[d, "H(\Sigma^{n+1} g)"'] \\ H(\Sigma^n X') \arrow[r] & H(\Sigma^n Y') \arrow[r] & H(\Sigma^n Z') \arrow[r] & H(\Sigma^{n+1} X') \arrow[r] & H(\Sigma^{n+1} Y') \end{tikzcd}$$

and it follows from the 5-lemma (DCAC,Lemma 4) that $H(\Sigma^n h)$ is an isomorphism. $\square$

Lemma 5. *Let $\mathcal{T}$ be a pretriangulated category and suppose we have a morphism of pretriangles*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

If f, g are isomorphisms, then so is h.

Proof. If f, g are isomorphisms then so are $\Sigma^n f, \Sigma^n g$ for any $n \in \mathbb{Z}$, so it follows from Lemma 4 that $H(h)$ is an isomorphism for any decent homological functor H. But the functors $Hom(Z, -)$ and $Hom(Z', -)$ are decent homological, so it is easy to check that h is an isomorphism. $\square$

Proposition 6. *Let $\mathcal{T}$ be a pretriangulated category and suppose we have a morphism of triangles*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

If any two of f, g, h are isomorphisms, then so is the remaining morphism.

Proof. If f, g are isomorphisms then Lemma 5 implies that h is an isomorphism. The other two cases follow by applying Lemma 5 to the "rotated" triangles obtained from TR2. $\square$

Remark 8. Let $\mathcal{T}$ be a pretriangulated category and $u : X \longrightarrow Y$ a morphism. By TR1 it may be completed to a triangle. Suppose we have two distinguished triangles "completing" u

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$$

Using TR3 we produce a morphism $h : Z \longrightarrow Z'$ making the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, Rightarrow] & Y \arrow[r, "v"] \arrow[d, Rightarrow] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, Rightarrow] \\
X \arrow[r, "u"'] & Y \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X
\end{tikzcd}$$

By Lemma 5 the morphism h must be an isomorphism, so any two distinguished triangles completing u are isomorphic (but not canonically).

Proposition 7. *Let $\mathcal{T}$ be a pretriangulated category, Λ a nonempty set and suppose that for every $\lambda \in \Lambda$ we have a triangle*

$$X_\lambda \longrightarrow Y_\lambda \longrightarrow Z_\lambda \longrightarrow \Sigma X_\lambda$$

Suppose further that products exist for the families $\{X_\lambda\}_{\lambda\in\Lambda}, \{Y_\lambda\}_{\lambda\in\Lambda}, \{Z_\lambda\}_{\lambda\in\Lambda}$. Then the following pretriangle is a triangle

$$\prod X_\lambda \longrightarrow \prod Y_\lambda \longrightarrow \prod Z_\lambda \longrightarrow \Sigma\Big\{\prod X_\lambda\Big\} \tag{3}$$

Proof. By TR1 we can complete the morphism $\prod X_\lambda \longrightarrow \prod Y_\lambda$ to a triangle

$$\prod X_\lambda \longrightarrow \prod Y_\lambda \longrightarrow Q \longrightarrow \Sigma\Big\{\prod X_\lambda\Big\}$$

By TR3 we obtain for each $\lambda \in \Lambda$ a morphism $Q \longrightarrow Z_\lambda$ making the following diagram commute

$$\begin{tikzcd}
\prod X_\lambda \arrow[r] \arrow[d] & \prod Y_\lambda \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \Sigma\Big\{\prod X_\lambda\Big\} \arrow[d] \\
X_\lambda \arrow[r] & Y_\lambda \arrow[r] & Z_\lambda \arrow[r] & \Sigma X_\lambda
\end{tikzcd}$$

Inducing morphisms into the products in the bottom row, we have a commutative diagram in which both rows are pretriangles (the bottom row by Lemma 3)

$$\begin{tikzcd}
\prod X_\lambda \arrow[r] \arrow[d, Rightarrow] & \prod Y_\lambda \arrow[r] \arrow[d, Rightarrow] & Q \arrow[r] \arrow[d] & \Sigma\Big\{\prod X_\lambda\Big\} \arrow[d, Rightarrow] \\
\prod X_\lambda \arrow[r] & \prod Y_\lambda \arrow[r] & \prod Z_\lambda \arrow[r] & \Sigma\Big\{\prod X_\lambda\Big\}
\end{tikzcd}$$

We conclude from Lemma 5 that this is an isomorphism of candidate triangles, and since the top row is distinguished it follows from TR0 that the bottom row is also. $\square$

Remark 9. Using duality, it is easy to see that in a pretriangulated category any nonempty coproduct of triangles is a triangle.

Proposition 8. *Let $\mathcal{T}$ be a pretriangulated category and suppose we have candidate triangles*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{4}$$
$$X' \longrightarrow Y' \longrightarrow Z' \longrightarrow \Sigma X' \tag{5}$$

Suppose that the following candidate triangle is a triangle

$$X \oplus X' \longrightarrow Y \oplus Y' \longrightarrow Z \oplus Z' \longrightarrow \Sigma(X \oplus X')$$

Then (4) and (5) are also triangles.

Proof. It suffices to prove that (4) is a triangle. Since it is a direct summand of a pretriangle, it is at least a pretriangle. We can complete $X \longrightarrow Y$ to a triangle

$$X \longrightarrow Y \longrightarrow Q \longrightarrow \Sigma X$$

And the injections $X \longrightarrow X \oplus X', Y \longrightarrow Y \oplus Y'$ induce a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ X \oplus X' \arrow[r] & Y \oplus Y' \arrow[r] & Z \oplus Z' \arrow[r] & \Sigma(X \oplus X') \end{tikzcd}$$

Composing with the projections, we have a morphism of pretriangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d, "h"'] & \Sigma X \arrow[d, equal] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

By Lemma 5 this is an isomorphism of candidate triangles, and since the top row is a triangle it follows by TR0 that the bottom row is also a triangle. □

Remark 10. In the loose intuitive sense of Remark 3 the axiom TR0 for a pretriangulated category says that the identity has zero homotopy kernel and cokernel. The next result shows that in fact this property characterises isomorphisms.

Lemma 9. *Let $\mathcal{T}$ be a pretriangulated category. A morphism $u : X \longrightarrow Y$ is an isomorphism if and only if the following candidate triangle is distinguished*

$$X \xrightarrow{u} Y \longrightarrow 0 \longrightarrow \Sigma X \tag{6}$$

Proof. If u is an isomorphism, then the following diagram is an isomorphism of candidate triangles

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "u"'] & Y \arrow[r] & 0 \arrow[r] & \Sigma X \end{tikzcd} \tag{7}$$

which shows that the bottom row is distinguished. Conversely, suppose that the candidate triangle (6) is distinguished. Then (7) is a morphism of triangles with isomorphisms in the first and third column. Proposition 6 now implies that u is an isomorphism, as required. □

Remark 11. Let $\mathcal{T}$ be a pretriangulated category. Then for any object X the morphism $-1 : X \longrightarrow X$ is an isomorphism, so using TR0 and TR2 we conclude that the following are triangles

$$0 \longrightarrow X \xrightarrow{1} X \longrightarrow 0$$

$$X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{1} \Sigma X$$

Using TR0 and Proposition 3 one checks that for any objects X, Z the following candidate triangle (with the morphisms $X \longrightarrow X \oplus Z, X \oplus Z \longrightarrow Z$ being the injection and projection respectively) is a triangle

$$X \longrightarrow X \oplus Z \longrightarrow Z \xrightarrow{0} \Sigma X$$

Definition 8. Let $\mathcal{T}$ be a pretriangulated category and $u : X \longrightarrow Y$ a morphism. We say that a morphism $v : Y \longrightarrow Z$ a *homotopy cokernel* or *hocokernel* of u if there exists a distinguished triangle in $\mathcal{T}$ of the following form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \tag{8}$$

So by definition $vu = 0$. We say that a morphism $t : T \longrightarrow X$ is a *homotopy kernel* or *hokernel* of u if there exists a distinguished triangle in $\mathcal{T}$ of the following form

$$X \xrightarrow{u} Y \xrightarrow{v} \Sigma T \xrightarrow{\Sigma t} \Sigma X$$

In particular given a triangle (8) the morphism $\Sigma^{-1}w : \Sigma^{-1}Z \longrightarrow X$ is a hokernel and every hokernel occurs in this way. By definition $ut = 0$. It follows from Remark 8 that any two hocokernels (resp. hokernels) of u are related by a (noncanonical) isomorphism. In particular statements like "the hocokernel is zero" are well-defined. It follows from TR1 that every morphism in $\mathcal{T}$ has a homotopy kernel and cokernel.

We say that u is a *homonomorphism* if its hokernel is zero, and a *hoepimorphism* if its hocokernel is zero. It is obvious that any monomorphism is a homonomorphism, and any epimorphism is a hoepimorphism.

Remark 12. Let $\mathcal{T}$ be a pretriangulated category. Using Remark 2 we observe that if $u : X \longrightarrow Y$ is a morphism with hocokernel $v : Y \longrightarrow Z$ then v is also a hocokernel of $-u$. Similarly if $t : T \longrightarrow X$ is a hokernel of u then it is also a hokernel of $-u$. This shows that homotopy kernels and cokernels are "sign agnostic", just like the usual kernel and cokernel.

Remark 13. With these new concepts, we can read a remarkable amount of information from a single triangle. Let $\mathcal{T}$ be a pretriangulated category and suppose we have a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

Using TR2 one can verify the following

- v is a hocokernel of u, w is a hocokernel of v and u is a hocokernel of $\Sigma^{-1}w$.
- u is a hokernel of v, v is a hokernel of w and $\Sigma^{-1}w$ is a hokernel of u.

This makes it clear that given any two morphisms $u : X \longrightarrow Y, v : Y \longrightarrow Z$, u is a hokernel of v *if and only if* v is a hocokernel of u.

Lemma 10. *Let $\mathcal{T}$ be a pretriangulated category. A morphism $u : X \longrightarrow Y$ is an isomorphism if and only if it is both a homonomorphism and a hoepimorphism.*

Proof. If u is an isomorphism then Lemma 9 yields a certain triangle which shows that the hokernel and hocokernel are zero. Conversely, if u has zero hokernel and hocokernel then we have a triangle of the following form for some object Z

$$X \xrightarrow{u} Y \xrightarrow{0} Z \xrightarrow{0} \Sigma X$$

We have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, Rightarrow, no head] & Y \arrow[r, "0"] \arrow[d, Rightarrow, no head] & Z \arrow[r, "0"] \arrow[d] & \Sigma X \arrow[d, Rightarrow, no head] \\
X \arrow[r, "u"'] & Y \arrow[r] & 0 \arrow[r] & \Sigma X
\end{tikzcd}$$

where the bottom row is a triangle by TR0. It now follows from Proposition 6 that the third column is an isomorphism, so $Z = 0$ and Lemma 9 implies that u is an isomorphism. □

Lemma 11. *Let $\mathcal{T}$ be a pretriangulated category, $u : X \longrightarrow Y$ a morphism and $v : Y \longrightarrow Z, t : T \longrightarrow X$ a hocokernel and hokernel of u respectively. Then*

(i) A morphism $m : Y \longrightarrow Q$ factors through v if and only if $mu = 0$.

(ii) A morphism $m : Q \longrightarrow X$ factors through t if and only if $um = 0$.

Proof. (i) If m factors through v then it is clear that $mu = 0$. Conversely, suppose that $mu = 0$. Then we have a commutative diagram with triangles in the rows

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d, "m"] & Z \arrow[r, "w"] & \Sigma X \\ 0 \arrow[r] & Q \arrow[r, "1"'] & Q \arrow[r] & 0 \end{tikzcd}$$

By TR3 there is a morphism $h : Z \longrightarrow Q$ with $hv = m$, as required. One proves (ii) in much the same way. One should observe that we are *not* claiming the factorisations are unique. □

Remark 14. With the notation of Lemma 11, it is now clear that if v were an epimorphism, it would be the cokernel of u (respectively, if t were a monomorphism, it would be the kernel of u). Although this is not true in general, in some important cases the notions of homotopy kernel and ordinary kernel coincide (and similarly for cokernels).

The following result shows just how special pretriangulated categories are among the additive categories. Among other things, we will show that *every monomorphism splits*.

Proposition 12. *Let $\mathcal{T}$ be a pretriangulated category, $u : X \longrightarrow Y$ a morphism. Then the following conditions are equivalent:*

(i) u is a monomorphism;

(ii) u is a homonomorphism;

(iii) u is a coretraction.

(iv) There is a triangle of the following form

$$X \xrightarrow{u} Y \longrightarrow Z \xrightarrow{0} \Sigma X$$

Proof. Since the implications $(i) \Rightarrow (ii), (iii) \Rightarrow (i), (ii) \Leftrightarrow (iv)$ are trivial, it suffices to show $(ii) \Rightarrow (iii)$. Suppose that u is a homonomorphism. Then we have a commutative diagram in which the rows are triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "1"'] & Y \arrow[r, "v"] & Z \arrow[r, "0"] \arrow[d] & \Sigma X \arrow[d, "1"'] \\ X \arrow[r, "1"'] & X \arrow[r] & 0 \arrow[r] & \Sigma X \end{tikzcd}$$

Therefore by TR3' there is a morphism $Y \longrightarrow X$ making the diagram commute, which shows that u is a coretraction. □

We record the dual result for convenience.

Corollary 13. *Let $\mathcal{T}$ be a pretriangulated category, $u : X \longrightarrow Y$ a morphism. Then the following conditions are equivalent:*

(i) u is an epimorphism;

(ii) u is a hoepimorphism;

(iii) u is a retraction.

(iv) There is a triangle of the following form

$$X \xrightarrow{u} Y \xrightarrow{0} Z \longrightarrow \Sigma X$$

Lemma 14. *Let $\mathcal{T}$ be a pretriangulated category, and $u : X \longrightarrow Y$ a morphism. Then*

(i) If u is a monomorphism, then any hocokernel of u is a cokernel.

(ii) If u is an epimorphism, then any hokernel of u is a kernel.

Proof. Suppose that u is a monomorphism, and let $v : Y \longrightarrow Z$ be any hocokernel. Then we have a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{0} \Sigma X$$

twisting we have another triangle

$$Y \xrightarrow{-v} Z \xrightarrow{0} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

This shows that $-v$ is a hoepimorphism, therefore an epimorphism. But then v is an epimorphism, which by Remark 14 is enough to show that v is the cokernel of u. The proof of (ii) is similar. □

Remark 15. Let $\mathcal{T}$ be a pretriangulated category. Proposition 12, its dual and Lemma 14 have the following interesting consequences:

- The category $\mathcal{T}$ is balanced. That is, a morphism is an isomorphism if and only if it is both a monomorphism and an epimorphism (this follows from Lemma 10).
- Every object in $\mathcal{T}$ is both injective and projective.
- Every monomorphism in $\mathcal{T}$ has a cokernel, and every epimorphism has a kernel.

Definition 9. Let $\mathcal{T}$ be a pretriangulated category. Suppose we are given a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

It follows from Proposition 12 and its dual together with Lemma 14 that u is a coretraction if and only if v is a retraction, and in this situation we say that the triangle is *split* (clearly then $w = 0$). If the triangle is split there is an isomorphism $X \oplus Z \cong Y$. In fact there are many such isomorphisms:

- If $v' : Z \longrightarrow Y$ is such that $vv' = 1$ then $(u \;\; v') : X \oplus Z \longrightarrow Y$ is an isomorphism.
- If $u' : Y \longrightarrow X$ is such that $u'u = 1$ then $\binom{u'}{v} : Y \longrightarrow X \oplus Z$ is an isomorphism.

Lemma 15. *Let $\mathcal{T}$ be a pretriangulated category and suppose we have a commutative diagram in which the rows are triangles*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

If $Hom(\Sigma X, Z') = 0$ or $Hom(Z, Y') = 0$ then there is a unique *morphism $h : Z \longrightarrow Z'$ completing this to a morphism of triangles.*

Proof. By TR3 there certainly exists such a morphism h, so it suffices to show uniqueness. Suppose that $Hom(\Sigma X, Z') = 0$ and that two morphisms $h, h' : Z \longrightarrow Z'$ both make the above diagram commute. Then $(h - h')v = 0$ and since w is a homotopy cokernel of v we deduce that $h - h'$ factors through ΣX. But as the only possible factorisation $\Sigma X \longrightarrow Z'$ is the zero morphism, we deduce that $h = h'$, as required. If instead $Hom(Z, Y') = 0$ then we proceed in the same way, using the fact that v' is a homotopy kernel of w'. $\square$

Remark 16. Let $\mathcal{T}$ be a pretriangulated category and suppose we have a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

It follows easily from the same arguments used in the proof of Lemma 15 that if $Hom(\Sigma X, Z) = 0$ then w is the *unique* morphism $Z \longrightarrow \Sigma X$ making this diagram a triangle. That is, if $w' : Z \longrightarrow \Sigma X$ is another morphism making (u, v, w') a triangle, then $w = w'$.

Remark 17. Before we give the next result, recall the following: if $\mathcal{A}$ is a preadditive category, $u : X \longrightarrow A$ a coretraction with $pu = 1_X$, and $v : Y \longrightarrow A$ a kernel of p then u, v is a coproduct in $\mathcal{A}$. It is therefore a biproduct, with p being the projection onto X

$$\begin{tikzcd} X \arrow[r, bend left, "u"] & A \arrow[l, bend left, "p"] \arrow[r, bend left, "q"] & Y \arrow[l, bend left, "v"] \end{tikzcd}$$

We can write the four morphisms as matrices in the obvious way

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, p = \begin{pmatrix} 1 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, q = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Suppose we are given a morphism $\alpha : Y \longrightarrow X$ and define $p' = \begin{pmatrix} 1 & \alpha \end{pmatrix} : A \longrightarrow X, v' = \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} : Y \longrightarrow A$. It is easy to check that v' is the kernel of p' and the morphisms u, p', v', q define the same object A as the biproduct of X, Y in a slightly different way (one must be careful to observe that $p' = \begin{pmatrix} 1 & \alpha \end{pmatrix}$ is the matrix representation of p' with respect to the original biproduct, not the new one).

Example Let R be a ring and let $\mathcal{A} = R\mathbf{Mod}$. Set $X = Y = R$ and let A be the canonical direct sum $R \oplus R$. Let u, p, v, q be the canonical injections and projections respectively. Given $\lambda \in R$ let $\alpha : R \longrightarrow R$ be the corresponding morphism, and set $p' = \begin{pmatrix} 1 & \alpha \end{pmatrix}$. In other words, $p'(r, s) = r + \lambda s$. Then $v' : R \longrightarrow A$ is defined by $v'(s) = (-\lambda s, s)$ and the morphisms u, p', v', q write the plane A as a direct sum of the x-axis and the line $x = -\lambda y$. The original u, p, v, q correspond to the case $\lambda = 0$.

Alternatively, suppose we are given a morphism $\beta : X \longrightarrow Y$ and define $u' = \begin{pmatrix} 1 \\ \beta \end{pmatrix} : X \longrightarrow A$. We still have $pu' = 1_X$ and v remains unchanged, but q (which was the factorisation of $1 - up$ through v) will become $q' = \begin{pmatrix} -\beta & 1 \end{pmatrix}$. So we have written the same object A as the biproduct of X, Y but with morphisms u', p, v, q'.

Example With the same notation as above, let $\beta : X \longrightarrow Y$ be multiplication by λ so that $u'(r) = (r, \lambda r)$. Then the morphisms u', p, v, q' write the plane A as a direct sum of the line $y = \lambda x$ (the image of u') with the y-axis.

Combining TR0 and Remark 11 we have three special triangles associated to any object X. Next we show that if a candidate triangle looks like it might admit one of these triangles as a direct summand, then it actually does (loosely speaking). We give a proof in each case for the sake of having a quick reference (and avoiding confusion about signs), but one could certainly be more efficient.

Lemma 16. *Let $\mathcal{T}$ be a pretriangulated category and suppose we are given a candidate triangle of the following form*

$$X \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} A \oplus Y \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} A \oplus Z \xrightarrow{\begin{pmatrix} c & d \end{pmatrix}} \Sigma X \tag{9}$$

This can be written as a direct sum of the following two candidate triangles

$$0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0 \tag{10}$$

$$X \xrightarrow{b} Y \xrightarrow{\gamma-\beta\alpha} Z \xrightarrow{d} \Sigma X$$

Proof. Let u_1, p_1, v_1, q_1 be the morphisms for the biproduct $A \oplus Y$ and u_2, p_2, v_2, q_2 the morphisms for $A \oplus Z$. Define $p_1' = \begin{pmatrix} 1 & \alpha \end{pmatrix} : A \oplus Y \longrightarrow A$ and $v_1' = \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} : Y \longrightarrow A \oplus Y$. This writes the object $A \oplus Y$ as a biproduct of A, Y with morphisms u_1, p_1', v_1', q_1. We also define $u_2' = \begin{pmatrix} 1 \\ \beta \end{pmatrix} : A \longrightarrow A \oplus Z$ and $q_2' = \begin{pmatrix} -\beta & 1 \end{pmatrix}$ so that u_2', p_2, v_2, q_2' writes $A \oplus Z$ as a biproduct of A, Z. With respect to these new biproduct structures on its domain and codomain, the matrix in (9) becomes $\begin{pmatrix} 1 & 0 \\ 0 & \gamma-\beta\alpha \end{pmatrix}$.

Since (9) is a candidate triangle, we deduce the following equations $a + \alpha b = 0$ and $c + d\beta = 0$. So with respect to the new biproduct structures, the morphisms have the following matrices

$$\begin{pmatrix} 0 \\ b \end{pmatrix} : X \longrightarrow A \oplus Y, \qquad \begin{pmatrix} 0 & d \end{pmatrix} : A \oplus Z \longrightarrow \Sigma X$$

Using the various other equations one deduces from the fact that (9) is a candidate triangle, one checks that both sequences in (10) are candidate triangles. It is not clear that (9) is the direct sum of these two candidate triangles (using the new biproduct stuctures, not the original ones). ☐

Lemma 17. *Let $\mathcal{T}$ be a pretriangulated category and suppose we are given a candidate triangle of the following form*

$$Q \oplus X \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} Y \xrightarrow{\begin{pmatrix} c \\ d \end{pmatrix}} \Sigma Q \oplus Z \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} \Sigma(Q \oplus X) \tag{11}$$

This can be written as a direct sum of the following two candidate triangles

$$Q \longrightarrow 0 \longrightarrow \Sigma Q \xrightarrow{1} \Sigma Q$$

$$X \xrightarrow{b} Y \xrightarrow{d} Z \xrightarrow{\gamma-\beta\alpha} \Sigma X$$

Proof. Let u_1, p_1, v_1, q_1 be the morphisms for the biproduct $Q \oplus X$ and u_2, p_2, v_2, q_2 the morphisms for $\Sigma X \oplus Z$. Define the following morphisms

$$\begin{aligned} u_1' &= \begin{pmatrix} 1 \\ \Sigma^{-1}\beta \end{pmatrix} : Q \longrightarrow Q \oplus X \\ q_1' &= \begin{pmatrix} -\Sigma^{-1}\beta & 1 \end{pmatrix} : Q \oplus X \longrightarrow X \\ p_2' &= \begin{pmatrix} 1 & \alpha \end{pmatrix} : \Sigma Q \oplus Z \longrightarrow \Sigma Q \\ v_2' &= \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} : Z \longrightarrow \Sigma Q \oplus Z \end{aligned}$$

One checks that u_1', p_1, v_1, q_1' and u_2, p_2', v_2', q_2 are biproducts, and using these biproducts the candidate triangle (11) can be written as the direct sum of our two new candidate triangles. ☐

Lemma 18. *Let $\mathcal{T}$ be a pretriangulated category and suppose we are given a candidate triangle of the following form*

$$Q \oplus X \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} Q \oplus Y \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} Z \xrightarrow{\begin{pmatrix} c \\ d \end{pmatrix}} \Sigma(Q \oplus X)$$

This can be written as a direct sum of the following two candidate triangles

$$Q \xrightarrow{1} Q \longrightarrow 0 \longrightarrow \Sigma Q$$

$$X \xrightarrow{\gamma-\beta\alpha} Y \xrightarrow{b} Z \xrightarrow{d} \Sigma X$$

Proof. Use the same arguments as above. □

1.2 Triangulated Categories

Definition 10. Let $\mathcal{T}$ be a pretriangulated category, and suppose we have a morphism of candidate triangles (call the first row T and the second row T')

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

Then we have a third candidate triangle

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma(Y \oplus X')$$

This new candidate triangle is called the *mapping cone* of the morphism $F = (f, g, h)$ of candidate triangles, and is sometimes denoted C_F. There is a canonical morphism of candidate triangles $T' \longrightarrow C_F$

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\ Y \oplus X' \arrow[r] & Z \oplus Y' \arrow[r] & \Sigma X \oplus Z' \arrow[r] & \Sigma(Y \oplus X') \end{tikzcd}$$

Definition 11. Let $\mathcal{T}$ be a pretriangulated category, and suppose we have two morphisms $F = (f, g, h)$ and $F' = (f', g', h')$ of the same candidate triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] & Y \arrow[r, "v"] \arrow[d, bend right, "g"'] \arrow[d, bend left, "g'"] & Z \arrow[r, "w"] \arrow[d, bend right, "h"'] \arrow[d, bend left, "h'"] & \Sigma X \arrow[d, bend right, "\Sigma f"'] \arrow[d, bend left, "\Sigma f'"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

A *homotopy* $(\Theta, \Phi, \Psi) : F \longrightarrow F'$ is a collection of three morphisms as depicted in the following diagram

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[dl, "\Theta"'] & Z \arrow[r, "w"] \arrow[dl, "\Phi"'] & \Sigma X \arrow[dl, "\Psi"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

which satisfy

$$\begin{aligned} f - f' &= \Theta u + \Sigma^{-1}(w'\Psi) \\ g - g' &= \Phi v + u'\Theta \\ h - h' &= \Psi w + v'\Phi \end{aligned}$$

If such a homotopy exists then we say that the morphisms of candidate triangles are *homotopic* and if necessary write $F \simeq F'$. If (Θ, Φ, Ψ) is a homotopy as above, then $(-\Theta, -\Phi, -\Psi) : F' \longrightarrow F$ is another homotopy. One checks that the relation of homotopy is an equivalence relation on the set of morphisms between the two candidate triangles.

Lemma 19. *Let $\mathcal{T}$ be a pretriangulated category and suppose we two morphisms of candidate triangles $F, F' : T \longrightarrow T'$. For each homotopy $F \simeq F'$ there is a canonical isomorphism $C_F \cong C_{F'}$ of the mapping cones fitting into the following commutative diagram*

$$\begin{tikzcd} T' \arrow[r] \arrow[dr] & C_F \arrow[d, Rightarrow] \\ & C_{F'} \end{tikzcd}$$

Proof. Let a homotopy $(\Theta, \Phi, \Psi) : (f, g, h) \longrightarrow (f', g', h')$ be given. Then one checks easily that the following diagram is the required canonical isomorphism of the mapping cones

$$\begin{tikzcd} Y \oplus X' \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \Theta & 1 \end{pmatrix}"'] & Z \oplus Y' \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \Phi & 1 \end{pmatrix}"'] & \Sigma X \oplus Z' \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \Psi & 1 \end{pmatrix}"'] & \Sigma(Y \oplus X') \arrow[d, "\begin{pmatrix} 1 & 0 \\ \Sigma\Theta & 1 \end{pmatrix}"'] \\ Y \oplus X' \arrow[r] & Z \oplus Y' \arrow[r] & \Sigma X \oplus Z' \arrow[r] & \Sigma(Y \oplus X') \end{tikzcd}$$

which clearly fits into the given commutative diagram. □

Lemma 20. *Let $\mathcal{T}$ be a pretriangulated category and suppose $F, F' : T \longrightarrow T'$ are homotopic morphisms of candidate triangles. Then for any morphism of candidate triangles $G : S \longrightarrow T$ we have $F \circ G \simeq F' \circ G$. Similarly if $H : T' \longrightarrow S$ is a morphism of candidate triangles, $H \circ F \simeq H \circ F'$.*

Proof. Suppose $F = (f, g, h), F' = (f', g', h'), G = (q, r, s)$ and let a homotopy $(\Theta, \Phi, \Psi) : F \longrightarrow F'$ be given. Then the morphisms $(\Theta r, \Phi s, \Psi \circ \Sigma q)$ are a homotopy $F \circ G \simeq F' \circ G$. Similarly if $H = (a, b, c)$ then the morphisms $(a\Theta, b\Phi, c\Psi)$ are a homotopy $H \circ F \simeq H \circ F'$. □

Definition 12. Let $\mathcal{T}$ be a pretriangulated category. A candidate triangle C is called *contractible* if the idenity morphism $1 : C \longrightarrow C$ is homotopic to the zero morphism $0 : C \longrightarrow C$.

Example 2. Let $\mathcal{T}$ be a pretriangulated category with objects X, Y. Then the following triangle is easily checked to be contractible

$$X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{0} \Sigma X$$

Lemma 21. *Let $\mathcal{T}$ be a pretriangulated category. If C is a contractible candidate triangle, then any morphism $C \longrightarrow D$ or $D \longrightarrow C$ of candidate triangles is homotopic to the zero morphism.*

Lemma 22. *Let $\mathcal{T}$ be a pretriangulated category. If C is a contractible candidate triangle, then C is a pretriangle.*

Proof. Let $H : \mathcal{T} \longrightarrow \mathcal{A}$ be any additive functor into an abelian category and suppose we have a contractible candidate triangle C

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

By definition we have morphisms $\Theta : Y \longrightarrow X, \Phi : Z \longrightarrow Y, \Psi : \Sigma X \longrightarrow Z$ with

$$\begin{aligned} 1_X &= \Theta u + \Sigma^{-1}(w\Psi) \\ 1_Y &= \Phi v + u\Theta \\ 1_Z &= \Psi w + v\Phi \end{aligned}$$

Applying H we deduce that the identity morphism on the following infinite sequence is chain homotopic to the zero morphism

$$\cdots \longrightarrow H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}w)} H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(\Sigma X) \longrightarrow \cdots$$

It is therefore exact, which completes the proof. □

Proposition 23. *Let $\mathcal{T}$ be a pretriangulated category and C a contractible candidate triangle. Then C is a distinguished triangle.*

Proof. Given our contractible candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

we can complete $u : X \longrightarrow Y$ to a triangle

$$X \xrightarrow{u} Y \xrightarrow{v'} Q \xrightarrow{w'} \Sigma X$$

Since $Hom(\Sigma X, -)$ is a homological functor, we have an exact sequence

$$Hom(\Sigma X, Q) \longrightarrow Hom(\Sigma X, \Sigma X) \longrightarrow Hom(\Sigma X, \Sigma Y)$$

Let (Θ, Φ, Ψ) be a homotopy of 1_C to the zero morphism, and consider the morphism $w\Psi : \Sigma X \longrightarrow \Sigma X$. We have $\Sigma u \circ (w\Psi) = 0$ and so by exactness of the above sequence there is a morphism $\Psi' : \Sigma X \longrightarrow Q$ with $w'\Psi' = w\Psi$. One now checks that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "1"'] & Y \arrow[r, "v"] \arrow[d, "1"'] & Z \arrow[r, "w"] \arrow[d, "\mu"'] & \Sigma X \arrow[d, "1"'] \\
X \arrow[r, "u"'] & Y \arrow[r, "v'"'] & Q \arrow[r, "w'"'] & \Sigma X
\end{tikzcd}$$

where $\mu = \Psi' w + v'\Phi$. From Lemma 5 we infer that this is an isomorphism of candidate triangles, which shows that C is a distinguished triangle. $\square$

Remark 18. Since any contractible candidate triangle is a triangle, we refer to them simply as *contractible triangles.*

Lemma 24. *Let $\mathcal{T}$ be a pretriangulated category. The mapping cone on the zero morphism between triangles is a triangle.*

Proof. Consider the zero morphism on the following triangle

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "0"'] & Y \arrow[r, "v"] \arrow[d, "0"'] & Z \arrow[r, "w"] \arrow[d, "0"'] & \Sigma X \arrow[d, "0"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

The mapping cone is the following sequence

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ 0 & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ 0 & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ 0 & w' \end{pmatrix}} \Sigma(Y \oplus X')$$

This is just the direct sum of the following two triangles

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

and is therefore itself a triangle by Proposition 3. $\square$

Corollary 25. *Let $\mathcal{T}$ be a pretriangulated category and $F : T \longrightarrow T'$ a morphism of triangles homotopic to zero. Then the mapping cone C_F is a triangle.*

Proof. This follows from Lemma 24 and Lemma 19. $\square$

Corollary 26. *Let $\mathcal{T}$ be a pretriangulated category and suppose we have a morphism of triangles*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

in which at least one row is contractible. Then the mapping cone is a triangle.

Proof. Since at least one row is contractible, it follows from Lemma 21 that the morphism is homotopic to zero, and therefore by Corollary 25 the mapping cone is a triangle. $\square$

This is as far as one gets without further assumptions. Now we come to the main definition of this section:

Definition 13. Let $\mathcal{T}$ be a pretriangulated category. Then $\mathcal{T}$ is *triangulated* if it satisfies the following condition

TR4': Given any commutative diagram in which the rows are triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd} \tag{12}$$

there is by TR3 a morphism $h : Z \longrightarrow Z'$ making the diagram commute. This h may be chosen so that the mapping cone is a triangle

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma(Y \oplus X')$$

Definition 14. Let $\mathcal{T}$ be a pretriangulated category. A morphism of triangles is *good* if its mapping cone is a triangle. One checks that good morphisms are stable under composition with isomorphisms of candidate triangles on either end.

Remark 19. Any twist of a good morphism of triangles is also good, in the following sense. Given a good morphism of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd} \tag{13}$$

The following morphisms of triangles are also good

$$\begin{tikzcd} Y \arrow[r, "-v"] \arrow[d, "-g"'] & Z \arrow[r, "-w"] \arrow[d, "-h"'] & \Sigma X \arrow[r, "-\Sigma u"] \arrow[d, "-\Sigma f"'] & \Sigma Y \arrow[d, "-\Sigma g"'] \\ Y' \arrow[r, "-v'"'] & Z' \arrow[r, "-w'"'] & \Sigma X' \arrow[r, "-\Sigma u'"'] & \Sigma Y' \end{tikzcd} \qquad \begin{tikzcd} \Sigma^{-1} Z \arrow[r, "-\Sigma^{-1} w"] \arrow[d, "-\Sigma^{-1} h"'] & X \arrow[r, "-u"] \arrow[d, "-f"'] & Y \arrow[r, "-v"] \arrow[d, "-g"'] & Z \arrow[d, "-h"'] \\ \Sigma^{-1} Z' \arrow[r, "-\Sigma^{-1} w'"'] & X' \arrow[r, "-u'"'] & Y' \arrow[r, "-v'"'] & Z' \end{tikzcd}$$

One can also alternate signs on any two of the morphisms f, g, h and still have a good morphism of triangles, provided one also alternates the signs on the triangles in a consistent way. To be precise, the following three morphisms of triangles are good

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "-v"] \arrow[d, "-g"'] & Z \arrow[r, "-w"] \arrow[d, "-h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\
X' \arrow[r, "-u'"'] & Y' \arrow[r, "-v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "-u"] \arrow[d, "-f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "-w"] \arrow[d, "-h"'] & \Sigma X \arrow[d, "-\Sigma f"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "-v'"'] & Z' \arrow[r, "-w'"'] & \Sigma X'
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "-u"] \arrow[d, "-f"'] & Y \arrow[r, "-v"] \arrow[d, "-g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "-\Sigma f"'] \\
X' \arrow[r, "-u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "-w'"'] & \Sigma X'
\end{tikzcd}$$

Remark 20. In this language, TR4' can be restated as saying that any commutative diagram (12) in which the rows are triangles, may be completed to a good morphism of triangles. One checks that the pretriangulated category $\mathcal{T}^{\mathrm{op}}$ is actually triangulated.

Next we discuss homotopy cartesian squares. First we review the usual construction of pushouts in an additive category, to motivate the new definition. Let $\mathcal{A}$ be an additive category, and suppose we have morphisms $f : Y \longrightarrow Z, g : Y \longrightarrow Y'$. Take the biproduct $Y' \oplus Z$ and let $\gamma : Y' \oplus Z \longrightarrow Z'$ be the cokernel of the morphism $\left(\begin{smallmatrix} g \\ -f \end{smallmatrix}\right) : Y \longrightarrow Y' \oplus Z$. We write $\gamma = (f' \;\; g')$ for some morphisms $f' : Y' \longrightarrow Z', g' : Z \longrightarrow Z'$. Then it is easily checked that the following square is a pushout

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\
Y' \arrow[r, "f'"'] & Z'
\end{tikzcd}$$

Using the by now familiar idea of replacing the usual cokernel with a homotopy cokernel, we can define homotopy pushouts. In fact we will see (as was the case with homotopy kernels and cokernels) the homotopy pushout and pullback can be defined on the same objects, which is why we introduce the idea of a *homotopy cartesian square*.

Definition 15. Let $\mathcal{T}$ be a triangulated category. A commutative square

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\
Y' \arrow[r, "f'"'] & Z'
\end{tikzcd} \tag{14}$$

is *homotopy cartesian* if there is a distinguished triangle of the following form

$$Y \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z' \longrightarrow \Sigma Y \tag{15}$$

This is equivalent to saying that $(f' \;\; g')$ is the homotopy cokernel of $\left(\begin{smallmatrix} g \\ -f \end{smallmatrix}\right)$ (which is the same as being the homotopy cokernel of $\left(\begin{smallmatrix} -g \\ f \end{smallmatrix}\right)$ by Remark 12). In this situation we call the triple (Y, f, g)

the *homotopy pullback* of (Z', f', g'), and we call the latter triple the *homotopy pushout* of the former. It follows from Remark 2 that if you replace f', g' by $-f', -g'$ (or f, g by $-f, -g$) the diagram is still homotopy cartesian.

Remark 21. Let $\mathcal{T}$ be a triangulated category. Then the first commutative diagram below is homotopy cartesian in $\mathcal{T}$ if and only if the second diagram is homotopy cartesian in $\mathcal{T}^{\mathrm{op}}$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\ Y' \arrow[r, "f'"'] & Z' \end{tikzcd} \qquad \begin{tikzcd} Y & Z \arrow[l, "f"'] \\ Y' \arrow[u, "-g"] & Z' \arrow[l, "-f'"] \arrow[u, "g'"'] \end{tikzcd}$$

Remark 22. Let $\mathcal{T}$ be a triangulated category. Given a pair of morphisms

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \\ Y' & \end{tikzcd}$$

it is clear that a homotopy pushout exists, since we can complete the morphism $\left(\begin{smallmatrix} g \\ -f \end{smallmatrix}\right)$ to a distinguished triangle. Dually, every pair of morphisms

$$\begin{tikzcd} & Z \arrow[d, "g'"] \\ Y' \arrow[r, "f'"'] & Z' \end{tikzcd}$$

must have a homotopy pullback. It follows from Remark 8 that homotopy pullbacks and pushouts are unique up to (non-canonical) isomorphism.

Lemma 27. *Let $\mathcal{T}$ be a triangulated category and suppose we have a homotopy cartesian square*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\ Y' \arrow[r, "f'"'] & Z' \end{tikzcd}$$

Then we have

(i) Given morphisms $\alpha : P \longrightarrow Y', \beta : P \longrightarrow Z$ with $g'\beta = f'\alpha$ there is a morphism $\gamma : P \longrightarrow Y$ such that $f\gamma = \beta, g\gamma = \alpha$.

(ii) Given morphisms $\alpha : Y' \longrightarrow Q, \beta : Z \longrightarrow Q$ with $\alpha g = \beta f$ there is a morphism $\gamma : Z' \longrightarrow Q$ such that $\gamma g' = \beta, \gamma f' = \alpha$.

Proof. By duality it suffices to prove (ii). We have a triangle (15) which the cohomological functor $Hom(-, P) : \mathcal{T} \longrightarrow \mathbf{Ab}$ takes to a long exact sequence. We are given a morphism $(\alpha \ \ \beta)$ in $Hom(Y' \oplus Z, P)$ whose image in $Hom(Y, P)$ is zero. We conclude that there is a morphism $\gamma : Y \longrightarrow P$ with the required property. $\square$

Lemma 28. *Let $\mathcal{T}$ be a triangulated category and suppose we have a commutative diagram with triangles for rows*

$$\begin{tikzcd} & Y \arrow[r, "v"] \arrow[dd, "g"] & Z \arrow[dr, "w"] & \\ X \arrow[ur, "f"] \arrow[dr] & & & \Sigma X \\ & Y' \arrow[r, "v'"'] & Z' \arrow[ur, "w'"'] & \end{tikzcd} \tag{16}$$

It may be completed to a morphism of triangles such that the middle square is homotopy cartesian.

Proof. By TR4' we may complete the diagram by a morphism $h : Z \longrightarrow Z'$ to a good morphism of triangles. Then the following mapping cone is a triangle

$$X \oplus Y \longrightarrow Y' \oplus Z \longrightarrow \Sigma X \oplus Z' \longrightarrow \Sigma(X \oplus Y)$$

It follows from Lemma 17 that we can write this triangle as a direct sum of the following two candidate triangles (which are therefore by Proposition 8 themselves triangles)

$$X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{1} \Sigma X$$

$$Y \xrightarrow{\begin{pmatrix} g \\ -v \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} v' & h \end{pmatrix}} Z' \xrightarrow{(\Sigma f)w} \Sigma Y$$

The fact that the second candidate triangle is distinguished shows precisely that the middle square in (16) is homotopy cartesian. $\square$

Lemma 29. *Let $\mathcal{T}$ be a triangulated category and suppose we have a commutative diagram in which the top row is a triangle and the square is homotopy cartesian*

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "r"'] & Y' \arrow[r, "k"] \arrow[d, "s"] & Y'' \arrow[r, "l"] & \Sigma Y \\
Z \arrow[r, "h"'] & Z' & &
\end{tikzcd}$$

Then there is a triangle

$$Z \xrightarrow{h} Z' \longrightarrow Y'' \longrightarrow \Sigma Z$$

making the following diagram commute

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "r"'] & Y' \arrow[r, "k"] \arrow[d, "s"] & Y'' \arrow[r, "l"] \arrow[d, "1"] & \Sigma Y \arrow[d, "\Sigma r"] \\
Z \arrow[r, "h"'] & Z' \arrow[r] & Y'' \arrow[r] & \Sigma Z
\end{tikzcd}$$

Proof. By assumption we have a commutative diagram in which both rows are triangles

$$\begin{tikzcd}
Y \arrow[r, "\begin{pmatrix} g \\ -r \end{pmatrix}"] \arrow[d, "1"'] & Y' \oplus Z \arrow[r, "\begin{pmatrix} s & h \end{pmatrix}"] \arrow[d] & Z' \arrow[r] & \Sigma Y \arrow[d, "1"] \\
Y \arrow[r, "g"'] & Y' \arrow[r] & Y'' \arrow[r] & \Sigma Y
\end{tikzcd}$$

By TR4' there exists a morphism $m : Z' \longrightarrow Y''$ completing this to a good morphism of triangles. Using Lemma 17 and Lemma 18 we can write the mapping cone (which is a triangle) as a direct sum of the following three candidate triangles

$$Y \longrightarrow 0 \longrightarrow \Sigma Y \xrightarrow{1} \Sigma Y$$

$$Y' \xrightarrow{1} Y' \longrightarrow 0 \longrightarrow \Sigma Y'$$

$$Z \xrightarrow{-h} Z' \xrightarrow{m} Y'' \xrightarrow{-(\Sigma r)l} \Sigma Z$$

which by Proposition 8 must themselves be triangles. The last triangle remains distinguished if we change the sign on the outer two morphisms, so we have constructed a triangle

$$Z \xrightarrow{h} Z' \xrightarrow{m} Y'' \xrightarrow{(\Sigma r)l} \Sigma Z$$

which clearly makes the required diagram commute. $\square$

Proposition 30. *Let $\mathcal{T}$ be a triangulated category, $f : X \longrightarrow Y$ and $g : Y \longrightarrow Y'$ morphisms, and suppose we are given triangles*

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "a"] & Z \arrow[r, "b"] & \Sigma X \\
X \arrow[r, "gf"] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X \\
Y \arrow[r, "g"] & Y' \arrow[r] & Y'' \arrow[r] & \Sigma Y
\end{tikzcd}
$$

Then we can complete this to a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1"'] & Y \arrow[r, "a"] \arrow[d, "g"'] \arrow[dr, phantom, "\circledast"] & Z \arrow[r, "b"] \arrow[d] & \Sigma X \arrow[d, "1"'] \\
X \arrow[r, "gf"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
0 \arrow[r] \arrow[d] & Y'' \arrow[r, "1"] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma a"] & \Sigma Z \arrow[r, "-\Sigma b"] & \Sigma^2 X
\end{tikzcd}
\tag{17}
$$

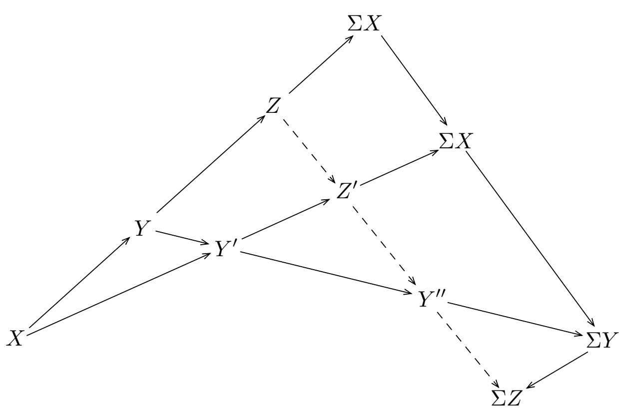
where every row and column is a distinguished triangle and $\circledast$ is homotopy cartesian.

Proof. The only morphisms we need to add are the vertical morphisms in the third column. It is clear that every row and every other column is already a triangle. By Lemma 28 we can find a morphism $h : Z \longrightarrow Z'$ making the first row (of squares) into a good morphism of triangles such that $\circledast$ is homotopy cartesian. Then by applying Lemma 29 to the following diagram

$$
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "a"'] \arrow[dr, phantom, "\circledast"] & Y' \arrow[r] \arrow[d] & Y'' \arrow[r] & \Sigma Y \\
Z \arrow[r] & Z'
\end{tikzcd}
$$

we find that there exists a morphism $m : Z' \longrightarrow Y''$ with the required properties. $\square$

Remark 23. This result is also known as the Octahedral Axiom. A more emotive way to draw the involved triangles is shown in the following diagram



From Proposition 30 we know that given the three solid triangles we can construct a fourth triangle fitting into the above commutative diagram. Commutativity of every cell except for the square $Z', \Sigma X, \Sigma Y, \Sigma Z$ is immediate from the statement of Proposition 30 and commutativity of the remaining square follows from a careful analysis of the proof.

We deduce immediately the dual result.

Corollary 31. *Let $\mathcal{T}$ be a triangulated category, $f : X \longrightarrow Y$ and $g : Y \longrightarrow Y'$ morphisms, and suppose we are given triangles as in Proposition 30. Then there exists a commutative diagram*

$$\begin{tikzcd}
\Sigma^{-1}Z \arrow[r] \arrow[d, "-\Sigma^{-1}b"'] & 0 \arrow[r] \arrow[d] & Z \arrow[r, "1"] \arrow[d] & Z \arrow[d, "b"] \\
X \arrow[r, "gf"] \arrow[d, "f"'] & Y' \arrow[r] \arrow[d, "1"] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, "g"] \arrow[d, "a"'] & Y' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & \Sigma Y \arrow[d, "\Sigma a"] \\
Z \arrow[r] & 0 \arrow[r] & \Sigma Z \arrow[r, "1"] & \Sigma Z
\end{tikzcd}$$

where every row and column is a distinguished triangle.

Corollary 32. *Let $\mathcal{T}$ be a triangulated category. Any commutative diagram*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

can be extended to a diagram of the form

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circledast"] & \Sigma X'' \arrow[d, "-"] \\
\Sigma X' \arrow[r] & \Sigma Y' \arrow[r] & \Sigma Z' \arrow[r, "-"'] & \Sigma^2 X'
\end{tikzcd}$$

where every row and column is a distinguished triangle and every square commutes, except for $\circledast$ which anticommutes.

Proof. Choose arbitrary extensions to triangles of the sides of the original commutative diagram as well as the equal composites $X' \longrightarrow Y$, as follows

$$X' \longrightarrow Y' \longrightarrow Z' \longrightarrow \Sigma X' \tag{18}$$
$$X' \longrightarrow Y \longrightarrow A \longrightarrow \Sigma X' \tag{19}$$
$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X' \tag{20}$$
$$Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow \Sigma Y' \tag{21}$$
$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{22}$$

First apply Remark 23 to the three triangles (18), (19), (21) to obtain a triangle

$$Z' \longrightarrow A \longrightarrow Y'' \longrightarrow \Sigma Z' \tag{23}$$

which yields a second shifted triangle

$$A \longrightarrow Y'' \longrightarrow \Sigma Z' \xrightarrow{-} \Sigma A \tag{24}$$

Now apply Remark 23 to the triangles (19), (19), (22) to obtain a triangle

$$X'' \longrightarrow A \longrightarrow Z \longrightarrow \Sigma X'' \tag{25}$$

This defines morphisms $e : A \longrightarrow Y''$ and $f : X'' \longrightarrow A$. Extend $ef : X'' \longrightarrow Y''$ to a triangle

$$X'' \longrightarrow Y'' \longrightarrow Z'' \longrightarrow \Sigma X'' \tag{26}$$

and use Remark 23 one last time on (25), (26), (24) to obtain a triangle $Z' \longrightarrow Z \longrightarrow Z'' \longrightarrow \Sigma Z'$. This constructs all the necessary triangles and one checks they fit into the diagram above as claimed. $\square$

1.3 Triangulated Subcategories

Definition 16. Let $\mathcal{T}$ be a triangulated category. A full additive subcategory $\mathcal{S}$ in $\mathcal{T}$ is called a *triangulated subcategory* if it is replete, if $\Sigma\mathcal{S} = \mathcal{S}$, and if for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{27}$$

such that X, Y are in $\mathcal{S}$, the object Z is also in $\mathcal{S}$ (we refer to this last condition by saying $\mathcal{S}$ is *closed under mapping cones*).

Remark 24. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory, with $i : \mathcal{S} \longrightarrow \mathcal{T}$ the inclusion functor. Since i is additive it preserves finite products and coproducts, which means that $\mathcal{S}$ is closed under finite products and coproducts of its objects in $\mathcal{T}$.

By definition the additive automorphism Σ on $\mathcal{T}$ restricts to an automorphism on $\mathcal{S}$. If we define a candidate triangle in $\mathcal{S}$ to be distinguished if it is distinguished in $\mathcal{T}$, then it is easily checked that $\mathcal{S}$ is indeed a triangulated category. If $\mathcal{S}'$ is a triangulated subcategory of $\mathcal{S}$, then it is clearly also a triangulated subcategory of $\mathcal{T}$.

Remark 25. One checks easily that if $\mathcal{S}$ is a triangulated subcategory of $\mathcal{T}$ and we have a distinguished triangle (27) in $\mathcal{T}$, and if any two of the objects X, Y, Z are in $\mathcal{S}$, then so is the third.

Remark 26. Let $\mathcal{T}$ be a triangulated category with triangulated subcategory $\mathcal{S}$. Then $\mathcal{S}^{\mathrm{op}}$ is a triangulated subcategory of $\mathcal{T}^{\mathrm{op}}$. A nonempty intersection of triangulated subcategories is still a triangulated subcategory.

Lemma 33. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full replete subcategory. Then $\mathcal{S}$ is a triangulated subcategory if and only if it is closed under Σ^{-1} and mapping cones.*

Proof. The conditions are certainly necessary. Suppose that $\mathcal{S}$ is full, replete and closed under mapping cones and Σ^{-1}. By convention all our categories are nonempty, so choose some $X \in \mathcal{S}$. From the following triangles

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

$$X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{1} \Sigma X$$

we deduce that $\mathcal{S}$ contains all the zero objects of $\mathcal{T}$, and that $\mathcal{S}$ is closed under Σ. Given objects $X, Z \in \mathcal{T}$ we have by Remark 11 the following triangle in $\mathcal{T}$

$$\Sigma^{-1}Z \xrightarrow{0} X \longrightarrow X \oplus Z \longrightarrow Z$$

from which we deduce that $\mathcal{S}$ is closed under finite products and coproducts in $\mathcal{T}$, and is therefore additive. This completes the verification that $\mathcal{S}$ is a triangulated subcategory. $\square$

Definition 17. Let $\mathcal{T}$ be a triangulated category, $\mathcal{S}$ a triangulated subcategory. We define a class of morphisms $Mor_{\mathcal{S}}$ of $\mathcal{T}$ by the following rule. A morphism $f : X \longrightarrow Y$ belongs to $Mor_{\mathcal{S}}$ if and only if there exists a triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

with Z an object of $\mathcal{S}$. Every morphism of $\mathcal{S}$ belongs to $Mor_{\mathcal{S}}$, and it is also clear that the subclass $Mor_{\mathcal{S}^{\mathrm{op}}}$ of $\mathcal{T}^{\mathrm{op}}$ consists of the same morphisms as $Mor_{\mathcal{S}}$. A morphism $f : X \longrightarrow Y$ becomes to $Mor_{\mathcal{S}}$ if and only if $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ does.

Lemma 34. *Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $\mathcal{T}$. Then every isomorphism $f : X \longrightarrow Y$ belongs to $Mor_{\mathcal{S}}$.*

Proof. This follows from Lemma 9 and the fact that since $\mathcal{S}$ is closed under finite products and coproducts, it in particular contains every zero object of $\mathcal{T}$. $\square$

Lemma 35. *Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $\mathcal{T}$ and let $f : X \longrightarrow Y, g : Y \longrightarrow Y'$ be morphisms in $\mathcal{T}$. If any two of f, g, gf belong to $Mor_{\mathcal{S}}$, then so does the third.*

Proof. By Proposition 30 there is a diagram of triangles (17). Now f is in $Mor_{\mathcal{S}}$ if and only if Z is in $\mathcal{S}$, gf is in $Mor_{\mathcal{S}}$ if and only if Z' is in $\mathcal{S}$, and g is in $Mor_{\mathcal{S}}$ if and only if Y'' is in $\mathcal{S}$. From the triangle

$$Z \longrightarrow Z' \longrightarrow Y'' \longrightarrow \Sigma Z$$

we learn that if any two of Z, Z', Y'' lie in $\mathcal{S}$, then so does the third. $\square$

Lemma 36. *If $\mathcal{S}$ is a triangulated subcategory of a triangulated category $\mathcal{T}$ then there is a subcategory of $\mathcal{T}$ whose objects are all the objects of $\mathcal{T}$, and whose morphisms are the ones in $Mor_{\mathcal{S}}$.*

Proof. All the identities of $\mathcal{T}$ belong to $Mor_{\mathcal{S}}$ by Lemma 34. It follows from Lemma 35 that $Mor_{\mathcal{S}}$ is closed under composition, so we can consider $Mor_{\mathcal{S}}$ as a subcategory of $\mathcal{T}$. $\square$

Lemma 37. *Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $\mathcal{T}$, and suppose we have a homotopy cartesian square*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\ Y' \arrow[r, "f'"'] & Z' \end{tikzcd}$$

Then g' belongs to $Mor_{\mathcal{S}}$ if and only if g does. That is, morphisms in $Mor_{\mathcal{S}}$ are stable under homotopy pushout and pullback.

Proof. By Lemma 29 the homotopy cartesian square may be completed to a morphism of triangles

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f"'] & Y' \arrow[r] \arrow[d, "f'"'] & Y'' \arrow[r] \arrow[d, "1"'] & \Sigma Y \arrow[d, "\Sigma f"'] \\ Y' \arrow[r, "g'"'] & Z' \arrow[r] & Y'' \arrow[r] & \Sigma Z \end{tikzcd}$$

Now Y'' lies in $\mathcal{S}$ precisely if both morphisms g, g' are in $Mor_{\mathcal{S}}$, which proves that g is in $Mor_{\mathcal{S}}$ if and only if g' is. $\square$

2 Triangulated Functors

Definition 18. Let $\mathcal{D}, \mathcal{E}$ be triangulated categories. A *triangulated functor* $F : \mathcal{D} \longrightarrow \mathcal{E}$ is an additive functor together with a natural equivalence $\phi : F\Sigma \longrightarrow \Sigma F$, where Σ denotes the respective automorphisms on $\mathcal{D}, \mathcal{E}$, with the property that for any distinguished triangle in $\mathcal{D}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the following candidate triangle is distinguished in $\mathcal{E}$

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\phi_X \circ F(w)} \Sigma F(X)$$

By abuse of notation we simply say the functor F is triangulated, and drop ϕ from the notation, whenever this will not cause confusion. Note that whenever we say two triangulated functors are *equal*, we mean the underlying functors are equal *and* the natural equivalences are the same. If we simply mean they are equal as additive functors, we will say so explicitly.

The identity functor $1 : \mathcal{D} \longrightarrow \mathcal{D}$ together with the identity transformation $\phi : 1\Sigma \longrightarrow \Sigma 1$ is a triangulated functor $\mathcal{D} \longrightarrow \mathcal{D}$. Given triangulated functors $F : \mathcal{D} \longrightarrow \mathcal{E}, G : \mathcal{E} \longrightarrow \mathcal{F}$ with corresponding natural equivalences ϕ, ψ we define a new triangulated functor $GF : \mathcal{D} \longrightarrow \mathcal{F}$ whose underlying functor is the usual composite, and whose natural transformation $\gamma : (GF)\Sigma \longrightarrow \Sigma(GF)$ is defined for an object X by $\gamma_X = \psi_{FX} G(\phi_X)$. This composition of triangulated functors is associative, and the identity triangulated functors clearly act as identities.

Definition 19. Let $(F, \phi), (G, \psi) : \mathcal{D} \longrightarrow \mathcal{E}$ be triangulated functors. A *trinatural transformation* $\eta : (F, \phi) \longrightarrow (G, \psi)$ is a natural transformation $\eta : F \longrightarrow G$ of the additive functors which has the additional property that for every object $X \in \mathcal{D}$ the following diagram commutes

$$\begin{tikzcd}
F\Sigma(X) \arrow[r, "\phi_X"] \arrow[d, "\eta_{\Sigma X}"'] & \Sigma F(X) \arrow[d, "\Sigma \eta_X"] \\
G\Sigma(X) \arrow[r, "\psi_X"'] & \Sigma G(X)
\end{tikzcd}$$

If $\eta, \eta' : (F, \phi) \longrightarrow (G, \psi)$ are two trinatural transformations then so is their sum and difference, so the trinatural transformations $(F, \phi) \longrightarrow (G, \psi)$ form an abelian group (this is a subgroup of the natural transformations $F \longrightarrow G$). Given trinatural transformations $\eta : (F, \phi) \longrightarrow (G, \psi)$ and $\tau : (G, \psi) \longrightarrow (H, \kappa)$ the composite $\tau \circ \eta$ is a trinatural transformation $(F, \phi) \longrightarrow (H, \kappa)$. The identity transformation $1 : (F, \phi) \longrightarrow (F, \phi)$ is certainly a trinatural transformation, and we call a trinatural transformation $\eta : (F, \phi) \longrightarrow (G, \psi)$ a *trinatural equivalence* if there is a trinatural transformation $\kappa : (G, \psi) \longrightarrow (F, \phi)$ with $\eta\kappa = 1$ and $\kappa\eta = 1$.

This notation is unambiguous, since it is clear that a trinatural transformation η is a trinatural equivalence if and only if is a natural equivalence (that is, η_X is an isomorphism for every X).

Remark 27. Let $F : \mathcal{D} \longrightarrow \mathcal{E}$ and $G, H : \mathcal{Q} \longrightarrow \mathcal{D}$ be triangulated functors, and suppose we have a trinatural transformation $\eta : G \longrightarrow H$. Then the natural transformation $F\eta : FG \longrightarrow FH$ is a trinatural transformation. In particular if $G \cong H$ are trinaturally equivalent then $FG \cong FH$ are trinaturally equivalent.

Similarly if $G, H : \mathcal{E} \longrightarrow \mathcal{Q}$ are triangulated functors and $\eta : G \longrightarrow H$ a trinatural transformation, then the natural transformation $\eta F : GF \longrightarrow HF$ is trinatural. In particular if $G \cong H$ are trinaturally equivalent then $GF \cong HF$ are trinaturally equivalent.

Remark 28. Let $\mathcal{D}, \mathcal{E}$ be triangulated categories. If a pair $(F, \phi) : \mathcal{D} \longrightarrow \mathcal{E}$ is a triangulated functor then the induced functor $F^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{E}^{\mathrm{op}}$ and natural equivalence $\phi^{\mathrm{op}} : F^{\mathrm{op}}\Sigma^{-1} \longrightarrow \Sigma^{-1}F^{\mathrm{op}}$ defined by $\phi^{\mathrm{op}}_X = \Sigma^{-1}\phi_{\Sigma^{-1}X}$ is a triangulated functor between the duals. Obviously dualising twice yields the original pair (F, ϕ). Given triangulated functors $F : \mathcal{D} \longrightarrow \mathcal{E}, G : \mathcal{E} \longrightarrow \mathcal{F}$ we have $(GF)^{\mathrm{op}} = G^{\mathrm{op}} \circ F^{\mathrm{op}}$, and the dual of the identity triangulated functor is the identity triangulated functor on the dual.

If $F, G : \mathcal{D} \longrightarrow \mathcal{E}$ are triangulated functors and $\eta : F \longrightarrow G$ a trinatural transformation, then $\eta^{\mathrm{op}} : G^{\mathrm{op}} \longrightarrow F^{\mathrm{op}}$ defined by $\eta^{\mathrm{op}}_X = \eta_X$ is a trinatural transformation. It is clear that $(\eta\psi)^{\mathrm{op}} = \psi^{\mathrm{op}}\eta^{\mathrm{op}}$ and $(\eta H)^{\mathrm{op}} = \eta^{\mathrm{op}} H^{\mathrm{op}}$, $(H\eta)^{\mathrm{op}} = H^{\mathrm{op}}\eta^{\mathrm{op}}$.

Definition 20. Let $\mathcal{T}$ be a triangulated category. A *fragile triangulated subcategory* is a triangulated functor $j : \mathcal{S} \longrightarrow \mathcal{T}$ which is a full subcategory with the additional property that the suspension functor on $\mathcal{S}$ is the restriction of the suspension functor on $\mathcal{T}$, and the natural equivalence $j\Sigma \longrightarrow \Sigma j$ is the identity. One checks that a candidate triangle in $\mathcal{S}$ is distinguished in $\mathcal{S}$ if and only if it is distinguished in $\mathcal{T}$, so a fragile triangulated subcategory is specificed completely by its class of objects.

Example 3. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. Then the inclusion $i : \mathcal{S} \longrightarrow \mathcal{T}$ together with the identity $1 : i\Sigma \longrightarrow \Sigma i$ is a triangulated functor, and this functor is a fragile triangulated subcategory. In fact, triangulated subcategories and replete fragile triangulated subcategories are the same thing. If $\mathcal{T}'$ is a fragile triangulated subcategory of $\mathcal{T}$ then it is easy to check that $\mathcal{S} \cap \mathcal{T}'$ is a triangulated subcategory of $\mathcal{T}'$.

Example 4. For examples of fragile triangulated subcategories which are not replete, see the results (DTC,Remark 7) and (DTC,Proposition 34).

Definition 21. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. The full subcategory of $\mathcal{D}$ consisting of objects X isomorphic to $F(Y)$ for some $Y \in \mathcal{C}$ is called the *essential image* of F.

Remark 29. Let $F : \mathcal{D} \longrightarrow \mathcal{E}$ be a triangulated functor and $\mathcal{S} \subseteq \mathcal{E}$ a fragile triangulated subcategory containing $F(X)$ for every $X \in \mathcal{D}$. Then there is a unique triangulated functor $\mathcal{D} \longrightarrow \mathcal{S}$ making the following diagram commute

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "F"] \arrow[dr] & \mathcal{E} \\
& \mathcal{S} \arrow[u]
\end{tikzcd}$$

On the other hand, given a *full* triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{E}$ the essential image $\mathcal{S}$ is a triangulated subcategory of $\mathcal{E}$, through which F certainly factors.

Definition 22. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor. The *kernel* of F is the full subcategory $\mathcal{C}$ of $\mathcal{D}$ consisting of those objects X with $F(X) = 0$.

Lemma 38. *Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor. Then the kernel $\mathcal{C}$ of F is a triangulated subcategory of $\mathcal{D}$.*

Proof. The kernel $\mathcal{C}$ is clearly a full additive replete subcategory of $\mathcal{D}$. For any object X we have an isomorphism $F\Sigma(X) \cong \Sigma F(X)$, so X belongs to $\mathcal{C}$ if and only if ΣX does. Now suppose we are given a triangle in $\mathcal{D}$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

Then

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow \Sigma F(X)$$

is a triangle in $\mathcal{T}$. If $F(X), F(Y)$ are zero then by Remark 8 the object $F(Z)$ is also zero. This shows that Z is in $\mathcal{C}$, which is therefore a triangulated subcategory. $\square$

Lemma 39. *Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor with kernel $\mathcal{C}$. If $X \oplus Y$ is an object of $\mathcal{C}$, then so are the direct summands X, Y.*

Definition 23. A triangulated subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$ is *thick* if it contains all direct summands of its objects. The kernel of any triangulated functor is thick. A triangulated subcategory $\mathcal{C}$ is thick if and only if $\mathcal{C}^{\mathrm{op}}$ is thick. A nonempty intersection of thick subcategories is thick.

Definition 24. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor. We say that F *reflects triangles* if it has the property that given any candidate triangle in $\mathcal{D}$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{28}$$

such that the following candidate triangle is a triangle in $\mathcal{T}$

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow \Sigma F(X)$$

then (28) is a triangle in $\mathcal{D}$.

Remark 30. Let $F, G : \mathcal{D} \longrightarrow \mathcal{E}$ be triangulated functors and $\psi : F \longrightarrow G$ a trinatural transformation. If we define $\mathcal{S} = \{X \in \mathcal{D} \,|\, \psi_X \text{ is an isomorphism}\}$ then one checks that $\mathcal{S}$ is a triangulated subcategory of $\mathcal{D}$. If F, G preserve coproducts then $\mathcal{S}$ is closed under coproducts in $\mathcal{D}$.

2.1 Triadjoints

The reader should consult (AC,Definition 28) (AC,Definition 30) for the definition of reflections and coreflections along a functor.

Definition 25. A triangulated functor $L : \mathcal{B} \longrightarrow \mathcal{A}$ is *left triadjoint* to a triangulated functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ if there exists a trinatural transformation $\eta : 1_{\mathcal{B}} \longrightarrow FL$ such that for every $B \in \mathcal{B}$ the pair $(L(B), \eta_B)$ is a reflection of B along F. A trinatural transformation η with this property is called a *left triadjunction* of L to F. Clearly if L is left triadjoint to F then it is left adjoint to F, although the converse is not true in general.

Remark 31. Observe that a natural transformation $\eta : 1 \longrightarrow FL$ is trinatural if and only if for every $B \in \mathcal{B}$ we have $\phi_{F,GB} F(\phi_{G,B}) \eta_{\Sigma B} = \Sigma \eta_B$ where ϕ_F, ϕ_G are the natural equivalences given as part of the data of the triangulated functors F, G.

Definition 26. A triangulated functor $R : \mathcal{B} \longrightarrow \mathcal{A}$ is *right triadjoint* to a triangulated functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ if there exists a trinatural transformation $\varepsilon : FR \longrightarrow 1_{\mathcal{B}}$ such that for every $B \in \mathcal{B}$ the pair $(R(B), \varepsilon_B)$ is a coreflection of B along F. A trinatural transformation ε with this property is called a *right triadjunction* of F to R. Clearly if R is right triadjoint to F then it is right adjoint to F, although the converse is not true in general.

Remark 32. Observe that a natural transformation $\varepsilon : FR \longrightarrow 1$ is trinatural if and only if for every $B \in \mathcal{B}$ we have $\Sigma(\varepsilon_B) \phi_{F,RB} F(\phi_{R,A}) = \varepsilon_{\Sigma B}$ where ϕ_F, ϕ_G are the natural equivalences given as part of the data of the triangulated functors F, G.

Lemma 40. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be triangulated functors. There is a bijection between left triadjunctions of G to F and right triadjunctions of G to F. In particular G is left triadjoint to F if and only if F is right triadjoint to G.*

Proof. Let $\eta : 1 \longrightarrow FG$ be a left triadjunction of G to F. Then it is in particular a left adjunction of G to F, and therefore corresponds to a right adjunction ε of G to F as in the proof of (AC,Lemma 12). We simply check that ε is a trinatural transformation. We have to check that for every $A \in \mathcal{A}$ the following diagram commutes

$$\begin{tikzcd}
GF\Sigma A \arrow[rr, "\phi_{G,FA} G(\phi_{F,A})"] \arrow[dr, "\varepsilon_{\Sigma A}"'] & & \Sigma GFA \arrow[dl, "\Sigma \varepsilon_A"] \\
& \Sigma A &
\end{tikzcd}$$

where ϕ_F, ϕ_G are the natural equivalences given as part of the data of the triangulated functors F, G. By applying F, composing with the isomorphism $\phi_{F,A}$ and using the universal property of η, this is not difficult.

Conversely if we are given a right triadjunction $\varepsilon : GF \longrightarrow 1$ of G to F we construct the left adjunction $\eta : 1 \longrightarrow FG$ as in (AC,Lemma 12) and check that η is trinatural by an argument dual to the one given above. We already know these two assignments are mutually inverse, so the proof is complete. $\square$

Definition 27. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be triangulated functors. A *triadjunction* $G \longrightarrow\!\!\!\!\!| \; F$ is a pair (η, ε) consisting of a left triadjunction η of G to F and a right triadjunction ε of G to F with η, ε corresponding under the bijection of Lemma 40. Any triadjunction is in particular an adjunction (AC,Definition 32).

Remark 33. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be triangulated functors and suppose that we have an *adjunction* $(\eta, \varepsilon) : G \longrightarrow\!\!\!\!\!| \; F$. It follows from the proof of Lemma 40 that if η is trinatural, so is ε, and vice versa. In other words, to prove that an adjunction is a triadjunction, it suffices to show that either the unit or the counit is trinatural.

Lemma 41. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be triangulated functors and let (η, ε) be a triadjunction $G \longrightarrow\!\!\!\!\!| \; F$. The pair $(\varepsilon^{op}, \eta^{op})$ is a triadjunction $F^{op} \longrightarrow\!\!\!\!\!| \; G^{op}$.*

Theorem 42. *Consider triangulated functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$. The following conditions are equivalent*

1. *G is left triadjoint to F.*

2. *F is right triadjoint to G.*

3. *There exist trinatural transformations $\eta : 1_{\mathcal{B}} \longrightarrow FG$ and $\varepsilon : GF \longrightarrow 1_{\mathcal{A}}$ such that*

$$F\varepsilon \circ \eta F = 1_F, \qquad \varepsilon G \circ G\eta = 1_G$$

4. *There exists a family of bijections $\{\theta_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}}$*

$$\theta_{A,B} : Hom_{\mathcal{A}}(GB, A) \longrightarrow Hom_{\mathcal{B}}(B, FA)$$

which is natural in both variables and commutes with suspension. That is, for any objects $A \in \mathcal{A}, B \in \mathcal{B}$ the following diagram commutes

$$\begin{tikzcd}
Hom(G\Sigma B, A) \arrow[r, "\theta_{A,\Sigma B}"] \arrow[d] & Hom(\Sigma B, FA) \arrow[d] \\
Hom(\Sigma GB, A) \arrow[d] & Hom(B, \Sigma^{-1}FA) \arrow[d] \\
Hom(GB, \Sigma^{-1}A) \arrow[r, "\theta_{\Sigma^{-1}A,B}"'] & Hom(B, F\Sigma^{-1}A)
\end{tikzcd} \tag{29}$$

Proof. In fact we will show that there is a bijection between (a) triadjunctions $G \longrightarrow\!\!\!\!\!| \; F$, (b) pairs of trinatural transformations η, ε with the property of (3) and (c) families of bijections θ with the property of (4).

Given a triadjunction (η, ε) it is clear that η, ε satisfy the condition of (3). Conversely, suppose that a pair of trinatural transformations η, ε is given satisfying this condition. From the proof of (AC,Theorem 14) we know that (η, ε) is an adjunction, and since by assumption both natural transformations are trinatural, it is also a triadjunction. This proves the bijection $(a) \Leftrightarrow (b)$.

Let $(\eta, \varepsilon) : G \longrightarrow\!\!\!\!\!| \; F$ be a triadjunction. Given a morphism $a : GB \longrightarrow A$ we define $\theta_{A,B}(a) = F(a)\eta_B$. Given $b : B \longrightarrow FA$ we define $\tau_{A,B}(b) = \varepsilon_A G(b)$. One checks in the usual way that $\tau_{A,B} = \theta_{A,B}^{-1}$ and the map θ is natural in both variables. It only remains to show that θ commutes with suspension.

Let a morphism $a : G\Sigma B \longrightarrow A$ be given. To check that (29) commutes on a we have to show

$$F\Sigma^{-1}(a\phi_{G,B}^{-1})\eta_B = \kappa_A \Sigma^{-1}(F(a)\eta_{\Sigma B})$$

where κ denotes the natural equivalence $\Sigma^{-1}\phi_F\Sigma^{-1} : \Sigma^{-1}F \longrightarrow F\Sigma^{-1}$. Since η is trinatural we deduce that

$$\Sigma^{-1}\eta_{\Sigma B} = \Sigma^{-1}F(\phi_{G,B}^{-1})\kappa_{\Sigma GB}^{-1}\eta_B$$

while by naturality of $\phi_F : F\Sigma \longrightarrow \Sigma F$ we have $\kappa_A \Sigma^{-1} F(a) = F\Sigma^{-1}(a)\kappa_{G\Sigma B}$. Therefore

$$\begin{aligned} F\Sigma^{-1}(a\phi_{G,B}^{-1})\eta_B &= F\Sigma^{-1}(a\phi_{G,B}^{-1})\kappa_{\Sigma GB}\kappa_{\Sigma GB}^{-1}\eta_B \\ &= \kappa_A \Sigma^{-1} F(a\phi_{G,B}^{-1})\kappa_{\Sigma GB}^{-1}\eta_B \\ &= \kappa_A \Sigma^{-1} F(a)\Sigma^{-1}\eta_{\Sigma B} \end{aligned}$$

as required. So any triadjunction (η, ε) gives rise to a family of bijections θ with the properties of (4). Conversely, suppose we are given the family of bijections θ. As in the proof of (AC,Theorem 14) we deduce an adjunction (η, ε). To show that this is a triadjunction, it suffices by Remark 33 to show that η is trinatural. But this follows from commutativity of (29) on the identity $1 : G\Sigma B \longrightarrow G\Sigma B$. We have defined the required bijection $(a) \Leftrightarrow (c)$, and therefore the proof is complete. □

Remark 34. With the notation of Theorem 42 it is also true that the *dual* of the compatibility diagram (29) commutes. In other words, for any objects $A \in \mathcal{A}, B \in \mathcal{B}$ the following diagram commutes

$$\begin{tikzcd} Hom(G\Sigma^{-1}B, A) \arrow[r, "\theta_{A,\Sigma^{-1}B}"] \arrow[d] & Hom(\Sigma^{-1}B, FA) \arrow[d] \\ Hom(\Sigma^{-1}GB, A) \arrow[d] & Hom(B, \Sigma FA) \arrow[d] \\ Hom(GB, \Sigma A) \arrow[r, "\theta_{\Sigma A,B}"'] & Hom(B, F\Sigma A) \end{tikzcd} \tag{30}$$

Lemma 43. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a triangulated functor, and suppose $G_1, G_2 : \mathcal{B} \longrightarrow \mathcal{A}$ are both left triadjoint to F, with triadjunctions η_1, η_2. Then there is a canonical trinatural equivalence $\rho : G_1 \longrightarrow G_2$.*

Proof. By (AC,Lemma 15) we have a natural equivalence $\rho : G_1 \longrightarrow G_2$ and it is clear from the construction that this natural transformation and its inverse are both trinatural. □

Lemma 44. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a triangulated functor, and suppose $G_1, G_2 : \mathcal{B} \longrightarrow \mathcal{A}$ are both right triadjoint to F, with triadjunctions $\varepsilon_1, \varepsilon_2$. Then there is a canonical trinatural equivalence $\rho : G_1 \longrightarrow G_2$.*

Proof. By (AC,Lemma 16) we have a natural equivalence $\rho : G_1 \longrightarrow G_2$ and it is clear from the construction that this natural transformation and its inverse are both trinatural. □

Lemma 45. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be triangulated functors with F left triadjoint to G. If F is trinaturally equivalent to F' then F' is left triadjoint to G, and if G is trinaturally equivalent to G' then G' is right triadjoint to F.*

Proof. The proof is the same as (AC,Lemma 17), we simply have to observe that one can compose trinatural transformations, and apply triangulated functors on either side, and the result is still trinatural. □

Proposition 46. *Consider the following diagram of triangulated functors*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right, "F"'] & \mathcal{B} \arrow[l, bend right, "G"'] \arrow[r, bend right, "H"'] & \mathcal{C} \arrow[l, bend right, "K"'] \end{tikzcd}$$

where G is left triadjoint to F and K is left triadjoint to H. Then GK is left triadjoint to HF.

Proof. Choose triadjunctions $G \dashv F$ and $K \dashv H$ represented by natural families of bijections μ and θ which are compatible with suspension. Then as in (AC,Proposition 18) we define a natural family of bijections

$$\varrho_{A,C} = \theta_{FA,C}\mu_{A,KC} : Hom_{\mathcal{A}}(GKC, A) \longrightarrow Hom_{\mathcal{C}}(C, HFA)$$

One checks this family of bijections commutes with suspension, and therefore defines the required triadjunction. $\square$

Proposition 47. *Let $F : \mathcal{D} \longrightarrow \mathcal{E}$ be a triangulated functor, $G : \mathcal{E} \longrightarrow \mathcal{D}$ an ordinary functor which is left adjoint to F with unit and counit*

$$\eta : 1 \longrightarrow FG, \quad \varepsilon : GF \longrightarrow 1$$

Then G becomes a triangulated functor in a canonical way, such that η, ε are trinatural. That is, so that G is left triadjoint to F. Dually if G is right adjoint to F with unit and counit

$$\nu : 1 \longrightarrow GF, \quad \rho : FG \longrightarrow 1$$

Then G becomes a triangulated functor in a canonical way, such that ν, ρ is a triadjunction.

Proof. For clarity, let Σ be the suspension functor on $\mathcal{D}$ and Λ the suspension functor on $\mathcal{E}$. We have the following natural transformation ϕ_G

$$G\Lambda \xrightarrow{G\Lambda\eta} G\Lambda FG \xrightarrow{G\phi_F^{-1}G} GF\Sigma G \xrightarrow{\varepsilon\Sigma G} \Sigma G$$

We have the following diagram of functors

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\Sigma^{-1}"] & \mathcal{D} \arrow[l, bend left, "\Sigma"] \arrow[r, bend left, "F"] & \mathcal{E} \arrow[l, bend left, "G"] \arrow[r, bend left, "\Lambda^{-1}"] & \mathcal{E} \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$

As isomorphisms are both left and right adjoint to their inverse, we can compose adjunctions to deduce that $\Sigma G \dashv F\Sigma^{-1}$ and $G\Lambda \dashv \Lambda^{-1}F$. If $\phi : F\Sigma \longrightarrow \Lambda F$ is the natural equivalence associated to F then $\Lambda^{-1}\phi\Sigma^{-1}$ is a natural equivalence $\Lambda^{-1}F \longrightarrow F\Sigma^{-1}$. Therefore $\Sigma G, G\Lambda$ are left adjoints of the same functor, and are consequently naturally equivalent. In explicit terms, if one works out the details we arrive at a canonical natural equivalence $G\Lambda \longrightarrow \Sigma G$ defined for $E \in \mathcal{E}$ by

$$\varepsilon_{\Sigma GE} \circ G(\phi_{F,GE})^{-1} \circ G\Lambda(\eta_E)$$

That is, our natural transformation ϕ_G is a natural equivalence. Observe that the functor G must be additive, since it has a right adjoint and therefore preserves finite products. We claim that the pair (G, ϕ_G) is a triangulated functor (observe that η is now a trinatural transformation). Henceforth we drop the notation Λ and just write Σ for both suspension functors. Suppose we are given a triangle in $\mathcal{E}$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

Complete the morphism $GX \longrightarrow GY$ to a triangle in $\mathcal{D}$

$$GX \longrightarrow GY \longrightarrow C \longrightarrow \Sigma GX$$

applying F we have a triangle in $\mathcal{E}$

$$FGX \longrightarrow FGY \longrightarrow FC \longrightarrow \Sigma FGX$$

The unit morphisms induce a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\eta_X"'] & Y \arrow[r] \arrow[d, "\eta_Y"'] & Z \arrow[r] \arrow[d, "\theta"'] & \Sigma X \arrow[d, "\Sigma\eta_X"'] \\ FGX \arrow[r] & FGY \arrow[r] & FC \arrow[r] & \Sigma FGX \end{tikzcd}$$

We define a morphism of abelian groups $\Theta : Hom_{\mathcal{D}}(C, R) \longrightarrow Hom_{\mathcal{E}}(Z, FR)$ by $\Theta(m) = F(m)\theta$. For any $R \in \mathcal{D}$ we have a commutative diagram with exact rows

$$\begin{tikzcd}
Hom_{\mathcal{D}}(GX,R) \arrow[d, Rightarrow] & Hom_{\mathcal{D}}(GY,R) \arrow[l] \arrow[d, Rightarrow] & Hom_{\mathcal{D}}(C,R) \arrow[l] \arrow[d, "\Theta"'] & Hom_{\mathcal{D}}(\Sigma GX,R) \arrow[l] \arrow[d, Rightarrow] & Hom_{\mathcal{D}}(\Sigma GY,R) \arrow[l] \arrow[d, Rightarrow] \\
Hom_{\mathcal{E}}(X,FR) & Hom_{\mathcal{E}}(Y,FR) \arrow[l] & Hom_{\mathcal{E}}(Z,FR) \arrow[l] & Hom_{\mathcal{E}}(\Sigma X,FR) \arrow[l] & Hom_{\mathcal{E}}(\Sigma Y,FR) \arrow[l]
\end{tikzcd}$$

In other words, given $R \in \mathcal{D}$ and a morphism $\tau : Z \longrightarrow FR$ there is a *unique* morphism $m : C \longrightarrow R$ making the following diagram commute

$$\begin{tikzcd}
Z \arrow[r, "\theta"] \arrow[dr, "\tau"'] & FC \arrow[d, "Fm"] \\
 & FR
\end{tikzcd}$$

This is the universal property of the unit η, from which we deduce that the morphism $\psi : GZ \longrightarrow C$ corresponding to θ is an isomorphism. Since we have a commutative diagram in $\mathcal{D}$

$$\begin{tikzcd}
GX \arrow[r] \arrow[d, "1"'] & GY \arrow[r] \arrow[d, "1"'] & GZ \arrow[r] \arrow[d, "\psi"'] & \Sigma GX \arrow[d, "1"'] \\
GX \arrow[r] & GY \arrow[r] & C \arrow[r] & \Sigma GX
\end{tikzcd}$$

we conclude that the first row is a triangle in $\mathcal{D}$, as required. Therefore (G, ϕ_G) is a triangulated functor, and by Remark 33 the natural transformation ε is trinatural and the pair (η, ε) is a triadjunction $G \longrightarrow\!\!\!\!\!\dashv F$.

Dually if G is right adjoint to F with unit ν and counit ρ then we use duality and the above to see that there is a canonical natural equivalence $\phi_G : G\Sigma \longrightarrow \Sigma G$ defined by

$$\phi_{G,X} = \Sigma G \Sigma^{-1}(\rho_{\Sigma X}) \circ \Sigma G(\Sigma^{-1}\phi_{F,\Sigma^{-1}G\Sigma X})^{-1} \circ \Sigma \nu_{\Sigma^{-1}G\Sigma X}$$

such that (G, ϕ_G) is a triangulated functor, and ν, ρ are trinatural. The pair (ν, ρ) is therefore a triadjunction $F \longrightarrow\!\!\!\!\!\dashv G$, as claimed. □

Corollary 48. *A triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{E}$ has a left (right) triadjoint if and only if it has a left (right) adjoint.*

Remark 35. Let $\mathcal{C}$ be an additive functor together with an additive automorphism $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$. Suppose we are given two classes $\mathscr{T}_1 \subseteq \mathscr{T}_2$ of candidate triangles such that $\mathcal{C}_1 = (\mathcal{C}, \Sigma, \mathscr{T}_1)$ and $\mathcal{C}_2 = (\mathcal{C}, \Sigma, \mathscr{T}_2)$ are triangulated categories. Then the identity functor $1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is triangulated, and therefore so is its left adjoint $1 : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$. Therefore $\mathscr{T}_1 = \mathscr{T}_2$, so you can't add (or remove) triangles from a triangulated category and still have a triangulated category.

Lemma 49. *Given a triangulated functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ the following conditions are equivalent:*

(i) F is an equivalence of categories.

(ii) There is a triangulated functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ and trinatural equivalences $1 \cong FG, GF \cong 1$.

And similarly the following conditions are equivalent:

(i) F is an isomorphism of categories.

(ii) There is a triangulated functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ such that $1 = FG, GF = 1$ (as triangulated functors).

Proof. We prove the first statement, with the second being an easy consequence. $(ii) \Rightarrow (i)$ is trivial. Suppose that F is an equivalence. That is, F is fully faithful and every object of $\mathcal{E}$ is isomorphic to $F(X)$ for some $X \in \mathcal{D}$. There exists an additive functor $G : \mathcal{E} \longrightarrow \mathcal{D}$ and natural equivalences $\eta : 1 \longrightarrow FG, \varepsilon : GF \longrightarrow 1$. In fact we can choose these natural transformations so that $F\varepsilon \circ \eta F = 1$ and $G\eta \circ \varepsilon G = 1$, that is, so that they are the unit and counit of an adjunction $G \dashv F$. Then we have a natural equivalence

$$G\Sigma \cong G\Sigma FG \cong GF\Sigma G \cong \Sigma G$$

which we denote by $\phi_G = \varepsilon\Sigma G \circ G\phi_F^{-1}G \circ G\Sigma\eta$. By the construction of Proposition 47 this makes G into a triangulated functor such that η, ε are trinatural. In other words we have trinatural equivalences $FG \cong 1, GF \cong 1$ which completes the proof. ☐

Definition 28. A triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{E}$ is called a *triequivalence* if it is an equivalence of categories, and a *triisomorphism* if it is an isomorphism of categories. Any triequivalence reflects triangles.

Remark 36. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a triequivalence of triangulated categories, so that there exists a triangulated functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ and trinatural equivalences $s : 1 \longrightarrow FG$ and $t : GF \longrightarrow 1$. It is not difficult to check that s is a left adjunction of G to F and s^{-1} is a right adjunction of F to G (although the corresponding counit need not be t or t^{-1}). Therefore F is left triadjoint *and* right triadjoint to G.

2.2 Verdier Quotients

Remark 37. In the remainder of this section we develop the construction of the verdier quotient $\mathcal{D}/\mathcal{C}$ of a triangulated $\mathfrak{U}$-category $\mathcal{D}$ by a triangulated $\mathfrak{U}$-subcategory $\mathcal{C}$. This construction will produce a *portly category* (AC,Section 1), that without some assumptions on $\mathcal{C}$ is not necessarily a category. In later sections we will indicate what these conditions are, and address some related set-theoretic difficulties.

To define a *portly triangulated category* one simply reads Definition 3 and Definition 13 with "portly category" replacing "category" throughout. The only other modification is that we do not require the distinguished triangles to form a class (that is, we may have a conglomerate of distinguished triangles). One defines triangulated functors between portly triangulated categories in the same way. Observe that any triangulated category is in particular a portly triangulated category, so it makes sense to talk about triangulated functors between triangulated categories and portly triangulated categories. All the basic results about triangulated categories hold in this generality, but the reader is directed to Section 7 for a careful elaboration on this point.

Definition 29. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory. A *verdier quotient of $\mathcal{D}$ by $\mathcal{C}$* is a portly triangulated category $\mathcal{T}$ together with a triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$ which satisfies $\mathcal{C} \subseteq Ker(F)$ and is universal with this property. That is, given any other triangulated functor $G : \mathcal{D} \longrightarrow \mathcal{S}$ into a portly triangulated category with $\mathcal{C} \subseteq Ker(G)$ there is a *unique* triangulated functor $H : \mathcal{T} \longrightarrow \mathcal{S}$ making the following diagram commute

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{T} \arrow[d, "H"] \\ & \mathcal{S} \end{tikzcd}$$

The functor $F : \mathcal{D} \longrightarrow \mathcal{T}$ is called the *verdier localisation functor*. It is clearly an epimorphism of triangulated categories, in the sense that given triangulated functors $P, Q : \mathcal{T} \longrightarrow \mathcal{T}'$ if we have $PF = QF$ then $P = Q$. It is also clear that the triangulated functor $F^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ is a verdier quotient of $\mathcal{D}^{\mathrm{op}}$ by $\mathcal{C}^{\mathrm{op}}$.

Remark 38. Given $\mathcal{D}, \mathcal{C}$ as in Definition 29, if a verdier quotient $F : \mathcal{D} \longrightarrow \mathcal{T}$ exists it must be unique up to a canonical isomorphism of portly triangulated categories. That is, if two triangulated functors $F : \mathcal{D} \longrightarrow \mathcal{T}$ and $F' : \mathcal{D} \longrightarrow \mathcal{T}'$ satisfy the properties of a verdier quotient, there is a unique isomorphism of portly triangulated categories $H : \mathcal{T} \longrightarrow \mathcal{T}'$ making the following diagram commute

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"] \arrow[dr, "F'"'] & \mathcal{T} \arrow[d, Rightarrow, "H"] \\ & \mathcal{T}' \end{tikzcd}$$

Ideally we would like the verdier quotient to be a category, not just a portly category, but in general this is not possible. Throughout the rest of this section, we work with a fixed triangulated category $\mathcal{D}$ and triangulated subcategory $\mathcal{C}$. The objects of the portly category $\mathcal{D}/\mathcal{C}$ will just be the objects of $\mathcal{D}$. It remains to define the morphisms.

Suppose for a moment that the verdier quotient $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ existed, and recall the subcategory $Mor_{\mathcal{C}}$ of $\mathcal{D}$ of Definition 17. Using Lemma 9 it is easy to check that if a morphism $f : X \longrightarrow Y$ belongs to $Mor_{\mathcal{C}}$ then $F(f)$ is an isomorphism. It is therefore natural to define

Definition 30. For any two objects X, Y in $\mathcal{D}$ let $\alpha(X, Y)$ be the class of all pairs of morphisms $f : Z \longrightarrow X, g : Z \longrightarrow Y$ with $f \in Mor_{\mathcal{C}}$. These are diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

which we think of as the "morphism" $X \longrightarrow Y$ given by $g \circ f^{-1}$. We say two pairs $(f, g), (f', g') \in \alpha(X, Y)$ are *equivalent* if there exists $(f'', g'') \in \alpha(X, Y)$ and morphisms u, v making the following diagram commute

$$\begin{tikzcd} & Z' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & Z'' \arrow[l, "f''"'] \arrow[r, "g''"] \arrow[u, "v"] \arrow[d, "u"'] & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} \tag{31}$$

In this situation it is clear from Lemma 35 that u, v belong to $Mor_{\mathcal{C}}$. This notion of equivalence defines a relation $R(X, Y)$ on the class $\alpha(X, Y)$.

Remark 39. In the context of (31) if we write "backwards" morphisms inverted, we have

$$gf^{-1} = guf''^{-1} = g''f''^{-1} = g''(f'v)^{-1} = g''v^{-1}f'^{-1} = g'f'^{-1}$$

which motivates the identification of the fractions gf^{-1} and $g'f'^{-1}$.

Lemma 50. *For any objects X, Y in $\mathcal{D}$ the relation $R(X, Y)$ is an equivalence relation.*

Proof. The relation is trivially reflexive and symmetric, so it only remains to show it is transitive. Let $(f_1, g_1), (f_2, g_2), (f_3, g_3)$ be three elements of $\alpha(X, Y)$ with the first equivalent to the second and the second equivalent to the third. Let this be expressed by the following commutative diagrams

$$\begin{tikzcd} & Z_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ X & Z \arrow[l] \arrow[r] \arrow[u, "u"] \arrow[d, "v"'] & Y \\ & Z_2 \arrow[ul, "f_2"] \arrow[ur, "g_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & Z_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ X & Z' \arrow[l] \arrow[r] \arrow[u, "u'"] \arrow[d, "v'"'] & Y \\ & Z_3 \arrow[ul, "f_3"] \arrow[ur, "g_3"'] & \end{tikzcd}$$

Form the following homotopy pullback diagram in $\mathcal{D}$

$$\begin{tikzcd}
Z'' \arrow[r, "w"] \arrow[d, "w'"'] & Z \arrow[d, "v"] \\
Z' \arrow[r, "u'"'] & Z_2
\end{tikzcd}$$

From Lemma 37 we infer that w, w' belong to $Mor_{\mathcal{C}}$. Therefore the pair (f_2vw, g_2vw) is an element of $\alpha(X,Y)$ and the morphisms $Z'' \longrightarrow Z \longrightarrow Z_1, Z'' \longrightarrow Z' \longrightarrow Z_3$ make the following diagram commute

$$\begin{tikzcd}
& Z_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
X & Z'' \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\
& Z_3 \arrow[ul, "f_3"] \arrow[ur, "g_3"'] &
\end{tikzcd}$$

This shows that (f_1, g_1) is equivalent to (f_3, g_3), as required. $\square$

Definition 31. Given objects X, Y in $\mathcal{D}$ we write $Hom_{\mathcal{D}/\mathcal{C}}(X,Y)$ for the conglomerate of equivalence classes of the class $\alpha(X,Y)$ under the equivalence relation $R(X,Y)$. Given a pair $(f,g) \in \alpha(X,Y)$ we denote its image in $Hom_{\mathcal{D}/\mathcal{C}}(X,Y)$ by $[f,g]$.

Let pairs $(f,g) \in \alpha(X,Y)$ and $(f',g') \in \alpha(Y,Z)$ be given. Suppose we have morphisms $v : W'' \longrightarrow W, u : W'' \longrightarrow W'$ with $v \in Mor_{\mathcal{C}}$ such that the following diagram commutes (such morphisms exist, since we can always take the homotopy pullback)

$$\begin{tikzcd}
& & W'' \arrow[dl, "v"'] \arrow[dr, "u"] & & \\
& W \arrow[dl, "f"'] \arrow[dr, "g"] & & W' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\
X & & Y & & Z
\end{tikzcd}$$

We therefore have a pair $(fv, g'u) \in \alpha(X,Z)$. One checks easily that this is independent of the chosen morphisms v, u up to equivalence in $\alpha(X,Z)$, so we have a well-defined map

$$\begin{aligned}
c : \alpha(Y,Z) \times \alpha(X,Y) &\longrightarrow Hom_{\mathcal{D}/\mathcal{C}}(X,Z) \\
((f',g'),(f,g)) &\mapsto (fv, g'u)
\end{aligned}$$

Lemma 51. *For any three objects X, Y, Z of $\mathcal{D}$ there is a canonical map of conglomerates*

$$Hom_{\mathcal{D}/\mathcal{C}}(Y,Z) \times Hom_{\mathcal{D}/\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}/\mathcal{C}}(X,Z)$$

calculated on representatives using the map c above.

Proof. We have to show that the value of map c defined above is fixed on equivalence classes. Let $(f,g) \in \alpha(X,Y)$ and equivalent pairs $(f',g'), (f'',g'') \in \alpha(Y,Z)$ be given. Let the following diagram express this equivalence

$$\begin{tikzcd}
& W'' \arrow[dl, "f''"'] \arrow[dr, "g''"] & \\
Y & Z' \arrow[l, "p"'] \arrow[r, "q"] \arrow[u] \arrow[d] & Z \\
& W' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

Let Q be a homotopy pullback of g, f', Q' a homotopy pullback of g, f'' and T a homotopy pullback of g, p. There are induced morphisms into the homotopy pullbacks $u : T \longrightarrow Q'$ and $v : T \longrightarrow Q$ making the following diagram commute

$$\begin{tikzcd}
 & & Q' \arrow[dddl] \arrow[dr] & & \\
 & & T \arrow[u, dotted] \arrow[d, dotted] \arrow[ddl] \arrow[dr] & W'' \arrow[dddl, "f''"'] \arrow[dddr, "g''"] & \\
 & & Q \arrow[dl] \arrow[dr] & Z' \arrow[u] \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & W \arrow[dl, "f"'] \arrow[dr, "g"] & & W' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\
X & & Y & & Z
\end{tikzcd}$$

Studying this diagram once concludes that $c((f', g'), (f, g)) = c((f'', g''), (f, g))$. In other words, c is fixed on equivalence classes in the left variable. One checks the right variable similarly, and together these facts show that we have a map on the Hom sets of the required form. $\square$

Given an object X, the pair $(1_X, 1_X)$ determines an equivalence class in $Hom_{\mathcal{D}/\mathcal{C}}(X, X)$ which is a left and right identity under the composition map of Lemma 51. Associativity of this composition is also easily checked, so we have completed the definition of the portly category $\mathcal{D}/\mathcal{C}$. We define a functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ to be the identity on objects, and to send a morphism $f : X \longrightarrow Y$ to the elements $[1_X, f] \in Hom_{\mathcal{D}/\mathcal{C}}(X, Y)$. That is, the equivalence class of following diagram

$$\begin{tikzcd}
 & X \arrow[dl, "1"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

Lemma 52. *Let $f : X \longrightarrow Y$ be a morphism in $Mor_{\mathcal{C}}$. Then in the portly category $\mathcal{D}/\mathcal{C}$ the morphism $F(f) = [1_X, f] : X \longrightarrow Y$ is an isomorphism with inverse $[f, 1_X] : Y \longrightarrow X$.*

Lemma 53. *Suppose we are given the following pair, with $f \in Mor_{\mathcal{C}}$*

$$\begin{tikzcd}
 & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}$$

Then in $\mathcal{D}/\mathcal{C}$ we have $[f, g] = [1_W, g] \circ [f, 1_W]$. In pictures

$$\begin{tikzcd}
 & W \arrow[dl, "f"'] \arrow[dr, "g"] & & & W \arrow[dl, "f"'] \arrow[dr, "1"] & & W \arrow[dl, "1"'] \arrow[dr, "g"] & \\
X & & Y & X & & W & & Y
\end{tikzcd}$$

Proposition 54. *The functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ is universal among all functors into portly categories sending morphisms in $Mor_{\mathcal{C}}$ to isomorphisms.*

Proof. Let $G : \mathcal{D} \longrightarrow \mathcal{E}$ be another functor into a portly category with the propertly that $G(f)$ is an isomorphism for every $f \in Mor_{\mathcal{C}}$. Given objects X, Y and a pair $(f, g) \in \alpha(X, Y)$ we define $z'(f, g) : G(X) \longrightarrow G(Y)$ to be the composite $G(g) \circ G(f)^{-1}$. If two pairs (f, g) and (f', g') are equivalent it is clear that $z'(f, g) = z'(f', g')$, so there is an induced map $z : Hom_{\mathcal{D}/\mathcal{C}}(X, Y) \longrightarrow Hom_{\mathcal{E}}(G(X), G(Y))$. If we define the functor $H : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{E}$ to agree with G on objects and be defined by the maps z on morphisms, then it is clear that H is unique satisfying $H \circ F = G$, as required. $\square$

Remark 40. The triangulated category $\mathcal{D}^{\text{op}}$ has a triangulated subcategory $\mathcal{C}^{\text{op}}$, so we can form the portly category $\mathcal{D}^{\text{op}}/\mathcal{C}^{\text{op}}$. This shares the same universal property as $(\mathcal{D}/\mathcal{C})^{\text{op}}$ so there is a unique isomorphism of portly categories $\mu : \mathcal{D}^{\text{op}}/\mathcal{C}^{\text{op}} \longrightarrow (\mathcal{D}/\mathcal{C})^{\text{op}}$ making the following diagram commute

$$\begin{tikzcd} & \mathcal{D}^{\text{op}} \arrow[dl] \arrow[dr] & \\ \mathcal{D}^{\text{op}}/\mathcal{C}^{\text{op}} \arrow[rr, Rightarrow, "\mu"'] & & (\mathcal{D}/\mathcal{C})^{\text{op}} \end{tikzcd}$$

Remark 41. Taken together, Lemma 52 and Lemma 53 say that every morphism $X \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ can be written in the form $F(g)F(f)^{-1}$ for some object W and morphisms $f : W \longrightarrow X, g : W \longrightarrow Y$ with $f \in Mor_{\mathcal{C}}$. Using homotopy pushouts, it follows that every morphism $X \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ can also be written in the form $F(b)^{-1}F(a)$ for some object W and $a : X \longrightarrow Z, b : Y \longrightarrow Z$ with $b \in Mor_{\mathcal{C}}$. We repesent this morphism by the following diagram

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "a"] & & Y \arrow[ul, "b"'] \end{tikzcd}$$

Using the duality of Remark 40 one checks that two such diagrams determine the same morphism in $\mathcal{D}/\mathcal{C}$ (in other words, $F(b)^{-1}F(a) = F(b')^{-1}F(a')$) if and only if there is a commutative diagram of the following form in $\mathcal{D}$

$$\begin{tikzcd} & Z \arrow[d] & \\ X \arrow[ur, "a"] \arrow[r] \arrow[dr, "a'"'] & Z'' & Y \arrow[ul, "b"'] \arrow[l] \arrow[dl, "b'"] \\ & Z' \arrow[u] & \end{tikzcd}$$

with $Y \longrightarrow Z''$ belonging to $Mor_{\mathcal{C}}$.

Lemma 55. *Let $f, g : X \longrightarrow Y$ be morphisms in $\mathcal{D}$. Then the following conditions are equivalent*

(i) $F(f) = F(g)$.

(ii) There exists a morphism $\alpha : W \longrightarrow X$ in $Mor_{\mathcal{C}}$ with $f\alpha = g\alpha$.

(iii) The morphism $f - g : X \longrightarrow Y$ factors through some object of $\mathcal{C}$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is easily checked. Let us prove $(ii) \Leftrightarrow (iii)$. Let $\alpha : W \longrightarrow X$ be any morphism, which we can complete to a triangle

$$W \xrightarrow{\alpha} X \longrightarrow C \longrightarrow \Sigma W$$

Because $Hom(-, Y)$ is a cohomological functor, we have $(f - g)\alpha = 0$ if and only if $f - g$ factors through C. But α belongs to $Mor_{\mathcal{C}}$ if and only if $C \in \mathcal{C}$. Consequently, there exists $\alpha \in Mor_{\mathcal{C}}$ with $f\alpha = g\alpha$ if and only if $f - g$ factors through an object of $\mathcal{C}$. □

Lemma 56. *Any zero object in $\mathcal{D}$ is also a zero object in $\mathcal{D}/\mathcal{C}$.*

Remark 42. Thinking of morphisms in $\mathcal{D}/\mathcal{C}$ as fractions, the following observations are intuitively obvious. Firstly, given two morphisms $f : W \longrightarrow X, g : W \longrightarrow Y$ in $\mathcal{D}$ with $f \in Mor_{\mathcal{C}}$, we have $[f, g] = [fu, gu]$ for any morphism $u : Q \longrightarrow W$ in $Mor_{\mathcal{C}}$ (multiplying numerator and denominator by the same thing does not change the fraction). One checks easily that any two morphisms $\alpha, \beta : X \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ can be represented by pairs $[f, g], [f, g']$ with the same morphism $f : W \longrightarrow X$ in the first position. Similarly α, β can be written as "left" fractions $\alpha = F(a)^{-1}F(b)$ and $\beta = F(a)^{-1}F(b')$ with the same denominator.

Lemma 57. *Given two morphisms $\alpha : X \longrightarrow Y, \beta : Y \longrightarrow Z$ in $\mathcal{D}/\mathcal{C}$ there are morphisms $a : X' \longrightarrow Y', b : Y' \longrightarrow Z'$ in $\mathcal{D}$ and morphisms $s : X' \longrightarrow X, t : Y' \longrightarrow Y, q : Z' \longrightarrow Z$ in $Mor_{\mathcal{C}}$ making the following diagram commute in $\mathcal{D}/\mathcal{C}$*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\
X \arrow[r, "\alpha"'] & Y \arrow[r, "\beta"'] & Z
\end{tikzcd}$$

In fact we can always choose $Z = Z'$ and $q = 1$.

Proof. Write $\alpha = [f, g]$ and $\alpha' = [f', g']$ for $f : W \longrightarrow X, g : W \longrightarrow Y, f' : W' \longrightarrow Y, g' : W' \longrightarrow Z$ and take a homotopy pullback

$$\begin{tikzcd}
Q \arrow[r, "n"] \arrow[d, "m"'] & W' \arrow[d, "f'"] \\
W \arrow[r, "g"'] & Y
\end{tikzcd}$$

Then setting $X' = Q, Y' = W', Z' = Z$ and $a = n, b = g', s = fm, t = f', q = 1$ has the desired effect. □

Lemma 58. *Any commutative square in $\mathcal{D}/\mathcal{C}$ is isomorphic to the image of a commutative square in $\mathcal{D}$. More precisely, given a commutative square in $\mathcal{D}/\mathcal{C}$*

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

There is a commutative square in $\mathcal{D}$

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & X' \arrow[d] \\
Y' \arrow[r] & Z'
\end{tikzcd} \tag{32}$$

and morphisms $W' \longrightarrow W, X' \longrightarrow X, Y' \longrightarrow Y$ and $Z' \longrightarrow Z$ in $Mor_{\mathcal{C}}$ which make the following diagram commute in $\mathcal{D}/\mathcal{C}$

$$\begin{tikzcd}
W' \arrow[rr] \arrow[dr, Rightarrow] \arrow[dd] & & X' \arrow[dr, Rightarrow] \arrow[dd] & \\
& W \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
Y' \arrow[rr] \arrow[dr, Rightarrow] & & Z' \arrow[dr, Rightarrow] & \\
& Y \arrow[rr] & & Z'
\end{tikzcd}$$

Proof. By Lemma 57 we can lift the composites $W \longrightarrow X \longrightarrow Z$ and $W \longrightarrow Y \longrightarrow Z$ of $\mathcal{D}/\mathcal{C}$ to morphisms $W_1 \longrightarrow X' \longrightarrow Z$ and $W_2 \longrightarrow Y' \longrightarrow Z$. By taking the homotopy pullback of W_1, W_2 we may assume $W_1 = W_2$ are the same object W''. Commutativity in $\mathcal{D}/\mathcal{C}$ of the following diagram

$$\begin{tikzcd}
W'' \arrow[r] \arrow[d] & X' \arrow[d] \\
Y' \arrow[r] & Z
\end{tikzcd}$$

implies by Lemma 55 that there is a morphism $W' \longrightarrow W''$ in $Mor_{\mathcal{C}}$ which equalises the two composites in $\mathcal{D}$. This defines the required commutative square (32) and the four morphisms in $Mor_{\mathcal{C}}$. We may even take $Z' \longrightarrow Z$ to be the identity. □

Lemma 59. *The functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ preserves biproducts. That is, given objects X, Y in $\mathcal{D}$ and a biproduct $X \oplus Y$ with morphisms u, p, v, q the images of these morphisms under F are a biproduct in $\mathcal{D}/\mathcal{C}$.*

Proof. By duality and Remark 40 it suffices to show that F preserves binary coproducts. So let $u : X \longrightarrow X \oplus Y, v : Y \longrightarrow X \oplus Y$ be a coproduct in $\mathcal{D}$ and let morphisms $[\alpha, f] : X \longrightarrow Q$ and $[\alpha', g] : Y \longrightarrow Q$ in $\mathcal{D}/\mathcal{C}$ be given. Since α, α' lie in $Mor_{\mathcal{C}}$, they fit into triangles

$$P \xrightarrow{\alpha} X \longrightarrow Z \longrightarrow \Sigma P$$

$$P' \xrightarrow{\alpha'} Y \longrightarrow Z' \longrightarrow \Sigma P'$$

with Z, Z' in $\mathcal{C}$. By Proposition 7 the direct sum of these triangles is a triangle

$$P \oplus P' \xrightarrow{\alpha \oplus \alpha'} X \oplus Y \longrightarrow Z \oplus Z' \longrightarrow \Sigma(P \oplus P')$$

But $Z \oplus Z'$ is in $\mathcal{C}$, and therefore $\alpha \oplus \alpha'$ belongs to $Mor_{\mathcal{C}}$. This means we have a morphism $[\alpha \oplus \alpha', (f \quad g)] : X \oplus Y \longrightarrow Q$ in $\mathcal{D}/\mathcal{C}$ which is clearly a factorisation of the pair $[\alpha, f], [\alpha', g]$ through $F(u), F(v)$. It only remains to show that this factorisation is unique.

Suppose we are given two morphisms $\varphi, \psi : X \oplus Y \longrightarrow Q$ in $\mathcal{D}/\mathcal{C}$ which agree on the injections $F(u), F(v)$. We can write these morphisms as $\varphi = F(a)^{-1}F(b)$ and $\psi = F(a)^{-1}F(b')$ for some $a : Q \longrightarrow P$ in $Mor_{\mathcal{C}}$ and $b, b' : X \oplus Y \longrightarrow P$. By assumption $\varphi F(u) = \psi F(u)$ and $\varphi F(v) = \psi F(v)$, so we have

$$F(a)^{-1}F(bu) = F(a)^{-1}F(b'u)$$
$$F(a)^{-1}F(bv) = F(a)^{-1}F(b'v)$$

multiplying through by $F(a)$ we have $F(bu) = F(b'u)$ and $F(bv) = F(b'v)$. By Lemma 55, this means that $(b - b')u$ factors through $C \in \mathcal{C}$ and $(b - b')v$ factors through $C' \in \mathcal{C}$. This means that $b - b'$ factors through $C \oplus C'$, and again by Lemma 55 we deduce that $F(b) = F(b')$. This implies $\alpha = \beta$, proving uniqueness of the factorisation. □

Lemma 60. *The portly category $\mathcal{D}/\mathcal{C}$ is additive, and the functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ is an additive functor.*

Proof. By Lemma 59 the portly category $\mathcal{D}/\mathcal{C}$ has binary biproducts, and also a zero object by Lemma 56. It follows from a standard result of category theory that there is a unique "semiadditive" structure on $\mathcal{D}/\mathcal{C}$. That is, for every pair of objects X, Y the conglomerate $Hom_{\mathcal{D}/\mathcal{C}}(X, Y)$ becomes a commutative monoid, in such a way that composition is bilinear and composition with additive identities yields additive identities. To show that $\mathcal{D}/\mathcal{C}$ is additive, it suffices to show each of these commutative monoids is actually an abelian group (that is, additive inverses exist). Since F preserves zero objects and binary biproducts, it is certainly additive.

Let $\alpha : X \longrightarrow Y$ be an arbitrary morphism in $\mathcal{D}/\mathcal{C}$, which can be written as $F(a)^{-1}F(b)$. It is not hard to check that $F(a)^{-1}F(-b)$ is an additive inverse for α, as required. To calculate the sum of two morphisms $\alpha, \beta : X \longrightarrow Y$ of $\mathcal{D}/\mathcal{C}$ you take either of the composites

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{(\alpha\ \beta)} Y \qquad X \xrightarrow{\binom{\alpha}{\beta}} Y \oplus Y \xrightarrow{\Delta} Y$$

which yield the same morphism $\alpha + \beta : X \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$. □

Remark 43. Let $G : \mathcal{D} \longrightarrow \mathcal{E}$ be an additive functor into an additive portly category which sends morphisms in $Mor_{\mathcal{C}}$ to isomorphisms. Then the unique factorisation $H : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{E}$ of Proposition 54 is easily checked to be an additive functor. In particular the additive functor $F\Sigma : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$

induces a unique additive functor $\Sigma : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{D}/\mathcal{C}$ making the following diagram commute

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"] \arrow[d, "\Sigma"'] & \mathcal{D}/\mathcal{C} \arrow[d, "\Sigma"] \\ \mathcal{D} \arrow[r, "F"'] & \mathcal{D}/\mathcal{C} \end{tikzcd}$$

To be explicit, the functor $\Sigma : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{D}/\mathcal{C}$ is defined on objects in the same way as the original Σ, and for a morphism $[f,g] : X \longrightarrow Y$ we have $\Sigma([f,g]) = [\Sigma f, \Sigma g]$. In the same we obtain an additive functor $\Sigma^{-1} : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{D}/\mathcal{C}$ defined on morphisms by $\Sigma^{-1}([f,g]) = [\Sigma^{-1} f, \Sigma^{-1} g]$. This is clearly an inverse for Σ, which is consequently an additive automorphism of $\mathcal{D}/\mathcal{C}$.

Lemma 61. *If a morphism $[f,g] : X \longrightarrow X$ in $\mathcal{D}/\mathcal{C}$ is equal to the identity, then $g \in Mor_{\mathcal{C}}$.*

Proof. This follows immediately from Lemma 35. □

Lemma 62. *A morphism in $\mathcal{D}/\mathcal{C}$ of the form*

$$\begin{tikzcd} & P \arrow[dl, "\alpha"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

is an isomorphism if and only if there exist morphisms f, h in $\mathcal{D}$ such that $gf, hg \in Mor_{\mathcal{C}}$.

Proof. If there exist morphisms f, h such that $gf, hg \in Mor_{\mathcal{C}}$ then $F(g)$ has a right and left inverse in $\mathcal{D}/\mathcal{C}$, and therefore so does $F(g)F(\alpha)^{-1}$. For the converse, suppose that $F(g)F(\alpha)^{-1}$ is an isomorphism in $\mathcal{D}/\mathcal{C}$. Then $F(g)$ must also be an isomorphism. We wish to produce f, h with $hg, gf \in Mor_{\mathcal{C}}$. Let the diagram

$$\begin{tikzcd} & Q \arrow[dl, "\beta"'] \arrow[dr, "f"] & \\ Y & & P \end{tikzcd}$$

be a right inverse in $\mathcal{D}/\mathcal{C}$ to $F(g)$. Then in $\mathcal{D}/\mathcal{C}$ we have $[\beta, gf] = 1$, so Lemma 61 implies that $gf \in Mor_{\mathcal{C}}$. We can write the left inverse $Y \longrightarrow P$ of $F(g)$ in the form $F(a)^{-1}F(b)$. Then $F(a)^{-1}F(h)F(g) = 1$ implies $F(hg) = F(a)$ and therefore by Lemma 55, $hgt = at$ for some $t \in Mor_{\mathcal{C}}$. It follows from Lemma 35 that $hg \in Mor_{\mathcal{C}}$, as required. □

Lemma 63. *A morphism $X \longrightarrow 0$ in $\mathcal{D}$ becomes an isomorphism in $\mathcal{D}/\mathcal{C}$ if and only if there exists $Y \in \mathcal{D}$ with $X \oplus Y \in \mathcal{C}$.*

Proof. Suppose $g : X \longrightarrow 0$ is a morphism in $\mathcal{D}$ taken to an isomorphism in $\mathcal{D}/\mathcal{C}$. By Lemma 62 there exists $h : 0 \longrightarrow \Sigma Y$ so that the composite $X \longrightarrow 0 \longrightarrow \Sigma Y$ is in $Mor_{\mathcal{C}}$. But taking a direct sum of triangles of the form given in Remark 11, we have a triangle

$$X \xrightarrow{0} \Sigma Y \longrightarrow \Sigma(X \oplus Y) \longrightarrow \Sigma X \tag{33}$$

from which it follows that $X \oplus Y$ is in $\mathcal{C}$. Conversely, suppose there exists Y with $X \oplus Y$ in $\mathcal{C}$. Let $h : 0 \longrightarrow \Sigma Y$ and $f : 0 \longrightarrow X$ be the zero morphisms, so that gf is an isomorphism and $hg : X \longrightarrow \Sigma Y$ is the zero morphism, and fits into a triangle (33) with $\Sigma(X \oplus Y) \in \mathcal{C}$. Hence both hg, gf are in $Mor_{\mathcal{C}}$, and g is an isomorphism in $\mathcal{D}/\mathcal{C}$. □

Remark 44. Later we will see that not only does there exist some Y with $X \oplus Y \in \mathcal{C}$, but Y may be chosen to be ΣX. That is, whenever X becomes zero in $\mathcal{D}/\mathcal{C}$ we have $X \oplus \Sigma X \in \mathcal{C}$.

Proposition 64. *Let $g : Y \longrightarrow Y'$ be a morphism in $\mathcal{D}$. Then $F(g)$ is an isomorphism if and only if for any triangle in $\mathcal{D}$*

$$Y \xrightarrow{g} Y' \longrightarrow Z \longrightarrow \Sigma Y$$

the object Z is a direct summand of an object of $\mathcal{C}$.

Proof. Suppose that we are given such a triangle, together with an object Z' such that $Z \oplus Z' \in \mathcal{C}$. We have to show that $F(g)$ is an isomorphism, and by Lemma 62 it suffices to find h, f with $gf, hg \in Mor_{\mathcal{C}}$. Starting with the two triangles

$$\begin{tikzcd} Y \arrow[r, "g"] & Y' \arrow[r] & Z \arrow[r] & \Sigma Y \\ 0 \arrow[r] & Z' \arrow[r] & Z' \arrow[r] & 0 \end{tikzcd}$$

we can form the direct sum, which is the following triangle

$$\begin{tikzcd} Y \arrow[r, "\binom{g}{0}"] & Y' \oplus Z' \arrow[r] & Z \oplus Z' \arrow[r] & \Sigma Y \end{tikzcd}$$

Since $Z \oplus Z'$ is in $\mathcal{C}$, the morphism $Y \longrightarrow Y' \oplus Z'$ is in $Mor_{\mathcal{C}}$. But it factors as g followed by the injection $h : Y' \longrightarrow Y' \oplus Z'$, which produces the required morphism h with $hg \in Mor_{\mathcal{C}}$. One defines f similarly using the triangle $\Sigma^{-1}Z' \longrightarrow 0 \longrightarrow Z' \longrightarrow Z'$. Thus $F(g)$ is an isomorphism.

Conversely, suppose that $F(g)$ is an isomorphism. Then there exists $h : Y' \longrightarrow Y''$ with $hg \in Mor_{\mathcal{C}}$. Consider the following morphism of triangles

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "hg"'] & Y' \arrow[r, "\alpha"] \arrow[d, "\binom{h}{\alpha}"'] & Z \arrow[r] \arrow[d, "1"'] & \Sigma Y \arrow[d, "\Sigma(hg)"'] \\ Y'' \arrow[r] & Y'' \oplus Z \arrow[r] & Z \arrow[r, "0"'] & \Sigma Y'' \end{tikzcd}$$

The bottom triangle is contractible, and therefore by Corollary 26 this is a good morphism of triangles. Using Remark 19 and the proof of Lemma 28 one checks that the following square is homotopy cartesian

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "hg"'] & Y' \arrow[d, "\binom{h}{\alpha}"] \\ Y'' \arrow[r] & Y'' \oplus Z \end{tikzcd}$$

But hg is in $Mor_{\mathcal{C}}$, so this is also true of $\binom{h}{\alpha}$. In particular this morphism maps to an isomorphism in $\mathcal{D}/\mathcal{C}$. Composing with the isomorphism $F(g)$ we see that the image of $\binom{hg}{0} : Y \longrightarrow Y'' \oplus Z$ is an isomorphism in $\mathcal{D}/\mathcal{C}$. Since this matrix is the composite $\binom{1}{0}hg$ and hg also maps to an isomorphism, the injection $Y'' \longrightarrow Y'' \oplus Z$ is an isomorphism in $\mathcal{D}/\mathcal{C}$. Its inverse can only be the image of the projection $Y'' \oplus Z \longrightarrow Y''$, which means that in particular the composite

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} : Y'' \oplus Z \longrightarrow Y'' \oplus Z$$

maps to the identity in $\mathcal{D}/\mathcal{C}$. Since F preserves biproducts we must conclude that the identity matrix (for the biproducts in $\mathcal{D}/\mathcal{C}$) on $Y'' \oplus Z$ is equal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and therefore the identity $1 : Z \longrightarrow Z$ maps to the zero morphism in $\mathcal{D}/\mathcal{C}$. But in an additive category, zero objects are characterised by the equality $1 = 0$ in their endomorphism ring, so we have $Z = 0$ in $\mathcal{D}/\mathcal{C}$. From Lemma 63 we conclude that there exists $Z' \in \mathcal{D}$ with $Z \oplus Z' \in \mathcal{C}$. $\square$

Lemma 65. *Suppose we are given a commutative diagram in $\mathcal{D}$ with triangles for rows*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] & \Sigma X \arrow[d, "1"'] \\ X \arrow[r, "gf"'] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X \end{tikzcd}$$

and suppose that $F(g)$ is an isomorphism. Then there exists a morphism h with $F(h)$ an isomorphism, such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & \Sigma X \arrow[d, "1"'] \\
X \arrow[r, "gf"'] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd} \tag{34}$$

Proof. By Proposition 30 it is possible to extend to a commutative diagram in which all columns and rows are triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & \Sigma X \arrow[d, "1"'] \\
X \arrow[r, "gf"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
0 \arrow[r] \arrow[d] & Y'' \arrow[r, "1"] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

Since $F(g)$ is an isomorphism, Proposition 64 for the second column implies that Y'' is a direct summand of an object in $\mathcal{C}$, which by Proposition 64 for the third column implies that $F(h)$ is also an isomorphism, as required. ☐

Remark 45. In fact, we know from Lemma 28 and the proof of Proposition 30 that we can always choose h so that the morphism of triangles (34) is good with the middle square homotopy cartesian (in addition to the property that $F(h)$ is an isomorphism).

Remark 46. The dual of Lemma 65 says that if we are given a commutative diagram in $\mathcal{D}$ with triangles for rows and $F(h)$ an isomorphism

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "1"'] & Y \arrow[r] & Z \arrow[r, "fh"] \arrow[d, "h"'] & \Sigma X \arrow[d, "1"'] \\
X \arrow[r] & Y' \arrow[r] & Z' \arrow[r, "f"'] & \Sigma X
\end{tikzcd}$$

Then there exists $g : Y \longrightarrow Y'$ with $F(g)$ an isomorphism making the diagram commute.

Lemma 66. *Suppose we are given two triangles in $\mathcal{D}$*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

and isomorphisms $\alpha : X \longrightarrow X', \beta : Y \longrightarrow Y'$ in $\mathcal{D}/\mathcal{C}$ making the induced square commute. Then there is an isomorphism $\gamma : Z \longrightarrow Z'$ in $\mathcal{D}/\mathcal{C}$ making the following diagram commute in $\mathcal{D}/\mathcal{C}$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

Proof. Here $\Sigma\alpha$ denotes the action of the additive automorphism Σ defined in Remark 43. We claim it suffices to prove the result in the case where $Y = Y'$ and β is the identity. To see this, write $\beta = [f, a]$ for morphisms $a : Y'' \longrightarrow Y'$ and $f : Y'' \longrightarrow Y$ and extend the composites $v'a, vf$ to triangles as in the following commutative diagrams in $\mathcal{D}$

$$\begin{tikzcd} X'' \arrow[r] & Y'' \arrow[r, "v'a"] \arrow[d, "a"'] & Z' \arrow[r] \arrow[d, "1"'] & \Sigma X'' \\ X' \arrow[r] & Y' \arrow[r, "v'"'] & Z' \arrow[r] & \Sigma X' \end{tikzcd} \qquad \begin{tikzcd} \overline{X}'' \arrow[r] & Y'' \arrow[r, "vf"] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, "1"'] & \Sigma\overline{X}'' \\ X \arrow[r] & Y \arrow[r, "v"'] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

We can apply Remark 46 to produce morphisms $m : X'' \longrightarrow X', n : \overline{X}'' \longrightarrow X$ in $\mathcal{D}$ completing these diagrams to isomorphisms in $\mathcal{D}/\mathcal{C}$. We have a new isomorphism $\tau : F(m)^{-1}\alpha F(n) : \overline{X}'' \longrightarrow X''$ in $\mathcal{D}/\mathcal{C}$ and to complete the proof it would suffice to extend the following diagram

$$\begin{tikzcd} \overline{X}'' \arrow[r] \arrow[d, "\tau"'] & Y'' \arrow[r] \arrow[d, "1"'] & Z \arrow[r] & \Sigma\overline{X}'' \arrow[d, "\Sigma\tau"'] \\ X'' \arrow[r] & Y'' \arrow[r] & Z' \arrow[r] & \Sigma X'' \end{tikzcd}$$

to an isomorphism of the rows in $\mathcal{D}/\mathcal{C}$. Therefore, as claimed, we can reduce to the case where $Y = Y'$ and β is the identity. If we write $\alpha = [g, b]$ for morphisms $b : X'' \longrightarrow X', g : X'' \longrightarrow X$ then the equality $\beta F(u) = F(u')\alpha$ in $\mathcal{D}/\mathcal{C}$ translates to $F(ug) = F(u'b)$ in $\mathcal{D}$. Therefore by Lemma 55 we have $ugt = u'bt$ for some $t : W \longrightarrow X''$ in $Mor_{\mathcal{C}}$. If we extend the morphism $ugt : W \longrightarrow Y$ to a triangle, it fits into the following commutative diagrams

$$\begin{tikzcd} W \arrow[r, "ugt"] \arrow[d, "gt"'] & Y \arrow[r] \arrow[d, "1"'] & Z'' \arrow[r] & \Sigma W \arrow[d, "\Sigma(gt)"'] \\ X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "ugt"] \arrow[d, "bt"'] & Y \arrow[r] \arrow[d, "1"'] & Z'' \arrow[r] & \Sigma W \arrow[d, "\Sigma(bt)"'] \\ X' \arrow[r, "u'"'] & Y \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

By Remark 46 there exist morphisms $h : Z'' \longrightarrow Z$ and $c : Z'' \longrightarrow Z'$ in $\mathcal{D}$ making these diagrams commute, with $F(h), F(c)$ isomorphisms. The isomorphism $\gamma = F(c)F(h)^{-1} : Z \longrightarrow Z'$ in $\mathcal{D}/\mathcal{C}$ is the morphism we are looking for. $\square$

We are ready to put the structure of a portly triangulated category on $\mathcal{D}/\mathcal{C}$. We know this portly category is additive, and we have defined an additive automorphism Σ. Observe that by definition we have an *equality* of functors $\Sigma F = F\Sigma$. For each distinguished triangle in $\mathcal{D}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

we have the following candidate triangle in $\mathcal{D}/\mathcal{C}$

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{F(w)} \Sigma F(X)$$

A candidate triangle in $\mathcal{D}/\mathcal{C}$ is to be distinguished if it is isomorphic as a candidate triangle to a candidate triangle in $\mathcal{D}/\mathcal{C}$ of this form. This defines the conglomerate of distinguished triangles in $\mathcal{D}/\mathcal{C}$.

Proposition 67. *The additive portly category $\mathcal{D}/\mathcal{C}$, together with the additive automorphism Σ and conglomerate of distinguished triangles defined above, is a portly triangulated category. The canonical functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ is a triangulated functor.*

Proof. The axioms TR0 and TR2 are easily verified. For TR1, let a morphism $\alpha = [f, u] : X \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ be given with $f : P \longrightarrow X, u : P \longrightarrow Y$. Complete u to a triangle in $\mathcal{D}$

$$P \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma P \tag{35}$$

Consider the following candidate triangle in $\mathcal{D}/\mathcal{C}$

$$F(X) \xrightarrow{\alpha} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{F(\Sigma f)F(w)} \Sigma F(X)$$

It is isomorphic as a candidate triangle in $\mathcal{D}/\mathcal{C}$ to the image of (35) under F, and is therefore distinguished. It only remains to prove TR4' (which in particular implies TR3). We need to show that, given a commutative diagram in $\mathcal{D}/\mathcal{C}$ with triangles for rows

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circledast"] & Y \arrow[r] \arrow[d] & Z \arrow[r] & \Sigma X \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd} \tag{36}$$

there is a way to choose a morphism $Z \longrightarrow Z'$ making this diagram a good morphism of triangles, that is, so that the mapping cone is a triangle. Observe that by Lemma 58 the commutative square $\circledast$ can be lifted from $\mathcal{D}/\mathcal{C}$ to an isomorphic commutative square in $\mathcal{D}$, say

$$\begin{tikzcd} \overline{X} \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ \overline{X}' \arrow[r] & \overline{Y}' \end{tikzcd}$$

The rows can be extended to triangles in $\mathcal{D}$, and the diagram

$$\begin{tikzcd} \overline{X} \arrow[r] \arrow[d] & \overline{Y} \arrow[r] \arrow[d] & \overline{Z} \arrow[r] & \Sigma\overline{X} \arrow[d] \\ \overline{X}' \arrow[r] & \overline{Y}' \arrow[r] & \overline{Z}' \arrow[r] & \Sigma\overline{X}' \end{tikzcd}$$

can be extended to a good morphism of triangles in $\mathcal{D}$, hence also in $\mathcal{D}/\mathcal{C}$. But by Lemma 66 the commutative square with vertical isomorphisms

$$\begin{tikzcd} \overline{X} \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

extends to an isomorphism of candidate triangles in $\mathcal{D}/\mathcal{C}$

$$\begin{tikzcd} \overline{X} \arrow[r] \arrow[d] & \overline{Y} \arrow[r] \arrow[d] & \overline{Z} \arrow[r] \arrow[d] & \Sigma\overline{X} \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

Similarly we obtain an isomorphism of the following candidate triangles in $\mathcal{D}/\mathcal{C}$

$$\begin{tikzcd} \overline{X}' \arrow[r] \arrow[d] & \overline{Y}' \arrow[r] \arrow[d] & \overline{Z}' \arrow[r] \arrow[d] & \Sigma\overline{X}' \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

Since good morphisms are stable under composition with isomorphisms, the composite $Z \longrightarrow \overline{Z} \longrightarrow \overline{Z}' \longrightarrow Z'$ in $\mathcal{D}/\mathcal{C}$ completes the diagram (36) to a good morphism of triangles, as required. This completes the proof that $\mathcal{D}/\mathcal{C}$ is a portly triangulated category. The additive functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ (together with the identity $\phi : F\Sigma \longrightarrow \Sigma F$) is clearly a triangulated functor. $\square$

We are now ready to prove the theorem stated at the beginning of this section.

Theorem 68. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory. Then there is a canonical portly triangulated category $\mathcal{D}/\mathcal{C}$ and triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ with $\mathcal{C} \subseteq Ker(F)$ which has the following universal property: given any triangulated functor $G : \mathcal{D} \longrightarrow \mathcal{S}$ into a portly triangulated category with $\mathcal{C} \subseteq Ker(G)$ there is a* unique *triangulated functor $H : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{S}$ such that $HF = G$.*

Proof. We have constructed the canonical portly triangulated category $\mathcal{D}/\mathcal{C}$ and triangulated functor F. It follows from Lemma 63 that $\mathcal{C} \subseteq Ker(F)$. If $G : \mathcal{D} \longrightarrow \mathcal{S}$ is any triangulated functor into a portly triangulated category with the property that $\mathcal{C} \subseteq Ker(G)$, then G must send $Mor_{\mathcal{C}}$ to isomorphisms and therefore factors uniquely through $\mathcal{D}/\mathcal{C}$ via some additive functor $H : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{S}$. We need to show that this functor is triangulated.

Let $\psi : G\Sigma \longrightarrow \Sigma G$ be the natural equivalence associated with G. Since $H(X) = G(X)$ for every object $X \in \mathcal{D}/\mathcal{C}$ one can check that the isomorphisms ψ_X are also a natural equivalence $H\Sigma \longrightarrow \Sigma H$. With the additive functor H this defines a triangulated functor $\mathcal{D}/\mathcal{C} \longrightarrow \mathcal{S}$, so the proof is complete. □

Remark 47. It follows that the isomorphism $\mathcal{D}^{\rm op}/\mathcal{C}^{\rm op} \cong (\mathcal{D}/\mathcal{C})^{\rm op}$ of Remark 40 is an isomorphism of triangulated categories. To be precise, by Theorem 68 we induce two triangulated functors μ, λ which are unique making their respective triangles commute in the following diagram

$$\begin{tikzcd} & \mathcal{D}^{\rm op} \arrow[dl] \arrow[dr] & \\ \mathcal{D}^{\rm op}/\mathcal{C}^{\rm op} \arrow[rr, bend left, "\mu"] & & (\mathcal{D}/\mathcal{C})^{\rm op} \arrow[ll, bend left, "\lambda"] \end{tikzcd}$$

It follows that $\mu\lambda = 1$ and $\lambda\mu = 1$ so this is an isomorphism of triangulated categories.

Remark 48. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory. The kernel of $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ is a thick subcategory of $\mathcal{D}$. From Lemma 63 we learn that the kernel contains $\mathcal{C}$, and can be described as the full subcategory whose objects are the direct summands of objects of $\mathcal{C}$. We will call this category the *thick closure* of $\mathcal{C}$ and denote it $\widehat{\mathcal{C}}$.

Because the kernel of a triangulated functor is always a triangulated subcategory, we deduce that for any triangulated subcategory $\mathcal{C} \subseteq \mathcal{D}$, the thick closure $\widehat{\mathcal{C}}$ is a triangulated subcategory. A triangulated subcategory $\mathcal{C}$ is thick if and only if $\mathcal{C} = \widehat{\mathcal{C}}$.

Remark 49. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor which is the verdier quotient of $\mathcal{D}$ by a triangulated subcategory $\mathcal{C}$ in the sense of the universal property given in Definition 29. Since the verdier quotient is unique up to isomorphism and we understand in detail one canonical example, we can deduce that $\widehat{\mathcal{C}} = Ker(F)$. It is also clear that F is the verdier quotient of $\mathcal{D}$ by the triangulated subcategory $Ker(F)$. In other words, a triangulated functor is a verdier quotient if and only if it is the verdier quotient of its kernel.

Remark 50. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}, \mathcal{C}'$ two triangulated subcategories with $\mathcal{C} \subseteq \mathcal{C}'$. Then the canonical functor $\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}'$ sends $\mathcal{C}$ to zero objects, so there is a unique triangulated functor $\mathcal{D}/\mathcal{C} \longrightarrow \mathcal{D}/\mathcal{C}'$ making the following diagram commute

$$\begin{tikzcd} & \mathcal{D} \arrow[dl] \arrow[dr] & \\ \mathcal{D}/\mathcal{C} \arrow[rr] & & \mathcal{D}/\mathcal{C}' \end{tikzcd}$$

Now suppose we have triangulated subcategories $\mathcal{C} \subseteq \mathcal{D}' \subseteq \mathcal{D}$. The composite $\mathcal{D}' \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ sends objects of $\mathcal{C}$ to zero, so there is a unique triangulated functor $\mathcal{D}'/\mathcal{C} \longrightarrow \mathcal{D}/\mathcal{C}$ making the following diagram commute

$$\begin{tikzcd} \mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{D}'/\mathcal{C} \arrow[r] & \mathcal{D}/\mathcal{C} \end{tikzcd}$$

Proposition 69. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ an essentially small triangulated subcategory. Then the portly category $\mathcal{D}/\mathcal{C}$ has small morphism conglomerates.*

Proof. For every pair of objects $X, Y \in \mathcal{D}/\mathcal{C}$ the morphisms $Hom_{\mathcal{D}/\mathcal{C}}(X, Y)$ in general only form a conglomerate. We are claiming that if $\mathcal{C}$ is essentially small then these conglomerates are all small (but they are not necessarily sets). This does *not* mean that $\mathcal{D}/\mathcal{C}$ is a "genuine" category.

For the proof we fix two objects $X, Y \in \mathcal{D}$ and a small subclass $C \subseteq \mathcal{C}$ with the property that every object of $\mathcal{C}$ is isomorphic to some object of C. Let $\mathcal{Z}$ be the class of all morphisms with domain X and codomain an object of C. This class is small. For every $g \in \mathcal{Z}$ choose a particular homotopy kernel $k_g : K_g \longrightarrow X$. Let $\mathcal{Z}'$ be the class of pairs (k_g, α) where $g \in \mathcal{Z}$ and $\alpha : K_g \longrightarrow Y$ is any morphism of $\mathcal{D}$. This class is also small. It is easy to check that every morphism $X \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ is of the form $[k_g, \alpha]$ for some pair $(k_g, \alpha) \in \mathcal{Z}'$, which shows that $Hom_{\mathcal{D}/\mathcal{C}}(X, Y)$ is a small conglomerate, as required. $\square$

Proposition 70. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{D}'$ a fragile triangulated subcategory and $\mathcal{C}$ a triangulated subcategory of $\mathcal{D}$. Assume that at least one of the following conditions holds*

(a) For any morphism $s : X \longrightarrow Y$ in $Mor_{\mathcal{C}}$ with $Y \in \mathcal{D}'$, there is a morphism $f : X' \longrightarrow X$ with $sf \in Mor_{\mathcal{C}}$ and $X' \in \mathcal{D}'$.

(b) For any morphism $t : Y \longrightarrow X$ in $Mor_{\mathcal{C}}$ with $Y \in \mathcal{D}'$, there is a morphism $f : X \longrightarrow X'$ with $fs \in Mor_{\mathcal{C}}$ and $X' \in \mathcal{D}'$.

Then the canonical functor $\mathcal{D}'/(\mathcal{C} \cap \mathcal{D}') \longrightarrow \mathcal{D}/\mathcal{C}$ is a full embedding that reflects triangles.

Proof. Let $\mathcal{C}'$ denote the triangulated subcategory $\mathcal{C} \cap \mathcal{D}'$ of $\mathcal{D}'$. First of all, we have two triangulated categories $\mathcal{D}, \mathcal{D}'$ with respective triangulated subcategories $\mathcal{C}, \mathcal{C}'$, so it makes sense to form the verdier quotients $\mathcal{D}/\mathcal{C}$ and $\mathcal{D}'/\mathcal{C}'$. The inclusion $\mathcal{D}' \longrightarrow \mathcal{D}$ is by assumption a triangulated functor, and the composite $\mathcal{D}' \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ clearly contains $\mathcal{C}'$ in its kernel, so by Theorem 68 there is a unique triangulated functor $q : \mathcal{D}'/\mathcal{C}' \longrightarrow \mathcal{D}/\mathcal{C}$ making the following diagram commute

$$\begin{tikzcd}
\mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{D}'/\mathcal{C}' \arrow[r, "q"'] & \mathcal{D}/\mathcal{C}
\end{tikzcd}$$

It is easy to check that $Mor_{\mathcal{C}'} = Mor_{\mathcal{C}} \cap \mathcal{D}'$. We claim that, provided at least one of $(a), (b)$ is satisfied, this functor q is a full embedding. It is certainly distinct on objects, so it suffices to show q is fully faithful. By duality it is enough to prove this in the case where (a) is satisfied. The functor q is defined by $q([f, g]) = [f, g]$ for any morphism in $\mathcal{D}'/\mathcal{C}'$ represented by a diagram of the form

$$\begin{tikzcd}
 & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}$$

with $f \in Mor_{\mathcal{C}'}$. Suppose that we have morphisms $[f, g], [f', g'] : X \longrightarrow Y$ in $\mathcal{D}'/\mathcal{C}'$ which become equal in $\mathcal{D}/\mathcal{C}$. That is, we have a commutative diagram of the form

$$\begin{tikzcd}
 & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & Q \arrow[l, "u"'] \arrow[r, "v"] \arrow[u, "s"] \arrow[d, "t"] & Y \\
 & W \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

with Q not necessariy in $\mathcal{D}'$ and $u \in Mor_{\mathcal{C}}$. By (a) we can find a morphism $m : Q' \longrightarrow Q$ with $um \in Mor'_{\mathcal{C}}$ (and in particular $Q' \in \mathcal{D}'$). Replacing Q by Q' and all the inner morphisms by their composite with m, we have a commutative diagram in $\mathcal{D}'$ expressing an equality $[f,g] = [f',g']$. Therefore q is faithful. One shows that q is full in much the same way, so it only remains to show that q reflects triangles. Suppose we are given a candidate triangle in $\mathcal{D}'/\mathcal{C}'$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \tag{37}$$

which becomes a triangle in $\mathcal{D}/\mathcal{C}$. We can extend u to a triangle in $\mathcal{D}'/\mathcal{C}'$, which of course maps to a triangle in $\mathcal{D}/\mathcal{C}$, necessarily isomorphic as a candidate triangle to the image of (37). We deduce from the fact that q is fully faithful that (37) must have been a triangle to begin with. □

In the special case of derived categories, one often shows that a particular morphism of complexes is a quasi-isomorphism (in our current notation, a morphism of $Mor_{\mathcal{C}}$ for a special choice of $\mathcal{C}$) by fitting it into a morphism of exact sequences in which the other two morphisms are quasi-isomorphisms. The desired conclusion then follows from consideration of the long exact cohomology sequence. The next result is an analogue of this argument for general verdier quotients.

Lemma 71. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a thick triangulated subcategory, and suppose that we have a commutative diagram in $\mathcal{D}$ with triangles for rows*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & \Sigma X \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

If any two of f, g, h belong to $Mor_{\mathcal{C}}$ then so does the third.

Proof. Denoting by $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ the verdier quotient, we have in $\mathcal{D}/\mathcal{C}$ the following morphism of triangles

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "F(f)"'] & F(Y) \arrow[r] \arrow[d, "F(g)"'] & F(Z) \arrow[r] \arrow[d, "F(h)"'] & \Sigma F(X) \arrow[d] \\ F(X') \arrow[r] & F(Y') \arrow[r] & F(Z') \arrow[r] & \Sigma F(X') \end{tikzcd}$$

If two of f, g, h belong to $Mor_{\mathcal{C}}$ then two of $F(f), F(g), F(h)$ are isomorphisms, and so by Proposition 6 so is the third. Since $\mathcal{C}$ is thick we conclude from Proposition 64 that the remaining morphism in the triple f, g, h must belong to $Mor_{\mathcal{C}}$. □

Definition 32. Given a commutative ring k a *k-linear triangulated category* is a triangulated category $\mathcal{T}$ which is also a k-linear category in the sense of (AC,Definition 35), so that $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ is a k-linear functor.

Remark 51. Let k be a commutative ring, $\mathcal{D}$ a k-linear triangulated category and $\mathcal{C}$ a triangulated subcategory. Then the verdier quotient $\mathcal{D}/\mathcal{C}$ is k-linear with action $r \cdot [f,g] = [f, r \cdot g]$. The canonical triangulated functor $\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ is clearly k-linear.

2.3 Weak Verdier Quotients

The results of this section are a technical weakening of verdier quotients that we will use in our study of bousfield subcategories of derived categories (DTC,Section 6). The reader can probably skip this section and refer back to it as needed. The definition of a weak verdier quotient is modeled on Definition 29, but we soften the factorisation so that it only works up to triequivalence.

Definition 33. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory. A *weak verdier quotient of $\mathcal{D}$ by $\mathcal{C}$* is a portly triangulated category $\mathcal{T}$ together with a triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$ which satisfies $\mathcal{C} \subseteq Ker(F)$ and is "weakly" universal with this property. That is, given any other triangulated functor $G : \mathcal{D} \longrightarrow \mathcal{S}$ into a portly triangulated category with

$\mathcal{C} \subseteq Ker(G)$ there is a triangulated functor $H : \mathcal{T} \longrightarrow \mathcal{S}$ making the following diagram commute up to trinatural equivalence

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{T} \arrow[d, "H"] \\ & \mathcal{S} \end{tikzcd}$$

Moreover we require that any two such factorisations H, H' be trinaturally equivalent. It is clear that the weak verdier quotient is unique up to triequivalence, and that $F^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ is a weak verdier quotient of $\mathcal{D}^{\mathrm{op}}$ by $\mathcal{C}^{\mathrm{op}}$.

Remark 52. With the notation of Definition 33 it is clear that given triangulated functors $H, H' : \mathcal{T} \longrightarrow \mathcal{S}$ into a portly triangulated category, there is a trinatural equivalence $H \cong H'$ if and only if there is a trinatural equivalence $HF \cong H'F$.

Lemma 72. *Suppose we have a diagram of functors*

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \mathcal{D} \arrow[rr, "H"'] & & \mathcal{E} \end{tikzcd}$$

in which F is the identity on objects and there is a natural equivalence $\alpha : G \longrightarrow HF$. Then there is a functor $H' : \mathcal{D} \longrightarrow \mathcal{E}$ naturally equivalent to H with $H'F = G$.

Proof. In other words, given a diagram of functors of this special type which commutes up to natural equivalence, we can "perturb" the bottom functor to make it actually commute. We define H' as follows: on objects it is defined to agree with G. For a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ we have the following diagram

$$\begin{tikzcd} G(X) \arrow[r, Rightarrow, "\alpha_X"] & HF(X) = H(X) \arrow[d, "H(f)"] \\ G(Y) \arrow[r, Rightarrow, "\alpha_Y"'] & HF(Y) = H(Y) \end{tikzcd}$$

and we define $H'(f) = \alpha_Y^{-1} H(f) \alpha_X$. It is easily checked that H' is naturally equivalent to H and $H'F = G$, as required. ☐

Remark 53. Let $(F, \phi) : \mathcal{A} \longrightarrow \mathcal{B}$ be a triangulated functor and $G : \mathcal{A} \longrightarrow \mathcal{B}$ any functor. Suppose we have a natural equivalence $\alpha : F \longrightarrow G$. Then G is additive, and $\psi = (\Sigma\alpha)\phi(\alpha\Sigma)^{-1} : G\Sigma \longrightarrow \Sigma G$ is a natural equivalence. One checks easily that $(G, \psi) : \mathcal{A} \longrightarrow \mathcal{B}$ is a triangualted functor and α is a trinatural equivalence.

Lemma 73. *Suppose we have a diagram of triangulated functors*

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \mathcal{D} \arrow[rr, "H"'] & & \mathcal{E} \end{tikzcd}$$

in which F is the identity on objects and there is a trinatural equivalence $\alpha : G \longrightarrow HF$. Then there is a triangulated functor $H' : \mathcal{D} \longrightarrow \mathcal{E}$ trinaturally equivalent to H with $H'F = G$.

Proof. If we define H' as before, then H' is naturally equivalent to the triangulated functor H and therefore becomes a triangulated functor itself. One checks easily that $H'F = G$ as triangulated functors. ☐

Proposition 74. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory. The usual verdier quotient $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ is a weak verdier quotient.*

Proof. We need only show that given triangulated functors $H_1, H_2 : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{S}$ into a portly triangulated category with the property that $H_1 F, H_2 F$ are trinaturally equivalent, there is a trinatural equivalence $H_1 \cong H_2$. Taking $G = H_1 F$ in Lemma 73 we deduce that there is a triangulated functor $H_2' : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{S}$ with $H_1 F = H_2' F$ and a trinatural equivalence $H_2 \cong H_2'$. The universal property of the verdier quotient implies that $H_1 = H_2'$, so we have the desired trinatural equivalence $H_1 \cong H_2$. $\square$

Corollary 75. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $\mathcal{D} \longrightarrow \mathcal{T}$ a weak verdier quotient of $\mathcal{D}$ by $\mathcal{C}$. Then there is a canonical triequivalence $\mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ making the following diagram commute*

$$\begin{tikzcd} & \mathcal{D} \arrow[dl] \arrow[dr] & \\ \mathcal{D}/\mathcal{C} \arrow[rr] & & \mathcal{T} \end{tikzcd}$$

Proof. The triangulated functor $\mathcal{D} \longrightarrow \mathcal{T}$ contains $\mathcal{C}$ in its kernel, so there is certainly a triangulated functor $\mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ making this diagram commute. By Proposition 74 the functor $\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ is also a weak verdier quotient, so we deduce from the weak uniqueness of the weak verdier quotient that the bottom functor is a triequivalence. $\square$

Remark 54. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a weak verdier quotient of $\mathcal{D}$ by $\mathcal{C}$. Corollary 75 has the following consequences:

- Given objects $X, Y \in \mathcal{D}$ any morphism $F(X) \longrightarrow F(Y)$ can be written as $F(g)F(f)^{-1}$ for some morphisms $f : W \longrightarrow X, g : W \longrightarrow Y$ in $\mathcal{D}$, with $f \in Mor_{\mathcal{C}}$.
- Every object of $\mathcal{T}$ is isomorphic to $F(X)$ for some $X \in \mathcal{D}$.
- The kernel of F is the thick closure of $\mathcal{C}$.
- For objects $X, Y \in \mathcal{D}$ with $X \in {}^{\perp}\mathcal{C}$ or $Y \in \mathcal{C}^{\perp}$ the map $Hom_{\mathcal{D}}(X, Y) \longrightarrow Hom_{\mathcal{T}}(FX, FY)$ induced by F is an isomorphism.

Remark 55. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor which is the weak verdier quotient of $\mathcal{D}$ by a triangulated subcategory $\mathcal{C}$. Then as in Remark 49 one verifies that F is also the weak verdier quotient of $\mathcal{D}$ by $Ker(F)$.

Proposition 76. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a weak verdier quotient of $\mathcal{D}$ by $\mathcal{C}$. If $T : \mathcal{T} \longrightarrow \mathcal{T}'$ is a triequivalence of portly triangulated categories then TF is also a weak verdier quotient.*

Proof. By definition of a triequivalence there is a triangulated functor $S : \mathcal{T}' \longrightarrow \mathcal{T}$ and trinatural equivalences $TS \cong 1, ST \cong 1$. Suppose we are given a triangulated functor $G : \mathcal{D} \longrightarrow \mathcal{S}$ containing $\mathcal{C}$ in its kernel. There is an induced triangulated functor $H : \mathcal{T} \longrightarrow \mathcal{S}$ and a trinatural equivalence $HF \cong G$. Then we have trinatural equivalences

$$(HS)(TF) = H(ST)F \cong HF \cong G$$

so factorisations of the desired type exist. For the weak uniqueness, it suffices to show that given triangulated functors $H_1, H_2 : \mathcal{T}' \longrightarrow \mathcal{S}$ into a portly triangulated category together with a trinatural equivalence $H_1 TF \cong H_2 TF$ that there exists a trinatural equivalence $H_1 \cong H_2$. But F is a weak verdier quotient, so by Remark 52 there is a trinatural equivalence $H_1 T \cong H_2 T$ and therefore trinatural equivalences

$$H_1 \cong H_1 TS \cong H_2 TS \cong H_2$$

which completes the proof. $\square$

Lemma 77. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a weak verdier quotient of $\mathcal{D}$ by $\mathcal{C}$. If $F' : \mathcal{D} \longrightarrow \mathcal{T}$ is another triangulated functor trinaturally equivalent to F, then F' is also a weak verdier quotient.*

These results explain why we have introduced the concept of a weak verdier quotient: given a verdier quotient $\mathcal{D}/\mathcal{C}$ it often natural to consider triangulated categories $\mathcal{E}$ triequivalent to $\mathcal{D}/\mathcal{C}$. In general $\mathcal{E}$ will not be a verdier quotient in the sense of Defintion 29. However it is always a weak verdier quotient.

Lemma 78. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a weak verdier quotient of $\mathcal{D}$ by $\mathcal{C}$. Given triangulated functors $H, H' : \mathcal{T} \longrightarrow \mathcal{S}$ into a portly triangulated category and a trinatural transformation $\Phi : HF \longrightarrow H'F$, there is a unique trinatural transformation $\phi : H \longrightarrow H'$ with $\phi F = \Phi$.*

Proof. If F is the canonical verdier quotient $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ this is straightforward to check. An arbitrary weak verdier quotient factors via an equivalence through the canonical one, so we can always reduce to this case, and the proof is complete. □

Proposition 79. *Suppose we have a diagram of triangulated functors*

$$\mathcal{T} \xrightarrow{F} \mathcal{S} \xrightarrow{G} \mathcal{Q}$$

where F is a weak verdier quotient. If GF has a right triadjoint then G has a right triadjoint.

Proof. Let $H : \mathcal{Q} \longrightarrow \mathcal{T}$ be right triadjoint to GF with counit $\varepsilon : GFH \longrightarrow 1$, and say that F is a weak verdier quotient of $\mathcal{T}$ by the triangulated subcategory $\mathcal{C}$. We will show that the triangulated functor FH is right triadjoint to G. Firstly we observe that given $C \in \mathcal{C}$ and $Q \in \mathcal{Q}$ we have

$$Hom_{\mathcal{T}}(C, HQ) \cong Hom_{\mathcal{Q}}(GFC, Q) = 0$$

so the image of H is contained in $\mathcal{C}^{\perp}$. It follows from Remark 54 that for any $Q \in \mathcal{Q}$ and $X \in \mathcal{T}$ the following morphism induced by F is an isomorphism

$$Hom_{\mathcal{T}}(X, HQ) \longrightarrow Hom_{\mathcal{S}}(FX, FHQ) \tag{38}$$

We already have a trinatural transformation $\varepsilon : GFH \longrightarrow 1$, and we complete the proof by showing that it is a right triadjunction of G to FH. Suppose we are given an object $Y \in \mathcal{S}$ and a morphism $b : GY \longrightarrow Q$. By Remark 54 there is some object $X \in \mathcal{T}$ and an isomorphism $t : FX \longrightarrow Y$. The morphism $bG(t) : GFX \longrightarrow Q$ induces via the adjunction $GF \dashv H$ a morphism $a : X \longrightarrow HQ$, and it is clear that the following diagram commutes

$$\begin{tikzcd} GY \arrow[rr, "G(F(a)t^{-1})"] \arrow[dr, "b"'] & & GFHQ \arrow[dl, "\varepsilon_Q"] \\ & Q & \end{tikzcd}$$

If $m : Y \longrightarrow FHQ$ is another morphism with $\varepsilon_Q G(m) = b$ then $mt : FX \longrightarrow FHQ$ lifts by (38) to a morphism $a' : X \longrightarrow HQ$ which satisfies $\varepsilon_Q GF(a') = bG(t)$. But a is unique with this property, so we deduce $a = a'$ and therefore $m = F(a)t^{-1}$. This proves that ε is a right triadjunction of G to FH, as required. □

3 Homotopy Colimits

The homotopy colimit is a construction originating in algebraic topology, which was introduced to the algebraists in the work of Bökstedt and Neeman [BN93]. At the formal level of these notes the main application is to splitting idempotents. While this is important, the reader should see our notes on Derived Categories (DTC) and particularly Derived Categories of Quasi-coherent Sheaves (DCOQS) to appreciate the full utility of the construction. In this section we follow the presentation given in [Nee01] and originally in [BN93].

Definition 34. Let $\mathcal{T}$ be a triangulated category with countable coproducts. Suppose we are given a sequence of objects and morphisms in $\mathcal{T}$

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \xrightarrow{j_4} \cdots \tag{39}$$

Let $\mu : \bigoplus_{i=0}^{\infty} X_i \longrightarrow \bigoplus_{i=0}^{\infty} X_i$ be the morphism induced out of the first coproduct by the morphisms $j_{i+1} : X_i \longrightarrow X_{i+1}$. That is, $\mu u_i = u_{i+1} j_{i+1}$ where u_i is the injection of X_i into the coproduct. A *homotopy colimit* of the sequence, denoted $\underrightarrow{holim} X_i$, is a homotopy cokernel of $1-\mu$. That is, it is a morphism $v : \bigoplus_{i=0}^{\infty} X_i \longrightarrow \underrightarrow{holim} X_i$ fitting into a distinguished triangle

$$\bigoplus_{i=0}^{\infty} X_i \xrightarrow{1-\mu} \bigoplus_{i=0}^{\infty} X_i \xrightarrow{v} \underrightarrow{holim} X_i \xrightarrow{w} \Sigma\{ \bigoplus_{i=0}^{\infty} X_i \}$$

The homotopy colimit is unique up to (non-canonical) isomorphism, and as part of the definition there are morphisms $X_i \longrightarrow \underrightarrow{holim} X_i$ compatible with the sequence morphisms j_i. Suppose we have a morphism of sequences: that is, morphisms $f_i : X_i \longrightarrow Y_i$ fitting into a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "j_1"] \arrow[d, "f_0"'] & X_1 \arrow[r, "j_2"] \arrow[d, "f_1"'] & X_2 \arrow[r, "j_3"] \arrow[d, "f_2"'] & X_3 \arrow[r, "j_4"] \arrow[d, "f_3"'] & \cdots \\
Y_0 \arrow[r, "k_1"] & Y_1 \arrow[r, "k_2"] & Y_2 \arrow[r, "k_3"] & Y_3 \arrow[r, "k_4"] & \cdots
\end{tikzcd}$$

Then for any choice of the homotopy colimits, there is an induced morphism $\underrightarrow{holim} X_i \longrightarrow \underrightarrow{holim} Y_i$. In particular isomorphic sequences have isomorphic homotopy colimits.

Throughout the remainder of this section we work in a fixed triangulated category $\mathcal{T}$ with countable coproducts. Since homotopy colimits are only defined up to noncanonical isomorphism, one has to interpret statements like "$\underrightarrow{holim} X_i = 0$" or "$\underrightarrow{holim} X_i = \underrightarrow{holim} Y_i$" in an appropriately loose sense (that is, any particular constructions of both sides are isomorphic in $\mathcal{T}$).

Lemma 80. *Suppose we have two sequences in* $\mathcal{T}$

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots$$

$$Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots$$

Then $\underrightarrow{holim}(X_i \oplus Y_i) = (\underrightarrow{holim} X_i) \oplus (\underrightarrow{holim} Y_i)$.

Proof. By Proposition 7 the direct sum of two triangles is a triangle, so we have a triangle in $\mathcal{T}$

$$\begin{tikzcd}
\bigoplus_{i=0}^{\infty} X_i \oplus \bigoplus_{i=0}^{\infty} Y_i \arrow[r, "1-\mu"] & \bigoplus_{i=0}^{\infty} X_i \oplus \bigoplus_{i=0}^{\infty} Y_i \arrow[dl] \\
(\underrightarrow{holim} X_i) \oplus (\underrightarrow{holim} Y_i) \arrow[r] & \Sigma\{ \bigoplus_{i=0}^{\infty} X_i \oplus \bigoplus_{i=0}^{\infty} Y_i \}
\end{tikzcd}$$

From which we deduce the desired isomorphism. ☐

Remark 56. In the next result we will need to use a nice trick, which probably falls into the class of tricks known as an "Eilenberg swindle". Let $\mathcal{C}$ be an additive category with countable coproducts and suppose we are given an object X together with a coproduct $u_i : X_i \longrightarrow \bigoplus_{i=0}^{\infty} X_i$

of an infinite number of copies of X (so $X_i = X$ for all i). We claim that the following morphisms

$$u_0 : X_0 \longrightarrow \bigoplus_{i=0}^{\infty} X$$

$$u_0 - u_1 : X_1 \longrightarrow \bigoplus_{i=0}^{\infty} X$$

$$u_1 - u_2 : X_2 \longrightarrow \bigoplus_{i=0}^{\infty} X$$

$$u_2 - u_3 : X_3 \longrightarrow \bigoplus_{i=0}^{\infty} X$$

$$\vdots$$

are also a coproduct in $\mathcal{C}$. Let morphisms $\beta_i : X_i \longrightarrow B$ be given for each $i \geq 0$. Then we have a family of morphisms $\gamma_i : X_i \longrightarrow \bigoplus_{i=0}^{\infty} X_i$ defined recursively by $\gamma_0 = \beta_0$ and for $i \geq 1$

$$\gamma_i = \gamma_{i-1} - \beta_i$$

So $\gamma_1 = \beta_0 - \beta_1, \gamma_2 = \beta_0 - \beta_1 - \beta_2$, and so on. These induce a morphism out of the coproduct $\{u_i\}_{i \geq 0}$. That is, a morphism $\beta : \bigoplus_{i=0}^{\infty} X_i \longrightarrow B$ with $\beta u_i = \gamma_i$ for $i \geq 0$. One checks inductively that $\beta(u_{i-1} - u_i) = \beta_i$ for each $i \geq 1$, and β is unique with this property, so the morphisms $u_0, u_0 - u_1, u_1 - u_2, \ldots$ are a coproduct in $\mathcal{C}$.

Lemma 81. *Let X be an object of $\mathcal{T}$, and consider the sequence*

$$X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \cdots$$

Then $\underrightarrow{holim} X \cong X$.

Proof. Let our chosen coproduct of the objects in the sequence be $u_i : X \longrightarrow \bigoplus_{i=0}^{\infty} X$. In this case the morphism $1 - \mu : \bigoplus_{i=0}^{\infty} X \longrightarrow \bigoplus_{i=0}^{\infty} X$ has components $u_0 - u_1, u_1 - u_2, \ldots$ which means that by Remark 56 the two morphisms $u_0 : X_0 \longrightarrow \bigoplus_{i=0}^{\infty} X$ and $1 - \mu : \bigoplus_{i=0}^{\infty} X \longrightarrow \bigoplus_{i=0}^{\infty} X$ are a coproduct in $\mathcal{T}$, and therefore induce an isomorphism

$$(u_0 \;\; 1 - \mu) : X \oplus \bigoplus_{i=0}^{\infty} X \longrightarrow \bigoplus_{i=0}^{\infty} X \tag{40}$$

In particular $1 - \mu$ must be a monomorphism in $\mathcal{T}$, in which case any homotopy cokernel is an actual cokernel by Lemma 14. But the cokernel of $1 - \mu$ is the projection $\bigoplus_{i=0}^{\infty} X \longrightarrow X$, so we obtain the desired isomorphism $\underrightarrow{holim} X \cong X$. To be precise, any choice of homotopy colimit $\underrightarrow{holim} X$ comes with a morphism $X \longrightarrow \underrightarrow{holim} X$, and this is an isomorphism. □

Remark 57. More generally, suppose we have a sequence in which all the morphisms are isomorphisms

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \xrightarrow{j_4} \cdots$$

This is isomorphic in the obvious way to the sequence with all objects equal to X_0 and all morphisms identities. Therefore for any homotopy colimit $\underrightarrow{holim} X_i$ the canonical morphism $X_i \longrightarrow \underrightarrow{holim} X_i$ is an isomorphism for $i \geq 0$.

Lemma 82. *Suppose we have a sequence in which all the morphisms are zero*

$$X_0 \xrightarrow{0} X_1 \xrightarrow{0} X_2 \xrightarrow{0} X_3 \xrightarrow{0} \cdots$$

Then $\underrightarrow{holim} X_i = 0$.

Proof. In this case μ is the zero morphism, so any homotopy cokernel of $1 - \mu$ is a homotopy cokernel of an isomorphism, and is therefore zero. □

3.1 Splitting Idempotents

Definition 35. Let $\mathcal{C}$ be a category and $e : X \longrightarrow X$ an endomorphism of some object. We say that e *splits* if there are morphisms $f : X \longrightarrow Y, g : Y \longrightarrow X$ such that $e = gf$ and $fg = 1_Y$. We say that e is *idempotent* if $ee = e$.

Remark 58. Splitting idempotents are intimately connected with binary coproducts. Let $\mathcal{C}$ be an additive category and $e : A \longrightarrow A$ an idempotent. If $\mathcal{C}$ has kernels, then the kernels $u_1 : A_1 \longrightarrow A, u_2 : A_2 \longrightarrow A$ of e and $1-e$ respectively are a coproduct. If the corresponding projections are p_1, p_2 then we have $e = u_2p_2$ and $p_2u_2 = 1$ so e is a splitting idempotent. If $\mathcal{C}$ doesn't have kernels, then in general not every idempotent splits. But as we will see in a moment, triangulated categories are special in this respect.

Proposition 83. *Let $\mathcal{T}$ be a triangulated category with countable coproducts. Then in $\mathcal{T}$ any idempotent is split.*

Proof. Let $e : X \longrightarrow X$ be an idempotent in $\mathcal{T}$. Consider the two sequences

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots$$

$$X \xrightarrow{1-e} X \xrightarrow{1-e} X \xrightarrow{1-e} \cdots$$

Let Y be a homotopy colimit of the first sequence, and Z a homotopy colimit of the second. By Lemma 80, $Y \oplus Z$ is isomorphic to the homotopy colimit of the following sequence

$$X \oplus X \xrightarrow{\left(\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right)} X \oplus X \xrightarrow{\left(\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right)} X \oplus X \xrightarrow{\left(\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right)} \cdots$$

But the following commutative diagram shows that this sequence is isomorphic to the bottom row

$$\begin{tikzcd}
X \oplus X \arrow[r, "\left(\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right)"] \arrow[d, "\kappa"'] & X \oplus X \arrow[r, "\left(\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right)"] \arrow[d, "\kappa"'] & X \oplus X \arrow[r, "\left(\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right)"] \arrow[d, "\kappa"'] & \cdots \\
X \oplus X \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] & X \oplus X \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] & X \oplus X \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] & \cdots
\end{tikzcd} \tag{41}$$

$$\kappa = \begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix} \tag{42}$$

It follows that $Y \oplus Z$ is isomorphic to the homotopy colimit of the bottom row, which as the direct sum of the sequences in Lemma 81 and Lemma 82 is isomorphic to X. We therefore obtain an isomorphism $Y \oplus Z \cong X$.

To be a little more careful, let our infinite coproduct $\bigoplus_{i=0}^{\infty} X$ be chosen. Then we realise the homotopy colimit of the sequence in Lemma 81 as the projection $p : \bigoplus_{i=0}^{\infty} X \longrightarrow X$ from the binary coproduct defined by u_0 and $1-\mu$. Realise the homotopy colimit of Lemma 82 as the zero morphism $\bigoplus_{i=0}^{\infty} X \longrightarrow 0$. Then the morphism $(p \;\; 0) : \bigoplus_{i=0}^{\infty} X \oplus \bigoplus_{i=0}^{\infty} X \longrightarrow X$ is a homotopy colimit for the bottom row of (41). If $v_Y : \bigoplus_{i=0}^{\infty} X \longrightarrow Y$ and $v_Z : \bigoplus_{i=0}^{\infty} X \longrightarrow Z$ are the respective homotopy colimits, then we choose the morphism $v_Y \oplus v_Z$ as the homotopy colimit of the top row of (41). The induced isomorphism $(g \;\; g') : Y \oplus Z \longrightarrow X$ makes the following diagram commute

$$\begin{tikzcd}
\bigoplus_{i=0}^{\infty} X \oplus \bigoplus_{i=0}^{\infty} X \arrow[r, "v_Y \oplus v_Z"] \arrow[d, "\bigoplus_{i=0}^{\infty} \kappa"'] & Y \oplus Z \arrow[d, "(g \;\; g')"] \\
\bigoplus_{i=0}^{\infty} X \oplus \bigoplus_{i=0}^{\infty} X \arrow[r, "(p \;\; 0)"'] & X
\end{tikzcd}$$

Expanding this out, we find that $gv_Y u_i = pu_i e$ and $g' v_Z u_i = pu_i(1-e)$, which says that $g : Y \longrightarrow X$ and $g' : Z \longrightarrow X$ can play the role of the morphisms induced between the respective homotopy colimits by the following morphisms of sequences

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[r, "e"] \arrow[d, "e"'] & \cdots \\ X \arrow[r, "1"'] & X \arrow[r, "1"'] & X \arrow[r, "1"'] & \cdots \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1-e"] \arrow[d, "1-e"'] & X \arrow[r, "1-e"] \arrow[d, "1-e"'] & X \arrow[r, "1-e"] \arrow[d, "1-e"'] & \cdots \\ X \arrow[r, "1"'] & X \arrow[r, "1"'] & X \arrow[r, "1"'] & \cdots \end{tikzcd}$$

Let $f : X \longrightarrow Y$ be the morphism $v_Y u_0$ and $f' : X \longrightarrow Z$ the morphism $v_Z u_0$. Then since $pu_0 = 1$ we obtain $gf = e$ and $g'f' = 1 - e$. The composite

$$X \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} Y \oplus Z \xrightarrow{\begin{pmatrix} g & g' \end{pmatrix}} X$$

is the identity and since we already know $(g \quad g')$ is an isomorphism, it follows that the first morphism is the two-sided inverse of the latter. In particular $fg = 1$, which completes the proof that e is split. $\square$

Corollary 84. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory in which idempotents split. Then $\mathcal{S}$ is thick.*

Proof. Suppose $X \oplus Y$ belongs to $\mathcal{S}$ for some objects $X, Y \in \mathcal{T}$. Let the structural morphisms for the biproduct be u, p, v, q. Then $\theta = up$ is an idempotent, which must split in $\mathcal{S}$ by hypothesis. Let $g : X \oplus Y \longrightarrow Q$, $f : Q \longrightarrow X \oplus Y$ be such a splitting, so $\theta = fg$ and $gf = 1$. We deduce that $(1 - \theta)f = 0$, so there is a morphism $t : Q \longrightarrow X$ with $ut = f$. Since f is a monomorphism, so is t. It is also a retraction, since

$$tgu = pfgu = p\theta u = pupu = 1$$

It follows that t is an isomorphism, and since $\mathcal{S}$ is replete this implies $X \in \mathcal{S}$. By symmetry we have $Y \in \mathcal{S}$, so the proof is complete. $\square$

Corollary 85. *Let $\mathcal{T}$ be a triangulated category with countable coproducts, and $\mathcal{S}$ a triangulated subcategory closed under countable coproducts in $\mathcal{T}$. Then $\mathcal{S}$ is thick.*

Proof. It follows from the hypothesis that $\mathcal{S}$ is a triangulated category with countable coproducts. Therefore by Proposition 83 any idempotent in $\mathcal{S}$ splits, and we can apply Corollary 84. $\square$

Corollary 86. *Let $\mathcal{T}$ be a triangulated category with countable products, and $\mathcal{S}$ a triangulated subcategory closed under countable products in $\mathcal{T}$. Then $\mathcal{S}$ is thick.*

3.2 Totalising a Complex

Given a bicomplex of objects in an abelian category, one useful thing we can do is to pass to the total complex. Unfortunately this process of totalisation isn't always defined for a sequence of objects in a triangulated category. In special cases, however, there is still something useful to be said. The results of this section are taken from [BN93]. Since we will not use this material until (DTC2,Section 6) the reader can safely skip this section on a first reading.

Throughout this section let $\mathcal{T}$ be a triangulated category with countable coproducts. Suppose we have a complex

$$\cdots \longrightarrow X_3 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \tag{43}$$

To be clear, this is a sequence of objects and morphisms with consecutive morphisms composing to give zero. First we complete $X_1 \longrightarrow X_0$ to a triangle $X_1 \longrightarrow X_0 \longrightarrow Y_1 \longrightarrow \Sigma X_1$. Because the composite $X_2 \longrightarrow X_1 \longrightarrow X_0$ is zero, there is an induced morphism $\Sigma X_2 \longrightarrow Y_1$ and we have a sequence

$$\cdots \longrightarrow \Sigma X_3 \longrightarrow \Sigma X_2 \longrightarrow Y_1$$

If we are lucky, the composite $\Sigma X_3 \longrightarrow \Sigma X_2 \longrightarrow Y_1$ will be zero so that this is also a complex, and we can iterate to define an object Y_2 and morphism $Y_1 \longrightarrow Y_2$. Assuming that this iteration works at each stage, we produce a sequence in $\mathcal{T}$

$$Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots \tag{44}$$

The following diagram attempts to describe what the iteration process looks like (labels indicate the degree of morphisms)

$$\begin{tikzcd}
\cdots & Y_3 \arrow[l] & Y_2 \arrow[l] & Y_1 \arrow[l] & \\
\cdots \arrow[r] & X_3 \arrow[r] \arrow[ur, "(-2)"] & X_2 \arrow[r] \arrow[ur, "(-1)"] & X_1 \arrow[r] & X_0 \arrow[ul]
\end{tikzcd}$$

We define a *totalisation* of the complex (43) to be a homotopy colimit of the sequence (44). Of course this totalisation is wildly noncanonical.

Proposition 87. *Suppose we are given a sequence of objects and morphisms in $\mathcal{T}$*

$$\begin{tikzcd}
\cdots \arrow[r, shift left] & X_n \arrow[l, shift left] \arrow[r, shift left, "i_{n-1}"] & X_{n-1} \arrow[l, shift left, "j_{n-1}"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "i_1"] & X_1 \arrow[l, shift left, "j_1"] \arrow[r, "i_0"] & X_0
\end{tikzcd} \tag{45}$$

such that $i_k i_{k+1} = 0$ and $i_k j_k i_k = i_k$. Then this complex can be totalised in a functorial way.

Proof. To be clear, we mean that the sequence consisting of the morphisms $\ldots, i_2, i_1, i_0$ satisfies the necessary condition at each stage of the above iteration, so we can produce the sequence of Y_i's and therefore a totalisation. By functoriality we mean the following: given a morphism of complexes of the above form (that is, vertical morphisms $X_i \longrightarrow Y_i$ commuting with the i's *and* j's) there is a (noncanonical) morphism of any totalisations of the two complexes.

Extend $X_1 \longrightarrow X_0$ to a triangle

$$X_1 \longrightarrow X_0 \longrightarrow Y_1 \xrightarrow{\Sigma g} \Sigma X_1$$

and let $\alpha : X_2 \longrightarrow \Sigma^{-1} Y_1$ be the induced morphism, so $i_1 = g\alpha$. If we replace α by $\alpha' = \alpha j_1 i_1$ then still $i_1 = g\alpha'$ and moreover $\alpha' i_2 = 0$ (also $\alpha' j_1 i_1 = \alpha'$ which we will need in a moment). So the following sequence has the form of the original (45)

$$\begin{tikzcd}
\cdots \arrow[r, shift left] & \Sigma X_{n+1} \arrow[l, shift left] \arrow[r, shift left, "\Sigma i_n"] & \Sigma X_n \arrow[l, shift left, "\Sigma j_n"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "\Sigma i_2"] & \Sigma X_2 \arrow[l, shift left, "\Sigma j_2"] \arrow[r, "\Sigma \alpha'"] & Y_1
\end{tikzcd}$$

This means that we can iterate, so (45) has a totalisation as claimed. Now suppose we have a morphism $\{\psi_i : X_i \longrightarrow P_i\}_{i \geq 0}$ of diagrams of the form (45) (so that the ψ_i commute with every i_k and j_k). Fix a particular totalisation of each complex. Then from the following commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "\psi_1"'] & X_0 \arrow[r] \arrow[d, "\psi_0"'] & Y_1 \arrow[r] \arrow[d, dotted] & \Sigma X_1 \arrow[d, "\Sigma\psi_1"'] \\
P_1 \arrow[r] & P_0 \arrow[r] & Q_1 \arrow[r, "h"] & \Sigma P_1
\end{tikzcd} \tag{46}$$

induces a morphism $\beta : Y_1 \longrightarrow Q_1$ making the diagram commute. Of course there is no assurance that the diagram

$$\begin{tikzcd} X_2 \arrow[r, "\alpha'"] \arrow[d, "\psi_2"'] & \Sigma^{-1}Y_1 \arrow[d, "\Sigma^{-1}\beta"] \\ P_2 \arrow[r, "\alpha'"'] & \Sigma^{-1}Q_1 \end{tikzcd} \tag{47}$$

commutes for this choice of β. But if we compose with $\Sigma^{-1}Q_1 \longrightarrow P_1$ we get equality, which means that $\gamma = \Sigma^{-1}\beta\alpha' - \alpha'\psi_2$ vanishes on $\Sigma^{-1}Q_1 \longrightarrow P_1$. If we set $\beta' = \beta - \Sigma(\gamma j_1 g)$ then β' also makes (46) commute, and moreover it makes (47) commute as well. Iterating we can define a morphism of sequences

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & \cdots \\ Q_1 \arrow[r] & Q_2 \arrow[r] & Q_3 \arrow[r] & \cdots \end{tikzcd}$$

which induces the required morphism of the totalisations. It is clear that if each ψ_i is an isomorphism, then so is every morphism $Y_i \longrightarrow Q_i$, so isomorphisms of complexes yield isomorphisms of the totalisations. $\square$

Remark 59. Suppose we are given two sequences of objects and morphisms of the form (45)

$$\begin{tikzcd} \cdots \arrow[r, shift left] & X_n \arrow[l, shift left] \arrow[r, shift left, "i_{n-1}"] & X_{n-1} \arrow[l, shift left, "j_{n-1}"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "i_1"] & X_1 \arrow[l, shift left, "j_1"] \arrow[r, "i_0"] & X_0 \end{tikzcd} \tag{48}$$

and

$$\begin{tikzcd} \cdots \arrow[r, shift left] & P_n \arrow[l, shift left] \arrow[r, shift left, "r_{n-1}"] & P_{n-1} \arrow[l, shift left, "s_{n-1}"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "r_1"] & P_1 \arrow[l, shift left, "s_1"] \arrow[r, "r_0"] & P_0 \end{tikzcd} \tag{49}$$

Taking the direct sum, we have a third diagram

$$\begin{tikzcd} X_n \oplus P_n \arrow[r, shift left, "i_{n-1}\oplus r_{n-1}"] & X_{n-1} \oplus P_{n-1} \arrow[l, shift left, "j_{n-1}\oplus s_{n-1}"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "i_1\oplus r_1"] & X_1 \oplus P_1 \arrow[l, shift left, "j_1\oplus s_1"] \arrow[r, "i_0\oplus r_0"] & X_0 \oplus P_0 \end{tikzcd} \tag{50}$$

which also satisfies the two conditions given for (45). It can therefore be totalised. In fact, we claim that if we choose totalisations X for (48) and Y for (49) then there is a way to construct a totalisation Z of (50) such that the induced morphisms $X \longrightarrow Z, Y \longrightarrow Z$ are a coproduct in $\mathcal{T}$. In fact this is obvious. Given extensions of i_0, r_0 to triangles

$$X_1 \longrightarrow X_0 \longrightarrow Y_0 \longrightarrow \Sigma X_1$$
$$P_1 \longrightarrow P_0 \longrightarrow Q_0 \longrightarrow \Sigma P_1$$

we can take the direct sum, which is an extension of $i_0 \oplus r_0$ to a triangle. In this way, the sequence of objects constructed for (50) is just the direct sum of the sequences for (48) and (49). Homotopy colimits commute with direct sums, so we deduce the claim.

Lemma 88. *For any object X we have a sequence of the form (45)*

$$\begin{tikzcd} \cdots \arrow[r, shift left, "\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)"] & X \oplus X \arrow[l, shift left] \arrow[r, shift left, "\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)"] & X \oplus X \arrow[l, shift left, "1"] \arrow[r, shift left, "\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)"] & X \oplus X \arrow[l, shift left, "1"] \arrow[r, "\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)"] & X \oplus X \end{tikzcd}$$

This sequence can be totalised in such a way that the totalisation is X itself.

Proof. This sequence is clearly the direct sum of the following two sequences

$$\begin{tikzcd} \cdots \arrow[r, shift left, "0"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "1"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "0"] & X \arrow[l, shift left, "1"] \arrow[r, "1"] & X \end{tikzcd} \tag{51}$$

and

$$\begin{tikzcd} \cdots \arrow[r, shift left, "1"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "0"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "1"] & X \arrow[l, shift left, "1"] \arrow[r, "0"] & X \end{tikzcd} \tag{52}$$

we show how to totalise the first to give zero and the second to give X, after which the result follows from Remark 59. For the first sequence one checks easily that the induced sequence of Y's is of the form

$$0 \longrightarrow \Sigma^2 X \longrightarrow 0 \longrightarrow \Sigma^3 X \longrightarrow 0 \longrightarrow \cdots \tag{53}$$

and the homotopy colimit of this sequence is clearly zero. For the second sequence the induced sequence of Y's has the form

$$X \oplus \Sigma X \longrightarrow X \longrightarrow X \oplus \Sigma^3 X \longrightarrow X \longrightarrow X \oplus \Sigma^5 X \longrightarrow \cdots \tag{54}$$

where the morphisms are all injections or projections from the coproducts. This sequence is the direct sum of a sequence like (53) with the sequence

$$X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} X \longrightarrow \cdots$$

whose homotopy colimit is X by (TRC,Lemma 81). Therefore the homotopy colimit of (54) is X, which is therefore also the totalisation of our original sequence. $\square$

Now we give a slightly different proof of Proposition 83. The reason we give two proofs is that the second one will become useful in our notes on derived categories of rings.

Proposition 89. *Let $\mathcal{T}$ be a triangulated category with countable coproducts. Then in $\mathcal{T}$ any idempotent is split.*

Proof. Let $e : X \longrightarrow X$ be an idempotent in $\mathcal{T}$ and consider the three sequences

$$\begin{tikzcd} \cdots \arrow[r, shift left, "1-e"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "e"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "1-e"] & X \arrow[l, shift left, "1"] \arrow[r, "e"] & X \end{tikzcd}$$

$$\begin{tikzcd} \cdots \arrow[r, shift left, "e"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "1-e"] & X \arrow[l, shift left, "1"] \arrow[r, shift left, "e"] & X \arrow[l, shift left, "1"] \arrow[r, "1-e"] & X \end{tikzcd}$$

$$\begin{tikzcd} \cdots \arrow[r, shift left, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & X \oplus X \arrow[l, shift left] \arrow[r, shift left, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & X \oplus X \arrow[l, shift left, "1"] \arrow[r, shift left, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & X \oplus X \arrow[l, shift left, "1"] \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & X \oplus X \end{tikzcd}$$

As in the proof of Proposition 83 it is easy to see that the direct sum of the first two sequences is canonically isomorphic to the third. Let totalisations of the first two sequences be Y, Z respectively. We deduce an isomorphism $(g \;\; g') : Y \oplus Z \cong X$. Proceeding as in the proof of Proposition 83 we can now construct a morphism $f : X \longrightarrow Y$ such that $gf = e$ and $fg = 1$. Therefore e is split and the proof is complete. $\square$

3.3 Homotopy Limits

One defines homotopy limits in the obvious way, dual to the definition of homotopy colimits. For the reader's convenience and future reference, we write down the definition here.

Definition 36. Let $\mathcal{T}$ be a triangulated category with countable products. Suppose we are given a sequence of objects and morphisms in $\mathcal{T}$

$$\cdots \longrightarrow X_3 \xrightarrow{j_3} X_2 \xrightarrow{j_2} X_1 \xrightarrow{j_1} X_0$$

Let $\mu : \prod_{i=0}^{\infty} X_i \longrightarrow \prod_{i=0}^{\infty} X_i$ be the morphism induced into the second product by the morphisms $j_{i+1} : X_{i+1} \longrightarrow X_i$. That is, $p_i\mu = j_{i+1}p_{i+1}$ where p_i is the projection onto X_i from the product. A *homotopy limit* of the sequence, denoted $\underleftarrow{holim} X_i$, is a homotopy kernel of $1 - \mu$. That is, it is a morphism $k : \underleftarrow{holim} X_i \longrightarrow \prod_{i=0}^{\infty} X_i$ fitting into a distinguished triangle

$$\underleftarrow{holim} X_i \xrightarrow{k} \prod_{i=0}^{\infty} X_i \xrightarrow{1-\mu} \prod_{i=0}^{\infty} X_i \longrightarrow \Sigma \underleftarrow{holim} X_i$$

The homotopy limit is unique up to (non-canonical) isomorphism, and as part of the definition there are morphisms $\underleftarrow{holim} X_i \longrightarrow X_i$ compatible with the sequence morphisms j_i. A morphism of sequences gives rise to a morphism of the homotopy limits, and isomorphic sequences have isomorphic homotopy limits.

4 Localising Subcategories

Definition 37. A triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is called *localising* if it is closed under coproducts in $\mathcal{T}$. That is, given any nonempty family $\{S_i\}_{i \in I}$ of objects of $\mathcal{S}$, any coproduct $\bigoplus_{i \in I} S_i$ in $\mathcal{T}$ belongs to $\mathcal{S}$. Dually we say that $\mathcal{S}$ is *colocalising* if it is closed under products in $\mathcal{T}$. If $\mathcal{T}$ has coproducts then any localising subcategory $\mathcal{S}$ is a triangulated category with coproducts, and is in particular a thick subcategory of $\mathcal{T}$ by Corollary 85. Dually if $\mathcal{T}$ has products then any colocalising subcategory is thick.

Definition 38. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. We say that an object $X \in \mathcal{T}$ is $\mathcal{S}$-*local* if $Hom(Y, X) = 0$ for every $Y \in \mathcal{S}$. The full subcategory of $\mathcal{T}$ consisting of the $\mathcal{S}$-local objects is denoted $\mathcal{S}^{\perp}$. We have $\mathcal{S} \cap \mathcal{S}^{\perp} = 0$. That is, the only objects in both subcategories are the zero objects.

Dually we say that an object X is $\mathcal{S}$-*colocal* if $Hom(X, Y) = 0$ for every $Y \in \mathcal{S}$, and denote the full subcategory of $\mathcal{S}$-colocal objects by ${}^{\perp}\mathcal{S}$. As before $\mathcal{S} \cap {}^{\perp}\mathcal{S} = 0$. Clearly an object X is $\mathcal{S}$-local if and only if it is $\mathcal{S}^{\mathrm{op}}$-colocal in $\mathcal{T}^{\mathrm{op}}$. That is, $(\mathcal{S}^{\perp})^{\mathrm{op}} = {}^{\perp}(\mathcal{S}^{\mathrm{op}})$ and dually of course $({}^{\perp}\mathcal{S})^{\mathrm{op}} = (\mathcal{S}^{\mathrm{op}})^{\perp}$.

Lemma 90. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. Then $\mathcal{S}^{\perp}, {}^{\perp}\mathcal{S}$ are thick triangulated subcategories of $\mathcal{T}$ which are respectively colocalising and localising.*

Proof. By duality it suffices to prove the statement for $\mathcal{S}^{\perp}$. The full subcategory $\mathcal{S}^{\perp}$ is clearly additive and replete. If $X \in \mathcal{S}^{\perp}$ and $Y \in \mathcal{S}$ then $Hom(Y, \Sigma X) \cong Hom(\Sigma^{-1} Y, X)$ so we deduce that $\mathcal{S}^{\perp}$ is closed under Σ and its inverse. Suppose we are given a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

with $X, Y \in \mathcal{S}^{\perp}$. Then for any $S \in \mathcal{S}$ we deduce from the long exact sequence associated to the homological functor $Hom(S, -)$ that $Hom(S, Z) = 0$. Therefore $Z \in \mathcal{S}^{\perp}$ and $\mathcal{S}^{\perp}$ is a triangulated subcategory of $\mathcal{T}$. Thickness is easily checked. From the equality $Hom(X, \prod_i Y_i) = \prod_i Hom(X, Y_i)$ we infer that $\mathcal{S}^{\perp}$ is closed under arbitrary products in $\mathcal{T}$. $\square$

Lemma 91. *Let $\mathcal{D}$ be a triangulated category with coproducts and $\mathcal{C}$ a localising subcategory. Then the portly triangulated category $\mathcal{D}/\mathcal{C}$ has coproducts and the canonical functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ preserves coproducts.*

Proof. It clearly suffices to show that F preserves coproducts. Let $\{X_i\}_{i \in I}$ be a nonempty family of objects of $\mathcal{D}$ and suppose we are given a coproduct $u_i : X_i \longrightarrow \bigoplus_i X_i$ in $\mathcal{D}$. We have to show

(a) Given morphisms $f_i : X_i \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ there is a morphism $f : \bigoplus_i X_i \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ satisfying $f \circ F(u_i) = f_i$ for each $i \in I$.

(b) If a morphism $f : \bigoplus_i X_i \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ satisfies $f \circ F(u_i) = 0$ for every $i \in I$, then $f = 0$.

Proof of (a). We can represent each f_i as a diagram of the following form

$$\begin{tikzcd} & W_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & \\ X_i & & Y \end{tikzcd}$$

where α_i belongs to $Mor_{\mathcal{C}}$. Taking coproducts we obtain a diagram

$$\begin{tikzcd} & \bigoplus_i W_i \arrow[dl, "\oplus\alpha_i"'] \arrow[dr] & \\ \bigoplus_i X_i & & Y \end{tikzcd}$$

By Remark 9 arbitrary coproducts of triangles are triangles. Since $\mathcal{C}$ is localising, it follows that $\oplus\alpha_i$ is in $Mor_{\mathcal{C}}$, so this diagram is a morphism $f : \bigoplus_i X_i \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$. It is easily checked that $f \circ F(u_i) = f_i$, so the proof of (a) is complete.

Proof of (b). Suppose we are given a morphism f with the stated propertly. We can write f as $F(b)^{-1}F(a)$ for some diagram of the following form, with $b \in Mor_{\mathcal{C}}$

$$\begin{tikzcd} \bigoplus_i X_i \arrow[dr, "a"'] & & Y \arrow[dl, "b"] \\ & W & \end{tikzcd}$$

Then since $f \circ F(u_i) = 0$ for every $i \in I$, we deduce that $F(au_i) = 0$ for every $i \in I$. By Lemma 55 the morphism $au_i : X_i \longrightarrow W$ then factors through some object $C_i \in \mathcal{C}$. It follows that a factors through $\bigoplus_i C_i$, which is in $\mathcal{C}$ since this category is localising. Therefore $F(a) = 0$ and consequently $f = 0$, which completes the proof. □

Proposition 92. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. Then for every pair of objects $X, Y \in \mathcal{D}$ with $X \in {}^{\perp}\mathcal{C}$ or $Y \in \mathcal{C}^{\perp}$ the canonical functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ induces an isomorphism*

$$Hom_{\mathcal{D}}(X, Y) \longrightarrow Hom_{\mathcal{D}/\mathcal{C}}(X, Y) \tag{55}$$

Proof. By duality it suffices to prove the result in the case where $X \in \mathcal{D}, Y \in \mathcal{C}^{\perp}$. Suppose we are given a morphism $\gamma : X \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$, which we can write as $\gamma = [f, g]$ for some morphisms $f : W \longrightarrow X, g : W \longrightarrow Y$ with $f \in Mor_{\mathcal{C}}$. That is, there is a triangle with $Z \in \mathcal{C}$

$$W \xrightarrow{f} X \longrightarrow Z \longrightarrow \Sigma W$$

But then we have an exact sequence

$$Hom_{\mathcal{D}}(Z, Y) \longrightarrow Hom_{\mathcal{D}}(X, Y) \longrightarrow Hom_{\mathcal{D}}(W, Y) \longrightarrow Hom_{\mathcal{D}}(\Sigma^{-1}Z, Y)$$

in which the two endpoints are zero, so $Hom_{\mathcal{D}}(X, Y) \longrightarrow Hom_{\mathcal{D}}(W, Y)$ is an isomorphism. In other words, there is a unique morphism $t : X \longrightarrow Y$ in $\mathcal{D}$ with $tf = g$. Then $F(t) = [f, g] = \gamma$ so the map (55) is surjective. To see that it is injective, suppose $F(t) = 0$ for some morphism $t : X \longrightarrow Y$. Then by Lemma 55 the morphism t factors through some object $Z \in \mathcal{C}$, and the factorisation $Z \longrightarrow Y$ can only be zero, so $t = 0$ as required. □

Corollary 93. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. The canonical functors $\mathcal{C}^{\perp} \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ and ${}^{\perp}\mathcal{C} \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ are full embeddings of triangulated categories.*

Proof. The full subcategory $\mathcal{C}^\perp$ is a triangulated subcategory of $\mathcal{D}$ by Lemma 90, so the inclusion $\mathcal{C}^\perp \longrightarrow \mathcal{D}$ is a triangulated functor. We certainly have a triangulated functor $\mathcal{C}^\perp \longrightarrow \mathcal{D}/\mathcal{C}$ which is distinct on objects, and by Proposition 92 this functor must be fully faithful, therefore a full embedding. The argument for ${}^\perp\mathcal{C}$ is identical. $\square$

The property given in Proposition 92 actually characterises objects of the orthogonal subcategories ${}^\perp\mathcal{C}$ and $\mathcal{C}^\perp$.

Lemma 94. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. For an object $Y \in \mathcal{D}$ the following are equivalent:*

(i) $Y \in \mathcal{C}^\perp$.

(ii) For any $X \in \mathcal{D}$ the map $Hom_{\mathcal{D}}(X, Y) \longrightarrow Hom_{\mathcal{D}/\mathcal{C}}(X, Y)$ is an isomorphism.

(iii) For any diagram of morphisms in $\mathcal{D}$

$$\begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

with $s \in Mor_{\mathcal{C}}$, there is a morphism $g : X \longrightarrow Y$ in $\mathcal{D}$ such that $gs = f$ in $\mathcal{D}$.

(iv) Every morphism $Y \longrightarrow X$ in $Mor_{\mathcal{C}}$ is a coretraction in $\mathcal{D}$.

Proof. $(i) \Rightarrow (ii)$ is Proposition 92. $(ii) \Rightarrow (iii), (iii) \Rightarrow (iv)$ are trivial. See Proposition 12 for a list of conditions on a morphism in $\mathcal{D}$ which are equivalent to being a coretraction. Suppose that (iv) is satisfied and let $f : C \longrightarrow Y$ be a morphism in $\mathcal{D}$ with $C \in \mathcal{C}$. We can extend this to a triangle in $\mathcal{D}$

$$C \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma C$$

so that $Y \longrightarrow Z$ belongs to $Mor_{\mathcal{C}}$. By (iv) this is a coretraction, so $C \longrightarrow Y$ is zero as required. $\square$

Dually, we have

Lemma 95. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. For an object $X \in \mathcal{D}$ the following are equivalent:*

(i) $X \in {}^\perp\mathcal{C}$.

(ii) For any $Y \in \mathcal{D}$ the map $Hom_{\mathcal{D}}(X, Y) \longrightarrow Hom_{\mathcal{D}/\mathcal{C}}(X, Y)$ is an isomorphism.

(iii) For any diagram of morphisms in $\mathcal{D}$

$$\begin{tikzcd} & W & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

with $s \in Mor_{\mathcal{C}}$, there is a morphism $g : X \longrightarrow Y$ in $\mathcal{D}$ such that $sg = f$ in $\mathcal{D}$.

(iv) Every morphism $Y \longrightarrow X$ in $Mor_{\mathcal{C}}$ is a retraction in $\mathcal{D}$.

Example 5. The canonical example is the triangulated category $K(\mathcal{A})$ for an abelian category $\mathcal{A}$ with exact coproducts. Then $K(\mathcal{A})$ has coproducts and the exact complexes $\mathcal{Z}$ form a localising subcategory, whose category $\mathcal{Z}^\perp$ of local objects includes the bounded below complexes of injectives (DTC,Corollary 50).

Definition 39. Let $\mathcal{T}$ be a triangulated category. A *localisation in $\mathcal{T}$* is a pair (ℓ, η) where $\ell : \mathcal{T} \longrightarrow \mathcal{T}$ is a triangulated functor and $\eta : 1 \longrightarrow \ell$ is a trinatural transformation such that for any $X \in \mathcal{T}$, $\ell(\eta_X) = \eta_{\ell X}$ and this morphism $\ell X \longrightarrow \ell\ell X$ is an isomorphism. We will often refer to ℓ as a *localisation functor in $\mathcal{T}$*, leaving η implicit.

For the next few results we work with a fixed triangulated category $\mathcal{T}$ and localisation (ℓ, η).

Lemma 96. *For any object X of $\mathcal{T}$ we have $\ell X = 0$ if and only if $Hom(X, \ell Y) = 0$ for every $Y \in \mathcal{T}$.*

Proof. If $\ell X = 0$ then since every morphism $X \longrightarrow \ell Y$ composed with $\ell Y \cong \ell\ell Y$ factors through ℓX, we deduce that $Hom(X, \ell Y) = 0$. Conversely, setting $Y = \ell X$ implies the canonical isomorphism $\ell X \longrightarrow \ell\ell X$ is the zero morphism, which shows $\ell X = 0$. □

Proposition 97. *The kernel $\mathcal{L}$ of ℓ is a thick localising subcategory of $\mathcal{T}$.*

Proof. The kernel of any triangulated functor is a thick triangulated subcategory, so it suffices to show that $\mathcal{L}$ is closed under arbitrary coproducts in $\mathcal{T}$. Given a nonempty family $\{X_i\}_{i \in I}$ of objects of $\mathcal{L}$ and $Y \in \mathcal{T}$ we have (assuming the coproduct in $\mathcal{T}$ exists)

$$Hom(\bigoplus_i X_i, \ell Y) = \prod_{i \in I} Hom(X_i, \ell Y) = 0$$

so by Lemma 96 we have $\bigoplus_{i \in I} X_i \in \mathcal{L}$, as required. □

Corollary 98. *For an object $X \in \mathcal{T}$ the following conditions are equivalent*

(i) $X \in \mathcal{L}^{\perp}$.

(ii) The morphism $\eta_X : X \longrightarrow \ell X$ is an isomorphism.

(iii) X is in the essential image of ℓ. That is, $X \cong \ell Y$ for some $Y \in \mathcal{T}$.

Proof. $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are trivial, so it suffices to prove $(i) \Rightarrow (ii)$. We can extend the morphism η_X to a triangle in $\mathcal{T}$

$$X \longrightarrow \ell X \longrightarrow Z \longrightarrow \Sigma X \tag{56}$$

and to show η_X is an isomorphism it suffices by Lemma 9 to show that $Z = 0$. Since ℓ is a triangulated functor we have another triangle in $\mathcal{T}$

$$\ell X \longrightarrow \ell\ell X \longrightarrow \ell Z \longrightarrow \Sigma\ell X$$

Since the first morphism is an isomorphism, we deduce that $Z \in \mathcal{L}$. For any object $S \in \mathcal{L}$ we can apply $Hom(S, -)$ to the triangle (56) to obtain an exact sequence

$$Hom(S, \ell X) \longrightarrow Hom(S, Z) \longrightarrow Hom(S, \Sigma X)$$

The two outside groups are zero, the first by $(iii) \Rightarrow (i)$ and the last since $\mathcal{L}^{\perp}$ is a triangulated subcategory of $\mathcal{T}$. We conclude that $Hom(S, Z) = 0$ and therefore $Z \in \mathcal{L}^{\perp}$. But then $Z \in \mathcal{L} \cap \mathcal{L}^{\perp}$ must be a zero object, as required. □

Remark 60. Let $Im\ell$ denote the essential image of ℓ. Then by Corollary 98 we have $\mathcal{L}^{\perp} = Im\ell$ and one can interpret Lemma 96 as saying that $\mathcal{L} = {}^{\perp}Im\ell$, so we have $\mathcal{L} = {}^{\perp}(\mathcal{L}^{\perp})$.

We have seen that any localisation (ℓ, η) gives rise to a localising subcategory $\mathcal{L}$ which has some nice additional properties. It is useful to have a list of equivalent characterisations of those localising subcategories that arise in this way.

Proposition 99. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{L}$ a thick subcategory. Denote by $i : \mathcal{L} \longrightarrow \mathcal{T}$ and $j : \mathcal{L}^{\perp} \longrightarrow \mathcal{T}$ the inclusions and by $Q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{L}$ the verdier quotient. Then the following are equivalent*

(i) There is a localisation (ℓ, η) whose kernel is $\mathcal{L}$.

(ii) The functor Q has a right adjoint.

(iii) The composition Qj is an equivalence of categories.

(iv) The functor j has a left adjoint and ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$.

(v) The functor i has a right adjoint.

(vi) For every $M \in \mathcal{T}$ there is a distinguished triangle

$$N_M \longrightarrow M \longrightarrow B_M \longrightarrow \Sigma N_M$$

with $N_M \in \mathcal{L}$ and $B_M \in \mathcal{L}^{\perp}$.

Proof. $(i) \Rightarrow (ii)$ The functor ℓ certainly contains $\mathcal{L}$ in its kernel, so by the universal property of Q there is a unique triangulated functor $R : \mathcal{T}/\mathcal{L} \longrightarrow \mathcal{T}$ with $RQ = \ell$. We show that R is a right adjoint for Q with unit $\eta : 1 \longrightarrow \ell = RQ$. First observe that by the argument of Corollary 98 the morphism $\eta_X : X \longrightarrow \ell X$ is in $Mor_{\mathcal{L}}$, and consequently $Q(\eta_X)$ is an isomorphism, for any object $X \in \mathcal{T}$. Given a morphism $f : X \longrightarrow RY = \ell Y$ we define a morphism $g : X \longrightarrow Y$ in $\mathcal{T}/\mathcal{L}$ by $g = Q(\eta_Y)^{-1} Q(f)$. This is represented by the following diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "\eta_Y"] \\ & \ell Y & \end{tikzcd}$$

One checks that g is the unique morphism in $\mathcal{T}/\mathcal{L}$ for which the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & RQX \arrow[d, "R(g)"] \\ & RY \end{tikzcd}$$

which proves that η is the unit of an adjunction $Q \dashv R$.

$(ii) \Rightarrow (iii)$ From Corollary 93 we already know that Qj is a full embedding of triangulated categories. Suppose that Q has a right adjoint $R : \mathcal{T}/\mathcal{L} \longrightarrow \mathcal{T}$. The key observation is that for $X \in \mathcal{L}$

$$Hom_{\mathcal{T}}(X, RQY) \cong Hom_{\mathcal{T}/\mathcal{L}}(QX, QY) = 0$$

so R sends every object of $\mathcal{T}/\mathcal{L}$ into an object of $\mathcal{L}^{\perp} \subseteq \mathcal{T}$. We show that Qj is an equivalence by showing that the counit $\varepsilon_Y : QRY \longrightarrow Y$ is an isomorphism for any $Y \in \mathcal{T}/\mathcal{L}$. Using Proposition 92 we have for any $X, Y \in \mathcal{T}$

$$Hom_{\mathcal{T}/\mathcal{L}}(QX, QRY) \cong Hom_{\mathcal{T}}(X, RY) \cong Hom_{\mathcal{T}/\mathcal{L}}(QX, Y)$$

One checks that this map is just composition with ε_Y, and by a standard argument this implies that ε_Y is an isomorphism.

$(iii) \Rightarrow (iv)$ Suppose that Qj is an equivalence and let $r : \mathcal{T}/\mathcal{L} \longrightarrow \mathcal{L}^{\perp}$ be a functor together with specific natural equivalences $rQj \cong 1, Qjr \cong 1$. We claim that rQ is left adjoint to j. By assumption for any object $X \in \mathcal{T}$ we have in $\mathcal{T}/\mathcal{L}$ an isomorphism $X \longrightarrow Qjr(X)$. By Proposition 92 this is the image of a morphism $\eta_X : X \longrightarrow jr(X)$ in $\mathcal{T}$, and together these morphisms form a natural transformation $\eta : 1 \longrightarrow jrQ$. For $X \in \mathcal{T}, Y \in \mathcal{L}^{\perp}$ we have an isomorphism

$$\begin{aligned} Hom_{\mathcal{L}^{\perp}}(rQX, Y) &\cong Hom_{\mathcal{T}/\mathcal{L}}(QrQX, QY) \\ &= Hom_{\mathcal{T}/\mathcal{L}}(Qjr(QX), QY) \\ &\cong Hom_{\mathcal{T}/\mathcal{L}}(QX, QY) \\ &\cong Hom_{\mathcal{T}}(X, jY) \end{aligned}$$

which one checks is defined by composition with η. This shows that η is the unit of an adjunction $rQ \dashv j$, as required. It remains to show that ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$. The inclusion $\supseteq$ is trivial, so suppose $X \in {}^{\perp}(\mathcal{L}^{\perp})$. Then for any $Y \in \mathcal{T}$

$$Hom_{\mathcal{T}/\mathcal{L}}(QX, QY) \cong Hom_{\mathcal{L}^{\perp}}(rQX, rQY) \cong Hom_{\mathcal{T}}(X, jrQY) = 0$$

and therefore $QX = 0$. Since $\mathcal{L}$ was assumed to be thick, this implies $X \in \mathcal{L}$.

$(iv) \Rightarrow (v)$ Suppose that the functor $j' : \mathcal{T} \longrightarrow \mathcal{L}^\perp$ is left adjoint to j with unit $\eta : 1 \longrightarrow jj'$. Using this adjunction one checks that for any $Y \in \mathcal{L}^\perp$ and $X \in \mathcal{T}$ the following maps, induced respectively by η_X and $\Sigma\eta_X$, are isomorphisms

$$Hom_{\mathcal{T}}(jj'X, jY) \longrightarrow Hom_{\mathcal{T}}(X, jY) \tag{57}$$
$$Hom_{\mathcal{T}}(\Sigma jj'X, jY) \longrightarrow Hom_{\mathcal{T}}(\Sigma X, jY) \tag{58}$$

For each object $X \in \mathcal{T}$ we simultaneously choose a particular extension of η_X to a distinguished triangle

$$X \xrightarrow{\eta_X} jj'X \longrightarrow N_X \xrightarrow{w_X} \Sigma X \tag{59}$$

Firstly we claim that $N_X \in \mathcal{L}$. If $Y \in \mathcal{L}^\perp$ then we can apply $Hom(-, jY)$ to the triangle (59) and use the isomorphisms (57), (58) in the resulting long exact sequence to deduce that $Hom(N_X, jY) = 0$. Therefore $N_X \in {}^\perp(\mathcal{L}^\perp) = \mathcal{L}$, as required. Suppose we are given a morphism $f : X \longrightarrow Y$ in $\mathcal{T}$, so that we have a commutative diagram in $\mathcal{T}$ with triangles for rows

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & jj'X \arrow[r] \arrow[d, "jj'(f)"'] & N_X \arrow[r] & \Sigma X \arrow[d, "\Sigma f"'] \\
Y \arrow[r] & jj'Y \arrow[r] & N_Y \arrow[r] & \Sigma Y
\end{tikzcd}$$

Since $Hom(N_X, jj'Y) = 0$ there is by Lemma 15 a *unique* morphism $N_f : N_X \longrightarrow N_Y$ making this diagram commute. We define a functor $i' : \mathcal{T} \longrightarrow \mathcal{L}$ by $i'(X) = \Sigma^{-1}N_X$ and $i'(f) = \Sigma^{-1}N_f$. We define the natural transformation $\varepsilon' : ii' \longrightarrow 1$ by $\varepsilon'_X = -\Sigma^{-1}w_X$, and claim that this is the counit of an adjunction $i \dashv i'$.

Suppose are given $T \in \mathcal{L}, X \in \mathcal{T}$ and a morphism $\alpha : iT \longrightarrow X$. Then $\eta_X \alpha = 0$ since $Hom(iT, jj'X) = 0$. As ε'_X is a homotopy kernel of η_X, there is a morphism $q : T \longrightarrow i'X$ in $\mathcal{L}$ with $\varepsilon'_X q = \alpha$. We claim this factorisation is unique. Suppose that there are morphisms $q, q' : T \longrightarrow i'X$ with $\varepsilon'_X q = \alpha, \varepsilon'_X q' = \alpha$. Then $\varepsilon'_X(q - q') = 0$, and $q - q'$ must factor through the homotopy kernel of ε'_X. This is a morphism $\Sigma^{-1}jj'X \longrightarrow i'X$, so the factorisation belongs to $Hom(iT, \Sigma^{-1}jj'X) = 0$. This shows that $q = q'$, and from this we infer that ε' is the counit of the required adjunction. As an aside, observe that by construction we have the following triangle in $\mathcal{T}$ for every object X

$$ii'X \xrightarrow{\varepsilon'_X} X \xrightarrow{\eta_X} jj'X \longrightarrow \Sigma ii'X$$

$(v) \Rightarrow (vi)$ Let $i' : \mathcal{T} \longrightarrow \mathcal{L}$ be right adjoint to i, with counit $\varepsilon' : ii' \longrightarrow 1$. Then for $M \in \mathcal{T}$ we can extend ε'_M to a triangle

$$ii'M \xrightarrow{\varepsilon'_M} M \longrightarrow B \longrightarrow \Sigma ii'M \tag{60}$$

For every object $X \in \mathcal{L}$ the following maps, induced respectively by ε'_M and $\Sigma\varepsilon'_M$, are isomorphisms

$$Hom_{\mathcal{T}}(iX, ii'M) \longrightarrow Hom_{\mathcal{T}}(iX, M) \tag{61}$$
$$Hom_{\mathcal{T}}(iX, \Sigma ii'M) \longrightarrow Hom_{\mathcal{T}}(iX, \Sigma M) \tag{62}$$

Applying $Hom(iX, -)$ to the triangle (60) we deduce that $Hom(iX, B) = 0$, hence $B \in \mathcal{L}^\perp$, which completes the proof.

$(vi) \Rightarrow (i)$ Choose for every $M \in \mathcal{T}$ a specific triangle

$$N_M \xrightarrow{u_M} M \xrightarrow{v_M} B_M \xrightarrow{w_M} \Sigma N_M$$

with $N_M \in \mathcal{L}$ and $B_M \in \mathcal{L}^{\perp}$. The slightly amazing thing is that N_M, B_M are already functorial in M. Given a morphism $f : M \longrightarrow M'$ consider the following diagram

$$\begin{tikzcd}
N_M \arrow[r, "u_M"] \arrow[d, dotted, "N_f"'] & M \arrow[r, "v_M"] \arrow[d, "f"'] & B_M \arrow[r, "w_M"] \arrow[d, dotted, "B_f"'] & \Sigma N_M \arrow[d, dotted, "\Sigma N_f"'] \\
N_{M'} \arrow[r, "u_{M'}"'] & M' \arrow[r, "v_{M'}"'] & B_{M'} \arrow[r, "w_{M'}"'] & \Sigma N_{M'}
\end{tikzcd}$$

Since $v_{M'} f u_M \in Hom(N_M, B_{M'}) = 0$, we use the fact that v_M is a homotopy cokernel of u_M and that $u_{M'}$ is a homotopy kernel of $v_{M'}$ to induce morphisms $N_f : N_M \longrightarrow N_{M'}$ and $B_f : B_M \longrightarrow B_{M'}$ making the first two squares above commute. In fact, using Remark 13 and the arguments given in the proof of Lemma 15 one checks that these morphisms are *unique* making their respective diagrams commute. This uniqueness together with TR3 shows that the third square must also commute.

We define a functor $\ell : \mathcal{T} \longrightarrow \mathcal{T}$ by $\ell(M) = B_M$ and $\ell(f) = B_f$. This functor is certainly additive. We also define $\ell^a : \mathcal{T} \longrightarrow \mathcal{T}$ by $\ell^a(M) = N_M$ and $\ell^a(f) = N_f$, which is again an additive functor. Given an object $M \in \mathcal{T}$ we have the following diagram in which both rows are triangles

$$\begin{tikzcd}
N_{\Sigma M} \arrow[r, "u_{\Sigma M}"] \arrow[d, dotted, "\psi_M"'] & \Sigma M \arrow[r, "v_{\Sigma M}"] \arrow[d, Rightarrow, "1"'] & B_{\Sigma M} \arrow[r, "w_{\Sigma M}"] \arrow[d, dotted, "\phi_M"'] & \Sigma N_{\Sigma M} \arrow[d, dotted, "\Sigma \psi_M"'] \\
\Sigma N_M \arrow[r, "\Sigma u_M"'] & \Sigma M \arrow[r, "\Sigma v_M"'] & \Sigma B_M \arrow[r, "-\Sigma w_M"'] & \Sigma^2 N_M
\end{tikzcd}$$

Repeating the same arguments, we find there are unique morphisms $\psi_M : N_{\Sigma M} \longrightarrow \Sigma N_M$ and $\phi_M : B_{\Sigma M} \longrightarrow \Sigma B_M$ making this diagram commute (to be precise, ψ is unique making the first square commute, ϕ is unique making the second square commute). By symmetry these must be isomorphisms, and one checks that they are natural in M, so we have natural equivalences $\psi : \ell^a \Sigma \longrightarrow \Sigma \ell^a$ and $\phi : \ell \Sigma \longrightarrow \Sigma \ell$.

Given an object $M \in \mathcal{T}$ we say a morphism $j : M \longrightarrow B$ is a *$\mathcal{L}$-localisation* if $B \in \mathcal{L}^{\perp}$ and if every morphism from M to a $\mathcal{L}$-local object factors uniquely through j. Similarly we say $k : N \longrightarrow M$ is a *$\mathcal{L}$-acyclisation* if $N \in \mathcal{L}$ and if every morphism from an object of $\mathcal{L}$ to M factors uniquely through k. In particular one checks that $v_M : M \longrightarrow B_M$ is a $\mathcal{L}$-localisation and $u_M : N_M \longrightarrow M$ is a $\mathcal{L}$-acyclisation.

We claim that the pairs (ℓ, ϕ) and (ℓ^a, ψ) are triangulated functors $\mathcal{T} \longrightarrow \mathcal{T}$. That is, given a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

we have to show that the following two candidate triangles are distinguished

$$B_X \longrightarrow B_Y \longrightarrow B_Z \longrightarrow B_{\Sigma X} \cong \Sigma B_X$$
$$N_X \longrightarrow N_Y \longrightarrow N_Z \longrightarrow N_{\Sigma X} \cong \Sigma N_X$$

We give the proof for the functor ℓ, with the other proof being similar. Extend the morphism $B_X \longrightarrow B_Y$ to a triangle $B_X \longrightarrow B_Y \longrightarrow C \longrightarrow \Sigma B_X$ and induce a morphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "j"'] & \Sigma X \arrow[d] \\
B_X \arrow[r] & B_Y \arrow[r] & C \arrow[r] & \Sigma B_X
\end{tikzcd}$$

Since $\mathcal{L}^{\perp}$ is a triangulated subcategory, we have $C \in \mathcal{L}^{\perp}$. Given an object $B' \in \mathcal{L}^{\perp}$ apply the functor $Hom(-, B')$ to this diagram and use the 5-Lemma to deduce that $Hom(C, B') \longrightarrow Hom(Z, B')$ is an isomorphism. In other words, $j : Z \longrightarrow C$ is a $\mathcal{L}$-localisation. We deduce an isomorphism $C \cong B_Z$ compatible with the morphisms j, v_Z and one checks that the following diagram commutes (use the fact that the homotopy kernel of a $\mathcal{L}$-localisation must belong to $\mathcal{L}$)

$$\begin{tikzcd}
B_X \arrow[r] \arrow[d, Rightarrow, "1"'] & B_Y \arrow[r] \arrow[d, Rightarrow, "1"'] & C \arrow[r] \arrow[d, Rightarrow] & \Sigma B_X \arrow[d, Rightarrow, "1"'] \\
B_X \arrow[r] & B_Y \arrow[r] & B_Z \arrow[r, "\phi_X \ell(w)"'] & \Sigma B_X
\end{tikzcd}$$

From this we deduce that the bottom row is a triangle, so (ℓ, ϕ) is a triangulated functor as claimed. By construction we have trinatural transformations $u : \ell^a \longrightarrow 1$ and $v : 1 \longrightarrow \ell$ and one checks easily that (ℓ, v) is a localisation in $\mathcal{T}$ with kernel $\mathcal{L}$. $\square$

Remark 61. It follows from Proposition 99 that if $\mathcal{T}$ is a triangulated category with thick subcategory $\mathcal{L}$, such that the inclusion $\mathcal{L} \longrightarrow \mathcal{T}$ has a right adjoint, then $\mathcal{L}$ must be localising.

The next result follows formally from Proposition 99 and Corollary 48, but we will find it useful later to have actually worked through the proof of Proposition 99 and observed that the various adjoints were in fact triadjoints.

Corollary 100. *With the notation of Proposition 99 the following are equivalent*

(i) There is a localisation (ℓ, η) whose associated localising subcategory is $\mathcal{L}$.

(ii) The triangulated functor Q has a right triadjoint.

(iii) The triangulated functor Qj is a triequivalence.

(iv) The triangulated functor j has a left triadjoint and ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$.

(v) The triangulated functor i has a right triadjoint.

Proof. We simply go through Proposition 99 and observe how the various adjoints we constructed are actually triadjoints. $(i) \Rightarrow (ii)$ The right adjoint to Q was the triangulated functor R, and the unit was $\eta : 1 \longrightarrow \ell$ which is trinatural by assumption. Therefore R is right triadjoint to Q.

$(ii) \Rightarrow (iii)$ Let R be right triadjoint to Q with triadjunction (η, ε). We know from the proof of Proposition 99 that the image of R is actually contained in $\mathcal{L}^{\perp}$, so R factors as a triangulated functor $R' : \mathcal{T}/\mathcal{L} \longrightarrow \mathcal{L}^{\perp}$ followed by the inclusion. It is clear that R' is right triadjoint to Qj, with the adjunction (η', ε') just being the restriction of the original adjunction. We showed in the earlier proof that ε' is a trinatural equivalence $QjR' \longrightarrow 1$. Since $\varepsilon' Qj \circ Qj\eta' = 1$ we deduce that $Qj(\eta_L)$ is an isomorphism for every $L \in \mathcal{L}^{\perp}$. But Qj is fully faithful, so η_L is an isomorphism and η' is therefore a trinatural equivalence $1 \longrightarrow R'Qj$. Therefore Qj is a triequivalence, as claimed.

$(iii) \Rightarrow (iv)$ Suppose that Qj is a triequivalence with $r : \mathcal{T}/\mathcal{L} \longrightarrow \mathcal{L}^{\perp}$ a triangulated functor and trinatural equivalences $rQj \cong 1, Qjr \cong 1$. We showed in the proof of Proposition 99 that rQ is left adjoint to j, and it is clear from the construction that the natural transformation $\eta : 1 \longrightarrow jrQ$ is trinatural, so rQ is left triadjoint to j.

$(iv) \Rightarrow (v)$ Suppose that the triangulated functor $j' : \mathcal{T} \longrightarrow \mathcal{L}^{\perp}$ is left triadjoint to j with unit $\eta : 1 \longrightarrow jj'$ and that ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$. For each $X \in \mathcal{T}$ we have the following triangle (the shift of (59))

$$\Sigma^{-1} N_X \xrightarrow{-\Sigma^{-1} w_X} X \xrightarrow{\eta_X} jj'X \longrightarrow N_X \tag{63}$$

where $N_X \in \mathcal{L}$ and $jj'X \in \mathcal{L}^{\perp}$. In the proof of Proposition 99 part $(iv) \Rightarrow (v)$ we proceeded to construct a functor $i' : \mathcal{T} \longrightarrow \mathcal{L}$ defined by $i'(X) = \Sigma^{-1} N_X$. In fact it is clear that i' is the factorisation through $\mathcal{L}$ of the triangulated functor ℓ^a defined in the proof of Proposition 99 part $(vi) \Rightarrow (i)$, with respect to the triangles (63). Therefore i' is a triangulated functor in a canonical way, and moreover the right adjunction $\varepsilon' : ii' \longrightarrow 1$ of $(iv) \Rightarrow (v)$ agrees with the trinatural transformation $u : \ell^a \longrightarrow 1$ of $(vi) \Rightarrow (i)$. Therefore ε' is a right triadjunction and the proof is complete.

$(v) \Rightarrow (i)$ If i has a right triadjoint then in particular it has a right adjoint, so by Proposition 99 the condition (i) is satisfied. $\square$

Remark 62. With the notation of Proposition 99, let us extract one useful observation from the proof of Corollary 100. Given a localisation (ℓ, η) with kernel $\mathcal{L}$, the triangulated functor ℓ factors through $\mathcal{T}/\mathcal{L}$ via some triangulated functor R, which is right triadjoint to Q with unit η. Conversely, suppose we are given a right triadjoint R to Q with unit $\eta : 1 \longrightarrow RQ$. Let $\ell : \mathcal{T} \longrightarrow \mathcal{T}$ be the triangulated functor RQ.

Firstly one checks that any morphism $X \longrightarrow M$ in $\mathcal{T}$ with $M \in \mathcal{L}^{\perp}$ factors uniquely through $\eta_X : X \longrightarrow \ell X$. Given $X \in \mathcal{T}$ extend η_X to a triangle in $\mathcal{T}$

$$L \longrightarrow X \xrightarrow{\eta_X} \ell X \longrightarrow \Sigma L$$

To show that $L \in \mathcal{L} = {}^{\perp}(\mathcal{L}^{\perp})$ one applies $Hom(-, T)$ to this triangle for any $T \in \mathcal{L}^{\perp}$. Then the proof of part $(vi) \Rightarrow (i)$ of Proposition 99 proves that the pair (ℓ, η) is a localisation of $\mathcal{T}$ with kernel $\mathcal{L}$. If $\varepsilon : QR \longrightarrow 1$ is the counit of the triadjunction $Q \dashv R$ then we have also shown that ε is a trinatural equivalence. In particular R must be fully faithful.

Definition 40. Let $\mathcal{T}$ be a triangulated category and $\mathcal{L}$ a thick subcategory. We say that $\mathcal{L}$ is a *bousfield subcategory* if the canonical triangulated functor $\mathcal{T} \longrightarrow \mathcal{T}/\mathcal{L}$ has a right adjoint. In other words, the equivalent conditions of Proposition 99 and Corollary 100 are satisfied. Note that a bousfield subcategory is automatically localising.

Remark 63. Observe that if $\mathcal{S}$ is a thick triangulated subcategory of a triangulated category $\mathcal{T}$ for which the functor $\mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ has a right adjoint, then $\mathcal{S}$ must be localising and is therefore a bousfield subcategory. So there is no generality to be gained in considering subcategories which are not localising.

Proposition 101. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{L} \subseteq \mathcal{T}$ a triangulated subcategory. Suppose that we are given a triangle*

$$N \xrightarrow{\varepsilon} X \xrightarrow{\eta} B \xrightarrow{\ell} \Sigma N$$

such that $N \in \mathcal{L}$ and every morphism $Y \longrightarrow X$ with $Y \in \mathcal{L}$ factors uniquely through N. Then $B \in \mathcal{L}^{\perp}$.

Proof. Given a morphism $f : Y \longrightarrow B$ with $Y \in \mathcal{L}$ we can extend the composite ℓf to a triangle and then induce a morphism of triangles of the form

$$\begin{tikzcd} N \arrow[r, "h"] \arrow[d, Rightarrow, "1"'] & W \arrow[r] \arrow[d, "g"] & Y \arrow[r, "\ell f"] \arrow[d, "f"] & \Sigma N \arrow[d, "1"] \\ N \arrow[r, "\varepsilon"'] & X \arrow[r, "\eta"'] & B \arrow[r, "\ell"'] & \Sigma N \end{tikzcd}$$

By definition of a triangulated subcategory we have $W \in \mathcal{L}$. Therefore g factors uniquely through ε, and in particular h is a coretraction. Therefore $\ell f = 0$ so f factors through η. As $Y \in \mathcal{L}$ this factorisation then factors through N, from which it follows that $f = 0$ as required. $\square$

4.1 Colocalising Subcategories

Proposition 102. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{L}$ a colocalising subcategory. Denote by $i : \mathcal{L} \longrightarrow \mathcal{T}, j : {}^{\perp}\mathcal{L} \longrightarrow \mathcal{T}$ the inclusions and by $Q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{L}$ the verdier quotient. Then the following are equivalent*

(i) The functor Q has a left adjoint.

(ii) The composition Qj is an equivalence of categories.

(iii) The functor j has a right adjoint and $({}^{\perp}\mathcal{L})^{\perp} = \mathcal{L}$.

(iv) The functor i has a left adjoint.

(v) For every $M \in \mathcal{T}$ there is a distinguished triangle

$$X_M \longrightarrow M \longrightarrow Y_M \longrightarrow \Sigma X_M$$

with $X_M \in {}^{\perp}\mathcal{L}$ and $Y_M \in \mathcal{L}$.

Proof. This follows from Proposition 99 by duality. □

Definition 41. Let $\mathcal{T}$ be a triangulated category and $\mathcal{L}$ a thick subcategory. We say that $\mathcal{L}$ is a *cobousfield subcategory* if the canonical triangulated functor $\mathcal{T} \longrightarrow \mathcal{T}/\mathcal{L}$ has a left adjoint. In other words, the equivalent conditions of Proposition 102 are satisfied.

4.2 Localisation Sequences

In Proposition 99 we identified a certain class of triangulated subcategories of a triangulated category $\mathcal{T}$, the so-called *bousfield subcategories*. These subcategories possess many good properties and are also very abundant in applications (see for example (DTC,Theorem 117)). Intuitively the bousfield subcategories $\mathcal{L}$ play the role of triangulated subcategories which are "direct summands" of the ambient triangulated category, with the orthogonal ${}^{\perp}\mathcal{L} \cong \mathcal{T}/\mathcal{L}$ playing the role of the complement.

In this section we study sequences of triangulated functors $\mathcal{L} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{L}$ which are the inclusion and quotient respectively by a bousfield subcategory. It will be necessary for applications to work in the generality of a fully faithful triangulated functor $\mathcal{L} \longrightarrow \mathcal{T}$ whose essential image (see Definition 21) is a bousfield subcategory, and a quotient $\mathcal{T} \longrightarrow \mathcal{T}/\mathcal{L}$ that is only a *weak* verdier quotient in the sense of Section 2.3. This leads us to the notion of a *localisation sequence* (using the notation of Verdier [Ver96]) which is intuitively a "split exact sequence" of triangulated categories.

Our exposition of localisation sequences follows [Ver96], [Kra05] and we adopt the convenient notation used in [Kra05], where a right adjoint acquires a subscript $(-)_\rho$ and a left adjoint the subscript $(-)_\lambda$.

Remark 64. Given a triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ recall that the *kernel* $Ker(F)$ is a thick triangulated subcategory of $\mathcal{T}$ by Lemma 38, and provided F is full, the essential image $Im(F)$ is a triangulated subcategory of $\mathcal{S}$ by Remark 29.

Remark 65. Let $F : \mathcal{T} \longrightarrow \mathcal{S}$ be a triangulated functor. Intuitively we think of F as being a "monomorphism" if it is fully faithful, so that up to equivalence it is the inclusion of a triangulated subcategory. We think of F as being an "epimorphism" if it is a weak verdier quotient of $\mathcal{S}$ by some triangulated subcategory, in which case we simply say that F *is a weak verdier quotient*. Note that F is a weak verdier quotient if and only if it is the weak verdier quotient of $\mathcal{T}$ by its own kernel.

Observe that if F is a verdier quotient in the sense of Definition 29 then it really is an epimorphism among morphisms of triangulated categories. A weak verdier quotient has the lesser property described in Remark 52.

Definition 42. We say that a sequence of triangulated functors

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{64}$$

is an *exact sequence* if the following holds

(E1) The functor F is fully faithful.

(E2) The functor G is a weak verdier quotient.

(E3) There is an equality of triangulated subcategories $Im(F) = Ker(G)$.

In particular G is a weak verdier quotient of $\mathcal{T}$ by $Im(F)$. It is clear that the sequence (64) is exact if and only if the dual sequence

$$(\mathcal{T}')^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{T}^{\mathrm{op}} \xrightarrow{G^{\mathrm{op}}} (\mathcal{T}'')^{\mathrm{op}}$$

is an exact sequence.

In applications the most useful kind of exact sequence are those that are "split exact". These are known as *localisation sequences* in [Ver96]. We begin with the definition given there, and then later show that a localisation sequence is a special type of exact sequence.

Definition 43. We say that a sequence of triangulated functors

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{65}$$

of is a *localisation sequence* if the following holds

(L1) The functor F is fully faithful and has a right adjoint.

(L2) The functor G has a fully faithful right adjoint.

(L3) There is an equality of triangulated subcategories $Im(F) = Ker(G)$.

The sequence (F, G) of functors is called a *colocalisation sequence* if the sequence $(F^{\mathrm{op}}, G^{\mathrm{op}})$ of opposite functors is a localisation sequence. That is, we replace "right" by "left" in (L1) and (L2).

Remark 66. We make the following remarks

(i) By (AC,Proposition 21) the following are equivalent to (L1),(L2) respectively.

(L1') F has a right adjoint $F_\rho : \mathcal{T} \longrightarrow \mathcal{T}'$ whose unit $1 \longrightarrow F_\rho F$ is a natural equivalence.

(L2') G has a right adjoint $G_\rho : \mathcal{T}'' \longrightarrow \mathcal{T}$ whose counit $GG_\rho \longrightarrow 1$ is a natural equivalence.

(ii) Choose right adjoints F_ρ for F and G_ρ for G. Then $F_\rho \circ G_\rho$ is right adjoint to the composite $G \circ F = 0$, so we deduce $F_\rho \circ G_\rho = 0$ also.

(iii) Given a localisation sequence let $Im(F)$ be the essential image of F and $Im(G_\rho)$ the essential image of the right adjoint of G (this does not depend on the specific choice of adjoint). These are triangulated subcategories of $\mathcal{T}$. If we choose right adjoints F_ρ, G_ρ then

$$X \in Im(F) \Leftrightarrow \varepsilon_X : FF_\rho(X) \longrightarrow X \text{ is an isomorphism}$$
$$X \in Im(G_\rho) \Leftrightarrow \eta_X : X \longrightarrow G_\rho G(X) \text{ is an isomorphism}$$

and one checks that $Im(F) = {}^\perp Im(G_\rho)$.

(iv) Localisation sequences are stable under composition with a triequivalence on either end. Given a pair of functors as in (65) and a triequivalence $U : \mathcal{T}'' \longrightarrow \mathcal{S}''$, the pair (F, UG) is a localisation sequence if and only if the pair (F, G) is a localisation sequence.

Similarly if $V : \mathcal{S}' \longrightarrow \mathcal{T}'$ is a triequivalence the pair (FV, G) is a localisation sequence if and only if (F, G) is a localisation sequence. The same statements hold for colocalisation sequences. If you have a pair of triangulated functors (65) and trinatural equivalences $F' \cong F, G' \cong G$ then (G, F) is a (co)localisation sequence if and only if (G', F') is.

It is a surprising but very useful fact that any adjunction in which one functor is triangulated can be upgraded to a triadjunction between triangulated functors (see Proposition 47). In particular an arbitrary right adjoint to a triangulated functor commutes with translation.

Lemma 103. *Suppose we have a localisation sequence (65) and fix right adjoints F_ρ, G_ρ. Then for every $X \in \mathcal{T}$ there is a canonical triangle in $\mathcal{T}$ natural in X*

$$FF_\rho(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} G_\rho G(X) \longrightarrow \Sigma FF_\rho(X) \tag{66}$$

Proof. We can extend the unit $X \longrightarrow G_\rho G(X)$ to a triangle

$$C \longrightarrow X \longrightarrow G_\rho G(X) \longrightarrow \Sigma C$$

Since any morphism $X \longrightarrow G_\rho T$ factors uniquely through $X \longrightarrow G_\rho G(X)$ one checks by applying $Hom_{\mathcal{T}}(-, G_\rho T)$ to this triangle that $C \in {}^\perp Im(G_\rho) = Im(F)$. Therefore the unit $FF_\rho(C) \longrightarrow C$ is an isomorphism. Applying F_ρ to this triangle we deduce that $F_\rho(C) \longrightarrow F_\rho(X)$ is an isomorphism, so from the commutative diagram

$$\begin{tikzcd} FF_\rho(C) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & FF_\rho(X) \arrow[d] \\ C \arrow[r] & X \end{tikzcd}$$

we conclude that there exists a triangle (66) in $\mathcal{T}$. Observe that the third morphism of the triangle is uniquely determined by the other two (TRC,Lemma 15). Naturality in X is easily checked. ☐

Remark 67. Given a localisation sequence (65) we use Lemma 103 to deduce the following

(i) Given right triadjoints F_ρ, G_ρ the pair

$$\mathcal{T}' \xleftarrow{F_\rho} \mathcal{T} \xleftarrow{G_\rho} \mathcal{T}''$$

is a colocalisation sequence.

(ii) Let $Im(F)$ be the essential image of F and $Im(G_\rho)$ the essential image of the right adjoint of G. Then $Im(G_\rho) = Im(F)^\perp$. In particular $Im(F) = {}^\perp(Im(F)^\perp)$ and $Im(G_\rho) = ({}^\perp Im(G_\rho))^\perp$.

Lemma 104. *Any localisation sequence or colocalisation sequence of triangulated functors is an exact sequence.*

Proof. It is clearly enough to prove that any localisation sequence

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{67}$$

is an exact sequence. Choose a right triadjoint G_ρ of G, so that the unit $\eta : 1 \longrightarrow G_\rho G$ and counit $\varepsilon : GG_\rho \longrightarrow 1$ are trinatural transformations. By definition the kernel of G contains $Im(F)$, so there is certainly a triangulated functor $M : \mathcal{T}/Im(F) \longrightarrow \mathcal{T}''$ making the following diagram commute

$$\begin{tikzcd} \mathcal{T} \arrow[r, "G"] \arrow[dr, "Q"'] & \mathcal{T}'' \\ & \mathcal{T}/Im(F) \arrow[u, "M"'] \end{tikzcd}$$

We show that M is an equivalence. Let N be the composite $Q \circ G_\rho$. Then we have a trivial trinatural equivalence $MN = MQG_\rho = GG_\rho \cong 1$. After applying Q to the triangle of Lemma 103 it is clear that $Q\eta : Q \longrightarrow QG_\rho G$ is also a trinatural equivalence. Therefore

$$NMQ = NG = QG_\rho G \cong Q$$

and since Q is itself a weak verdier quotient we deduce from (TRC,Remark 52) a trinatural equivalence $NM \cong 1$, as claimed. It follows from Proposition 76 that G is a weak verdier quotient of $\mathcal{T}$ by $Im(F)$, which means that our localisation sequence is exact. ☐

The next result classifies the localisation sequences as those exact sequences which "split". As with an exact sequence in an abelian category, an exact sequence splits at both ends if it splits at either end.

Proposition 105. *Suppose we have an exact sequence of triangulated functors*

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{68}$$

Then the following are equivalent:

(a) The sequence (68) is a localisation sequence.

(b) The functor F has a right adjoint.

(c) The functor G has a right adjoint.

Proof. $(a) \Rightarrow (b)$ is trivial. $(b) \Rightarrow (c)$ From Proposition 99 we deduce that $\mathcal{T} \longrightarrow \mathcal{T}/Im(F)$ has a right adjoint, and $\mathcal{T}''$ is equivalent to this quotient, hence G has a right adjoint. $(c) \Rightarrow (a)$ Consulting Proposition 99 we see that F has a right adjoint and that the right adjoint to G is fully faithful. Hence (68) is a localisation sequence. $\square$

The following result gives a useful criterion for finding localisation sequences.

Lemma 106. *Let $F : \mathcal{T} \longrightarrow \mathcal{S}$ be a triangulated functor with a fully faithful right adjoint. Then $Ker(F)$ is bousfield and we have a localisation sequence*

$$Ker(F) \longrightarrow \mathcal{T} \xrightarrow{F} \mathcal{S}$$

In particular there is a canonical triequivalence $\mathcal{T}/Ker(F) \longrightarrow \mathcal{S}$.

Proof. Given $X \in \mathcal{T}$ extend the unit morphism $X \longrightarrow F_\rho F(X)$ to a triangle in $\mathcal{T}$

$$Y \longrightarrow X \longrightarrow F_\rho F(X) \longrightarrow \Sigma Y$$

Since F_ρ is fully faithful the counit of the adjunction is an isomorphism, so applying F to this triangle we deduce $F(Y) = 0$. That is, $Y \in Ker(F)$. One checks that $F_\rho F(X) \in Ker(F)^\perp$ so it follows from Proposition 99 that $Ker(F)$ is bousfield. The other claims are follow immediately. $\square$

Definition 44. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}, \mathcal{Q}$ triangulated subcategories. Let $\mathcal{S} \star \mathcal{Q}$ denote the full subcategory of $\mathcal{T}$ consisting of objects $X \in \mathcal{T}$ that fit into triangles

$$S \longrightarrow X \longrightarrow Q \longrightarrow \Sigma S$$

with $S \in \mathcal{S}$ and $Q \in \mathcal{Q}$. This is a replete subcategory that contains both $\mathcal{S}, \mathcal{Q}$ and is closed under Σ, Σ^{-1} and we call it the *Verdier sum* of $\mathcal{S}, \mathcal{Q}$.

Lemma 107. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}, \mathcal{Q}$ triangulated subcategories such that $Hom_{\mathcal{T}}(\mathcal{S}, \mathcal{Q}) = 0$. Then $\mathcal{S} \star \mathcal{Q}$ is a triangulated subcategory of $\mathcal{T}$.*

Proof. When we write $Hom_{\mathcal{T}}(\mathcal{S}, \mathcal{Q}) = 0$ we mean that $\mathcal{Q} \subseteq \mathcal{S}^\perp$. It suffices to show that $\mathcal{S} \star \mathcal{Q}$ is closed under mapping cones, so let a morphism $f : X \longrightarrow Y$ in $\mathcal{S} \star \mathcal{Q}$ be given and find triangles

$$S_X \longrightarrow X \longrightarrow Q_X \longrightarrow \Sigma S_X$$
$$S_Y \longrightarrow Y \longrightarrow Q_Y \longrightarrow \Sigma S_Y$$

with $S_X, S_Y \in \mathcal{S}$ and $Q_X, Q_Y \in \mathcal{Q}$. Since $Hom_{\mathcal{T}}(S_X, Q_Y) = 0$ we deduce a morphism $S_X \longrightarrow S_Y$ making a commutative diagram

$$\begin{tikzcd} S_X \arrow[r] \arrow[d] & S_Y \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

which by Corollary 32 we can extend to a large diagram, whose third column shows that any mapping cone on f belongs to $\mathcal{S} \star \mathcal{Q}$. $\square$

Lemma 108. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}, \mathcal{Q}$ triangulated subcategories such that*

$$Hom_{\mathcal{T}}(\mathcal{S}, \mathcal{Q}) = 0 \quad \textit{and} \quad Hom_{\mathcal{T}}(\mathcal{Q}, \mathcal{S}) = 0$$

Then $\mathcal{S} \star \mathcal{Q} = \mathcal{Q} \star \mathcal{S}$ is a triangulated subcategory of $\mathcal{T}$, and an object $X \in \mathcal{T}$ belongs to this subcategory if and only if it can be written as $X = S \oplus Q$ for some $S \in \mathcal{S}, Q \in \mathcal{Q}$.

Proof. By Lemma 107 both $\mathcal{S} \star \mathcal{Q}$ and $\mathcal{Q} \star \mathcal{S}$ are triangulated subcategories. If $X \in \mathcal{T}$ belongs to $\mathcal{S} \star \mathcal{Q}$ then we have a triangle

$$S \longrightarrow X \longrightarrow Q \longrightarrow \Sigma S$$

with $S \in \mathcal{S}$ and $Q \in \mathcal{Q}$. By hypothesis $Hom(Q, \Sigma S) = 0$ so this triangle splits, and therefore $X = S \oplus Q$. A similar argument applies if $X \in \mathcal{Q} \star \mathcal{S}$. The converse is trivial since any direct sum in $\mathcal{T}$ yields a triangle. $\square$

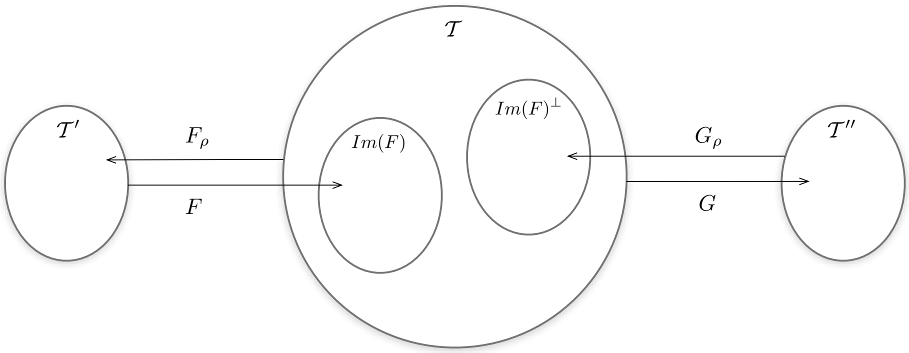
Remark 68. Let $\mathcal{T}$ be a triangulated category and $\mathcal{L}$ a thick, localising subcategory. It follows from Proposition 99 that $\mathcal{L}$ is bousfield if and only if $\mathcal{L} \star \mathcal{L}^{\perp} = \mathcal{T}$, which reinforces the intuition that the bousfield subcategories are precisely the direct summands.

Remark 69. Putting everything together, we arrive at our final understanding of bousfield subcategories and localisation sequences. Suppose we have a localisation sequence

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{69}$$

Fix right triadjoints F_ρ, G_ρ for F, G respectively with units η_F, η_G and counits $\varepsilon_F, \varepsilon_G$. We have a diagram of triangulated functors

$$\begin{tikzcd} \mathcal{T}' \arrow[r, bend right, "F"'] & \mathcal{T} \arrow[l, bend right, "F_\rho"'] \arrow[r, bend right, "G"'] & \mathcal{T}'' \arrow[l, bend right, "G_\rho"'] \end{tikzcd}$$



This diagram consists of a localisation sequence going to the right and a colocalisation sequence going to the left. The triangulated subcategories $Im(F)$ and $Im(G_\rho) = Im(F)^\perp$ are respectively bousfield and cobousfield and the functors F, G_ρ factor through canonical triequivalences $\mathcal{T}' \longrightarrow Im(F)$ and $\mathcal{T}'' \longrightarrow Im(F)^\perp$. The induced triangulated functors

$$\mathcal{T}/Im(F) \longrightarrow \mathcal{T}'', \quad \mathcal{T}/Im(F)^\perp \longrightarrow \mathcal{T}'$$

are equivalences and we have $\mathcal{T} = Im(F) \star Im(F)^\perp$. The analogy with a direct sum

$$\mathcal{T} = \mathcal{T}' \oplus \mathcal{T}'' \cong Im(F) \oplus Im(F)^\perp$$

is obvious. We have the following additional observations:

- Let $\widehat{F} : \mathcal{T}' \longrightarrow Im(F)$ be the factorisation of F and $i : Im(F) \longrightarrow \mathcal{T}$ the inclusion. If we set $i_\rho = \widehat{F} F_\rho$ the morphisms $\varepsilon_{F,X} : i i_\rho(X) \longrightarrow X$ define the counit of a triadjunction $i \dashv i_\rho$.

- Let $\widehat{G}_\rho : \mathcal{T}'' \longrightarrow Im(F)^\perp$ be the factorisation of G_ρ and $j : Im(F)^\perp \longrightarrow \mathcal{T}$ the inclusion. If we set $j_\lambda = \widehat{G}_\rho G$ the morphisms $\eta_{G,X} : X \longrightarrow j j_\lambda(X)$ define the unit of a triadjunction $j_\lambda \dashv j$.
- The composite $Im(F)^\perp \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}''$ is an equivalence.

4.3 Colocalisation Sequences

In this section we collect for later reference the duals of the results of Section 4.2. We also give the definition of a *recollement* of triangulated categories. Suppose we are given a colocalisation sequence

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{70}$$

Choose left triadjoints F_λ, G_λ. By duality we can make the following remarks:

- We have $Im(F) = Im(G_\lambda)^\perp$ and $Im(G_\lambda) = {}^\perp Im(F)$.
- The pair $\mathcal{T}' \xleftarrow{F_\lambda} \mathcal{T} \xleftarrow{G_\lambda} \mathcal{T}''$ is a localisation sequence.

Lemma 109. *Suppose we have a colocalisation sequence (70) and fix left adjoints F_λ, G_λ. Then for every $X \in \mathcal{T}$ there is a canonical triangle in $\mathcal{T}$ natural in X*

$$G_\lambda G(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} FF_\lambda(X) \longrightarrow \Sigma G_\lambda G(X) \tag{71}$$

Proposition 110. *Suppose we have an exact sequence of triangulated functors*

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{72}$$

Then the following are equivalent:

(a) The sequence (72) is a colocalisation sequence.

(b) The functor F has a left adjoint.

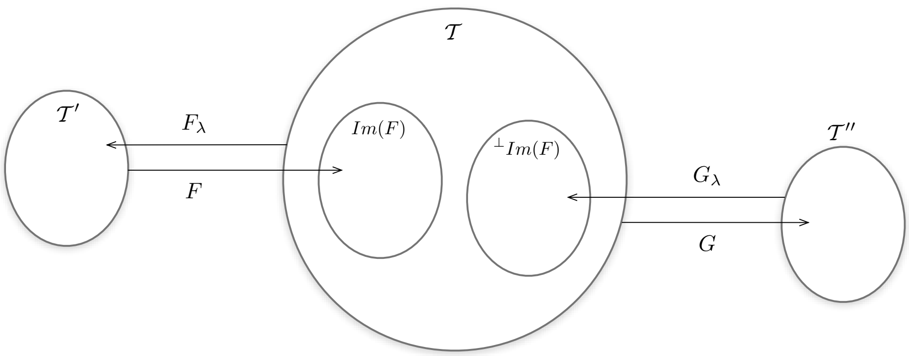
(c) The functor G has a left adjoint.

Remark 70. Putting everything together, we arrive at our final understanding of cobousfield subcategories and colocalisation sequences. Suppose we have a colocalisation sequence

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{73}$$

Fix left triadjoints F_λ, G_λ for F, G respectively with units η_F, η_G and counits $\varepsilon_F, \varepsilon_G$. We have a diagram of triangulated functors

$$\begin{tikzcd} \mathcal{T}' \arrow[r, bend right, "F"'] & \mathcal{T} \arrow[l, bend right, "F_\lambda"'] \arrow[r, bend right, "G"'] & \mathcal{T}'' \arrow[l, bend right, "G_\lambda"'] \end{tikzcd}$$



This diagram consists of a colocalisation sequence going to the right and a localisation sequence going to the left. The triangulated subcategories $Im(F)$ and $Im(G_\lambda) = {}^\perp Im(F)$ are respectively cobousfield and bousfield and the functors F, G_λ factor through canonical triequivalences $\mathcal{T}' \longrightarrow Im(F)$ and $\mathcal{T}'' \longrightarrow {}^\perp Im(F)$. The induced triangulated functors

$$\mathcal{T}/Im(F) \longrightarrow \mathcal{T}'', \quad \mathcal{T}/{}^\perp Im(F) \longrightarrow \mathcal{T}'$$

are equivalences and we have $\mathcal{T} = {}^\perp Im(F) \star Im(F)$. We have the following additional observations:

- Let $\widehat{G}_\lambda : \mathcal{T}'' \longrightarrow {}^\perp Im(F)$ be the factorisation of $\widehat{G}_\lambda$ and $k : {}^\perp Im(F) \longrightarrow \mathcal{T}$ the inclusion. If we set $k_\rho = \widehat{G}_\lambda G$ the morphisms $\varepsilon_{G,X} : kk_\rho(X) \longrightarrow X$ define the counit of a triadjunction $k \longrightarrow\!\!\!\!\!\dashv k_\rho$.
- Let $\widehat{F} : \mathcal{T}' \longrightarrow Im(F)$ be the factorisation of F and $i : Im(F) \longrightarrow \mathcal{T}$ the inclusion. If we set $i_\lambda = \widehat{F} F_\lambda$ the morphisms $\eta_{F,X} : X \longrightarrow ii_\lambda(X)$ define the unit of a triadjunction $i_\lambda \longrightarrow\!\!\!\!\!\dashv i$.
- The composite ${}^\perp Im(F) \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}''$ is an equivalence.

The intuition is that a localisation sequence and colocalisation sequence both write $\mathcal{T}$ as a direct sum $\mathcal{T} = \mathcal{T}' \oplus \mathcal{T}/\mathcal{T}'$, but in the first case the quotient embeds as the orthogonal $Im(F)^\perp$ and in the second case as ${}^\perp Im(F)$.

Definition 45. We say that a sequence of triangulated functors

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{74}$$

is a *recollement* if it is both a localisation sequence and a colocalisation sequence. Fixing right triadjoints F_ρ, G_ρ and left triadjoints F_λ, G_λ we have a diagram

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "F"] & \mathcal{T} \arrow[l, bend right, "F_\lambda"'] \arrow[l, bend left, "F_\rho"] \arrow[r, "G"] & \mathcal{T}'' \arrow[l, bend right, "G_\lambda"'] \arrow[l, bend left, "G_\rho"] \end{tikzcd} \tag{75}$$

In which G_λ, G_ρ are fully faithful and we have sequences

$$\mathcal{T}' \xleftarrow{F_\rho} \mathcal{T} \xleftarrow{G_\rho} \mathcal{T}'' \tag{76}$$

$$\mathcal{T}' \xleftarrow{F_\lambda} \mathcal{T} \xleftarrow{G_\lambda} \mathcal{T}'' \tag{77}$$

which are respectively a colocalisation and localisation sequence.

Remark 71. We make the following remarks

(i) The triangulated subcategory $Im(F)$ is both bousfield and cobousfield in $\mathcal{T}$.

(ii) We have $Im(G_\rho) = Im(F)^\perp$ and $Im(G_\lambda) = {}^\perp Im(F)$ and induced triequivalences

$$\mathcal{T}/Im(F)^\perp \longrightarrow \mathcal{T}', \quad \mathcal{T}/{}^\perp Im(F) \longrightarrow \mathcal{T}'$$
$$\mathcal{T}'' \longrightarrow Im(F)^\perp, \quad \mathcal{T}'' \longrightarrow {}^\perp Im(F)$$

5 Right Derived Functors

Definition 46. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory with verdier quotient $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$. Given a triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$ a *right derived functor of F with respect to $\mathcal{C}$* is a pair consisting of a triangulated functor $\mathbb{R}F : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ and a trinatural transformation $\zeta : F \longrightarrow \mathbb{R}F \circ Q$, with the following universal property: given any triangulated functor $G : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ and trinatural transformation $\rho : F \longrightarrow G \circ Q$ there is a *unique* trinatural transformation $\eta : \mathbb{R}F \longrightarrow G$ making the following diagram commute

$$\begin{tikzcd} & F \arrow[dl, "\zeta"'] \arrow[dr, "\rho"] & \\ \mathbb{R}F \circ Q \arrow[rr, "\eta Q"'] & & G \circ Q \end{tikzcd}$$

Clearly if a right derived functor exists it is unique up to canonical trinatural equivalence. We will often abuse notation by dropping the subcategory $\mathcal{C}$ and the transformation ζ from the notation, and saying simply that $\mathbb{R}F$ is a right derived functor of F.

Remark 72. Right derived functors are stable under trinatural equivalence. With the notation of Definition 46 suppose that $G : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ is a triangulated functor and that $\tau : \mathbb{R}F \longrightarrow G$ is a trinatural equivalence. Then the pair $(G, \tau Q \circ \zeta)$ is easily checked to be a right derived functor of F. Similarly if $E : \mathcal{T} \longrightarrow \mathcal{T}'$ is a triisomorphism the pair $(E \circ \mathbb{R}F, E\zeta)$ is a right derived functor of $E \circ F$.

Lemma 111. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor containing $\mathcal{C}$ in its kernel. Then the induced triangulated functor $H : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ together with the identity transformation is a right derived functor of F.*

Proof. Suppose we are given a triangulated functor $G : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ and a trinatural transformation $\rho : F \longrightarrow G \circ Q$. Using the fact that morphisms in $\mathcal{D}/\mathcal{C}$ can be written in the form $Q(g)Q(f)^{-1}$ for morphisms f, g of $\mathcal{D}$, it is easily checked that $\eta_X = \rho_X$ defines a trinatural transformation $\eta : H \longrightarrow G$ which is unique such that $\eta Q = \rho$, so the proof is complete. $\square$

Definition 47. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a triangulated functor. A fragile triangulated subcategory $\mathcal{A} \subseteq \mathcal{D}$ is *right adapted for F and $\mathcal{C}$* if it satisfies the following conditions

(i) Any object in $\mathcal{C} \cap \mathcal{A}$ belongs to the kernel of F.

(ii) For any object $X \in \mathcal{D}$ there exists a morphism $f : X \longrightarrow M$ with $M \in \mathcal{A}$ and $f \in Mor_{\mathcal{C}}$.

In other words, the restriction of F to $\mathcal{A}$ factors through the quotient $\mathcal{A}/(\mathcal{C} \cap \mathcal{A})$ and every object of $\mathcal{D}$ admits a "resolution" by an object of $\mathcal{A}$ (one should think of morphisms in $Mor_{\mathcal{C}}$ as being quasi-isomorphisms, in which case (ii) is akin to the existence of hoinjective resolutions (DTC,Definition 25)). We say that F is *right $\mathcal{C}$-adaptable* if there exists a fragile triangulated subcategory $\mathcal{A}$ which is right adapted for F and $\mathcal{C}$.

Theorem 112. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a triangulated functor. If F is right $\mathcal{C}$-adaptable then it has a right derived functor with respect to $\mathcal{C}$.*

Proof. Let $\mathcal{A}$ be a fragile triangulated subcategory of $\mathcal{D}$ which is right adapted for F and $\mathcal{C}$. We observe that $Mor_{\mathcal{C} \cap \mathcal{A}} = Mor_{\mathcal{C}} \cap \mathcal{A}$, so the condition (ii) of adaptability mean that the conditions of Proposition 70 are satisfied, and the canonical triangulated functor

$$T : \mathcal{A}/(\mathcal{C} \cap \mathcal{A}) \longrightarrow \mathcal{D}/\mathcal{C}$$

is a full embedding. In fact, the hypothesis (ii) means that every object of $\mathcal{D}/\mathcal{C}$ is isomorphic to an object of $\mathcal{A}$, so this functor is an equivalence and hence by Lemma 49 a triequivalence. We can construct a triangulated functor $(S, \phi_S) : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{A}/(\mathcal{C} \cap \mathcal{A})$ together with trinatural equivalences $\eta : 1 \longrightarrow TS, \varepsilon : ST \longrightarrow 1$ such that ε is the identity (that is, $ST = 1$), and $\phi_S = S\Sigma\eta$. The composite $\mathcal{A} \longrightarrow \mathcal{D} \longrightarrow \mathcal{T}$ sends objects of $\mathcal{C} \cap \mathcal{A}$ to zero, and therefore factors uniquely through the verdier quotient $\mathcal{A}/(\mathcal{C} \cap \mathcal{A})$. If the factorisation is $H : \mathcal{A}/(\mathcal{C} \cap \mathcal{A}) \longrightarrow \mathcal{T}$ then we claim the composite $\mathbb{R}F = HS : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ is a right derived functor of F with respect to $\mathcal{C}$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "Q"'] \arrow[dr, bend left, "F"] & \\
\mathcal{A}/(\mathcal{C} \cap \mathcal{A}) \arrow[r, shift left] \arrow[drr, bend right, "H"'] & \mathcal{D}/\mathcal{C} \arrow[l, shift left] & \\
 & & \mathcal{T}
\end{tikzcd}$$

Observe that the functor S is essentially an assignment of resolutions: for every object $X \in \mathcal{D}$ it chooses an object $SX \in \mathcal{A}$ together with an isomorphism $\eta_X : X \longrightarrow SX$ in $\mathcal{D}/\mathcal{C}$. This morphism can be written as a diagram in $\mathcal{D}$

$$\begin{tikzcd}
 & W & \\
X \arrow[ur, "b"] & & SX \arrow[ul, "a"']
\end{tikzcd} \tag{78}$$

with $a \in Mor_{\mathcal{C}}$. That is, $\eta_X = Q(a)^{-1}Q(b)$. By taking a morphism $W \longrightarrow A$ in $Mor_{\mathcal{C}}$ with $A \in \mathcal{A}$ we can assume that $W \in \mathcal{A}$ and consequently that $a \in Mor_{\mathcal{C} \cap \mathcal{A}}$. Therefore $F(a)$ is an isomorphism in $\mathcal{T}$ and we can define a morphism $\zeta_X : F(X) \longrightarrow FS(X)$ in $\mathcal{T}$ by $\zeta_X = F(a)^{-1}F(b)$. One checks that ζ_X is canonical: that is, it does not depend on the choice of diagram (78) used to represent η_X. By construction of H we have $HS(X) = FS(X)$ for any $X \in \mathcal{D}$, and it is a little tedious but not difficult to check that ζ is a trinatural transformation $F \longrightarrow \mathbb{R}F \circ Q$.

It remains to show that $(\mathbb{R}F, \zeta)$ possesses the universal property of a right derived functor. Suppose we are given a triangulated functor $G : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ and a trinatural transformation $\rho : F \longrightarrow G \circ Q$. We want to construct a trinatural transformation $\theta : \mathbb{R}F \longrightarrow G$, which consists for $X \in \mathcal{D}$ of a morphism $\theta_X : FS(X) \longrightarrow G(X)$. Since $\eta_X : X \longrightarrow SX$ is an isomorphism in $\mathcal{D}/\mathcal{C}$ we can define θ_X to be the composite $\theta_X = G(\eta_X)^{-1}\rho_{SX}$

$$FS(X) \xrightarrow{\rho_{SX}} GS(X) \xrightarrow{G(\eta_X)^{-1}} G(X)$$

It is fairly straightforward to check that this definition makes θ a trinatural transformation, and that the following diagram commutes

$$\begin{tikzcd}
 & F \arrow[dl, "\zeta"'] \arrow[dr, "\rho"] & \\
\mathbb{R}F \circ Q \arrow[rr, "\theta Q"'] & & G \circ Q
\end{tikzcd}$$

To see that θ is unique with this property, let $\theta' : \mathbb{R}F \longrightarrow G$ be another trinatural transformation making this diagram commute. We deduce immediately that $\theta'_X = \theta_X$ for any $X \in \mathcal{A}$. For arbitrary $X \in \mathcal{D}$ the following diagram commutes

$$\begin{tikzcd}
HSX \arrow[r, "HS(\eta_X)"] \arrow[d, "\theta'_X"'] & HSX \arrow[d, "\theta'_{SX}"] \\
GX \arrow[r, "G(\theta_X)"'] & GSX
\end{tikzcd}$$

and therefore $G(\eta_X)\theta'_X = \theta_{SX}HS(\eta_X) = G(\eta_X)\theta_X$ from which we deduce that $\theta_X = \theta'_X$ since η_X is an isomorphism. This shows that $(\mathbb{R}F, \zeta)$ is a right derived functor, and completes the proof. $\square$

Remark 73. With the notation of Theorem 112 and its proof, the right derived functor $\mathbb{R}F$ was defined by "choosing resolutions" in $\mathcal{A}$ for every object of $\mathcal{D}$ (that is, defining the functor S) and then setting $\mathbb{R}F(X) = F(SX)$. For convenience in the proof we chose S to be of a special form, which chooses the identity resolution for every object of $\mathcal{A}$. We will now show that this point is not crucial.

We know that every object of $\mathcal{D}/\mathcal{C}$ is isomorphic to an object of $\mathcal{A}$. Choose for every $X \in \mathcal{D}$ a particular object $A_X \in \mathcal{A}$ and isomorphism $\mu_X : X \longrightarrow A_X$ in $\mathcal{D}/\mathcal{C}$. As in Lemma 49 we can construct a triangulated functor $A : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{A}/(\mathcal{C} \cap \mathcal{A})$ with $A(X) = A_X$ such that $\mu : 1 \longrightarrow TA$ is a trinatural equivalence.

Let S and η be constructed as in Theorem 112 (i.e. by the same process, but with the restricted choices of resolutions). By Remark 36 the functors S, A are left triadjoint to T with respective units η, μ. It follows from Lemma 43 that there is a canonical trinatural equivalence $\gamma : S \longrightarrow A$ such that $T\gamma \circ \eta = \mu$. We have therefore a trinatural equivalence $H\gamma : \mathbb{R}F \longrightarrow HA$. Remark 72 implies that the pair $(HA, H\gamma Q \circ \zeta)$ is a right derived functor of F with respect to $\mathcal{C}$. Given $X \in \mathcal{D}/\mathcal{C}$ write μ_X as a diagram of the form

$$\begin{tikzcd} & W & \\ X \arrow[ur, "b"] & & A_X \arrow[ul, "a"'] \end{tikzcd}$$

with $W \in \mathcal{A}$ and $a \in Mor_{\mathcal{C}}$. Then it is easy to check that $(H\gamma Q \circ \zeta)_X$ is the morphism $F(a)^{-1}F(b) : F(X) \longrightarrow \mathbb{R}FQ(X)$, so our right derived functor is calculated in the same way as before but with arbitrary choices of resolutions.

Remark 74. With the same notation, suppose that for each $X \in \mathcal{D}$ the chosen isomorphism $X \longrightarrow A_X$ in $\mathcal{D}/\mathcal{C}$ is of the form $Q(\mu_X)$ for some morphism $\mu_X : X \longrightarrow A_X$ in $Mor_{\mathcal{C}}$. Let $(\mathbb{R}F, \zeta)$ be the right derived functor canonically constructed from these choices. By construction we have $\zeta_X = F(\mu_X)$ for every $X \in \mathcal{D}$. Given a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ suppose we have a diagram in $\mathcal{D}$

$$\begin{tikzcd} & W & \\ A_X \arrow[ur, "a"] & & A_Y \arrow[ul, "b"'] \end{tikzcd}$$

with $W \in \mathcal{A}$ and $b \in Mor_{\mathcal{C}}$, which represents a morphism $\gamma : A_X \longrightarrow A_Y$ of $\mathcal{D}/\mathcal{C}$ making the following diagram commute

$$\begin{tikzcd} X \arrow[r, "Q(\eta_X)"] \arrow[d, "Q(f)"'] & A_X \arrow[d, "\gamma"] \\ Y \arrow[r, "Q(\eta_Y)"'] & A_Y \end{tikzcd}$$

Then $\mathbb{R}FQ(f) = F(b)^{-1}F(a)$. This is enough to calculate $\mathbb{R}F$ on morphisms of $\mathcal{D}/\mathcal{C}$. It remains to give explicitly the natural equivalence $\phi_{\mathbb{R}F} : \mathbb{R}F \circ \Sigma \longrightarrow \Sigma \circ \mathbb{R}F$. For $X \in \mathcal{D}$ we have

$$\phi_{\mathbb{R}F,X} = \phi_{F,A_X} F(\mu_{\Sigma A_X})^{-1} \mathbb{R}FQ(\Sigma \mu_X)$$

In practice this is awkward, because of the occurrence of the functor $\mathbb{R}F$ itself. But we can usually reduce to the case where $X \in \mathcal{A}$, and in that case we have

$$\phi_{\mathbb{R}F,X} = \phi_{F,A_X} F(\Sigma \mu_X) F(\mu_{\Sigma X})^{-1}$$

This discussion defines $(\mathbb{R}F, \zeta)$ using only the chosen resolutions $\mu_X : X \longrightarrow A_X$.

Let us set up some notation for the next result. Suppose we have a diagram of triangulated categories and triangulated functors

$$\begin{tikzcd}
\mathcal{C} \arrow[d] & \mathcal{C}' \arrow[d] & \\
\mathcal{D} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D}' \arrow[r, "G"] \arrow[d, "Q'"'] & \mathcal{T} \\
\mathcal{D}/\mathcal{C} \arrow[r, "\mathbb{R}(Q'F)"'] & \mathcal{D}'/\mathcal{C}' \arrow[ur, "\mathbb{R}G"'] &
\end{tikzcd} \tag{79}$$

where the columns are verdier quotients of thick triangulated subcategories, and we assume that there exist right derived functors $(\mathbb{R}(Q'F), \zeta), (\mathbb{R}G, \omega)$. Then we have a trinatural transformation

$$GF \xrightarrow{\omega F} \mathbb{R}(G)Q'F \xrightarrow{\mathbb{R}(G)\zeta} \mathbb{R}(G)\mathbb{R}(Q'F)Q$$

which we denote by $\mu : GF \longrightarrow \mathbb{R}(G)\mathbb{R}(Q'F)Q$. If a right derived functor $(\mathbb{R}(GF), \xi)$ exists, then there is a unique trinatural transformation $\theta : \mathbb{R}(GF) \longrightarrow \mathbb{R}(G)\mathbb{R}(Q'F)$ making the following diagram commute

$$\begin{tikzcd}
GF \arrow[r, "\omega F"] \arrow[d, "\xi"'] & \mathbb{R}(G)Q'F \arrow[d, "\mathbb{R}(G)\zeta"] \\
\mathbb{R}(GF)Q \arrow[r, "\theta Q"'] & \mathbb{R}(G)\mathbb{R}(Q'F)Q
\end{tikzcd}$$

In general the pair consisting of the triangulated functor $\mathbb{R}(G)\mathbb{R}(Q'F)$ and μ is not a right derived functor of GF (that is, θ is not a trinatural equivalence). But it is true under some natural hypothesis. With this notation,

Theorem 113. *Assume that $\mathcal{A} \subseteq \mathcal{D}$ is right adapted for $Q'F$ and $\mathcal{C}$, and that $\mathcal{A}' \subseteq \mathcal{D}'$ is right adapted for G and $\mathcal{C}'$. If F sends objects of $\mathcal{A}$ to objects of $\mathcal{A}'$, then the pair $(\mathbb{R}(G)\mathbb{R}(Q'F), \mu)$ is a right derived functor of GF. That is, the canonical trinatural transformation*

$$\theta : \mathbb{R}(GF) \longrightarrow \mathbb{R}(G)\mathbb{R}(Q'F)$$

is a trinatural equivalence.

Proof. To be clear, we assume that we are given a diagram (79) and two arbitrary right derived functors $(\mathbb{R}(Q'F), \zeta), (\mathbb{R}G, \omega)$, and we prove that the pair $(\mathbb{R}(G)\mathbb{R}(Q'F), \mu)$ is a right derived functor of GF. It follows that given any other right derived functor $\mathbb{R}(GF)$ the induced trinatural transformation θ is a trinatural equivalence.

The hypothesis that $F(\mathcal{A}) \subseteq \mathcal{A}'$ together with thickness of $\mathcal{C}'$ means that the subcategory $\mathcal{A}$ is right adapted for GF and $\mathcal{C}$, so a right derived functor of GF exists. Choose triangulated functors $S : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{A}/(\mathcal{C} \cap \mathcal{A})$ and $S' : \mathcal{D}'/\mathcal{C}' \longrightarrow \mathcal{A}'/(\mathcal{C}' \cap \mathcal{A}')$ as in the proof of Theorem 112. We can immediately reduce to the case where our right derived functors

$$(\mathbb{R}(Q'F), \zeta), (\mathbb{R}(G), \omega), (\mathbb{R}(GF), \xi)$$

are the canonical ones defined using the functors S, S'. For $X \in \mathcal{A}$ the morphism θ_X is the following equality

$$\mathbb{R}(GF)(X) = GF(X) = G(F(X)) = \mathbb{R}(G)(F(X)) = \mathbb{R}(G)\mathbb{R}(Q'F)(X)$$

For arbitrary $X \in \mathcal{D}$ we have the isomorphism $\eta_X : X \longrightarrow SX$ in $\mathcal{D}/\mathcal{C}$ and a therefore a commutative diagram in $\mathcal{T}$ with vertical isomorphisms

$$\begin{tikzcd}
\mathbb{R}(GF)(X) \arrow[r, "\theta_X"] \arrow[d, Rightarrow, "\mathbb{R}(GF)(\eta_X)"'] & \mathbb{R}(G)\mathbb{R}(Q'F)(X) \arrow[d, Rightarrow, "\mathbb{R}(G)\mathbb{R}(Q'F)(\eta_X)"] \\
\mathbb{R}(GF)(SX) \arrow[r, "\theta_{SX}"'] & \mathbb{R}(G)\mathbb{R}(Q'F)(SX)
\end{tikzcd}$$

Since the bottom row is an isomorphism it follows that θ_X is also an isomorphism, which is what we wanted to show. $\square$

5.1 Acyclic Objects

Definition 48. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor and $\mathcal{C}$ a triangulated subcategory of $\mathcal{D}$. An object $X \in \mathcal{D}$ is *right F-acyclic with respect to $\mathcal{C}$* if whenever there is a morphism $s : X \longrightarrow Y$ in $Mor_{\mathcal{C}}$ there is another morphism $t : Y \longrightarrow Z$ in $Mor_{\mathcal{C}}$ such that $F(ts)$ is an isomorphism. We simply say that X is *right F-acyclic* if there is no chance of confusion, and denote the full subcategory of $\mathcal{D}$ consisting of such objects by $\mathcal{A}_{F,\mathcal{C}}$.

Remark 75. With the above notation it is clear that any zero object is right F-acyclic with respect to $\mathcal{C}$, and that $\mathcal{A}_{F,\mathcal{C}}$ is a replete subcategory of $\mathcal{D}$.

Later we will see that objects which are right acyclic for a functor can be used as “resolutions” to calculate derived functors. The next observation is trivial, but it describes the most common type of acyclic objects that we we will encounter in applications.

Lemma 114. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. An object $X \in \mathcal{D}$ belongs to $\mathcal{C}^{\perp}$ if and only if it is right F-acyclic with respect to $\mathcal{C}$ for every triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$.*

Proof. Suppose that $X \in \mathcal{C}^{\perp}$ and let a triangulated functor F and morphism $s : X \longrightarrow Y$ in $Mor_{\mathcal{C}}$ be given. By Lemma 94 there exists a morphism $t : Y \longrightarrow X$ with $ts = 1$, so it is clear from Lemma 35 that X is right F-acyclic with respect to $\mathcal{C}$. For the converse it is enough that X be right acyclic for the identity $1 : \mathcal{D} \longrightarrow \mathcal{D}$. Then any morphism $s : X \longrightarrow Y$ in $Mor_{\mathcal{C}}$ admits $t : Y \longrightarrow Z$ with ts an isomorphism. In particular s is a coretraction, so by Lemma 94 we have $X \in \mathcal{C}^{\perp}$. $\square$

Proposition 115. *Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor and $\mathcal{C}$ a thick triangulated subcategory of $\mathcal{D}$. Then $\mathcal{A}_{F,\mathcal{C}}$ is a triangulated subcategory of $\mathcal{D}$ with the property that every object of $\mathcal{C} \cap \mathcal{A}_{F,\mathcal{C}}$ belongs to the kernel of F.*

Proof. It is easy to check that $\mathcal{A}_{F,\mathcal{C}}$ is closed under Σ^{-1}, so by Lemma 33 to show that $\mathcal{A}_{F,\mathcal{C}}$ is a triangulated subcategory it suffices to prove closure under mapping cones. It is enough to show that given a triangle in $\mathcal{D}$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

with Y, Z both in $\mathcal{A}_{F,C}$, then also $X \in \mathcal{A}_{F,C}$. Let $s : X \longrightarrow Q$ be a morphism in $Mor_{\mathcal{C}}$ and form the homotopy pushout

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "t"] & Z \arrow[r] & \Sigma X \\
Q \arrow[r] & T & &
\end{tikzcd}$$

By Lemma 37 we have $t \in Mor_{\mathcal{C}}$ and by Lemma 29 we can extend the bottom row to a triangle and then complete to a morphism of triangles of the following form

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "t"'] & Z \arrow[r] \arrow[d, "1"'] & \Sigma X \arrow[d, "\Sigma s"'] \\
Q \arrow[r] & T \arrow[r, "v"'] & Z \arrow[r] & \Sigma Q
\end{tikzcd} \tag{80}$$

Now let $t' : T \longrightarrow T'$ be a morphism in $Mor_{\mathcal{C}}$ with $F(t't)$ an isomorphism. Form the homotopy pushout of this morphism and $T \longrightarrow Z$ and complete to a morphism of triangles

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "s'"'] & T \arrow[r] \arrow[d, "t'"'] & Z \arrow[r] \arrow[d, "q"'] & \Sigma Q \arrow[d] \\
Q' \arrow[r] & T' \arrow[r, "v'"'] & Z' \arrow[r] & \Sigma Q'
\end{tikzcd} \tag{81}$$

where $q \in Mor_{\mathcal{C}}$ by Lemma 37. Composing with (80) we have a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s's"'] & Y \arrow[r] \arrow[d, "t't"'] & Z \arrow[r] \arrow[d, "q"'] & \Sigma X \arrow[d] \\ Q' \arrow[r] & T' \arrow[r, "v'"'] & Z' \arrow[r] & \Sigma Q' \end{tikzcd} \tag{82}$$

in which $t't, q$ belong to $Mor_{\mathcal{C}}$ and $F(t't)$ is an isomorphism. Let $q' : Z' \longrightarrow Z''$ be a morphism in $Mor_{\mathcal{C}}$ such that $F(q'q)$ is an isomorphism. Then we can complete the morphism $q'v'$ to a triangle and induce a morphism of triangles of the following form

$$\begin{tikzcd} Q' \arrow[r] \arrow[d, "s''"'] & T' \arrow[r, "v'"] \arrow[d, "1"'] & Z' \arrow[r] \arrow[d, "q'"'] & \Sigma Q' \arrow[d] \\ Q'' \arrow[r] & T' \arrow[r, "q'v'"'] & Z'' \arrow[r] & \Sigma Q'' \end{tikzcd} \tag{83}$$

Composing with (82) finally yields a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s''s's"'] & Y \arrow[r] \arrow[d, "t't"'] & Z \arrow[r] \arrow[d, "q'q"'] & \Sigma X \arrow[d] \\ Q'' \arrow[r] & T' \arrow[r] & Z'' \arrow[r] & \Sigma Q'' \end{tikzcd}$$

where $F(t't), F(q'q)$ are isomorphisms. Mapping this morphism of triangles into $\mathcal{T}$ and applying Proposition 6 we conclude that $F(s''s's)$ is an isomorphism. Since $\mathcal{C}$ is thick, we can invoke Lemma 71 on the triangles (81), (83) to see that $s', s'' \in Mor_{\mathcal{C}}$. We have therefore constructed a morphism $s''s' : Q \longrightarrow Q''$ in $Mor_{\mathcal{C}}$ with $F(s''s's)$ an isomorphism, which proves that X is right F-ayclic with respect to $\mathcal{C}$. Therefore $\mathcal{A}_{F,C}$ is a triangulated subcategory of $\mathcal{D}$. If an object X belongs to $\mathcal{C} \cap \mathcal{A}_{F,C}$ then the zero morphism $X \longrightarrow 0$ belongs to $Mor_{\mathcal{C}}$, so it is clear that $F(X) = 0$. □

Theorem 116. *Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor, $\mathcal{C}$ a thick triangulated subcategory of $\mathcal{D}$. Suppose that for every object $X \in \mathcal{D}$ there exists a morphism $\eta_X : X \longrightarrow A_X$ in $Mor_{\mathcal{C}}$ with A_X right F-acyclic. Then F admits a right derived functor $(\mathbb{R}F, \zeta)$ with the following properties*

(i) For any object $X \in \mathcal{D}$ we have $\mathbb{R}F(X) = F(A_X)$ and $\zeta_X = F(\eta_X)$.

(ii) An object $X \in \mathcal{D}$ is right F-acyclic if and only if ζ_X is an isomorphism in $\mathcal{T}$.

Proof. From Proposition 115 we know that $\mathcal{A}_{F,\mathcal{C}}$ is a triangulated subcategory of $\mathcal{D}$ with $\mathcal{C} \cap \mathcal{A}_{F,\mathcal{C}} \subseteq Ker(F)$. By hypothesis the other condition of Definition 47 is satisfied, so $\mathcal{A}_{F,C}$ is right adapted for F and $\mathcal{C}$. Choose for each $X \in \mathcal{D}$ a morphism $\eta_X : X \longrightarrow A_X$ in $Mor_{\mathcal{C}}$ with $A_X \in \mathcal{A}_{F,\mathcal{C}}$ and define $\mathbb{R}F$ to be the right derived functor constructed using these choices as in Remark 74, from which we deduce the desired properties.

(ii) Suppose that ζ_X is an isomorphism and let $s : X \longrightarrow Y$ be a morphism in $Mor_{\mathcal{C}}$. The following diagram commutes in $\mathcal{T}$

$$\begin{tikzcd} F(X) \arrow[r, "F(s)"] \arrow[d, Rightarrow, "F(\eta_X)"'] & F(Y) \arrow[d, "F(\eta_Y)"] \\ F(A_X) \arrow[r, Rightarrow, "\mathbb{R}FQ(s)"'] & F(A_Y) \end{tikzcd}$$

So $\eta_Y : Y \longrightarrow A_Y$ is a morphism in $Mor_{\mathcal{C}}$ such that $F(\eta_Y s)$ is an isomorphism, as required. It remains to show that $X \in \mathcal{A}_{F,\mathcal{C}}$ implies that ζ_X is an isomorphism.

As in the proof of Theorem 112 let S be a triangulated inverse to T constructed so that $ST = 1$ and let (HS, ζ') be the corresponding right derived functor of F. By construction ζ'_X

is an isomorphism if $X \in \mathcal{A}_{F,\mathcal{C}}$. Uniqueness of the right derived functor means that there is a canonical trinatural equivalence $\tau : HS \longrightarrow \mathbb{R}F$ such that $\tau Q \circ \zeta' = \zeta$, from which we deduce that (ii) is true for the pair $(\mathbb{R}F, \zeta)$. $\square$

Remark 76. With the notation of Proposition 116 let $\mathcal{A}$ be the smallest triangulated subcategory of $\mathcal{D}$ containing A_X for every $X \in \mathcal{D}$. Then $\mathcal{A} \subseteq \mathcal{A}_{F,\mathcal{C}}$ and it is not hard to see that $\mathcal{A}$ is right adapted for F and $\mathcal{C}$.

We know from (DF,Proposition 63) that taking right derived functors in the classical sense commutes with composition with an exact functor. The next result proves the analogue for the new version of derived functors.

Lemma 117. *In the situation of Theorem 116 suppose that $(\mathbb{R}F, \zeta)$ is a right derived functor of F and $G : \mathcal{T} \longrightarrow \mathcal{S}$ a triangulated functor. Then $(G \circ \mathbb{R}F, G\zeta)$ is a right derived functor of GF.*

Proof. When we say that $(\mathbb{R}F, \zeta)$ is a right derived functor of F, we mean that it is an *arbitrary* right derived functor, not necessarily one constructed as in Theorem 116. Of course to prove the result, it suffices to prove it in this case. That is, we choose morphisms $\eta_X : X \longrightarrow A_X$ and obtain a canonical right derived functor $(\mathbb{R}F, \zeta)$ defined by $\mathbb{R}F(X) = A_X$. Now it is clear that if $B \in \mathcal{D}$ is right F-acyclic then it is also right GF-acyclic, so we can use these same morphisms η_X to define a right derived functor $(\mathbb{R}(GF), \zeta')$. One checks from our explicit construction that in fact $\mathbb{R}(GF) = G \circ \mathbb{R}F$ as triangulated functors, and $\zeta' = G\zeta$, which completes the proof. $\square$

5.2 Derived Transformations

Definition 49. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory with verdier quotient $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$. Suppose that $F, G : \mathcal{D} \longrightarrow \mathcal{T}$ are triangulated functors with right derived functors $\mathbb{R}F, \mathbb{R}G$ and let $\alpha : F \longrightarrow G$ be a trinatural transformation. Then the composite $\zeta_G \alpha : F \longrightarrow \mathbb{R}G \circ Q$ induces a unique trinatural transformation $\mathbb{R}\alpha : \mathbb{R}F \longrightarrow \mathbb{R}G$ making the following diagram commute

$$\begin{tikzcd} F \arrow[rr, "\alpha"] \arrow[d, "\zeta_F"'] & & G \arrow[d, "\zeta_G"] \\ \mathbb{R}F \circ Q \arrow[rr, "(\mathbb{R}\alpha)Q"'] & & \mathbb{R}G \circ Q \end{tikzcd}$$

It is clear that given another trinatural transformation $\beta : G \longrightarrow H$ we have $\mathbb{R}(\beta\alpha) = \mathbb{R}\beta \circ \mathbb{R}\alpha$. Similarly $\mathbb{R}(\alpha + \alpha') = \mathbb{R}(\alpha) + \mathbb{R}(\alpha')$ and $\mathbb{R}1 = 1$.

Let $F, G : \mathcal{D} \longrightarrow \mathcal{T}$ be triangulated functors, $\mathcal{C}$ a thick triangulated subcategory of $\mathcal{D}$ and $\eta_X : X \longrightarrow A_X$ a morphism in $Mor_{\mathcal{C}}$ with A_X right F and G-acyclic for each $X \in \mathcal{D}$. Define the right derived functors $\mathbb{R}F, \mathbb{R}G$ using these resolutions as in Theorem 116. Then given a trinatural transformation $\alpha : F \longrightarrow G$ we can describe the trinatural transformation $\mathbb{R}\alpha$ explicitly.

Lemma 118. *For any trinatural transformation $\alpha : F \longrightarrow G$ and $X \in \mathcal{D}$ we have $(\mathbb{R}\alpha)_X = \alpha_{A_X}$.*

Proof. It suffices to show that $\gamma_X = \alpha_{A_X}$ defines a trinatural transformation $\mathbb{R}F \longrightarrow \mathbb{R}G$ such that $\gamma Q \zeta_F = \zeta_G \alpha$. Using the explicit calculations of Remark 74 this is not difficult. $\square$

6 Left Derived Functors

Definition 50. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory with verdier quotient $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$. Given a triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$ a *left derived functor of F with respect to $\mathcal{C}$* is a pair consisting of a triangulated functor $\mathbb{L}F : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ and a trinatural transformation $\zeta : \mathbb{L}F \circ Q \longrightarrow F$ with the following universal property: given any triangulated functor $G : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ and trinatural transformation $\rho : G \circ Q \longrightarrow F$ there is a *unique* trinatural transformation $\eta : G \longrightarrow \mathbb{L}F$ making the following diagram commute

$$\begin{tikzcd} G \circ Q \arrow[rr, "\eta Q"] \arrow[dr, "\rho"'] & & \mathbb{L}F \circ Q \arrow[dl, "\zeta"] \\ & F & \end{tikzcd}$$

Clearly if a left derived functor exists it is unique up to canonical trinatural equivalence. We will often abuse notation by dropping the subcategory $\mathcal{C}$ and the transformation ζ from the notation, and saying simply that $\mathbb{L}F$ is a left derived functor of F.

Remark 77. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory with verdier quotient $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$. Recall that $\mathcal{C}^{\mathrm{op}}$ is a triangulated subcategory of $\mathcal{D}^{\mathrm{op}}$ and the verdier quotient $Q' : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}/\mathcal{C}^{\mathrm{op}}$ fits into a commutative diagram of triangulated functors

$$\begin{tikzcd} & \mathcal{D}^{\mathrm{op}} \arrow[dl, "Q'"'] \arrow[dr, "Q^{\mathrm{op}}"] & \\ \mathcal{D}^{\mathrm{op}}/\mathcal{C}^{\mathrm{op}} \arrow[rr, "Z"'] & & (\mathcal{D}/\mathcal{C})^{\mathrm{op}} \end{tikzcd}$$

where Z is a triisomorphism. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor and $(\mathbb{L}F, \zeta)$ a left derived functor of F. Then we can compose Z with the triangulated functor $(\mathbb{L}F)^{\mathrm{op}} : (\mathcal{D}/\mathcal{C})^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ to obtain a triangulated functor $\mathbb{R}F^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}}/\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ and a trinatural transformation $\zeta^{\mathrm{op}} : F^{\mathrm{op}} \longrightarrow \mathbb{R}F^{\mathrm{op}} \circ Q'$. It is striaghtforward to check that the pair $(\mathbb{R}F^{\mathrm{op}}, \zeta^{\mathrm{op}})$ is a right derived functor of F^{op}. Similarly if $(\mathbb{R}F, \zeta)$ is a right derived functor of F then we set $\mathbb{L}F^{\mathrm{op}} = (\mathbb{R}F)^{\mathrm{op}}Z$ and $(\mathbb{L}F^{\mathrm{op}}, \zeta^{\mathrm{op}})$ is a left derived functor of F^{op}. This means that by duality arguments the results of Section 5 can be translated to statements about left derived functors.

Remark 78. Left derived functors are stable under trinatural equivalence. With the notation of Definition 50 suppose that $G : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ is a triangulated functor and that $\tau : G \longrightarrow \mathbb{L}F$ is a trinatural equivalence. Then the pair $(G, \zeta \circ \tau Q)$ is easily checked to be a left derived functor of F.

Lemma 119. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor containing $\mathcal{C}$ in its kernel. Then the induced triangulated functor $H : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$ together with the identity transformation is a left derived functor of F.*

Proof. Follows by duality from Lemma 111. □

Definition 51. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a triangulated functor. A fragile triangulated subcategory $\mathcal{A} \subseteq \mathcal{D}$ is *left adapted for F and $\mathcal{C}$* if it satisfies the following conditions

(i) Any object in $\mathcal{C} \cap \mathcal{A}$ belongs to the kernel of F.

(ii) For any object $X \in \mathcal{D}$ there exists a morphism $f : M \longrightarrow X$ with $M \in \mathcal{A}$ and $f \in Mor_{\mathcal{C}}$.

We say that F is *left $\mathcal{C}$-adaptable* if there exists a fragile triangulated subcategory $\mathcal{A}$ which is left adapted for F and $\mathcal{C}$. It is clear that $\mathcal{A}$ is left adapted for F and $\mathcal{C}$ if and only if $\mathcal{A}^{\mathrm{op}}$ is right adapted for F^{op} and $\mathcal{C}^{\mathrm{op}}$. In particular F is left $\mathcal{C}$-adaptable if and only if F^{op} is right $\mathcal{C}^{\mathrm{op}}$-adaptable.

Theorem 120. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a triangulated subcategory and $F : \mathcal{D} \longrightarrow \mathcal{T}$ a triangulated functor. If F is left $\mathcal{C}$-adaptable then it has a left derived functor with respect to $\mathcal{C}$.*

Proof. Follows by duality from Theorem 112. □

Remark 79. Following Remark 74 we can describe explicitly how to construct a left derived functor. With the notation of Theorem 120 let $\mathcal{A}$ be left adapted for F and $\mathcal{C}$, and choose for every $X \in \mathcal{D}$ a particular object $A_X \in \mathcal{A}$ and a morphism $\mu_X : A_X \longrightarrow X$ in $Mor_{\mathcal{C}}$. In $\mathcal{D}^{\mathrm{op}}$ this is a morphism $\mu_X : X \longrightarrow A_X$ in $Mor_{\mathcal{C}^{\mathrm{op}}}$ which allows us to define canonically a triangulated functor $A : \mathcal{D}^{\mathrm{op}}/\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{op}}/(\mathcal{C} \cap \mathcal{A})^{\mathrm{op}}$. Using this we construct a right derived functor of F^{op} which by duality yields a left derived functor $(\mathbb{L}F, \zeta)$ of F. For $X \in \mathcal{D}$ we have $\mathbb{L}F(X) = F(A_X)$ and $\zeta_X = F(\mu_X)$. Given a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ suppose we have a diagram in $\mathcal{D}$

$$\begin{tikzcd} & W \arrow[dl, "b"'] \arrow[dr, "a"] & \\ A_X & & A_Y \end{tikzcd}$$

with $W \in \mathcal{A}$ and $b \in Mor_{\mathcal{C}}$, which represents a morphism $\gamma : A_X \longrightarrow A_Y$ of $\mathcal{D}/\mathcal{C}$ making the following diagram commute

$$\begin{tikzcd} A_X \arrow[r, "Q(\mu_X)"] \arrow[d, "\gamma"'] & X \arrow[d, "Q(f)"] \\ A_Y \arrow[r, "Q(\mu_Y)"'] & Y \end{tikzcd}$$

Then $\mathbb{L}FQ(f) = F(a)F(b)^{-1}$. This is enough to calculate $\mathbb{L}F$ on morphisms of $\mathcal{D}/\mathcal{C}$. It remains to give explicitly the natural equivalence $\phi_{\mathbb{L}F} : \mathbb{L}F \circ \Sigma \longrightarrow \Sigma \circ \mathbb{L}F$. For $X \in \mathcal{D}$ we have

$$\phi_{\mathbb{L}F,X} = \Sigma\mathbb{L}FQ(\Sigma^{-1}\mu_{\Sigma X})\Sigma F(\mu_{\Sigma^{-1}A_{\Sigma X}})^{-1}\phi_{F,\Sigma^{-1}A_{\Sigma X}}$$

In the special case where $X \in \mathcal{A}$ this expression simplifies considerably to

$$\phi_{\mathbb{L}F,X} = \Sigma F(\mu_X)^{-1}\Sigma F(\Sigma^{-1}\mu_{\Sigma X})\phi_{F,\Sigma^{-1}A_{\Sigma X}}$$

This discussion defines $(\mathbb{L}F, \zeta)$ using only the chosen resolutions $\mu_X : A_X \longrightarrow X$.

Let us set up some notation for the next result. Suppose we have a diagram of triangulated categories and triangulated functors

$$\begin{tikzcd} \mathcal{C} \arrow[d] & \mathcal{C}' \arrow[d] & \\ \mathcal{D} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D}' \arrow[r, "G"] \arrow[d, "Q'"'] & \mathcal{T} \\ \mathcal{D}/\mathcal{C} \arrow[r, "\mathbb{L}(Q'F)"'] & \mathcal{D}'/\mathcal{C}' \arrow[ur, "\mathbb{L}G"'] & \end{tikzcd} \tag{84}$$

where the columns are verdier quotients of thick triangulated subcategories, and we assume that there exist left derived functors $(\mathbb{L}(Q'F), \zeta), (\mathbb{L}G, \omega)$. Then we have a trinatural transformation

$$\mathbb{L}(G)\mathbb{L}(Q'F)Q \xrightarrow{\mathbb{L}(G)\zeta} \mathbb{L}(G)Q'F \xrightarrow{\omega F} GF$$

which we denote by $\mu : \mathbb{L}(G)\mathbb{L}(Q'F)Q \longrightarrow GF$. If a left derived functor $(\mathbb{L}(GF), \xi)$ exists, then there is a unique trinatural transformation $\theta : \mathbb{L}(G)\mathbb{L}(Q'F) \longrightarrow \mathbb{L}(GF)$ making the following diagram commute

$$\begin{tikzcd} \mathbb{L}(G)Q'F \arrow[r, "\omega F"] & GF \\ \mathbb{L}(G)\mathbb{L}(Q'F)Q \arrow[u, "\mathbb{L}(G)\zeta"] \arrow[r, "\theta Q"'] & \mathbb{L}(GF)Q \arrow[u, "\xi"'] \end{tikzcd}$$

In general the pair consisting of the triangulated functor $\mathbb{L}(G)\mathbb{L}(Q'F)$ and μ is not a left derived functor of GF (that is, θ is not a trinatural equivalence). But it is true under some natural hypothesis. With this notation,

Theorem 121. *Assume that $\mathcal{A} \subseteq \mathcal{D}$ is left adapted for $Q'F$ and $\mathcal{C}$, and that $\mathcal{A}' \subseteq \mathcal{D}'$ is left adapted for G and $\mathcal{C}'$. If F sends objects of $\mathcal{A}$ to objects of $\mathcal{A}'$, then the pair $(\mathbb{L}(G)\mathbb{L}(Q'F), \mu)$ is a left derived functor of GF. That is, the canonical trinatural transformation*

$$\theta : \mathbb{L}(G)\mathbb{L}(Q'F) \longrightarrow \mathbb{L}(GF)$$

is a trinatural equivalence.

Proof. This follows by duality from Theorem 113. □

6.1 Derived Adjunctions

Let us set up some notation that we will use throughout this section. Suppose we have a diagram of triangulated categories and triangulated functors

$$\begin{tikzcd}
\mathcal{C} \arrow[d] & \mathcal{C}' \arrow[d] \\
\mathcal{D} \arrow[r, bend left, "F"] \arrow[d, "Q"'] & \mathcal{D}' \arrow[l, bend left, "G"] \arrow[d, "Q'"] \\
\mathcal{D}/\mathcal{C} \arrow[r, bend left, "\mathbb{R}(Q'F)"] & \mathcal{D}'/\mathcal{C}' \arrow[l, bend left, "\mathbb{L}(QG)"]
\end{tikzcd}$$

where the columns are verdier quotients of thick triangulated subcategories, and we assume that there exists a right derived functor $(\mathbb{R}(Q'F), \zeta)$ and left derived functor $(\mathbb{L}(QG), \omega)$. With this notation,

Theorem 122. *Assume that $Q'F$ is right $\mathcal{C}$-adaptable and that QG is left $\mathcal{C}'$-adaptable. If G is left triadjoint to F, then $\mathbb{L}(QG)$ is left triadjoint to $\mathbb{R}(Q'F)$. Moreover, given a triadjunction $(\eta, \varepsilon) : G \longrightarrow\!\!\!\!\dashv F$ there is a* canonical *triadjunction $\mathbb{L}(QG) \longrightarrow\!\!\!\!\dashv \mathbb{R}(Q'F)$ represented by its unit*

$$\eta^{\diamondsuit} : 1 \longrightarrow \mathbb{R}(Q'F)\mathbb{L}(QG)$$

which is the unique trinatural transformation making the following diagram commute

$$\begin{tikzcd}
Q' \arrow[r, "\eta^{\diamondsuit} Q'"] \arrow[d, "Q'\eta"'] & \mathbb{R}(Q'F)\mathbb{L}(QG)Q' \arrow[d, "\mathbb{R}(Q'F)\omega"] \\
Q'FG \arrow[r, "\zeta G"'] & \mathbb{R}(Q'F)QG
\end{tikzcd} \tag{85}$$

Proof. First we consider the special case where the derived functors are constructed from resolutions. Let $\mathcal{A} \subseteq \mathcal{D}$ be right adapted for $Q'F$ and $\mathcal{C}$, and $\mathcal{A}' \subseteq \mathcal{D}'$ left adapted for QG and $\mathcal{C}'$. Choose for every $Y \in \mathcal{D}$ an object $B_Y \in \mathcal{A}$ and a morphism $\mu_Y : Y \longrightarrow B_Y$ in $Mor_{\mathcal{C}}$. Similarly, for each $X \in \mathcal{D}'$ choose $A_X \in \mathcal{A}'$ and a morphism $\tau_X : A_X \longrightarrow X$ in $Mor_{\mathcal{C}'}$. Let $(\mathbb{R}(Q'F), \zeta)$ and $(\mathbb{L}(QG), \omega)$ be calculated with respect to these choices, as in Remark 73 and Remark 79. Suppose that G is left triadjoint to F, with unit and counit

$$\eta : 1 \longrightarrow FG, \quad \varepsilon : GF \longrightarrow 1 \tag{86}$$

We need to construct a natural bijection commuting with suspension

$$\Lambda : Hom_{\mathcal{D}/\mathcal{C}}(\mathbb{L}(QG)(X), Y) \longrightarrow Hom_{\mathcal{D}'/\mathcal{C}'}(X, \mathbb{R}(Q'F)(Y))$$

for any $X \in \mathcal{D}', Y \in \mathcal{D}$. Our first step is to define a natural bijection

$$\theta : Hom_{\mathcal{D}/\mathcal{C}}(G(X), Y) \longrightarrow Hom_{\mathcal{D}'/\mathcal{C}'}(X, F(Y))$$

in the special case where $X \in \mathcal{A}', Y \in \mathcal{A}$. Let a morphism $\alpha : G(X) \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ be given, and represent it by a diagram in $\mathcal{D}$ of the following form

$$\begin{tikzcd} & W & \\ G(X) \arrow[ur, "b"] & & Y \arrow[ul, "a"'] \end{tikzcd} \tag{87}$$

with $W \in \mathcal{A}$ and $a \in Mor_{\mathcal{C}}$. Under the adjunction of G and F, $b : G(X) \longrightarrow W$ corresponds to a morphism $b' : X \longrightarrow F(W)$, and we define $\theta(\alpha)$ to be the morphism in $\mathcal{D}'/\mathcal{C}'$ represented by the following diagram in $\mathcal{D}'$

$$\begin{tikzcd} & F(W) & \\ X \arrow[ur, "b'"] & & F(Y) \arrow[ul, "F(a)"'] \end{tikzcd}$$

That is, $\theta(\alpha) = Q'F(a)^{-1}Q'(b')$. Observe that since $\mathcal{A}$ is right adapted for $Q'F$, and $a \in Mor_{\mathcal{C}\cap\mathcal{A}}$, the morphism $F(a)$ belongs to $Mor_{\mathcal{C}'}$ (here we use thickness of $\mathcal{C}'$ and Proposition 64). One checks that this definition is independent of the choice of diagram (87) representing α, so θ is well-defined.

To see that θ is injective, suppose that $\theta(\alpha) = 0$. Then $Q'(b') = 0$, so by Lemma 55 there exists a morphism $m : T \longrightarrow X$ in $Mor_{\mathcal{C}'}$ with $b'm = 0$ in $\mathcal{D}'$. We may as well assume $T \in \mathcal{A}'$, in which case $G(m) \in Mor_{\mathcal{C}}$ (by thickness of $\mathcal{C}$ and Proposition 64) and by the adjunction we have $bG(m) = 0$ in $\mathcal{D}$. Applying Lemma 55 once more, we infer that $Q(b) = 0$ in $\mathcal{D}/\mathcal{C}$ and therefore $\alpha = 0$.

To see that θ is surjective, let a morphism $\beta : X \longrightarrow F(Y)$ in $\mathcal{D}'/\mathcal{C}'$ be given, and represent it by a diagram in $\mathcal{D}'$ of the following form

$$\begin{tikzcd} & T \arrow[dl, "r"'] \arrow[dr, "s"] & \\ X & & F(Y) \end{tikzcd}$$

with $T \in \mathcal{A}'$ and $r \in Mor_{\mathcal{C}'}$. In fact $r \in Mor_{\mathcal{C}'\cap\mathcal{A}'}$, from which we deduce that $G(r) \in Mor_{\mathcal{C}}$. Hence if $s' : G(T) \longrightarrow Y$ is adjoint to $s : T \longrightarrow F(Y)$, the following diagram in $\mathcal{D}$ represents a morphism $\alpha : G(X) \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$

$$\begin{tikzcd} & G(T) \arrow[dl, "G(r)"'] \arrow[dr, "s'"] & \\ G(X) & & Y \end{tikzcd} \tag{88}$$

Taking a homotopy pushout and replacing the bottom vertex by an element of $\mathcal{A}$, we have a commutative diagram

$$\begin{tikzcd} & G(T) \arrow[dl, "G(r)"'] \arrow[dr, "s'"] & \\ G(X) \arrow[dr, "b"'] & & Y \arrow[dl, "a"] \\ & Q & \end{tikzcd}$$

with $Q \in \mathcal{A}$ and $a \in Mor_{\mathcal{C}}$. Applying the definition of θ to this diagram, it is clear that $\theta(\alpha) = \beta$, so θ is surjective. As a corollary, observe that every morphism $G(X) \longrightarrow Y$ in $\mathcal{D}/\mathcal{C}$ can be represented by a diagram of the form (88). One checks that the bijection θ is natural in both variables.

Now let $X \in \mathcal{D}', Y \in \mathcal{D}$ be arbitrary. We have a bijection

$$\begin{aligned}\Lambda : Hom_{\mathcal{D}/\mathcal{C}}(\mathbb{L}(QG)(X), Y) &= Hom_{\mathcal{D}/\mathcal{C}}(G(A_X), Y)\\ &\cong Hom_{\mathcal{D}/\mathcal{C}}(G(A_X), B_Y)\\ &\cong Hom_{\mathcal{D}'/\mathcal{C}'}(A_X, F(B_Y))\\ &\cong Hom_{\mathcal{D}'/\mathcal{C}'}(X, F(B_Y))\\ &= Hom_{\mathcal{D}'/\mathcal{C}'}(X, \mathbb{R}(Q'F)(Y))\end{aligned}$$

which is defined on a morphism $\alpha : \mathbb{L}(QG)(X) \longrightarrow Y$ of $\mathcal{D}/\mathcal{C}$ by the formula

$$\Lambda(\alpha) = \theta(Q(\mu_Y)\alpha)Q'(\tau_X)^{-1}$$

To check naturality of Λ with respect to morphisms $X \longrightarrow X'$ of $\mathcal{D}'/\mathcal{C}'$ and $Y \longrightarrow Y'$ of $\mathcal{D}/\mathcal{C}$, it is enough to check naturality for morphisms in $\mathcal{D}'$ and $\mathcal{D}$ respectively. For this, we use the technique of Remark 74 and Remark 79 to expand the value of $\mathbb{R}(Q'F)$ and $\mathbb{L}(QG)$ on morphisms, and then check commutativity of several smaller diagrams.

We have shown so far that Λ is a bijection natural in both variables. That is, we have defined an adjunction $\mathbb{L}(QG) \dashv \mathbb{R}(Q'F)$. Using the explicit formula for Λ one checks that the unit of this adjunction is the following natural transformation

$$\begin{gathered}\eta^{\Diamond} : 1 \longrightarrow \mathbb{R}(Q'F)\mathbb{L}(QG)\\ \eta^{\Diamond}_X = Q'F(\mu_{G(A_X)}) \circ Q'(\eta_{A_X}) \circ Q'(\tau_X)^{-1}\end{gathered}$$

To proceed we must first make some observations. Suppose we are given a second assignment of resolutions $\mu'_Y : Y \longrightarrow B'_Y$ and $\tau'_X : A_X \longrightarrow X$ to the objects of $\mathcal{D}$ and $\mathcal{D}'$ respectively. Let $(\mathbb{R}'(Q'F), \zeta')$ and $(\mathbb{L}'(QG), \omega')$ be the derived functors calculated with these resolutions. We deduce canonical trinatural equivalences

$$\begin{gathered}\pi : \mathbb{R}(Q'F) \longrightarrow \mathbb{R}'(Q'F)\\ \sigma : \mathbb{L}(QG) \longrightarrow \mathbb{L}'(QG)\end{gathered}$$

Run the above proof to obtain a natural bijection Λ'. Since θ doesn't depend on the resolutions, we can use the explicit construction of Λ' from θ to check that the following diagram commutes for $X \in \mathcal{D}', Y \in \mathcal{D}$

$$\begin{tikzcd} Hom_{\mathcal{D}/\mathcal{C}}(\mathbb{L}(QG)(X), Y) \arrow[r, "\Lambda"] & Hom_{\mathcal{D}'/\mathcal{C}'}(X, \mathbb{R}(Q'F)(Y)) \arrow[d, Rightarrow, "\pi_Y \circ -"] \\ Hom_{\mathcal{D}/\mathcal{C}}(\mathbb{L}'(QG)(X), Y) \arrow[r, "\Lambda'"'] \arrow[u, Rightarrow, "- \circ \sigma_X"] & Hom_{\mathcal{D}'/\mathcal{C}'}(X, \mathbb{R}'(Q'F)(Y)) \end{tikzcd} \tag{89}$$

Therefore to check that Λ commutes with suspension in the sense of Theorem 42, we can reduce to the case where $\mu_Y = 1$ and $\tau_X = 1$ for any $X \in \mathcal{A}', Y \in \mathcal{A}$. It is enough to show that $\eta^{\Diamond}$ is trinatural. That is, we have to show that the following diagram commutes for $X \in \mathcal{D}'$

$$\begin{tikzcd} \Sigma X \arrow[r, "\eta^{\Diamond}_{\Sigma X}"] \arrow[dr, "\Sigma\eta^{\Diamond}_X"'] & \mathbb{R}(Q'F)\mathbb{L}(QG)\Sigma X \arrow[d] \\ & \Sigma\mathbb{R}(Q'F)\mathbb{L}(QG)X \end{tikzcd}$$

We can reduce to the case $X \in \mathcal{A}'$, which is straightforward using the explicit formulae of Remark 74 and Remark 79. This proves that Λ is a triadjunction $\mathbb{L}QG \dashv \mathbb{R}Q'F$. One checks that $\eta^{\Diamond}$ is the unique trinatural transformation making (85) commute.

Now we are ready for the general case. Instead of being defined using alternative resolutions, now we let $(\mathbb{R}'(Q'F), \zeta')$ and $(\mathbb{L}'(QG), \omega')$ denote arbitrary derived functors. We have the trinatural equivalences π, σ as above, and the diagram (89) allows us to *define* a bijection Λ', which is clearly a triadjunction $\mathbb{L}'(QG) \longrightarrow\!\!\!\!\dashv \mathbb{R}'(Q'F)$. If

$$\eta^{\diamondsuit'} : 1 \longrightarrow \mathbb{R}'(Q'F)\mathbb{L}'(QG)$$

is the unit of this triadjunction, then using the properties of $\eta^{\diamondsuit}$ above one checks that $\eta^{\diamondsuit'}$ makes the appropriate modification of (85) commute, and is unique with this property. Since the diagram (85) depends only on the triadjunction (η, ε) this also shows that the triadjunction Λ' is canonical (that is, it does not depend on the many choices we made to construct it). $\square$

Remark 80. With the above notation, we have for $X \in \mathcal{D}'$ and $Y \in \mathcal{D}$ two morphisms

$$\zeta_Y : F(Y) \longrightarrow \mathbb{R}(Q'F)(Y)$$
$$\omega_X : \mathbb{L}(QG)(X) \longrightarrow G(X)$$

and the unique property of the unit $\eta^{\diamondsuit}$ means that the following diagram commutes

$$\begin{tikzcd}
Hom_{\mathcal{D}}(G(X), Y) \arrow[rr] \arrow[d] & & Hom_{\mathcal{D}'}(X, F(Y)) \arrow[d] \\
Hom_{\mathcal{D}/\mathcal{C}}(G(X), Y) \arrow[d] & & Hom_{\mathcal{D}'\mathcal{C}'}(X, F(Y)) \arrow[d] \\
Hom_{\mathcal{D}/\mathcal{C}}(\mathbb{L}(QG)(X), Y) \arrow[rr] & & Hom_{\mathcal{D}'\mathcal{C}'}(X, \mathbb{R}(Q'F)(Y))
\end{tikzcd}$$

where the only nonobvious maps are composition with ζ_Y and ω_X. Setting $X = F(Y)$ and chasing the identity $1_{F(Y)}$ around the diagram, we deduce that the counit $\varepsilon^{\diamondsuit}$ of the triadjunction $\mathbb{L}(QG) \longrightarrow\!\!\!\!\dashv \mathbb{R}(Q'F)$ makes the following diagram commute

$$\begin{tikzcd}
\mathbb{L}(QG)Q'F \arrow[r, "\mathbb{L}(QG)\zeta"] \arrow[d, "\omega F"'] & \mathbb{L}(QG)\mathbb{R}(Q'F)Q \arrow[d, "\varepsilon^{\diamondsuit} Q"] \\
QGF \arrow[r, "Q\varepsilon"'] & Q
\end{tikzcd}$$

In fact $\varepsilon^{\diamondsuit} : \mathbb{L}(QG)\mathbb{R}(Q'F) \longrightarrow 1$ is the *unique* trinatural transformation making this diagram commute. If τ is another trinatural transformation making the diagram commute, then τ_Y is adjoint to some morphism $\rho_Y : \mathbb{R}(Q'F)(Y) \longrightarrow \mathbb{R}(Q'F)(Y)$ for every $Y \in \mathcal{D}'$. One checks that ρ defines a trinatural transformation with $\rho Q \circ \zeta = \zeta$, from which it follows that $\rho = 1$ and $\tau = \varepsilon^{\diamondsuit}$.

6.2 Acyclic Objects

Definition 52. Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor and $\mathcal{C}$ a triangulated subcategory of $\mathcal{D}$. An object $X \in \mathcal{D}$ is *left F-acyclic with respect to $\mathcal{C}$* if whenever there is a morphism $s : Y \longrightarrow X$ in $Mor_{\mathcal{C}}$ there is another morphism $t : Z \longrightarrow Y$ in $Mor_{\mathcal{C}}$ such that $F(st)$ is an isomorphism. We simply say that X is *left F-acyclic* if there is no chance of confusion, and denote the full subcategory of $\mathcal{D}$ consisting of such objects by ${}_{F,\mathcal{C}}\mathcal{A}$.

Remark 81. With the above notation it is clear that any zero object is left F-acyclic with respect to $\mathcal{C}$, and that ${}_{F,\mathcal{C}}\mathcal{A}$ is a replete subcategory of $\mathcal{D}$. An object X is left F-acyclic with respect to $\mathcal{C}$ if and only if it is right F^{op}-acyclic with respect to $\mathcal{C}^{\mathrm{op}}$.

Lemma 123. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. An object $X \in \mathcal{D}$ belongs to ${}^{\perp}\mathcal{C}$ if and only if it is left F-acyclic with respect to $\mathcal{C}$ for every triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$.*

Proof. Follows by duality from Lemma 123. $\square$

Proposition 124. *Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor and $\mathcal{C}$ a thick triangulated subcategory of $\mathcal{D}$. Then ${}_{F,\mathcal{C}}\mathcal{A}$ is a triangulated subcategory of $\mathcal{D}$ with the property that every object of $\mathcal{C} \cap {}_{F,\mathcal{C}}\mathcal{A}$ belongs to the kernel of F.*

Proof. Follows by duality from Proposition 115. $\square$

Theorem 125. *Let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor, $\mathcal{C}$ a thick triangulated subcategory of $\mathcal{D}$. Suppose that for every object $X \in \mathcal{D}$ there exists a morphism $\eta_X : A_X \longrightarrow X$ in $Mor_{\mathcal{C}}$ with A_X left F-acyclic. Then F admits a left derived functor $(\mathbb{L}F, \zeta)$ with the following properties*

(i) For any object $X \in \mathcal{D}$ we have $\mathbb{L}F(X) = F(A_X)$ and $\zeta_X = F(\eta_X)$.

(ii) An object $X \in \mathcal{D}$ is left F-acyclic if and only if ζ_X is an isomorphism in $\mathcal{T}$.

Proof. The assumptions mean that ${}_{F,\mathcal{C}}\mathcal{A}$ is left adapted for F and $\mathcal{C}$, so by Theorem 120 and Remark 79 we can construct a left derived functor with the desired properties for any assignment of morphisms $\eta_X : A_X \longrightarrow X$ in $Mor_{\mathcal{C}}$ with A_X left F-acyclic. $\square$

Lemma 126. *In the situation of Theorem 125 suppose that $(\mathbb{L}F, \zeta)$ is a left derived functor of F and $G : \mathcal{T} \longrightarrow \mathcal{S}$ a triangulated functor. Then $(G \circ \mathbb{L}F, G\zeta)$ is a left derived functor of GF.*

Proof. This follows by duality from Lemma 125. $\square$

6.3 Derived Transformations

Definition 53. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory with verdier quotient $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$. Suppose that $F, G : \mathcal{D} \longrightarrow \mathcal{T}$ are triangulated functors with left derived functors $\mathbb{L}F, \mathbb{L}G$ and let $\alpha : F \longrightarrow G$ be a trinatural transformation. Then the composite $\alpha\zeta_F : \mathbb{L}F \circ Q \longrightarrow G$ induces a unique trinatural transformation $\mathbb{L}\alpha : \mathbb{L}F \longrightarrow \mathbb{L}G$ making the following diagram commute

$$\begin{tikzcd} \mathbb{L}F \circ Q \arrow[r, "(\mathbb{L}\alpha)Q"] \arrow[d, "\zeta_F"'] & \mathbb{L}G \circ Q \arrow[d, "\zeta_G"] \\ F \arrow[r, "\alpha"'] & G \end{tikzcd}$$

It is clear that given another trinatural transformation $\beta : G \longrightarrow H$ we have $\mathbb{L}(\beta\alpha) = \mathbb{L}\beta \circ \mathbb{L}\alpha$. Similarly $\mathbb{L}(\alpha + \alpha') = \mathbb{L}(\alpha) + \mathbb{L}(\alpha')$ and $\mathbb{L}1 = 1$.

Let $F, G : \mathcal{D} \longrightarrow \mathcal{T}$ be triangulated functors, $\mathcal{C}$ a thick triangulated subcategory of $\mathcal{D}$ and $\eta_X : A_X \longrightarrow X$ a morphism in $Mor_{\mathcal{C}}$ with A_X left F and G-acyclic for each $X \in \mathcal{D}$. Define the left derived functors $\mathbb{L}F, \mathbb{L}G$ using these resolutions as in Theorem 125. Then given a trinatural transformation $\alpha : F \longrightarrow G$ we can describe the trinatural transformation $\mathbb{L}\alpha$ explicitly.

Lemma 127. *For any trinatural transformation $\alpha : F \longrightarrow G$ and $X \in \mathcal{D}$ we have $(\mathbb{L}\alpha)_X = \alpha_{A_X}$.*

Proof. This follows by duality from Lemma 118. $\square$

7 Portly Considerations

Throughout the previous sections our main object of study has been triangulated categories. Unfortunately it becomes necessary to allow portly triangulated categories in many of these results (see Remark 37 for the definition of a portly triangulated category). To be clear, we have collected here explicitly all the "portly" versions of results we will need. The proofs are all the same.

If $\mathcal{T}$ is a portly triangulated category, it is clear what we mean by a (co)homological functor between $\mathcal{T}$ and an abelian category $\mathcal{A}$. However, the functors $Hom(U, -)$ and $Hom(-, U)$ are *not* homological, because in general their values are not abelian groups. However, it is clear that for any triangle in $\mathcal{T}$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

the following sequences of (large) abelian groups are exact

$$\begin{aligned} Hom(U,X) &\longrightarrow Hom(U,Y) \longrightarrow Hom(U,Z) \\ Hom(Z,U) &\longrightarrow Hom(Y,U) \longrightarrow Hom(X,U) \end{aligned}$$

The notion of a decent homological functor on $\mathcal{T}$ makes sense, but is essentially useless without these central examples. In any case, we really only use the notion of decent homological functors to define pretriangles, and the definition we used in Section 1.1 is stronger than we really needed.

Definition 54. Let $\mathcal{T}$ be a portly pretriangulated category. A candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called a *portly pretriangle* if for every object $U \in \mathcal{T}$ the following sequence of (large) abelian groups is exact

$$\cdots \longrightarrow H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}w)} H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(\Sigma X) \longrightarrow \cdots$$

where $H(-) = Hom(U,-)$. Clearly any triangle is a portly pretriangle, and any direct summand of a portly pretriangle is a portly pretriangle.

If we replace “pretriangle” by “portly pretriangle” and “pretriangulated category” by “portly pretriangulated category” then the statements of Section 1.1 and Section 1.2 all remain true (even the proofs are the same) with the exception of Lemma 1 and Lemma 2 which no longer make sense.

Definition 55. Let $\mathcal{C}$ be a portly category. A *portly subcategory* is a functor $F : \mathcal{A} \longrightarrow \mathcal{C}$ between portly categories which on objects is the inclusion of a subconglomerate, and on morphisms is the inclusion of a subconglomerate $Hom_{\mathcal{A}}(A,B) \subseteq Hom_{\mathcal{C}}(A,B)$ for every pair of objects A, B.

Definition 56. Let $\mathcal{T}$ be a portly triangulated category. A full additive portly subcategory $\mathcal{S}$ in $\mathcal{T}$ is called a *portly triangulated subcategory* if it is replete, $\Sigma\mathcal{S} = \mathcal{S}$, and if for every distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

such that X, Y are in $\mathcal{S}$, the object Z is also in $\mathcal{S}$. As usual there is an induced structure of a portly triangulated category on $\mathcal{S}$ so that the inclusion is a triangulated functor.

With “triangulated category” and “triangulated subcategory” replaced by their portly equivalents, the statements of Section 1.3 hold, with the exception that $Mor_{\mathcal{S}}$ will in general be a conglomerate, not necessarily a class.

We defined in Remark 37 what we mean by a triangulated functor between portly triangulated categories, and by triadjunctions between such functors. One composes such triangulated functors in the same way as before. A *trinatural transformation* is defined as before, with the exception that the trinatural transformations form a “large” abelian group (that is, an abelian group whose underlying conglomerate is not necessarily a set). It is clear what we mean by a *triequivalence* and a *triisomorphism* of portly triangulated categories. Remark 27 and Remark 28 still hold in this context. One defines a *fragile portly triangulated subcategory* of a portly triangulated category in the obvious way, so that a inclusion of a portly triangulated subcategory is a fragile portly triangulated subcategory. We define a *thick portly triangulated subcategory* in the obvious way, and the kernel of a triangulated functor between portly triangulated categories is a thick portly triangulated subcategory. We define *triadjunctions* of triangulated functors between portly triangulated categories in the obvious way, and with this definition all the results of Section 2.1 are true.

If we fix a portly triangulated category $\mathcal{D}$ and a portly triangulated subcategory $\mathcal{C}$ then the construction of the *verdier quotient* $\mathcal{D}/\mathcal{C}$ goes through as before. This is a portly triangulated category and there is a canonical triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$. All the results of Section 2.2 are true in our current context, with the exception of Proposition 69. In particular we have the universal property of Theorem 68. The definition of a *weak verdier quotient* and all the results of Section 2.3 remain valid when we replace triangulated categories by portly triangulated categories (and triangulated subcategories by portly triangulated subcategories).

One checks that the contents of Section 3 work for portly triangulated categories. In particular Corollary 85 is true with $\mathcal{T}$ a portly triangulated category and $\mathcal{S}$ a portly triangulated subcategory. Given a portly triangulated category $\mathcal{T}$ is clearly what we mean if we say a portly triangulated subcategory is *localising* or *colocalising*. Given a portly triangulated category $\mathcal{T}$ and a portly triangulated subcategory $\mathcal{S}$ we define the thick portly triangulated subcategories $\mathcal{S}^{\perp}, {}^{\perp}\mathcal{S}$ of $\mathcal{S}$-*local* and $\mathcal{S}$-*colocal* objects as before. These are respectively colocalising and localising. If we define a *localisation*, a *portly bousfield subcategory* and a *portly cobousfield subcategory* of a portly triangulated category in the obvious way, then the results of Section 4 are still true.

The definitions of Section 5 and Section 6 have obvious translations for portly triangulated categories and portly triangulated subcategories, and all the results remain correct.

References

[ATJLSS00] Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio, *Localization in categories of complexes and unbounded resolutions*, Canad. J. Math. **52** (2000), no. 2, 225–247. MR MR1755776 (2001i:18019)

[BN93] Marcel Bökstedt and Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234. MR MR1214458 (94f:18008)

[Kra05] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162. MR MR2157133 (2006e:18019)

[Nee01] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR MR1812507 (2001k:18010)

[Ver96] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. MR MR1453167 (98c:18007)